Smith-Treumann Theory for Sheaves

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Abstract

In classical algebraic topology for p a prime and X a finite CW complex with a $\mathbb{Z}/p$-action there is a theorem of P. A. Smith which states that if the cohomology $H^i(X, \mathbb{Z}/p) = 0$ for $i > 0$, then the cohomology $H^i(X^{\mathbb{Z}/p}, \mathbb{Z}/p) = 0$ of the fixed points vanishes. This was eventually reformulated with the Borel construction leading to the localization theorem of Borel-Atiyah-Segal-Quillen and more generally has had many applications in algebraic topology. These results are often referred to as Smith theory. In [Tre19], Treumann defines a sheaf theoretic variant of Smith theory appropriately dubbed Smith theory for sheaves. This is accomplished by defining a functor called the Smith operation from the equivariant derived category of sheaves to a particular Verdier quotient. This functor commutes with the six operations leading to similar results as in classical Smith Theory. The goal of this project is to gain an understanding of this operation, its definition, and the various categories involved.

Contents

Introduction

The main topic of this thesis is to gain an understanding of the Smith operation functor defined in Section 4.2 of [Tre19]. The setting for this functor is the following. Let p be a prime, k a field of characteristic p, and X a space with $G = \mathbb{Z}/p$ action.

There is an equivariant derived category of constructible sheaves of k-modules, denoted $D^b_{G,c}(X, k)$. Let $\mathrm{Perf}(X^G, kG)$ denote the subcategory spanned by complexes of sheaves of kG-modules with stalks complexes of finitely generated free kG-modules. We may take the Verdier quotient of $D^b(X^G, kG)$ by $\mathrm{Perf}(X^G, kG)$ which we denote by $\mathrm{Shv}(X^G, k^{tG})$. The Smith operation is then a functor

$$\mathbf{Psm} : D^b_{G,c}(X, k) \to \mathrm{Shv}(X^G, k^{tG}).$$

Appropriate versions of the six operations for derived categories of sheaves descend to the category $\mathrm{Shv}(X^G, k^{tG})$.

In classical algebraic topology Quillen [Qui71] extends the original result of Smith [Smi34] relating the cohomology of a finite dimensional space with $\mathbb{Z}/p$ coefficients to the cohomology of $X^{\mathbb{Z}/p}$ with $\mathbb{Z}/p$ coefficients. Furthermore, in [Bre73, CS72] it is shown that if X satisfies Poincare duality with $\mathbb{Z}/p$-coefficients, then $X^{\mathbb{Z}/p}$ satisfied Poincare duality with $\mathbb{Z}/p$-coefficients. The fact that the Smith operation commutes with the six operations recovers these results.

The category $\mathrm{Shv}(Y, k^{tG})$ for which the Smith operation is defined is interesting in that for $X = *$ a point this becomes (the finitely generated part of) the stable module category of the group ring kG and is in fact also equivalent to the category of compact objects over a particular $\mathbb{E}_\infty$-ring spectrum k^{tG} called the Tate spectrum. This spectrum is constructed as the cofiber of a map $k_{hG} \to k^{hG}$ from the homotopy orbits to the homotopy fixed points of the Eilenberg-Maclane spectrum for the field k.

We begin by discussing the construction of the stable module category of a general Frobenius ring R via a model category structure on $\mathbf{Mod}(R)$, the category of modules over R. This construction can also be found in [Hov07]. We then discuss the triangulated structure on the homotopy category as well as several other properties. We then turn to the construction of this category as a particular Verdier quotient of the bounded derived category of finitely generated modules by perfect complexes and conclude by discussing the construction of the $\mathbb{E}_\infty$-ring spectrum k^{tG}. In the case $G = \mathbb{Z}/p$ and k a field of characteristic p all these constructions coincide.

After the stable module category we turn towards the definition of equivariant sheaves with an eye towards finite groups and the equivariant derived category. For a G-space with an action by a finite group we construct a site giving an appropriate definition of equivariant sheaf. We prove several properties of the corresponding Grothendieck topos. Using this site we define equivariant sheaves on a space for a finite group in the ∞-categorical context in the hopes of defining the Smith operation with ∞-categories. While we were unsuccessful in defining the Smith operation in this context include some discussion of sheaves valued in compactly generated ∞-categories as well as the relation between the derived ∞-category of sheaves and sheaves valued in the derived ∞-category. As the definition of our site does not work well with the six operations we consider the classical construction of the equivariant derived category and the corresponding six operations following [BL06], providing the necessary definitions for the Smith operation.

In the final section we cover the definition and properties of the Smith operation as defined in [Tre19, §4]. We conclude by discussing some statements we had hoped to be able to prove or disprove in the course of this project.

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Notation and Conventions

Here a list of notation and conventions used:

- We denote a 1-category using bold face $\mathbf{C}$ letters while we denote ∞-categories with calligraphic $\mathcal{C}$ letters.
- If $\mathbf{C}$ is a 1-category, then $N(\mathbf{C})$ denotes its nerve.
- We let $\mathcal{S}$ denote the ∞-category of spaces.
- Given a category $\mathbf{C}$ we let $\mathrm{PSh}(\mathbf{C})$ denote the 1-category $\mathrm{Fun}(\mathbf{C}^{op}, \mathbf{Set})$ of presheaves of sets on $\mathbf{C}$. If $\mathcal{C}$ is an ∞-category, then $\mathrm{PSh}(\mathcal{C})$ instead denotes the ∞-category $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ of presheaves of spaces. Although this notation is conflicting it should generally be clear from context which is being used.
- Similar to the above we let $\mathrm{PSh}(\mathbf{C}, \mathbf{D})$ denote the functor category $\mathrm{Fun}(\mathbf{C}^{op}, \mathbf{D})$ of $\mathbf{D}$-valued presheaves and $\mathrm{PSh}(\mathcal{C}, \mathcal{D})$ the functor ∞-category $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{D})$ of $\mathcal{D}$-valued presheaves.
- Let R be a ring, then $\mathbf{mod}(R)$ denotes the category of *finitely generated* (left) modules while $\mathbf{Mod}(R)$ denotes the category of *all* (left) modules. If R is a ring spectrum, then we denote the ∞-category of (left) modules by $Mod(R)$.
- We let **Top** denote the category of topological spaces and **GTop** the category of spaces with G-action.
- If $\mathbf{C}$ is a model category we let $h\mathbf{C}$ denote the homotopy category. Similarly, if $\mathcal{C}$ is an ∞-category we let $h\mathcal{C}$ denote the corresponding homotopy category.

1 The Stable Module Category

1.1 The Stable Module Category

We begin by constructing a model structure on the category of modules over a Frobenius ring. This appears in [Hov07, Section. 2.2]. However, our notation and terminology is that of [MP11] and [Rie14] explanations of which can be found in Appendix A.

Definition 1.1.1. Let R be a ring, then we say that R is a *Frobenius ring* if a (left) R-module M is projective if and only if it is injective.

Proposition 1.1.2. *If G is a finite group and k a field, then the group algebra kG is a Frobenius ring.*

Given a Frobenius ring R we construct a cofibrantly generated model structure on $\mathbf{Mod}(R)$ whose homotopy category is the *stable module category.* We let $\underline{\mathbf{Mod}}(R)$ denote this category with its model structure and $\mathbf{StMod}(R)$ the homotopy category. For $R = kG$ with G a p-group and k a field of characteristic p this will coincide with the category of modules over the Tate spectrum k^{tG}.

Definition 1.1.3. Let R be a ring and $f, g : M \to N$ maps of R-modules. We say that f is *stably equivalent* to g, denoted $f \sim g$, if $f - g$ factors through a projective R-module, that is, there is a projective R-module P and morphisms $p_1 : M \to P$ and $p_2 : P \to N$ making the diagram commute.

$$\begin{tikzcd} M \arrow[rr, "f-g"] \arrow[dr, "p_1"'] & & N \\ & P \arrow[ur, "p_2"'] & \end{tikzcd}$$

We will say that $f : M \to N$ is a *stable equivalence* if there exists $g : N \to M$ such that $g \circ f \sim \mathbf{1}_M$ and $f \circ g \sim \mathbf{1}_N$.

Lemma 1.1.4. *Stable equivalence is an equivalence relation on* $\mathrm{Hom}_R(M, N)$ *compatible with composition in the sense that if $f, g : M \to N$ are stably equivalent and $h : N \to L$, $k : K \to M$ are morphisms, then $h \circ f \sim h \circ g$ and $f \circ k \sim g \circ k$.*

Proof. We begin by showing that stable equivalence is preserved under compositions. Let $f, g : M \to N$ be such that $f \sim g$ and $h : N \to L$. Since $f \sim g$, then we have a commuting diagram

$$\begin{tikzcd} M \arrow[rr, "f-g"] \arrow[dr, "p_1"'] & & N \\ & P \arrow[ur, "p_2"'] & \end{tikzcd}$$

with P projective. Now we claim that $h \circ f \sim h \circ g$ which is clear since $h \circ (f - g) = h \circ f - h \circ g$ as then the diagram

$$\begin{tikzcd} M \arrow[rr, "h\circ f-h\circ g"] \arrow[dr, "p_1"'] & & L \\ & P \arrow[ur, "h\circ p_2"'] & \end{tikzcd}$$

commutes. The argument for $f \circ k \sim g \circ k$ is similar.

Now it is clear that stable equivalence is reflexive as the zero module is projective and $f - f = 0$. For $f \sim g$ symmetry follows from the diagram

$$\begin{tikzcd} M \arrow[rr, "g-f"] \arrow[dr, "p_1"'] & & N. \\ & P \arrow[ur, "-p_2"'] & \end{tikzcd}$$

For transitivity let $f, g, h \in \mathrm{Hom}_R(M, N)$ with factorizations,

$$\begin{tikzcd} M \arrow[rr, "f-g"] \arrow[dr, "p_1"'] & & N \\ & P \arrow[ur, "p_2"'] & \end{tikzcd} \qquad \begin{tikzcd} M \arrow[rr, "g-h"] \arrow[dr, "p_1'"'] & & N \\ & P' \arrow[ur, "p_2'"'] & \end{tikzcd}$$

then we obtain a factorization of $f - h$

$$\begin{tikzcd} M \arrow[rr, "f-h=(f-g)+(g-h)"] \arrow[dr, "p_1 \oplus p_1'"'] & & L \\ & P \oplus P' \arrow[ur, "p_2 \oplus p_2'"'] & \end{tikzcd}$$

where $P \oplus P'$ is projective since direct sums of projectives are projective. $\square$

Remark 1.1.5. We make the following two observations:

- Every isomorphism is a stable equivalence.
- If P is a projective, then for any R-module M the inclusion $i_M : M \hookrightarrow M \oplus P$ is a stable equivalence with stable inverse $p_M : M \oplus P \to M$. This is clear as $p_M \circ i_M = \mathbf{1}_M$ and $i_M \circ p_M$ factors through P.

Definition 1.1.6. Let R be a ring. Define the *stable module category* for R to be the category with objects (left) R-modules and morphisms stable equivalence classes of R-module maps. Denote this category by $\mathbf{StMod}(R)$. Further, let $\mathbf{stmod}(R)$ denote the full subcategory of $\mathbf{StMod}(R)$ spanned by finitely generated modules.

Note that the isomorphisms in $\mathbf{StMod}(R)$ are the stable equivalences. We now show that $\mathbf{StMod}(R)$ arises as the homotopy category of a cofibrantly generated model structure on $\mathbf{Mod}(R)$.

Theorem 1.1.7. *There is a cofibrantly generated model structure* $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ *on* $\mathbf{Mod}(R)$ *such that*

- *(Weak Equivalences)* $\mathcal{W}$ *is the collection of stable equivalences,*
- *(Cofibrations)* $\mathcal{C}$ *is the collection of injections,*
- *(Fibrations)* $\mathcal{F}$ *is the collection of surjections.*

By the definition of model category (A.0.6) in order to show that $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ makes $\mathbf{Mod}(R)$ into a cofibrantly generated model category we need to show that $\mathcal{W}$ satisfies the 2-of-3 property and that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ form weak factorization systems. However, by the small object argument (A.0.8) it is sufficient to show that there are small sets I and J of generating cofibrations and generating trivial fibrations such that

(1) $\mathcal{F} = J^{\boxslash}$,

(2) $\mathcal{F} \cap \mathcal{W} = I^{\boxslash}$,

(3) $\mathcal{C} = {}^{\boxslash}(I^{\boxslash})$, and

(4) $\mathcal{C} \cap \mathcal{W} = {}^{\boxslash}(J^{\boxslash})$.

where $A^{\boxslash}$ denotes the collection of morphisms with the right lifting property against every element of some collection A and ${}^{\boxslash}A$ denotes the collection of morphisms with the left lifting property against every element of some collection A. Hence, we make the following definition.

Definition 1.1.8. Let I denote the set of inclusions $\mathfrak{a} \hookrightarrow R$ with $\mathfrak{a}$ a left ideal in R. Let J be the set containing the inclusion $0 \hookrightarrow R$. We call I the *generating cofibrations* and J the *generating trivial fibrations*.

We now show that

(C1) $\mathcal{W}$ satisfies the 2-of-3 property,

(C2) $J^{\boxslash} = \mathcal{F}$: $f \in J^{\boxslash}$ if and only if f is a surjection,

(C3) $\mathcal{F} \cap \mathcal{W} = I^{\boxslash}$: f is a surjection and stable equivalence if and only if $f \in I^{\boxslash}$,

(C4) ${}^{\boxslash}(I^{\boxslash}) = \mathcal{C}$: $i \in {}^{\boxslash}(I^{\boxslash})$ if and only if i is an injection, (C5) ${}^{\boxslash}(J^{\boxslash}) = \mathcal{C} \cap \mathcal{W}$: i is an injection and a stable equivalence if and only if $i \in {}^{\boxslash}(J^{\boxslash})$

so that by the small object argument (A.0.8) $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ defines a cofibrantly generated model structure on $\mathbf{Mod}(R)$.

Proposition 1.1.9 (Condition (C1))**.** *The stable equivalences satisfy the 2-of-3 property.*

Proof. This follows directly from the compatibility of stable equivalence with composition.

$\square$

Corollary 1.1.10. *If i is an injection with projective cokernel, then i is a stable equivalence. Dually, if f is a surjection with injective kernel, then f is a stable equivalence.*

Proof. For, the first statement we obtain a split short exact sequence

$$\begin{tikzcd}
 & & B \arrow[d, "p"] \arrow[dr] & & \\
0 \arrow[r] & A \arrow[ur, "i"] \arrow[r, "i_A"'] & \operatorname{coker} i \oplus A \arrow[r] & \operatorname{coker} i \arrow[r] & 0
\end{tikzcd}$$

where p is an isomorphism so a stable equivalence and i_A is a stable equivalence. Hence, by 2-of-3 for stable equivalences i is a stable equivalence. The second statement holds similarly. $\square$

We now turn to characterizing the fibrations and trivial fibrations as $J^{\boxslash}$ and $I^{\boxslash}$.

Proposition 1.1.11 (Condition (C2))**.** *If $f : M \to N$ is a map of R-modules, then $f \in J^{\boxslash}$ if and only if f is surjective so $J^{\boxslash} = \mathcal{F}$.*

Proof. First, if $f : M \to N$ is surjective, then we obtain a lift in the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & M \arrow[d, two heads, "f"] \\
R \arrow[r, "v"'] \arrow[ur, dashed, "\exists \tilde{f}"] & N
\end{tikzcd}$$

since R is a projective R-module.

Suppose $f : M \to N$ is a fibration so that we have a lift $\tilde{f} : R \to M$ as above for any morphism $v : R \to N$. In particular, if $y \in N$, then by the universal property of free modules we have a map $v_y : R \to N$ with $v(1) = y$ and $f(\tilde{f}(1)) = y = v_y(1)$ so f is surjective. $\square$

Lemma 1.1.12. *A map $f : M \to N$ of R-modules is in $J^{\boxslash} \cap \mathcal{W}$ if and only if f is a surjection with projective kernel.*

Proof. First, suppose f is a surjection with projective kernel, then $f \in J^{\boxslash}$ by Proposition 1.1.11. Further, since R is Frobenius, then $\ker f$ is injective so the short exact sequence

$$0 \to \ker f \hookrightarrow M \twoheadrightarrow N \to 0$$

splits. Let $h : M \to \ker f \oplus N$ be the splitting isomorphism so that $f = p_N \circ h$ where $p_N : \ker f \oplus N \to N$ is the projection. Then since $\ker f$ is projective p_N is a stable equivalence and h is an isomorphism so a stable equivalence. Therefore, by the 2-of-3 property for stable equivalences f is a stable equivalence.

Now suppose $f \in J^{\boxslash} \cap \mathcal{W}$, then by Proposition 1.1.11 f is a surjection so it only remains to show that f has projective kernel. Let $g : N \to M$ be a stable inverse for f so we have a factorization

$$\begin{tikzcd} N \arrow[rr, "fg-\mathbf{1}_N"] \arrow[dr, "u"'] & & N. \\ & P \arrow[ur, "v"'] & \end{tikzcd}$$

Consider the diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \ker f \arrow[r] \arrow[d, hook, "h"] & M \arrow[r, "f"] \arrow[d, hook, "i_M"] & N \arrow[r] \arrow[d, "\mathbf{1}_N"] & 0 \\ 0 \arrow[r] & K \arrow[r, "j_K"] & M \oplus P \arrow[r, "{(f,v)}"'] & N \arrow[r] & 0 \end{tikzcd}$$

where $i_M : M \to M \oplus P$ is the inclusion of M and $K = \ker(f, v)$. Now $(g, -u) : N \to M \oplus P$ is a section of (f, v) as $fg - vu = \mathbf{1}_N$. Hence, we obtain a splitting of the lower sequence. Further, since i_M is a stable equivalence and

f is a stable equivalence by assumption, then (f, v) is a stable equivalence by the 2-of-3 property. Thus, the inclusion $j_K : K \hookrightarrow M \oplus P$ is stably trivial in the sense that j_K is stably equivalent to the zero morphism and so factors through a projective. By the splitting of the sequence we obtain a retraction $r : M \oplus P \to K$ from which is follows that $\mathbf{1}_K$ factors through a projective. Therefore, K is projective as it is a retract of a projective. By the Snake lemma we get that

$$K/\ker f \cong \operatorname{coker} h \cong \operatorname{coker} i_M \cong P$$

so that $K/\ker f$ is projective. Hence, the short exact sequence

$$0 \to \ker f \to K \to K/\ker f \to 0$$

splits so $\ker f$ is a retract of the projective K and retracts of projectives are projective. □

We now recall Baer's criterion for injective modules which we use in the next result to characterize surjections with injective kernel.

Lemma 1.1.13 (Baer's Criterion)**.** *If Q is an R-module, then Q is injective if and only if for any left ideal $\mathfrak{a} \to R$ and any morphism $\mathfrak{a} \to Q$ there is an extension*

$$\begin{tikzcd} \mathfrak{a} \arrow[r] \arrow[d, hook] & Q. \\ R \arrow[ur, dashed] & \end{tikzcd}$$

to all of R.

Proposition 1.1.14. *A map $f : M \to N$ is in $I^{\boxslash}$ if an only if f is a surjection with injective kernel.*

Proof. First, if $f \in I^{\boxslash}$, then $f \in J^{\boxslash} = \mathfrak{F}$ as $J \subseteq I$ so f is surjective by Proposition 1.1.11. Hence, it remains to show that $\ker f$ is injective for which we will apply Baer's criterion (Lemma 1.1.13). Let $\mathfrak{a} \subseteq R$ be a left ideal of R, $g : \mathfrak{a} \to \ker f$ a map, and $j : \ker f \hookrightarrow M$ the inclusion. Then in the diagram below there is a lift

$$\begin{tikzcd} \mathfrak{a} \arrow[r, "j \circ g"] \arrow[d, hook] & M \arrow[d, "f"] \\ R \arrow[r, "0"] \arrow[ur, dashed] & N \end{tikzcd}$$

since $f \in I^{\boxslash}$ where such a lift is an extension of g to all of $\ker f$ so that $\ker f$ is injective by Baer's criterion.

Conversely, suppose f is a surjection with injective kernel and there is a commutative square

$$\begin{tikzcd} \mathfrak{a} \arrow[r, "h"] \arrow[d, hook, "i"'] & M \arrow[d, "f"] \\ R \arrow[r, "g"] & N. \end{tikzcd} \tag{1.1.1}$$

Since $\ker f$ is injective, then the short exact sequence

$$0 \to \ker f \xrightarrow{j} M \xrightarrow{f} N \to 0$$

splits so f has a section $s : N \to M$. Consider the map

$$sgi - h : \mathfrak{a} \to M$$

whose image is contained in $\ker f$ as $f(sgi - h) = gi - fh = 0$ by commutativity of the square. Thus, $sgi - h$ restricts to a map $\varphi : \mathfrak{a} \to \ker f$

$$j \circ \varphi = sgi - h : \mathfrak{a} \to M.$$

By injectivity of $\ker f$, then Baer's criterion says we obtain an extension of φ to R

$$\begin{tikzcd} \mathfrak{a} \arrow[r, "\varphi"] \arrow[d, hook, "i"'] & \ker f. \\ R \arrow[ur, dashed, "\tilde{\varphi}"'] & \end{tikzcd}$$

Now we claim that $sg - j\tilde{\varphi} : R \to M$ gives a lift in (1.1.1). Indeed we have

$$sgi - j\underbrace{\tilde{\varphi}i}_{\varphi} = sgi - \underbrace{j\varphi}_{sgi-h} = h$$

and

$$\underbrace{fs}_{\mathbf{1}_N}g - fj\tilde{\varphi} = g - \underbrace{fj}_{0}\tilde{\varphi} = g$$

and we are done. $\square$

Corollary 1.1.15 (Condition (C3))**.** *If $f : M \to N$ is a map of R-modules, then f is a surjection and stable equivalence if and only if $f \in I^{\boxslash}$.*

Proof. By Lemma 1.1.11 $J^{\boxslash} = \mathcal{F}$. Further, since R is a Frobenius ring a module is projective if and only if it is injective so applying Proposition 1.1.14 and Lemma 1.1.12 we get that

$$f \in \mathcal{F} \cap \mathcal{W} \iff f \in I^{\boxslash}$$

as desired. ◻

Proposition 1.1.16 (Condition (C4))**.** *If $i : M \to N$ is a map of R-modules, then i is an injection if and only if $i \in {}^{\boxslash}(I^{\boxslash})$.*

Proof. First, by Proposition 1.1.14 we have $i \in {}^{\boxslash}(I^{\boxslash})$ if and only if i has the left lifting property with respect to all surjections with injective kernel. Hence, to show that $i \in {}^{\boxslash}(I^{\boxslash})$ if and only if i is injective may be shown identically to the second half of the proof of Proposition 1.1.14.

Now suppose $i : A \to B$ is in ${}^{\boxslash}(I^{\boxslash})$ so i has the left lifting property with respect to all surjections with injective kernel. Let $j : A \hookrightarrow Q$ be an embedding of A into an injective Q. Then we get a commutative diagram

$$\begin{tikzcd}
A \arrow[r, hook, "j"] \arrow[d, "i"'] & Q \arrow[d] \\
B \arrow[r] \arrow[ur, dashed] & 0
\end{tikzcd}$$

since i has the left lifting property with respect to $Q \to 0$. In particular, it follows that i is injective since j is injective. ◻

Proposition 1.1.17. *If $i : A \to B$ is a map of R-modules, then $i \in {}^{\boxslash}(J^{\boxslash})$ if and only if i is an injection with projective cokernel.*

Proof. This is precisely dual to the proof of Proposition 1.1.14. ◻

Lemma 1.1.18. *If $i : A \to B$ is a map of R-modules, then i is an injection and stable equivalence if and only if i is an injection with projective cokernel.*

Proof. This follows in a similar manner to Lemma 1.1.12. ◻

Corollary 1.1.19 (Condition (C5))**.** *If $i : M \to N$ is a map of R-modules, then i is an injection and stable equivalence if and only if $i \in {}^{\boxslash}(J^{\boxslash})$*

Proof. Apply Proposition 1.1.17 and Lemma 1.1.18. ◻

Having verified conditions (C1), (C2), (C3), (C4), (C5) it immediately follows that Theorem 1.1.7 holds so we obtain a model structure on $\mathbf{Mod}(R)$. The fact that the homotopy category is equivalent to the stable module category follows from the fact that every object is both fibrant and cofibrant.

Remark 1.1.20. The category $\underline{\mathbf{Mod}(R)}$ is a stable model category. See [HPS97, 9.6] and [SS03, 2.4(v)].

As every object in $\underline{\mathbf{Mod}(R)}$ is both fibrant and cofibrant following [Lur, 1.3.4.15] we make the following definition.

Definition 1.1.21. The stable module ∞-category is the ∞-category

$$StMod(R) := N(\underline{\mathbf{Mod}(R)})[W^{-1}].$$

Let *stmod*(R) denote the full subcategory spanned by finitely generated modules.

The ∞-category $StMod(R)$ is a stable ∞-category as $\underline{\mathbf{Mod}(R)}$ is a stable model category. In fact more is true. It is a symmetric monoidal model category so the corresponding stable ∞-category is symmetric monoidal.

1.2 The Triangulated Structure on $\mathbf{StMod}(R)$

We turn to the triangulated structure on the stable module category $\mathbf{StMod}(R)$.

Define a functor $\Omega : \mathbf{StMod}(R) \to \mathbf{StMod}(R)$ in the following way. Let $M \in \mathbf{StMod}(R)$ be a module, then there is an epimorphism $\alpha : P \to M$ for some projective module P. Set

$$\Omega(M) := \ker(\alpha)$$

This is well-defined in $\mathbf{StMod}(R)$ as by Schanuel's Lemma [Zim14, Lemma 1.8.12] given epimorphisms $P_1 \xrightarrow{p_1} M$ and $P_2 \xrightarrow{p_2} M$ with P_1 and P_2 projective, then

$$\ker(p_1) \oplus P_1 \cong \ker(p_2) \oplus P_2$$

so $\ker(p_1) \cong \ker(p_2)$ in the stable module category. To see that Ω defines a functor we must also specify how it behaves on morphisms. Let $f : M \to N$ be a map of R-modules and define $\Omega(f) : \Omega(M) \to \Omega(N)$ by the following construction. Let $p_M : P_M \twoheadrightarrow M$ and $p_N : P_N \twoheadrightarrow N$ be epimorphisms representing $\Omega(M)$ and $\Omega(N)$. Then there is a map $g : P_M \to P_N$ obtained as a lift

$$\begin{tikzcd}
 & P_N \arrow[d, two heads, "p_N"] \\
P_M \arrow[ur, dashed, "g"] \arrow[r, "f\circ p_M"'] & N
\end{tikzcd}$$

since P_M is projective. Thus, we obtain a diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \Omega(M) \arrow[r] \arrow[d, "\Omega(f)"] & P_M \arrow[r, two heads, "p_M"] \arrow[d, "g"] & M \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \Omega(N) \arrow[r] & P_N \arrow[r, two heads, "p_N"] & N \arrow[r] & 0
\end{tikzcd}$$

We claim that $\Omega(f) := g_{|\Omega(M)}$ is well-defined. Let $g, g' : P_M \to P_N$ provide lifts of $p_M \circ f$ against p_N, then $g - g'$ provides a lift

$$\begin{tikzcd}
 & P_N \arrow[d, two heads, "p_N"] \\
P_M \arrow[ur, dashed, "g-g'"] \arrow[r, "0\circ p_M"'] & N
\end{tikzcd}$$

through the zero map in $\mathrm{Hom}(M, N)$. Hence, since $p_N \circ (g - g') = 0$ it follows that $g - g'$ factors through $\Omega(N) = \ker(p_N)$

$$\begin{tikzcd}
P_M \arrow[rr, "g-g'"] \arrow[dr, "h"'] & & P_N \\
 & \ker(p_N) \arrow[ur, "h'"'] &
\end{tikzcd}$$

so $g_{|\Omega(M)} - g'_{|\Omega(M)}$ factors as

$$\begin{tikzcd}
\Omega(M) \arrow[rr, "g_{|\Omega(M)}-g'_{|\Omega(M)}"] \arrow[dr] & & P_N \\
 & P_M \arrow[ur] &
\end{tikzcd}$$

where P_M is projective so $g_{|\Omega(M)} \sim g'_{|\Omega(M)}$ in $\mathbf{StMod}(R)$ and $\Omega(f)$ is well-defined.

Remark 1.2.1. If the ring R is Noetherian, then Ω restricts to a functor on $\mathbf{stmod}(R)$ as then we may choose a map $P_M \to M$ with P_M finitely generated projective and since R is Noetherian, then the kernel is finitely generated.

We define $\Sigma : \mathbf{stmod}(R) \to \mathbf{stmod}(R)$ on the finitely generated part. Let M be a finitely generated R-module and $\mathrm{Hom}_k(M, k)$ the k-dual of M. This fits into an exact sequence

$$0 \to \ker(p) \to P \xrightarrow{p} \mathrm{Hom}_k(M, k) \to 0$$

with P projective and p an epimorphism. Now the functor $\mathrm{Hom}_k(-, k)$ is exact so we obtain

$$0 \to \mathrm{Hom}_k(\mathrm{Hom}_k(M, k)) \to \mathrm{Hom}_k(P, k) \to \mathrm{Hom}_k(\ker(p), k) \to 0$$

where $\mathrm{Hom}_k(P, k)$ is injective and so projective. Furthermore, we have an isomorphism

$$\mathrm{Hom}_k(\mathrm{Hom}_k(M, k), k) \cong M$$

induced by the evaluation map as k is a field and M is finitely generated. Thus, the above sequence reduces to an exact sequence

$$0 \to M \to \mathrm{Hom}_k(P, k) \to \mathrm{Hom}_k(\ker(p), k) \to 0 \tag{1.2.1}$$

with $\mathrm{Hom}_k(P, k)$ projective. We define Σ by setting

$$\Sigma(M) := \mathrm{Hom}_k(\ker(p), k).$$

This defines a functor for the same reasons as for Ω. Furthermore, we have

$$\Omega \circ \Sigma = \mathbf{1}_{\mathbf{stmod}(R)} = \Sigma \circ \Omega.$$

This follows by construction from the exact sequence 1.2.1. We note that the fact that Σ extends to a self-equivalence on $\mathbf{StMod}(R)$ follows from the fact that $\mathbf{StMod}(R) \simeq \mathrm{Ind}(\mathbf{stmod}(R))$.

Remark 1.2.2. An alternative construction of Σ may be done in the following way. Since R is a Frobenius ring, then every projective module is injective. Let $M \overset{i}{\hookrightarrow} Q$ be an injective hull of M, that is, an embedding of M into an injective module M. Then set $\Sigma(M) := \mathrm{coker}(i)$. Then Σ becomes a well-defined functor by similar arguments as for Ω.

Now we have obtained a translation functor so to define the triangulated structure on $\mathbf{StMod}(R)$ we must specify the distinguished triangles. These are defined as coming from short exact sequences

$$0 \to M \to N \to L \to 0.$$

More precisely, given such a short exact sequence define a map $\Omega(L) \to M$ which by applying Σ gives a map $L \to \Sigma(M)$ and then the distinguished triangles are defined to be any sequence

$$M \to N \to L \to \Sigma(M)$$

coming from a short exact exact sequence

$$0 \to M \to N \to L \to 0.$$

We define the map $\Omega(L) \to M$ which is done by contemplating the diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, hook] & N \arrow[r, two heads, "f"] & L \arrow[r] & 0 \\ 0 \arrow[r] & \Omega(L) \arrow[r, hook] \arrow[u, dashed] & P_L \arrow[r, two heads, "p_L"] \arrow[u, dashed] & L \arrow[r] \arrow[u, "\mathbf{1}"] & 0. \end{tikzcd}$$

Here $P_L \to L$ is a surjective map with P_N projective defining $\Omega(L)$. A lift $P_L \to N$ exists making the diagram commute as P_N is projective and $N \twoheadrightarrow L$ is surjective. Thus, we obtain a map $\Omega(L) \to M$. To see that this map is well-defined in $\mathbf{StMod}(R)$ suppose we have two lifts $g, g' : P_L \to N$ of $f : N \to L$. Then $g - g'$ provides a lift of $0 \in \mathrm{Hom}(P_L, L)$. Thus, $f \circ (g - g') = 0$ and so $g - g'$ factors as

$$\begin{tikzcd} P_L \arrow[rr, "g-g'"] \arrow[dr, "h"'] & & N. \\ & M \arrow[ur, "h'"'] & \end{tikzcd}$$

as $M = \ker(f)$. It follows that the induced map $(g - g')_{|\Omega(L)} : \Omega(L) \to M$ factors through P_L which is projective and so $g_{|\Omega(L)} \sim g'_{|\Omega(L)}$ in $\mathbf{StMod}(R)$.

Remark 1.2.3. In [HD88, Chp. 2] the distinguished triangles

$$M \to N \to L \to \Sigma(M)$$

arise via diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xhookrightarrow{i} & Q & \longrightarrow & \Sigma(M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & L & \longrightarrow & \Sigma(M) & \longrightarrow & 0 \end{array}$$

where i exhibits Q as an injective hull of M, the rows are exact, and the left hand square is a pushout. We will take advantage of this in the next section.

1.3 Alternative Constructions of the Stable Module Category

We given an alternative construction of the stable module category as a Verdier quotient of the (bounded) derived ∞-category of the Frobenius ring R by the perfect complexes.

Let k be a field and R be a finite dimensional k-algebra which is Frobenius. Let $\mathcal{D}(\mathbf{Mod}(R))$ denote the (unbounded) derived ∞-category of all R-modules. We let $\mathcal{D}^b(\mathbf{Mod}(R))$ denote the full subcategory of $\mathcal{D}(\mathbf{Mod}(R))$ spanned by bounded complex which is a stable subcategory.

Definition 1.3.1. Let R be a ring. A complex $M \in \mathcal{D}(\mathbf{Mod}(R))$ is *perfect* if it is a compact object in $\mathcal{D}(\mathbf{Mod}(R))$. Equivalently, a complex M is perfect if it is quasi-isomorphic to a bounded complex of finitely generated projective modules. Denote by $\mathrm{Perf}(R)$ the full subcategory $\mathcal{D}(\mathbf{Mod}(R))$ spanned by perfect objects.

Lemma 1.3.2. *The inclusion* $i : \mathrm{Perf}(R) \subseteq \mathcal{D}^b(\mathbf{mod}(R))$ *exhibits* $\mathrm{Perf}(R)$ *as a stable subcategory of* $\mathrm{Perf}(R)$*.*

Proof. By [Lur, Lemma. 1.1.3.3] a full subcategory of a stable ∞-category is a stable subcategory if it is closed under cofibers and translations. Hence, it is clear that $\mathrm{Perf}(R)$ is a full subcategory as cofibers are given by mapping cones and translations by shifting. $\square$

Theorem 1.3.3 ([Ric89, Thm. 2.1])**.** *There is an equivalence of stable ∞-categories between the Verdier quotient* $\mathcal{D}^b(\mathbf{mod}(R))/\mathrm{Perf}(R)$ *and stmod*(R) *the finitely generated part of the stable module ∞-category.*

Proof. By Proposition B.0.3 it is sufficient to give a functor of stable ∞-categories and check it induces an equivalence on the triangulated homotopy categories. Furthermore, by Proposition B.0.2 the homotopy category of the Verdier quotient $\mathcal{D}^b(\mathbf{mod}(R))/\operatorname{Perf}(R)$ is the usual Verdier quotient of triangulated categories.

Let F denote the composition

$$N(\mathbf{mod}(R)) \xrightarrow{i} \mathcal{D}^b(\mathbf{mod}(R)) \xrightarrow{Q} \mathcal{D}^b(\mathbf{mod}(R))/\operatorname{Perf}(R).$$

of the inclusion of (the nerve of) finitely generated R-modules into $\mathcal{D}^b(\mathbf{mod}(R))$ followed by the Verdier quotient functor. If $P \in N(\mathbf{mod}(R))$ is projective, then $F(P) \cong 0$ by definition of $\operatorname{Perf}(R)$. Thus, by the universal property of the localization defining $stmod(R)$ it follows that F factors as

$$\begin{tikzcd} N(\mathbf{mod}(R)) \arrow[rr, "F"] \arrow[dr] & & \mathcal{D}^b(\mathbf{mod}(R))/\operatorname{Perf}(R). \\ & stmod(R) \arrow[ur, "\tilde{F}"'] & \end{tikzcd}$$

The claim is that $\tilde{F}$ is the desired equivalence.

We first show that $\tilde{F}$ is exact. Let

$$M \to N \to L \to \Sigma(M)$$

be a distinguished triangle in $\mathbf{stmod}(R)$. By Remark 1.2.3 this arises via a diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, hook, "i"] \arrow[d] & Q \arrow[r] \arrow[d] & \Sigma(M) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & N \arrow[r] & L \arrow[r] & \Sigma(M) \arrow[r] & 0 \end{tikzcd}$$

with $M \hookrightarrow Q$ an injective hull of Q and the left hand square a pushout. Thus, in $\mathcal{D}^b(\mathbf{mod}(R))$ we have distinguished triangles

$$M \to Q \to \Sigma(M) \to M[1]$$

and

$$N \to L \to \Sigma(M) \to N[1].$$

Since Q is injective it is projective and so $F(Q) \cong 0$ in the quotient. Hence, in the Verdier quotient we have a distinguished triangle

$$FM \to 0 \to F\Sigma(M) \to FM[1]$$

so that $F\Sigma(M) \cong FM[1]$ in the Verdier quotient. It follows that

$$FN \to FL \to FM[1] \to FN[1]$$

is a distinguished triangle in the Verdier quotient as it is isomorphic to

$$FN \to FL \to F\Sigma(M) \to FN[1]$$

which is distinguished. Thus, by shifting

$$FM \to FN \to FL \to FM[1]$$

is distinguished and the result follows.

Now we claim that $\tilde{F}$ is full. This follows as F is full and no non-projective finitely generated module M is mapped to zero since the only modules with finite projective resolutions over R are projective.

The functor $\tilde{F}$ is faithful. Indeed suppose $f : M \to N$ is the class of a map in $\mathbf{stmod}(R)$ such that $\tilde{F}(f) = 0$. Let

$$M \xrightarrow{f} N \xrightarrow{h} L \to M[1]$$

be an extension of f to a distinguished triangle. Now since $\tilde{F}f = 0$, then by considering the morphism of distinguished triangles

$$\begin{tikzcd}
\tilde{F}M \arrow[r, "\tilde{F}f=0"] \arrow[d, "0"'] & \tilde{F}N \arrow[r, "\tilde{F}h"] \arrow[d, "\mathbf{1}"'] & \tilde{F}L \arrow[d, dashed, "\tilde{F}h'"] \\
0 \arrow[r, "0"'] & \tilde{F}N \arrow[r, "\mathbf{1}"'] & \tilde{F}N
\end{tikzcd}$$

where the dotted arrow exists by TR3 for triangulated categories. Thus, the identity map for $\tilde{F}N$ factors through the map $\tilde{F}h : \tilde{F}N \to \tilde{F}L$. Hence, since $\tilde{F}$ is full there is a map $g : Y \to Y$ such that g factors through h

$$\begin{tikzcd}
Y \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \\
& Z \arrow[ur, "h'"'] &
\end{tikzcd}$$

and $\tilde{F}g$ is an equivalence. As previously seen if $M \in \mathbf{stmod}(R)$ is non-zero, then $\tilde{F}M \not\cong 0$ since no finitely generated non-projective module has a finite projective resolution. Hence, since $\tilde{F}(g)$ is an isomorphism, then $\tilde{F}\,\mathrm{cofib}(g) \cong \mathrm{cofib}(\tilde{F}g) \cong 0$ so by the previous observation $\mathrm{cofib}(g) \cong 0$. Therefore, g is an isomorphism and we see that h is a split monomorphism as $\mathbf{1}_N = g^{-1}g = (g^{-1}h')h$. It follows that we obtain a morphism

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & N \arrow[r, "\tilde{h}"] \arrow[d, "\mathbf{1}"'] & L \arrow[r] \arrow[d, "g^{-1}h'"] & M[1] \arrow[d] \\ 0 \arrow[r] & N \arrow[r, "\mathbf{1}"'] & N \arrow[r] & 0 \end{tikzcd}$$

of cofiber sequences which implies $f = 0$ as desired.

Now we claim that $\tilde{F}$ is essentially surjective. Let $C_\bullet \in \mathcal{D}^b(\mathbf{mod}(R))/\operatorname{Perf}(R)$ which as an object of $\mathcal{D}^b(\mathbf{mod}(R))$ is equivalent to a complex

$$P_\bullet : \cdots \to P_{r+1} \to P_r \to \cdots \to P_s \to 0$$

with homology bounded above r. Consider the chain map $f_\bullet$ below

$$\begin{tikzcd} P_\bullet & \cdots \arrow[r] & P_{r+1} \arrow[r] \arrow[d, "\mathbf{1}"] & P_r \arrow[r] \arrow[d, "\mathbf{1}"] & P_{r-1} \arrow[r] \arrow[d] & \cdots \\ \tilde{P}_\bullet & \cdots \arrow[r] & P_{r+1} \arrow[r] & P_r \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

to the complex $\tilde{P}_\bullet$. We claim that $\mathrm{cofib}(f)$ lies in $\operatorname{Perf}(R)$. Indeed the kernel of f is the complex

$$\cdots \to 0 \to P_{r-1} \to \cdots \to P_s \to 0$$

which lies in $\operatorname{Perf}(R)$) and by a general result since f is surjective, then $\ker(p)[1]$ is quasi-isomorphic to $\mathrm{cofib}(f)$ and so $\mathrm{cofib}(f) \in \operatorname{Perf}(R)$. Now we have a complex

$$Q_\bullet : \cdots \to P_{r+1} \to P_r \to Q_{r-1} \to \cdots \to Q_0 \to 0$$

which is the projective resolution of some module M in $\mathbf{stmod}(R)$ with the obvious map from $Q_\bullet \to \tilde{P}_\bullet$ an isomorphism in $\mathcal{D}^b(\mathbf{mod}(R))/\operatorname{Perf}(R)$ and therefore, $\tilde{F}M \cong P_\bullet \cong C_\bullet$. $\square$

Corollary 1.3.4. *The categories* $\mathbf{StMod}(R)$ *and* $\operatorname{Ind}(D^b(\mathbf{mod}(R))/\operatorname{Perf}(R))$ *are equivalent.*

1.4 The Tate Construction

In this section we give a third construction of the stable module category in the case that $R = kG$ for G a finite p-group and k a field of characteristic p.

If G is a finite group and A a G-module, then we have G-fixed points A^G and G-orbits. There is a map

$$\begin{aligned} Nm_G : A_G &\to A^G \\ [a] &\mapsto Na \end{aligned}$$

which sends a class $[a]$ to multiplication by the norm element $N = \sum_{g \in G} g$ of kG. Using this map we are able to splice group homology and cohomology together into a single object, Tate cohomology, defined by setting

$$\widehat{H}^n(G; A) := \begin{cases} H^n(G; A), & n \geq 1, \\ \operatorname{coker} Nm_G, & n = 0, \\ \ker Nm_G, & n = -1, \\ H_{-n-1}(G; A), & n \leq -2 \end{cases}$$

so we have an exact sequence

$$0 \to \widehat{H}^{-1}(G; A) \to A_G \xrightarrow{Nm_G} A^G \to \widehat{H}^0(G; A) \to 0.$$

In this section we discuss this construction in the setting of spectra with G-action to obtain a norm map

$$Nm_G : X_{hG} \to X^{hG}$$

from the homotopy orbits to the homotopy fixed points. Taking the cofiber of this map we obtain a spectrum X^{tG} called the Tate construction. We follow [NS$^+$18, I.1] although this material appears in [Lur, 6.1.6].

Let Sp denote the stable ∞-category of spectra and let Sp^{BG} denote the ∞-category $\mathrm{Fun}(BG, \mathrm{Sp})$ where BG is a classifying space for G. We call this ∞-category *spectra with G-action*.

Remark 1.4.1. We call Sp^{BG} spectra with G-action rather than G-equivariant spectra in order to distinguish this category from the category of genuine equivariant spectra. Although by [NS$^+$18, Thm. II.2.7] the category Sp^{BG} may be realized as a full subcategory of the ∞-category of genuine G-spectra spanned by those genuine spectra whose genuine fixed points and homotopy fixed points agree for all subgroups H of G.

Consider the map $f : BG \to *$ which gives rise to a precomposition functor $f^* : \mathrm{Sp} \to \mathrm{Sp}^{BG}$. Note that this simply gives a spectrum X the trivial action. Since f^* is a precomposition functor and Sp is complete and cocomplete we make take Kan extensions

$$\begin{tikzcd} \mathrm{Sp} \arrow[r, "f^*"] & \mathrm{Sp}^{BG} \arrow[l, bend right=50, "f_!"'] \arrow[l, bend left=50, "f_*"] \end{tikzcd}$$

by [Lur09b, 4.3.3.7]. Observe since these are Kan extensions they are precisely the (homotopy) orbit

$$\begin{aligned} f_! = (-)_{hG} : \mathrm{Sp}^{BG} &\to \mathrm{Sp} \\ X &\mapsto \varinjlim_{BG} X \end{aligned}$$

functor and a (homotopy) fixed point functor

$$\begin{aligned} f_* = (-)^{hG} : \mathrm{Sp}^{BG} &\to \mathrm{Sp} \\ X &\mapsto \varprojlim_{BG} X. \end{aligned}$$

Now we wish to construct a natural transformation

$$Nm_G : (-)_{hG} \to (-)^{hG}$$

just as in the case of G-modules.

Remark 1.4.2. Given a map $f : X \to Y$ of Kan complexes and an ∞-category $\mathcal{C}$ which is complete and cocomplete. We always let $f^* : \mathcal{C}^Y \to \mathcal{C}^X$ denote the precomposition with f functor, $f_!$ its left adjoint, and f_* its right adjoint obtained via Kan extension.

Construction 1.4.3. Consider the pullback of spaces

$$\begin{tikzcd} BG \times BG \arrow[r, "p_1"] \arrow[d, "p_2"'] & BG \arrow[d, "f"] \\ BG \arrow[r, "f"'] & * \end{tikzcd} \tag{1.4.1}$$

where p_1 and p_2 are the projection maps onto the first and second coordinates, respectively. Let $\delta : BG \to BG \times BG$ be the diagonal map, then by Kan extension we obtain functors

$$\begin{tikzcd} \mathrm{Sp}^{BG\times BG} \arrow[r, "\delta^*"] & \mathrm{Sp}^{BG} \arrow[l, bend right, "\delta_!"'] \arrow[l, bend left, "\delta_*"] \end{tikzcd}$$

where there is a map $Nm_\delta : \delta_! \to \delta_*$ which is an equivalence by [Lur, Construction. 6.1.6.19] as the map $f : BG \to *$ has fiber BG. Now we may combine the unit map $\mathbf{1} \to \delta_*\delta^*$ and the counit map $\delta_!\delta^* \to \mathbf{1}$ to obtain a map

$$p_1^* \to \delta_*\delta^*p_1^* \simeq \delta_* \xrightarrow[\simeq]{Nm_\delta^{-1}} \delta_! \simeq \delta_!\delta^*p_2^* \to p_2^*.$$

From the pullback diagram (1.4.1) consider the transformation

$$f^*f_* \to \mathbf{1} \to p_{1*}p_2^*.$$

It follows from [Lur, Lemma. 6.1.6.3] that this map is in fact an equivalence so we obtain a map $\mathbf{1} \to f^*f_*$ from the map $\mathbf{1} \to p_{1*}p_2^*$. Thus, as $f_!$ is left adjoint to f^* there is a map

$$Nm_G : f_! = (-)_{hG} \to (-)^{hG} = f_*$$

which is the desired transformation.

Definition 1.4.4. The *Tate Construction* for Sp^{BG} is the cofiber

$$(-)^{tG} = \mathrm{cofib}(Nm_G : (-)_{hG} \to (-)^{hG})$$

of the norm map.

For k a ring we let k^{tG} denote the Tate construction on its Eilenberg-Maclane spectrum with trivial action.

Remark 1.4.5. If M is a G-module, then the homotopy groups of HM^{tG} are the Tate cohomology

$$\pi_n(HM^{tG}) \cong \hat{H}^{-n}(G; M)$$

of G with coefficients in M where HM denotes the Eilenberg-Maclane spectrum of M.

Remark 1.4.6. It is shown in [NS⁺18, Thm. I.3.1] that the functor $(-)^{tG})$ has a unique (in a suitable ∞-categorical sense) lax symmetric monoidal structure.

Theorem 1.4.7 ([Kel94, 4.3])**.** *Let k denote a field of characteristic p and G be a finite p-group, then there is an equivalence of ∞-categories*

$$Mod(k^{tG}) \simeq StMod(kG)$$

between the ∞-category of modules over the $\mathbb{E}_\infty$-ring k^{tG}.

Remark 1.4.8. Another proof appears in [Mat15, Section 2].

1.5 Properties of the Stable Module Category

We discuss some additional structure and properties of the stable module category in the case that $R = kG$ for G a finite group and k a field.

Proposition 1.5.1 ([Hov07, Prop. 4.2.15])**.** *The model category $\underline{\mathbf{Mod}}(kG)$ is a symmetric monoidal model category. In particular, the underlying ∞-category $StMod(kG)$ is a symmetric monoidal ∞-category.*

Proof. The category $\mathbf{Mod}(kG)$ with the model structure of Section 1.1 is a closed symmetric monoidal model category in the sense of [Hov07, Def. 4.2.6] (See also [Lur09b, A.3.1]). The tensor product is the tensor product $M \otimes_k N$ of k-vector spaces with diagonal G-action. The internal hom is the usual internal hom for the category of representations. Namely, for kG-modules M and N we give $\mathrm{Hom}(M, N)$ the conjugation action so for $f : M \to N$, then $g.f$ is the map

$$g.f : x \mapsto gf(g^{-1}x).$$

It follows from [Lur, Example 4.1.7.6] that the underlying ∞-category of the model category is a symmetric monoidal ∞-category. $\square$

Recall, that every finitely generated kG-module is dualizable via the dual representation $Hom_k(-, k)$. This descends to a functor on $\mathbf{stmod}(kG)$ by the following proposition.

Proposition 1.5.2. *Let M and N be finitely generated kG-modules. If $f : M \to N$ is a stable equivalence with stable inverse $g : N \to M$, then* $\mathrm{Hom}_k(-, k)$ *sends f and g to isomorphisms.*

Proof. This follows from the fact that if P is projective, then $\mathrm{Hom}_k(P, k)$ is injective, but kG is Frobenius so it is projective. Thus, if $g \circ f - \mathbf{1}_M$ factors through a projective P, then the dual of $g \circ f - \mathbf{1}_M$ factors through $\mathrm{Hom}_k(P, k)$ which is projective. A similar argument applies to $f \circ g$ and so since $\mathrm{Hom}_k(-, k)$ respects the stable equivalences it descends to a duality functor on $\mathbf{stmod}(kG)$. $\square$

By definition an object in an ∞-category is dualizable if it is dualizable as an object in the homotopy category. Hence, we obtain the following corollary.

Corollary 1.5.3. *Every finitely generated module in the stable module ∞-category is dualizable using the k-linear dual functor* $\mathrm{Hom}_k(-, k)$.

Remark 1.5.4. In the ∞-category $\mathcal{D}^b(\mathbf{mod}(kG))/\operatorname{Perf}(kG)$ the symmetric monoidal structure is induced by the tensor product of chain complexes $M_\bullet \otimes_k N_\bullet$ over k with diagonal G-action. The duality functor is similarly described by applying $\mathrm{Hom}_k(-, k)$ to the chain complex.

2 Sheaves and Equivariant Sheaves

In this section we begin by discussing the definition of sheaves on a site in order to fix some notation. Then given a topological space X with an action by a group G we define the category of equivariant sheaves on the G-space X. We do this by constructing a particular site $\mathcal{U}_G(X)$ and define the category of equivariant sheaves to be the sheaves on this site. We prove a few properties of this category.

2.1 Sheaves

In order to fix notation we begin with the basic definitions of sheaves on a site with a Grothendieck topology. Our primary source is [MM12]. Although we also consulted [KS06] and the Stacks project [Sta21].

Given a category $\mathbf{C}$ we let $\mathrm{PSh}(\mathbf{C}) = \mathrm{Fun}(\mathbf{C}^{op}, \mathbf{Set})$ denote the category of $\mathbf{Set}$-valued presheaves.

Definition 2.1.1. Let $\mathbf{C}$ be a (small) category. A *sieve* on $\mathbf{C}$ is a full subcategory $S \subseteq \mathbf{C}$ such that if $f : C \to C'$ is a morphism in $\mathbf{C}$ and $C' \in S$, then $C \in S$. If $C \in \mathbf{C}$ is an object in $\mathbf{C}$, then a *sieve on C* is a sieve on the over category $\mathbf{C}_{/C}$.

In other words a sieve on an object $C \in \mathbf{C}$ is a collection of morphisms $S = \{f_i : C_i \to C\}_{i \in I}$ with codomain C such that if $f \in S$ and $g : D \to C_i$ is a morphism in $\mathbf{C}$, then $f \circ g \in S$. Observe that if $S = \{f_i : C_i \to C\}$ is a sieve on $C \in \mathbf{C}$ and $h : D \to C$ is any morphism, then

$$h^*(S) = \{g : D' \to D \mid h \circ g \in S\}$$

defines a sieve on D.

Remark 2.1.2. We make an important observation that a sieve S on an object $C \in \mathbf{C}$ is the same as a subfunctor $S \hookrightarrow \mathrm{Hom}(-, C)$ of the representable functor corresponding to C. In other words sieves on an object C correspond to equivalence class of monomorphisms into the representable functor $\mathrm{Hom}(-, C)$.

Definition 2.1.3. Let $\mathbf{C}$ be a category. A (Grothendieck) topology on $\mathbf{C}$ is a map J which to each $C \in \mathbf{C}$ associates a collection of sieves, $J(C)$, on C, called *covering sieves* such that

(T1) (Maximality) The sieve $\mathbf{C}_{/C}$ is in $J(C)$.

(T2) (Stability) if $f : D \to D$ is a morphism in $\mathbf{C}$ and $S \in J(C)$, then $f^*(S)$ is in $J(D)$.

(T3) (Transitivity) If $S \in J(C)$ and S' is any sieve on C such that for all $f : D \to C$ in S we have $f^*(S') \in J(D)$, then S' is in $J(C)$.

Definition 2.1.4. A *site* is a pair $(\mathbf{C}, J)$ where $\mathbf{C}$ is a category and J a Grothendieck topology on $\mathbf{C}$.

Example 2.1.5. Let X be a topological space and $\mathcal{U}(X)$ the poset of open subsets of X. Let $U \subseteq X$ be an open subset. Observe that since a sieve S on U is a full subcategory of $\mathcal{U}(X)_{/U}$ it is simply a collection of open subsets of X contained in U. Define a topology on $\mathcal{U}(X)$ by declaring a sieve S on U to be a covering sieve if $U = \cup_{V \in S} V$, that is, if U is covered by the elements of S. It is an easy exercise to check that this defines a Grothendieck topology on $\mathcal{U}(X)$.

In order to define a topology on a set X one often does not define a topology on X, but rather gives a basis for a topology on X and then considers the topology generated by this basis. It follows that properties of the topology on X may then be checked on the basis elements. A similar idea applies to Grothendieck topologies.

Definition 2.1.6. Let $\mathbf{C}$ be a category with pullbacks. A *basis* for a (Grothendieck) topology on $\mathbf{C}$ is a map K which for each $C \in \mathbf{C}$ gives a collection

$$K(C) = \{\{f_{ij} : C_{ij} \to C\}_{j \in J}\}_{i \in I}$$

of families of morphisms of the form $f : C' \to C$, called *covering families* such that

(B1) If $f : C' \to C$ is an isomorphism, then $f \in K(C)$.

(B2) If $\{f_i : C_i \to C\}_{i \in I} \in K(C)$, then for any $g : D \to C$ the pullback family

$$\{\mathbf{pr}_2 : C_i \times_C D \to D\}_{i \in I}$$

is in $K(D)$.

(B3) If $\{f_i : C_i \to C\}_{i \in I}$ is in $K(C)$ and for each $i \in I$ there is a collection $\{g_{ij} : D_{ij} \to C_i\}_{j \in I_i}$ in $K(C_i)$, then

$$\{f_i \circ g_{ij} : D_{ij} \to C\}_{(i,j) \in I \times I_i}$$

lies in $K(C)$.

Remark 2.1.7. Given a basis K we obtain a Grothendieck topology J *generated by* K by a declaring a sieve S on C to be a covering sieve of C if and only if there exists some covering family $S' \in K(C)$ such that $S' \subseteq S$. In other words the topology generated by K consists of those sieves which contain a K-covering family.

Conversely, given a Grothendieck topology J on $\mathbf{C}$ there is a maximal basis K which generates J by taking S to be a covering family for C if and only if S contains the sieve $\{f \circ g \mid f \in S\}$ for any g with the same codomain as f.

Example 2.1.8. We note a few more examples of Grothendieck topologies. Let

(1) The *indiscrete topology*, sometimes also called the chaotic or trivial topology, is the topology with the only sieve being the maximal one. In other words $J(C) = \{\mathbf{C}_{/C}\}$ for all objects C in $\mathbf{C}$.

(2) The *discrete topology* is the topology such that any (non-empty) sieve is declared to be a covering sieve.

We now turn to defining sheaves on a site $(\mathbf{C}, J)$. For simplicity we will assume that $\mathbf{C}$ has pullbacks.

Definition 2.1.9. Let $(\mathbf{C}, J)$ be a site where $\mathbf{C}$ has pullbacks. A presheaf $\mathcal{F} : \mathbf{C}^{op} \to \mathbf{Set}$ is said to be a J-*sheaf* or simply a sheaf if for every covering family $\{f_i : C_i \to C\}$ the diagram

$$F(C) \xrightarrow{\prod_i f_i^*} \prod_i F(C_i) \underset{\prod_j \mathbf{pr}_{2,i,j}^*}{\overset{\prod_i \mathbf{pr}_{1,i,j}^*}{\rightrightarrows}} \prod_{i,j} F(C_i \times_C C_j) \tag{2.1.1}$$

is an equalizer. Here the maps

$$\begin{aligned} f_i^* &: F(C) \to \mathcal{F}(C_i) \\ \mathbf{pr}_{1,i,j}^* &: \mathcal{F}(C_i) \to \mathcal{F}(C_i \times_C C_j) \\ \mathbf{pr}_{2,i,j}^* &: \mathcal{F}(C_j) \to \mathcal{F}(C_i \times_C C_j) \end{aligned}$$

are those obtained after applying $\mathcal{F}$ to f_i and the projections in $\mathbf{C}$. We let $\mathrm{Shv}_J(\mathbf{C})$ denote the full subcategory of $\mathrm{PSh}(\mathbf{C})$ spanned by the J-sheaves.

Remark 2.1.10. In other words $\mathcal{F}$ is a sheaf if given any family $s_i \in \mathcal{F}(C_i)$ such that for all i, j we have $\mathbf{pr}_{1,i,j}^*(s_i) = \mathbf{pr}_{2,i,j}^*(s_j)$ in $\mathcal{F}(C_i \times_C C_j)$, then there exists a unique $s \in \mathcal{F}(C)$ such that $f_i^*(s) = s_i$ for all i.

Remark 2.1.11. We have the following important alternative version of the sheaf condition. Let $j : \mathbf{C} \to \mathrm{PSh}(\mathbf{C})$ denote the Yoneda embedding. Recall, by Remark 2.1.2 a sieve S on $C \in \mathbf{C}$ is the same as a subfunctor of $j(C) := \mathrm{Hom}(-, C)$. Let $i_S : S \hookrightarrow j(C)$ denote the monomorphism corresponding to S. A presheaf $F : \mathbf{C}^{op} \to \mathbf{Set}$ is a sheaf if and only if for all covering sieves S of an object C the inclusion i_S induces an isomorphism

$$\mathrm{Hom}_{\mathrm{PSh}(\mathbf{C})}(j(C), F) \xrightarrow[\cong]{-\circ i_S} \mathrm{Hom}_{\mathrm{PSh}(\mathbf{C})}(S, F).$$

Definition 2.1.12. A category $\mathbf{X}$ is a (Grothendieck) topos if there exists a site $(\mathbf{C}, J)$ such that $\mathbf{X} \simeq \mathrm{Shv}_J(\mathbf{C})$.

Example 2.1.13. (1) Let $X = \{*\}$ denote the one point space. Then $\mathrm{Shv}(X) = \mathbf{Set}$ is a topos.

(2) If X is a topological space, then $\mathrm{Shv}(\mathcal{U}(X))$ is the usual category of sheaves on X and we simply denote this category $\mathrm{Shv}(X)$.

(3) Let $\mathbf{C}$ be a category. Then $\mathrm{PSh}(\mathbf{C})$ is a Grothendieck topos by endowing $\mathbf{C}$ with the indiscrete topology as then $\mathrm{PSh}(\mathbf{C}) = \mathrm{Shv}(\mathbf{C})$.

Example 2.1.14. A basis for a topology on $\mathbf{C}$ is said to be *subcanonical* if every representable presheaf is a sheaf. The *canonical* topology on $\mathbf{C}$ is the largest subcanonical topology on $\mathbf{C}$.

If $(\mathbf{C}, J)$ is a site, then there is an equivalence of categories

$$\mathrm{Shv}_J(\mathbf{C}) \simeq \mathrm{Shv}_{can}(\mathrm{Shv}_J(\mathbf{C}))$$

between the category of sheaves on $\mathbf{C}$ and the sheaves on $\mathrm{Shv}_J(\mathbf{C})$ with respect to the canonical topology. See [J$^+$02, Prop. C.2.2.7]. Note that one can describe the covering sieves for the canonical topology on a Grothendieck topos concretely. They are precisely the covering families consisting of jointly epimorphic maps.

We now briefly discuss morphisms of topoi. This discussion will be helpful for defining functors of equivariant sheaves induced by continuous equivariant maps $f : X \to Y$.

Definition 2.1.15. Let $\mathbf{X} = \mathrm{Shv}(\mathbf{C})$ and $\mathbf{Y} = \mathrm{Shv}(\mathbf{D})$ be (Grothendieck) topos. A geometric morphism of topoi $f : \mathbf{X} \to \mathbf{Y}$ is a pair $f = (f^*, f_*)$ of adjoint functors

$$\begin{tikzcd} \mathbf{X} \arrow[r, shift right, "f_*"'] & \mathbf{Y} \arrow[l, shift right, "f^*"'] \end{tikzcd}$$

such that $f^* \dashv f_*$ and f^* is left exact. We call the right adjoint f_* the *direct image* or *pushforward* functor and we call the left adjoint f^* the *inverse image* or *pullback* functor.

Given geometric morphisms

$$f = (f^*, f_*) : \mathbf{X} \to \mathbf{Y}$$
$$g = (g^*, g_*) : \mathbf{Y} \to \mathbf{Z}$$

of topoi the composition $g \circ f$ is defined as

$$g \circ f = (f^* \circ g^*, g_* \circ f_*) : \mathbf{X} \to \mathbf{Z}$$

as in the diagram

$$\begin{tikzcd} \mathbf{X} \arrow[r, shift right, "f_*"'] & \mathbf{Y} \arrow[l, shift right, "f^*"'] \arrow[r, shift right, "g_*"'] & \mathbf{Z}. \arrow[l, shift right, "g^*"'] \end{tikzcd}$$

Remark 2.1.16. Every Grothendieck topos is locally presentable (See [Bor94, Prop. 3.4.16]). In particular, we may apply the adjoint functor theorem so to give a geometric morphism $f = (f^*, f_*)$ it is sufficient to give either f_* and show it respects limits or give f^* and show it is left exact and respects colimits.

Definition 2.1.17. Let $\mathbf{X}$ be a topos, then a *point* of the topos $\mathbf{X}$ is a geometric morphism $x = (x^*, x_*) : \mathbf{Set} \to \mathbf{X}$.

Definition 2.1.18. Let $\mathbf{X}$ be a topos and $x = (x^*, x_*)$ a point of $\mathbf{X}$, then the *stalk of a sheaf* $\mathcal{F}$ *at* x is $x^*\mathcal{F}$ the inverse image of $\mathcal{F}$. If A is a set, then x_*A is the *skyscraper sheaf* of A at X.

We now describe a few important properties of the category $\mathrm{Shv}_J(\mathbf{C})$.

Proposition 2.1.19. *Let* $(\mathbf{C}, J)$ *be a site. If* $I \to \mathrm{PSh}(\mathbf{C})$ *is a diagram and every presheaf* $\mathcal{F}_i$ *in the diagram is a sheaf, then* $\varprojlim \mathcal{F}_i$ *is a sheaf.*

Proof. See [MM12, Prop. III.4.4]. ☐

Proposition 2.1.20. *The inclusion functor* $i : \mathrm{Shv}_J(\mathbf{C}) \hookrightarrow \mathrm{PSh}(\mathbf{C})$ *has a left exact left adjoint* $\mathbf{a} : \mathrm{PSh}(\mathbf{C}) \to \mathrm{Shv}_J(\mathbf{C})$ *called the associated sheaf or sheafification functor. In particular* $\mathbf{a}$ *commutes with finite limits and colimits.*

Proof. See [MM12, Thm. III.5.1]. ☐

Note that if $\mathcal{F}$ is a presheaf, then the unit map η of the adjunction $\mathbf{a} \dashv i$ gives a canonical map of presheaves $\mathcal{F} \xrightarrow{\eta_{\mathcal{F}}} \mathbf{a}(\mathcal{F})$. This map has the following universal property. If $\alpha : \mathcal{F} \to \mathcal{G}$ is a map of presheaves where $\mathcal{G}$ is a sheaf, then α factors uniquely through $\eta_{\mathcal{F}}$ as in

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, "\alpha"] \arrow[dr, "\eta_{\mathcal{F}}"'] & & \mathcal{G} \\
& \mathbf{a}(\mathcal{F}) \arrow[ur, dashed] &
\end{tikzcd}$$

Remark 2.1.21. The above proposition tells us that the category $\mathrm{Shv}_J(\mathbf{C})$ is a reflective localization of $\mathrm{PSh}(\mathbf{C})$. In particular, we may realize $\mathrm{Shv}_J(\mathbf{C})$ as the full subcategory of $\mathrm{PSh}(\mathbf{C})$ of S-local objects where S is collection of all monomorphisms $P \overset{i_P}{\hookrightarrow} \mathrm{Hom}(-, C)$ which correspond to covering sieves in the topology J.

Corollary 2.1.22. *If* $(\mathbf{C}, J)$ *is a site, then the category* $\mathrm{Shv}_J(\mathbf{C})$ *has all limits and colimits. Furthermore, colimits in* $\mathrm{Shv}_J(\mathbf{C})$ *may be calculated in* $\mathrm{PSh}(\mathbf{C})$ *followed by applying the sheafification functor* $\mathbf{a}$.

Example 2.1.23. This appears in [Sta21, Tag 09VK]. Let $X_\bullet : \Delta^{op} \to \mathbf{Top}$ be a simplicial topological space. Then we may associate a site X^{Zar} to $X_\bullet$ in the following way. The objects of X^{Zar} are the open subsets U of X_n for some n. A morphism $U \to V$ in X^{Zar} is induced by $\varphi : [m] \to [n]$ where $U \subseteq X_n$ and $V \subseteq X_m$ are open and φ is such that $X_\bullet(\varphi)(U) \subseteq V$. We define $\{f_i : U_i \to U\}_{i \in I}$ to be a covering family in X^{Zar} if

(1) the $U_i \subseteq X_n$ are open,

(2) all the morphisms $f_i = X_\bullet(\mathbf{1}_{[n]})$ are induced by the identity for $[n]$, and

(3) $\cup_{i \in I} U_i = U$ the U_i cover U.

A sheaf $\mathcal{F}$ on X^{Zar} consists of a pair $(\{\mathcal{F}_n\}_{n \geq 0}, \{\mathcal{F}(\varphi)\}_{\varphi \in \mathrm{Mor}(\Delta)})$ such that:

(1) Each $\mathcal{F}_n$ is a sheaf on X_n.

(2) If $\varphi : [m] \to [n]$ is a morphism in Δ, then φ induces a continuous map $f_\varphi := X(\varphi) : X_n \to X_m$. Then $\mathcal{F}(\varphi)$ is a map

$$\mathcal{F}(\varphi) : f_\varphi^* \mathcal{F}_m \to \mathcal{F}_n.$$

Furthermore, the structure maps $\mathcal{F}(\varphi)$ must be functorial in the following sense: if $\varphi : [m] \to [n]$ and $\psi : [l] \to [m]$ are morphisms in Δ, then the maps $\mathcal{F}(\varphi)$ and $\mathcal{F}(\psi)$ satisfy

$$\mathcal{F}(\psi \circ \varphi) = \mathcal{F}(\varphi) \circ f_\varphi^* \mathcal{F}(\psi)$$

as in the diagram

$$\begin{tikzcd}
f_{\psi\varphi}^* \mathcal{F}_l \arrow[rr, "\mathcal{F}(\psi\circ\varphi)"] \arrow[dr, "f_\varphi^*\mathcal{F}(\psi)"'] & & \mathcal{F}_n \\
& f_\varphi^* \mathcal{F}_m \arrow[ur, "\mathcal{F}(\varphi)"'] &
\end{tikzcd}$$

We conclude this section with a brief discussion of sheaves valued in other categories.

Definition 2.1.24. Let $\mathbf{A}$ be a category with (small) limits and $\mathbf{X} := \mathrm{Shv}_J(\mathbf{C})$ a Grothendieck topos.

(1) A functor $\mathcal{F} : \mathbf{C}^{op} \to \mathbf{A}$ is an $\mathbf{A}$-*valued sheaf* on $\mathbf{C}$ if for every $C \in \mathbf{C}$ and every covering sieve $S \subseteq \mathbf{C}_{/C}$ of C that $\mathcal{F}$ satisfies the sheaf condition of Definition 2.1.9 or Remark 2.1.11 as an $\mathbf{A}$-valued functor. Let $\mathrm{Shv}(\mathbf{C}, \mathbf{A})$ denote the full subcategory of $\mathrm{PSh}(\mathbf{C}, \mathbf{A})$ that are $\mathbf{A}$-valued sheaves.

(2) An $\mathbf{A}$-*valued sheaf on* $\mathbf{X}$ is a functor $F : \mathbf{X}^{op} \to \mathbf{C}$ which preserves small limits. Let $\mathrm{Shv}(\mathbf{X}, \mathbf{A})$ denote the full subcategory of $\mathrm{PSh}(\mathbf{X}, \mathbf{A})$ spanned by functors which preserves small limits.

Proposition 2.1.25. *Let $\mathbf{A}$ be a category with (small) limits and $\mathbf{X} = \mathrm{Shv}_J(\mathbf{C})$ be a Grothendieck topos. Then the categories $\mathrm{Shv}(\mathbf{C}, \mathbf{A})$ and $\mathrm{Shv}(\mathbf{X}, \mathbf{A})$ are equivalent.*

Proof. Let $j : \mathbf{C} \to \mathrm{PSh}(\mathbf{C})$ denote the Yoneda embedding. The universal property of the Yoneda embedding tells us that precomposition with j induces an equivalence

$$\mathrm{Fun}^L(\mathrm{PSh}(\mathbf{C}), \mathbf{D}) \xrightarrow{-\circ j} \mathrm{Fun}(\mathbf{C}, \mathbf{D})$$

where $\mathbf{D}$ is any category with all colimits and Fun^L denotes the full subcategory spanned by functors $\mathrm{PSh}(\mathbf{C}) \to \mathbf{D}$ which preserve colimits. Hence, by duality composition with $j : \mathbf{C}^{op} \to \mathrm{PSh}(\mathbf{C})^{op}$ induces and equivalence

$$\mathrm{Fun}^R(\mathrm{PSh}(\mathbf{C})^{op}, \mathbf{D}) \xrightarrow{-\circ j} \mathrm{Fun}(\mathbf{C}^{op}, \mathbf{D})$$

where $\mathbf{D}$ is any category with all limits and Fun^R consists of those functors $\mathrm{PSh}(\mathbf{C})^{op} \to \mathbf{D}$ which preserve limits. Let $\mathbf{a} : \mathrm{PSh}(\mathbf{C}) \to \mathrm{Shv}(\mathbf{C})$ denote the sheafification functor. Recall this is a localization at the class of monomorphisms $S \hookrightarrow j(C)$ which correspond to covering sieves of $C \in \mathbf{C}$. Hence, $\mathbf{a}$ has the following universal property: composition with $\mathbf{a}$ induces a fully faithful embedding

$$\mathrm{Fun}^L(\mathbf{X}, \mathbf{A}) \to \mathrm{Fun}^L(\mathrm{PSh}(\mathbf{C}), \mathbf{A})$$

with essential image those functors $F : \mathrm{PSh}(\mathbf{C}) \to \mathbf{A}$ such that for every monomorphism $S \to j(C)$ which corresponds to a covering sieve, then the induced map $F(S) \to F(j(C))$ is an isomorphism in $\mathbf{A}$. Again by duality we have that composition with $\mathbf{a} : \mathbf{X}^{op} \to \mathrm{PSh}(\mathbf{C})^{op}$ induces a fully faithful embedding

$$\mathrm{Fun}^R(\mathbf{X}^{op}, \mathbf{A}) \xrightarrow{-\circ \mathbf{a}} \mathrm{Fun}^R(\mathrm{PSh}(\mathbf{C})^{op}, \mathbf{A})$$

from the limit preserving functors $\mathbf{X}^{op} \to \mathbf{A}$ to the limit preserving functors $\mathrm{PSh}(\mathbf{C})^{op} \to \mathbf{A}$ such that for every monomorphism $S \to j(C)$ corresponding to a covering sieve the induced map $F(j(C)) \to F(S)$ is an equivalence, but this is precisely the sheaf condition from 2.1.11. Thus, the precomposition functor

$$\mathrm{Shv}(\mathbf{X}, \mathbf{A}) \xrightarrow{(\mathbf{a}\circ j)^*} \mathrm{Shv}(\mathbf{C}, \mathbf{A})$$

is an equivalence. □

For sheaves of algebraic structures we have some additional useful properties which we list here. Let $(\mathbf{C}, J)$ be a site and $\mathbf{A}$ a category.

(F1) Let $\mathbf{A}$ denote the category of abelian groups **Ab**, rings **Ring**, or commutative rings **CRing**. There is a canonical equivalence (isomorphism even) between the categories

$$\mathbf{A}(\mathrm{Shv}_J(\mathbf{C})) \simeq \mathrm{Shv}_J(\mathbf{C}, \mathbf{A})$$

between the categories of $\mathbf{A}$-objects in $\mathrm{Shv}_J(\mathbf{C})$ and $\mathbf{A}$-valued sheaves on $\mathbf{C}$.

(F2) Let R be a fixed ring and $\underline{R}$ denote the sheafification of the constant presheaf of rings $C \mapsto R$. This is a sheaf of rings. There is an equivalence

$$\mathrm{Shv}_J(\mathbf{C}, \mathbf{Mod}(R)) \simeq \mathbf{Mod}(\mathrm{Shv}_J(\mathbf{C}, \underline{R})$$

between the categories of sheaves with values in the category of (left) R-modules and the category of (left) modules over the ring object $\underline{R}$.

(F3) Let R be a ring and $\mathbf{A}$ denote any of the categories $\mathbf{Ab}, \mathbf{Ring}, \mathbf{CRing}, \mathbf{Mod}(R)$. Let $F : \mathbf{A} \to \mathbf{Set}$ denote the obvious forgetful functor. Then a presheaf $\mathcal{F} : \mathbf{C}^{op} \to \mathbf{A}$ is a sheaf if and only if $F\mathcal{F} : \mathbf{C}^{op} \to \mathbf{Set}$ is a sheaf.

(F4) Let $\mathbf{A}$ be as in (F3). Then there is a sheafification functor

$$\mathbf{a} : \mathrm{PSh}(\mathbf{C}, \mathbf{A}) \to \mathrm{Shv}_J(\mathbf{C}, \mathbf{A})$$

which is a left exact left adjoint to the inclusion

$$i : \mathrm{Shv}_J(\mathbf{C}, \mathbf{A}) \hookrightarrow \mathrm{PSh}(\mathbf{C}, \mathbf{A}).$$

Moreover, we have a commuting diagram

$$\begin{tikzcd} \mathrm{PSh}(\mathbf{C}, \mathbf{A}) \arrow[r, shift left, "\mathbf{a}"] \arrow[d, "F"'] & \mathrm{Shv}_J(\mathbf{C}, \mathbf{A}) \arrow[l, shift left, hook'] \arrow[d, "F"] \\ \mathrm{PSh}(\mathbf{C}) \arrow[r, shift left, "\mathbf{a}"] & \mathrm{Shv}_J(\mathbf{C}) \arrow[l, shift left, hook'] \end{tikzcd}$$

where F denote the obvious forgetful functor.

(F5) Let $\mathbf{A}$ denote any of the categories $\mathbf{Ab}$, $\mathbf{Ring}$, $\mathbf{CRing}$, $\mathbf{Mod}(R)$, then a geometric morphism $f = (f^*, f_*)$ from $\mathrm{Shv}(\mathbf{C})$ to $\mathrm{Shv}(\mathbf{D})$ induces an adjoint pair of functors

$$\begin{tikzcd} \mathrm{Shv}(\mathbf{C}, \mathbf{A}) \arrow[r, shift right=2, "f_*"'] & \mathrm{Shv}(\mathbf{D}, \mathbf{A}) \arrow[l, shift right=2, "f^*"'] \end{tikzcd}$$

compatible with the forgetful functor and the above results.

(F6) Let $\mathcal{O}$ be a ring object in $\mathrm{Shv}_J(\mathbf{C})$, then the category of modules over $\mathcal{O}$ is a Grothendieck abelian category.

We defer to [Sta21, Section 00YR] for details.

2.2 Sheaves on Spaces

In this section we recall the construction of the direct image f_* and inverse image functors f^* giving rise to a geometric morphism $f : \mathrm{Shv}(X) \to \mathrm{Shv}(Y)$ for a continuous map $f : X \to Y$ of topological spaces. Additionally, we discuss the equivalence between sheaves on a space and local homeomorphisms over the space. We then conclude with a second construction of the inverse image functor f^* using this equivalence. These ideas extend essentially directly to our definition of equivariant sheaves. A much more thorough discussion can be found in [MM12, II].

Let $f : X \to Y$ be a continuous map, then there is an obvious induced map $f^{-1} : \mathcal{U}(Y) \to \mathcal{U}(X)$ of posets. Moreover, this functor respects intersections and unions. In particular, it respects the Grothendieck topology in the following sense:

(G1) If $\{U_i \to U\}$ is a cover of U in Y, then $\{f^{-1}(U_i) \to f^{-1}(U)\}$ is a cover of $f^{-1}(U)$ in X.

(G2) For any $V \to U$ in $\mathcal{U}(Y)$ we have $f^{-1}(V \cap U_i) = f^{-1}(V) \cap f^{-1}(U_i)$.

In other words f^{-1} sends covering families to covering families and intersections (pullbacks) to intersections (pullbacks). Thus, we obtain a functor

$$\mathrm{PSh}(X) \xrightarrow{(f^{-1})^*} \mathrm{PSh}(Y)$$
$$\mathcal{F} \mapsto \mathcal{F} \circ f^{-1}$$

given by precomposition with f^{-1}. It follows from the fact that the functor f^{-1} respects covering families and pullbacks that $(f^{-1})^*$ sends sheaves to sheaves. Hence, we obtain a functor.

$$f_* : \mathrm{Shv}(X) \to \mathrm{Shv}(Y).$$

Now since the category of sets is complete and cocomplete we may take (pointwise) left $(f^{-1})_!$and right $(f^{-1})_*$ Kan extension of the functor $(f^{-1})^*$

$$\begin{tikzcd} \mathrm{PSh}(X) \arrow[r, "(f^{-1})^*"] & \mathrm{PSh}(Y) \arrow[l, bend right=50, "(f^{-1})_!"'] \arrow[l, bend left=50, "(f^{-1})_*"] \end{tikzcd}$$

Note that in this case the left Kan extension may be explicitly described as

$$(f^{-1})_! \mathcal{F} : U \mapsto \varinjlim_{f(U) \subseteq V} \mathcal{F}(V)$$

Consider the composition

$$f^* : \mathrm{Shv}(Y) \subseteq \mathrm{PSh}(Y) \xrightarrow{(f^{-1})_!} \mathrm{PSh}(X) \xrightarrow{\mathbf{a}} \mathrm{Shv}(X).$$

Lemma 2.2.1. *The functor f^* is a left exact left adjoint to f_*.*

Proof. The functor f^* is a left adjoint by the series of adjunctions:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Shv}(Y)}(\mathcal{G}, f_*\mathcal{F}) &\cong \mathrm{Hom}_{\mathrm{PSh}(Y)}(\mathcal{G}, f_*\mathcal{F}) \\ &\cong \mathrm{Hom}_{\mathrm{PSh}(X)}((f^{-1})_!\mathcal{G}, \mathcal{F}) \\ &\cong \mathrm{Hom}_{\mathrm{Shv}(X)}(\mathbf{a}(f^{-1})_!\mathcal{G}, \mathcal{F}) \\ &= \mathrm{Hom}_{\mathrm{Shv}(X)}(f^*\mathcal{G}, \mathcal{F}). \end{aligned} \tag{2.2.1}$$

For left exactness observe that the functor f^* is a composition of the right adjoint $\mathrm{Shv}(Y) \subseteq \mathrm{PSh}(Y)$ and a left exact functor $\mathbf{a} : \mathrm{PSh}(X) \to \mathrm{Shv}(X)$ so it is sufficient to show that the functor $(f^{-1})_!$ is left exact. The left Kan extension $(f^{-1})_!$ may be defined via a colimit over the comma category $f^{-1} \downarrow V$ or equivalently the category of elements of the functor

$$\mathrm{Hom}_{\mathcal{U}(X)^{op}}(f^{-1}(-), V) : \mathcal{U}(Y) \to \mathbf{Set}$$

for $V \in \mathcal{U}(X)$ an open subset of X ([Rie17, Thm. 6.2.1, Cor. 6.2.6]). These categories are filtered and filtered colimits commute with finite limits so the functor $(f^{-1})_!$ is left exact. □

Thus, given a continuous map $f : X \to Y$ we have obtained a geometric morphism $f = (f^*, f_*) : \mathrm{Shv}(X) \to \mathrm{Shv}(Y)$.

Remark 2.2.2. The construction of the inverse image functor f^* above gives us a definition of the stalk of a sheaf $\mathcal{F}$ at a point $x \in X$. Namely, let $x : \{*\} \hookrightarrow X$ denote the inclusion of x into X. Then since $\mathrm{Shv}(*) \simeq \mathbf{Set}$ we obtain a point of the topos $\mathrm{Shv}(X)$ and by definition the stalk of $\mathcal{F}$ at x is the inverse image $x^*\mathcal{F}$. Since x^* is constructed as left Kan extension we see that

$$\mathcal{F}_x := x^*\mathcal{F} = \varinjlim_{x \in U} \mathcal{F}(U)$$

where the colimit is taken over the full subcategory of $\mathcal{U}(X)^{op}$ spanned by opens containing x.

Remark 2.2.3. Much of the discussion above holds in a much more general setting. Specifically, suppose we have a functor $f : \mathbf{D} \to \mathbf{C}$ where $\mathbf{C}$ and $\mathbf{D}$ have Grothendieck topologies. Furthermore, suppose f respects the Grothendieck topology in the following sense: for every covering family $\{U_i \to U\}_{i \in I}$ in $\mathbf{D}$

(1) $\{f(U_i) \to f(U_i\}_{i \in I})$ is a covering family of $f(U_i)$ in $\mathbf{C}$

(2) for any $V \to U$ the canonical map $f(V \times_U U_i) \to f(T) \times_{f(U)} f(U_i)$ is an isomorphism for all $i \in I$.

Then the precomposition functor

$$\mathrm{PSh}(\mathbf{C}) \xrightarrow{f^*} \mathrm{PSh}(\mathbf{D})$$
$$\mathcal{F} \mapsto \mathcal{F} \circ f$$

restricts to a functor $f_* : \mathrm{Shv}(\mathbf{C}) \to \mathrm{Shv}(\mathbf{D})$. Additionally, if $f_! : \mathrm{PSh}(\mathbf{D}) \to \mathrm{PSh}(\mathbf{C})$ denotes the (pointwise) left Kan extension of the precomposition with f functor, then the composition

$$f^* : \mathrm{Shv}(Y) \subseteq \mathrm{PSh}(Y) \xrightarrow{(f^{-1})_!} \mathrm{PSh}(X) \xrightarrow{\mathbf{a}} \mathrm{Shv}(X)$$

defines a left adjoint to f_* by the same argument as in 2.2.1.

In order to obtain left exactness of f^* additional assumptions are needed. For example, if the left Kan extension is defined over filtered categories or if $\mathbf{C}$ has fiber products and a final object which f respects. See [Sta21, Section 00X0] or [KS06, Thm. 17.5.2].

We now recall the equivalence

$$\Lambda : \mathrm{Shv}(X) \simeq \mathbf{Etale}(X) : \Gamma$$

between sheaves on a topological space X and etale maps $p : E \to X$. For a more thorough discussion of this see [MM12, Chp. II]. We will have a similar equivalence in the case of equivariant sheaves.

Construction 2.2.4. Let X be a topological space and let $\mathbf{Bund}(X) := \mathbf{Top}_{/X}$ the category of spaces (or bundles) over X. We say that a bundle $p : E \to X$ is *etale* over X if it is a local homeomorphism in the sense that for each $e \in E$ there is an open neighborhood $V \subseteq E$ of e such that $p(V) \subseteq X$ is open and the restriction, $p_{|V} : V \to p(V)$, is a homeomorphism. Let $\mathbf{Etale}(X)$ denote the full subcategory of $\mathbf{Bund}(X)$ spanned by the etale maps. Then there is an adjoint pair of functors

$$\begin{tikzcd} \mathrm{PSh}(X) \arrow[r, shift left, "\Lambda"] & \mathbf{Bund}(X) \arrow[l, shift left, "\Gamma"] \end{tikzcd}$$

which restrict to an equivalence

$$\begin{tikzcd} \mathrm{Shv}(X) \arrow[r, shift left, "\Lambda"] & \mathbf{Etale}(X). \arrow[l, shift left, "\Gamma"] \end{tikzcd}$$

Given a bundle $p : Y \to X$ the functor Γ is defined by setting for each $U \subseteq X$ open

$$\Gamma(p)(U) = \{s : U \to Y \mid p \circ s = \mathbf{1}_U : U \subseteq X\}.$$

That is, $\Gamma(p)(U)$ is the set of sections of p over U. For each map $V \subseteq U$ in $\mathcal{U}(X)$ we simply map a section s over V to its restriction to V

$$\begin{aligned} \Gamma(p)(U) &\xrightarrow{\rho_{UV}} \Gamma(p)(V) \\ s &\mapsto s_{|V}. \end{aligned}$$

The functor Λ is defined in the following way. Let $\mathcal{F}$ be a presheaf and

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U)$$

be the stalk of $\mathcal{F}$ at x. We denote an equivalence class of elements of $\mathcal{F}_x$ by $\mathrm{germ}_x s$. Let

$$\Lambda(\mathcal{F}) = \coprod_{x \in X} \mathcal{F}_x$$

and define $p_{\mathcal{F}} : \Lambda(\mathcal{F}) \to X$ by $\Lambda(\mathcal{F})(\mathrm{germ}_x s) = x$. Each $s \in \mathcal{F}(U)$ defines a section

$$\begin{aligned} \tilde{s} : U &\to \Lambda(\mathcal{F}) \\ x &\mapsto \mathrm{germ}_x s. \end{aligned}$$

Finally, topologize $\Lambda(\mathcal{F})$ with the initial topology for $p_{\mathcal{F}}$ and $\tilde{s}$, that is, the minimal topology which makes $p_{\mathcal{F}}$ and all sections $\tilde{s}$ continuous. A basis for this topology is given by the images $\tilde{s}(U) \subseteq \Lambda(\mathcal{F})$.

The unit, $\eta : \mathbf{1} \to \Gamma\Lambda$, of the adjunction is

$$\begin{aligned} \eta_U : \mathcal{F}(U) &\to \Gamma\Lambda(\mathcal{F})(U) \\ s &\mapsto \tilde{s}. \end{aligned}$$

For the counit, $\epsilon : \Lambda\Gamma \to \mathbf{1}$ we observe that for $p : Y \to X$ a bundle $\Lambda\Gamma(p)$ consists of pairs $(\tilde{s}, x)$ where $s : U \to Y$ is a section and $x \in X$. Hence, the counit is defined as

$$\begin{aligned} \epsilon_Y : \Lambda\Gamma(p) &\to Y \\ (\tilde{s}, x) &\mapsto s(x). \end{aligned}$$

Using the equivalence $\mathbf{Etale}(X) \simeq \mathrm{Shv}(X)$ of Construction 2.2.4 there is an alternative definition for the inverse image functor $f^* : \mathrm{Shv}(Y) \to \mathrm{Shv}(X)$. Let $f : X \to Y$ be continuous and $p : E \to Y$ a map. Then there is a functor

$f^* : \mathbf{Etale}(Y) \to \mathbf{Etale}(X)$ defined by pullback of p along f. Then the composition

$$\mathrm{Shv}(Y) \xrightarrow{\Lambda} \mathbf{Etale}(Y) \xrightarrow{f^*} \mathbf{Etale}(X) \xrightarrow{\Gamma} \mathrm{Shv}(X) \tag{2.2.2}$$

is left adjoint to the direct image $f_* : \mathrm{Shv}(X) \to \mathrm{Shv}(Y)$. We will use a similar definition for equivariant sheaves. See [MM12, II.9] for details.

2.3 Equivariant Sheaves

We now turn to defining equivariant sheaves in the case of a discrete group. Our first definition is via etale bundles over a G-space X analogously to sheaves on spaces. We then give a site of definition for this category so that it becomes a Grothendieck topos. This notion of equivariant sheaf was considered in [Gro57, Section. 5]. We will additionally state the alternative definitions of equivariant sheaf in the case of a non-discrete group.

For a discrete group we consider the category of G-spaces as the category of functors $\mathbf{GTop} := \mathrm{Fun}(BG, \mathbf{Top})$ where BG is the category with a single object with morphism set G. Let $X \in \mathrm{Fun}(BG, \mathbf{Top})$ be a G-space, then a G-space over X is simply an object in the over category $\mathbf{Bund}_G(X) := \mathbf{GTop}_{/X}$. In other words a G-space over X consists of a continuous map $p : E \to X$ of topological spaces such that p is G-equivariant as in

$$\begin{tikzcd} G \times E \arrow[r, "\mu"] \arrow[d, "\mathbf{1}_G \times p"'] & E \arrow[d, "p"] \\ G \times X \arrow[r, "\mu"] & X \end{tikzcd}$$

where the horizontal maps are the action maps. We say that p is etale if it is a local homeomorphism as defined in Construction 2.2.4.

Definition 2.3.1. Let G be a (discrete) group and X a G-space. The category of G-sheaves or G-equivariant sheaves on X is the category of etale G-spaces over X, denoted $\mathbf{Etale}_G(X)$.

We now give an equivalent definition in terms of sheaves on a site.

Definition 2.3.2. Let G be a (discrete) group and X a G-space. Define a category $\mathcal{U}_G(X)$ with objects the open subsets of X and morphisms

$$\mathrm{Hom}(U, V) := \{\tilde{g} : U \to V \mid g \in G, g \cdot U \subseteq V\}.$$

Define a Grothendieck topology J on $\mathfrak{U}_G(X)$ by declaring a sieve $S = \{\tilde{g}_i : U_i \to V\}_{i \in I}$ to be a covering sieve if and only if

$$\bigcup_{i \in I} g_i \cdot U_i = V.$$

Remark 2.3.3. We use the notation $\tilde{g} : U \to V$ for the morphisms in $\mathfrak{U}_G(X)$ to emphasize the fact that a morphism is a triple (g, U, V) where $g \in G$, $U, V \subseteq X$ are open, and $g \cdot U \subseteq V$.

Additionally, observe that if X has a trivial G-action, then $\mathrm{Hom}(U, V) = G$ and every $\tilde{g} \in \mathrm{Hom}(U, U)$ is an isomorphism with inverse $\tilde{g}^{-1}$.

Remark 2.3.4. For a topological group G acting continuously, that is, so the action map $\mu : G \times X \to X$ is continuous the etale space definition of equivariant sheaf is still correct with the requirement that the action map $\mu : G \times E \to E$ is continuous for an etale space $p : E \to X$. This category is a Grothendieck topos by [MM12, Appendix Prop. 4]. Since it is a Grothendieck topos it has a site of definition. However, we are not aware of any site of definition for this topos which is as simple as the one in Definition 2.3.2.

Proposition 2.3.5. *The covering sieves of Definition 2.3.2 form a basis for a Grothendieck topology on $\mathfrak{U}_G(X)$.*

Proof. Pullbacks $U_1 \times_U U_2$ in $\mathfrak{U}_G(X)$ along $\tilde{g}_1 : U_1 \to U$ and $\tilde{g}_2 : U_2 \to U$ are given by

$$\begin{tikzcd} P \arrow[r, "\tilde{g}_1^{-1}"] \arrow[d, "\tilde{g}_2^{-1}"] & U_1 \arrow[d, "\tilde{g}_1"] \\ U_2 \arrow[r, "\tilde{g}_2"] & U \end{tikzcd}$$

where $P = g_1 U_1 \cap g_2 U_2$ and $\tilde{g}_i^{-1} : P \to U_i$ indicates the morphism induced by g_i^{-1}.

If $\tilde{g} : U \to V$ is an isomorphism, then it has an inverse $\tilde{h} : V \to U$ where in particular $\tilde{g} \circ \tilde{h} = \mathbf{1}_V$ is the identity function on V so $\tilde{g} : U \to V$ is surjective and (B1) is satisfied.

For (B2), by definition of the pullback and projection maps, we need to show that

$$\bigcup_{i \in I} h^{-1}(g_i U_i \cap hV) = V$$

which is clear as

$$h^{-1}(g_iU_i \cap hV) = \{x \in X \mid x = h^{-1}g_iu_i = h^{-1}hv = v, v \in V\}$$

and $\cup_{i\in I} g_iU_i$ covers U.

Finally, (B3) follows since if $\cup_{i\in I} g_iU_i = U$ and $\cup_{j\in J_i} h_{ij}V_{ij} = U_i$ for each $i \in I$, then certainly $\cup_{i,j} g_ih_{ij}V_{ij} = U$. $\square$

Definition 2.3.6. Let X be a G-space. Denote the category of presheaves on $\mathcal{U}_G(X)$ by $\mathrm{PSh}_G(X)$. The category of G-sheaves or G-equivariant sheaves on X is the category of sheaves on $(\mathcal{U}_G(X), J)$. Denote this category $\mathrm{Shv}_G(X)$.

Remark 2.3.7. In order to distinguish the above definition of equivariant sheaf from the definition of equivariant sheaf for a topological group we will usually refer to the definition of sheaf in 2.3.6 as G-sheaves rather than as G-equivariant sheaves.

Remark 2.3.8. Just as $\mathrm{Shv}(*) = \mathbf{Set}$ is the category of sets we have that $\mathrm{Shv}_G(*)$ is the category of G-sets where $*$ is the one-point space with necessarily trivial G-action. This follows since for a point the category $\mathcal{U}_G(*)$ has $\mathrm{Hom}(*, *) = G$.

Theorem 2.3.9. *If X is a G-space, then there is an equivalence of categories* $\mathbf{Etale}_G(X) \simeq \mathrm{Shv}_G(X)$

Proof. Recall from Construction 2.2.4 the adjoint functors

$$\begin{tikzcd} \mathrm{PSh}(X) \arrow[r, shift left, "\Lambda"] & \mathbf{Bund}(X) \arrow[l, shift left, "\Gamma"] \end{tikzcd}$$

which restrict to an equivalence between the category, $\mathrm{Shv}(X)$ of sheaves on X, and the category, $\mathbf{Etale}(X)$ of etale spaces over X. We claim that this directly generalizes to the desired equivalence.

Namely, let $p : E \to X$ denote a G-equivariant continuous map of G-spaces. Then using Γ we obtain an assignment

$$\Gamma(p) : U \mapsto \Gamma(p)(U) = \{s : U \to E \mid p \circ s = \mathbf{1}_U : U \subseteq X\}.$$

We extend $\Gamma(p)$ to a functor on $\mathcal{U}_G(X)^{op}$ by defining

$$\begin{aligned} \Gamma(p)(\tilde{g}) : \Gamma(p)(U) &\to \Gamma(p)(V) \\ s &\mapsto s^g. \end{aligned}$$

Here s^g denotes the composite

$$s^g : V \xrightarrow{\mu_{VU}(g)} U \xrightarrow{s} E \xrightarrow{\mu_E(g^{-1})} E$$

where $\mu_E(g^{-1})$ is the automorphism of E corresponding to g^{-1} and $\mu_{VU}(g)$ is the map from $V \to U$ induced by the action of g on E as $gV \subseteq U$ by assumption. It follows that s^g is a section of V as p is equivariant. Given a G-equivariant map $f : X \to Y$ the functor Γ induces a map $\Gamma(p) \to \Gamma(q)$ defined as precomposition with f. It follows by definition of s^g that if f is equivariant then the induced map is a natural transformation of presheaves. Thus, we have a functor $\Gamma_G : \mathbf{Bund}_G(X) \to \mathrm{PSh}_G(X)$. The functor Γ_G is in fact valued in sheaves as the sheaf condition holds for $\Gamma_G(p)$. Indeed given $s_i \in \Gamma_G(p)(U_i)$ for some covering family $\{\tilde{g}_i : U_i \to U\}_{i \in I}$ such that $s_i^{g_i^{-1}} = s_j^{g_j^{-1}}$ for all $(i,j) \in I \times I$. Then we need a unique section $s : U \to Y$ such that $\Gamma_G(p)(\tilde{g}_i)(s) = s_i$ for all $i \in I$. Now for any $u \in U$ we have $u \in g_i U_i$ for some i. Hence, consider

$$\begin{aligned} s : U &\to E \\ u &\mapsto g_i s_i(g_i^{-1} u). \end{aligned}$$

which is well-defined, a section, and continuous.

For the functor $\Lambda_G : \mathrm{PSh}_G(X) \to \mathbf{Bund}_G(X)$ we simply extend the functor Λ from Construction 2.2.4. Given a presheaf $\mathcal{F} \in \mathrm{PSh}_G(X)$ we may apply Λ to obtain a map $p_{\mathcal{F}} : \Lambda(\mathcal{F}) \to X$ which is etale if $\mathcal{F}$ is a sheaf. Now we give $\Lambda(\mathcal{F})$ a G-action in the following way. For $g \in G$ we let $\tilde{g}_U^{-1}$ denote the corresponding map from $gU \to U$ in $\mathcal{U}_G(X)$. Note that this is an isomorphism with inverse $\tilde{g} : U \to gU$. If $s \in \mathcal{F}(U)$ is a section, then let $gs := \mathcal{F}(\tilde{g}_U^{-1})(s)$. This allows us to define an action on $\Lambda(\mathcal{F})$ in the following way:

$$\begin{aligned} \mu : G \times \Lambda(\mathcal{F}) &\to \Lambda(\mathcal{F}) \\ (g, \mathrm{germ}_x s) &\mapsto \mathrm{germ}_{gx} gs. \end{aligned}$$

The action is well-defined since if $\mathrm{germ}_x s = \mathrm{germ}_x t$ for $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$, then there exists a neighborhood W of x with $W \subseteq U \cap V$ such that $s_{|W} = t_{|W}$. Now we have compositions

$$\mathcal{F}(U) \xrightarrow{\mathcal{F}(\tilde{g}_U^{-1})} \mathcal{F}(gU) \xrightarrow{\mathcal{F}(\tilde{e})} \mathcal{F}(gW)$$

and similarly for V. In particular, the composition is precisely the restriction of $\mathcal{F}(\tilde{g}_U^{-1})(s) = gs$ and $\mathcal{F}(\tilde{g}_V^{-1})(t) = gt$ to $\mathcal{F}(gW)$. Now the claim is that $gs_{|gW} = gt_{|gW}$. This follows from the fact that $\mathcal{F}(\tilde{g}_W)$ is an isomorphism and

$$\mathcal{F}(\tilde{g}_W)(gs_{|gW}) = \mathcal{F}(\tilde{g}_W) \circ \mathcal{F}(\tilde{e}) \circ \mathcal{F}(\tilde{g}_U^{-1})(s) = s_{|W}$$

and similarly for t. It is easily checked that this defines a group action using the fact that $\mathcal{F}$ is contravariant and composition in $\mathcal{U}_G(X)$ is induced by composition in G. It follows that we have bijections $\mu(g) : \Lambda(\mathcal{F}) \to \Lambda(\mathcal{F})$ for all $g \in G$. One readily checks that the maps $\mu(g)$ are all homeomorphisms of $\Lambda(\mathcal{F})$ and that $p_{\mathcal{F}}$ is equivariant.

We now check that this defines a functor $\Lambda_G : \mathrm{PSh}_G(X) \to \mathbf{Bund}_G(X)$. Namely, for a map of sheaves $\alpha : \mathcal{F} \to \mathcal{G}$ we need only ensure that the induced map $f_\alpha : \Lambda(\mathcal{F}) \to \Lambda(\mathcal{G})$ is equivariant. This follows from the fact that f_α is the disjoint union of the stalk maps for α and α is a natural transformation of functors on $\mathrm{PSh}_G(X)$. In other words α commutes with the maps $\mathcal{F}(\tilde{g})$ and taking stalks. Thus, we have a functor

$$\Lambda_G : \mathrm{PSh}_G(X) \to \mathbf{Bund}_G(X)$$

and if $\mathcal{F}$ is already a sheaf then $\Lambda(\mathcal{F})$ is etale over X.

We finally check that we have unit and counit maps so that we obtain an adjunction $\Gamma_G \dashv \Lambda_G$ which will restrict to an equivalence between $\mathrm{Shv}_G(X)$ and $\mathbf{Etale}_G(X)$. For the unit map $\eta : \mathbf{1} \to \Gamma_G \circ \Lambda_G$ we simply use the same map as in Construction 2.2.4. Namely, we have maps

$$\begin{aligned} \eta_U : \mathcal{F}(U) &\to \Gamma_G \Lambda_G(\mathcal{F})(U) \\ s &\mapsto (\tilde{s} : x \mapsto \mathrm{germ}_x s). \end{aligned}$$

From the definitions it is easily checked that these form a natural transformation of functors on $\mathcal{U}_G(X)^{op}$. Further, if $\mathcal{F}$ is already a sheaf, then this map is an isomorphism since it is an isomorphism in the non-equivariant setting. Similar reasoning applies in defining the counit map for a bundle $Y \to X$

$$\begin{aligned} \varepsilon_Y : \Lambda_G \Gamma_G Y &\to Y \\ \tilde{s}x &\mapsto s(x). \end{aligned}$$

$\square$

Corollary 2.3.10. *The inclusion* $\mathrm{Shv}_G(X) \subseteq \mathrm{PSh}_G(X)$ *has a left adjoint (the subcategory is reflective) and the inclusion* $\mathbf{Etale}_G(X) \subseteq \mathbf{Bund}_G(X)$ *has a right adjoint (is a coreflective subcategory).*

Proof. This follows from a general fact about adjunctions restricting to equivalences. See [MM12, Lemma. II.6.4]. □

2.4 Equivariant Sheaves and Geometric Morphisms

We now define direct and inverse image functors given a continuous G-equivariant map $f : X \to Y$ of G-spaces. First, observe that the induced functor $f^{-1} : \mathcal{U}(Y) \to \mathcal{U}(X)$ is equivariant in the sense that $gf^{-1}(U) = f^{-1}(gU)$.

Hence, if $gU \subseteq V$, then $gf^{-1}(U) = f^{-1}(gU) \subseteq f^{-1}(V)$. It follows we obtain an extension of the functor f^{-1} to a functor

$$\begin{aligned} f^{-1} : \mathcal{U}_G(Y) &\to \mathcal{U}_G(X) \\ U &\mapsto f^{-1}(U). \end{aligned}$$

As in Remark 2.2.3 this functor f^{-1} respects the Grothendieck topology in the following sense:

(1) If $\{V_i \xrightarrow{\tilde{g}_i} V\}_{i \in I}$ is a cover of V on Y, then $\{f^{-1}(V_i) \xrightarrow{\tilde{g}_i} f^{-1}(V)\}_{i \in I}$ is a cover of $f^{-1}(V)$ in X.

(2) f^{-1} respects pullbacks: since f^{-1} is equivariant, then

$$gf^{-1}(U) \cap g_i f^{-1}(V_i) = f^{-1}(gU) \cap f^{-1}(g_i V_i) = f^{-1}(gU \cap g_i V_i).$$

So we may define the direct image functor f_* by precomposition with f^{-1}.

Definition 2.4.1. Let $f : X \to Y$ be a continuous equivariant map of G-spaces. The *equivariant direct image* or *equivariant pushforward* is the functor

$$\begin{aligned} f_* : \mathrm{Shv}_G(X) &\to \mathrm{Shv}_G(Y) \\ \mathcal{F} &\mapsto \mathcal{F} \circ f^{-1} \end{aligned}$$

For the inverse image sheaf we may define it through a left Kan extension. However, due to the lack of certain properties of the category $\mathcal{U}_G(X)$, for example it has no terminal object, the general theorems which imply that the left Kan extension $(f^{-1})_!$ of $(f^{-1})^*$ is left exact do not apply. Hence, using the equivalence

$$\mathrm{Shv}_G(X) \simeq \mathbf{Etale}_G(X)$$

from Theorem 2.3.9 we instead define the inverse image via a composition

$$f^* : \mathrm{Shv}_G(Y) \xrightarrow{\Lambda_G} \mathbf{Etale}_G(Y) \xrightarrow{f^*} \mathbf{Etale}_G(X) \xrightarrow{\Gamma_G} \mathrm{Shv}_G(X)$$

as in 2.2.2. Once we see that this functor is a left adjoint to the usual definition of inverse image via left Kan extension it follows that they are naturally isomorphic.

We now make this precise. The category **GTop** has pullbacks and these may be calculated in **Top** with the obvious action. Namely,

$$\begin{aligned} A \times_X B &= \{(a,b) \subseteq A \times B \mid f(a) = g(b)\} \\ g(a,b) &= (ga, gb) \end{aligned}$$

with the initial topology with respect to the projection maps. It follows that given $f : X \to Y$ an equivariant map we obtain a base change functor

$$f^* : \mathbf{Bund}_G(Y) \to \mathbf{Bund}_G(X)$$

which always has a left adjoint ([MM12, Thm. I.9.4]) so in particular respects limits. Further, by Lemma [MM12, II.9.1] if $p : E \to Y$ is etale, then the induced map $f^*E \to X$ is etale over X so f^* restricts to a functor

$$f^* : \mathbf{Etale}_G(Y) \to \mathbf{Etale}_G(X).$$

Define a functor

$$f^* : \mathrm{Shv}_G(Y) \xrightarrow{\Lambda_G} \mathbf{Etale}_G(Y) \xrightarrow{f^*} \mathbf{Etale}_G(X) \xrightarrow{\Gamma_G} \mathrm{Shv}_G(X).$$

Proposition 2.4.2. *The functor $f^* : \mathrm{Shv}_G(Y) \to \mathrm{Shv}_G(X)$ is a left exact left adjoint to the equivariant direct image functor $f_* : \mathrm{Shv}_G(X) \to \mathrm{Shv}_G(Y)$.*

Proof. We begin with left exactness. To show this it is sufficient to show that $f^* : \mathbf{Etale}_G(Y) \to \mathbf{Etale}_G(X)$ commutes with products and equalizers in $\mathbf{Etale}_G(X)$. However, the inverse image functor f^* is simply the restriction of the base change functor f^* which is a right adjoint so it is sufficient to show that $\mathbf{Etale}_G(X) \subseteq \mathbf{Bund}_G(X)$ is closed under finite limits. Now if $p : A \to X$ and $q : B \to X$ are etale over X, then their pullback is their product in $\mathbf{Bund}_G(X)$. However, the pullback of an etale map is etale so $\mathbf{Etale}_G(X)$ is closed under products. The equalizer in $\mathbf{Bund}(X)$ is simply the equalizer in $\mathbf{Top}$ and it follows that the equalizer in $\mathbf{Bund}_G(X)$ is simply the equalizer in $\mathbf{Bund}(X)$ with the obvious action. By [MM12, Prop. II.9.3] this is etale and so we are done as the inclusion $\mathbf{Etale}_G(X) \subseteq \mathbf{Bund}_G(X))$ clearly also preserves the terminal object given by the identity map $\mathbf{1}_X : X \to X$.

The fact that the composition

$$f^* : \mathrm{Shv}_G(Y) \xrightarrow{\Lambda_G} \mathbf{Etale}_G(Y) \xrightarrow{f^*} \mathbf{Etale}_G(X) \xrightarrow{\Gamma_G} \mathrm{Shv}_G(X).$$

is left adjoint to f_* may be shown in precisely the same way as for the non-equivariant case. For details see [MM12, Thm. II.9]. $\square$

Corollary 2.4.3. *If $f : X \to Y$ is an equivariant map, then f induces a geometric morphism $f = (f^*, f_*) : \mathrm{Shv}_G(X) \to \mathrm{Shv}_G(Y)$.*

Proposition 2.4.4. *Let X be a G-space and let $\pi^* : \mathrm{Shv}_G(X) \to \mathrm{Shv}(X)$ denote the obvious forgetful functor. Then π^* determines a geometric morphism*

$$\pi = (\pi^*, \pi_*) : \mathrm{Shv}(X) \to \mathrm{Shv}_G(X).$$

Proof. Per Remark 2.1.16 a topos is locally presentable so it is sufficient to show that π^* is left exact and respects colimits. By the equivalence of Theorem 2.3.9 it will be sufficient to show that the forgetful functor $\mathbf{Etale}_G(X) \xrightarrow{\pi^*} \mathbf{Etale}(X)$ satisfies these properties. This functor is clearly left exact as the terminal object in $\mathbf{Etale}_G(X)$ is the identity map $\mathbf{1}_X$ which is the terminal object in $\mathbf{Etale}(X)$. An equalizer in $\mathbf{Etale}_G(X)$ is simply the equalizer in $\mathbf{Etale}(X)$ with the obvious action so is clearly the equalizer in $\mathbf{Etale}(X)$ after forgetting. Finally, products in $\mathbf{Etale}_G(X)$ and $\mathbf{Etale}(X)$ are calculated the same way. Thus, the forgetful functor respects the terminal object, equalizers, and products so it respects all finite limits.

By Corollary 2.3.10 since $\mathbf{Etale}_G(X)$ is a coreflective subcategory of $\mathbf{Bund}_G(X)$, then a colimit in $\mathbf{Etale}_G(X)$ is simply a colimit in $\mathbf{Bund}_G(X)$. However, a colimit in $\mathbf{Bund}_G(X) = \mathbf{GTop}_{/X}$ is simply a colimit in $\mathbf{GTop}$ with the induced map to X and colimits in $\mathbf{GTop}$ are simply the colimit in $\mathbf{Top}$ with an induced action. The result follows. $\square$

Corollary 2.4.5. *Let X be a G-space. Then every $x \in X$ determines a point of the topos $\mathrm{Shv}_G(X)$.*

Proof. Let $x \in X$, then the inclusion $x : \{x\} \hookrightarrow X$ determines a geometric morphism

$$x = (x^*, x_*) : \mathrm{Shv}_G(\{x\}) \to \mathrm{Shv}_G(X)$$

where $\{x\}$ is the space with trivial action. Using Proposition 2.4.4 we obtain a geometric morphism

$$\mathbf{Set} = \mathrm{Shv}(\{x\}) \xrightarrow{\pi=(\pi^*,\pi_*)} \mathrm{Shv}_G(\{x\}) \xrightarrow{x=(x^*,x_*)} \mathrm{Shv}_G(X).$$

□

Proposition 2.4.6. *Let X be a G-space and let $x : \mathrm{Shv}_G(\{x\}) \to \mathrm{Shv}_G(X)$ the geometric morphism corresponding to a point $x \in X$. If $\mathcal{F} \overset{\alpha}{\to} \mathcal{G}$ is a map of sheaves in $\mathrm{Shv}_G(X)$, then α is an isomorphism if and only if $x^*\alpha : x^*\mathcal{F} \to x^*\mathcal{G}$ is an isomorphism for all $x \in X$.*

Proof. This holds for the same reasons it holds for sheaves of spaces. □

Corollary 2.4.7. *The topos $\mathrm{Shv}_G(X)$ has enough points.*

Proof. We need to show that if for any $f : \mathcal{F} \to \mathcal{G}$ in $\mathrm{Shv}_G(X)$ it holds that for every (topos) point $x : \mathbf{Set} \to \mathrm{Shv}_G(X)$ that the morphism of stalks $x^*f : x^*\mathcal{F} \to x^*\mathcal{G}$ is an isomorphism, then f is an isomorphism. The result follows from Proposition 2.4.6 as this holds for the (topos) points which arise from points $x \in X$. □

Proposition 2.4.8. *If k is a commutative ring, G a finite group, and Y a space with trivial G-action, then there is an equivalence of categories*

$$\mathrm{Shv}(Y, kG) \simeq \mathrm{Shv}_G(Y, k)$$

between G-equivariant sheaves of k-modules and sheaves of kG-modules.

Remark 2.4.9. In fact in [Gro57, 5.1] a stronger claim is made. Namely, that for $\mathcal{O}$ a G-sheaf of rings on Y such that the group acts trivially on $\mathcal{O}$, then

$$\mathbf{Mod}(\mathrm{Shv}_G(Y), \mathcal{O}) \simeq \mathbf{Mod}(\mathrm{Shv}(Y), \mathcal{O}')$$

where $\mathcal{O}' = \mathcal{O} \otimes_{\underline{\mathbb{Z}}} \underline{\mathbb{Z}G}$ is the sheaf with $\underline{R}$ denoting the constant sheaf associated to a ring R. One should be able to prove this using the same method here, but we have not checked all the details.

Proof. We construct a pair of functors

$$\mathrm{Shv}(Y, kG) \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \mathrm{Shv}_G(Y, k)$$

along with natural isomorphisms $\eta : \mathbf{1}_{\mathrm{Shv}(Y,kG)} \xrightarrow{\cong} \Phi\Psi$ and $\varepsilon : \Psi\Phi \xrightarrow{\cong} \mathbf{1}_{\mathrm{Shv}_G(Y,k)}$.

We begin with the functor Φ. Let $\mathcal{F} \in \mathrm{Shv}_G(Y)$, then there is an obvious forgetful functor to $\mathrm{Shv}(Y)$ by letting $\Phi(\mathcal{F})(U) := \mathcal{F}(U)$ and restriction maps

$$\rho_{VU} := \mathcal{F}(\tilde{e}) : \Phi(\mathcal{F})(U) \to \Phi(\mathcal{F})(V)$$

for e the identity element of G. If $\mathcal{F} \in \mathrm{Shv}_G(Y, k)$, then this is also a sheaf of k-modules so it remains only to ensure that we have a G-action on each $\Phi(\mathcal{F})(U)$ and that the restriction maps are G-equivariant. Using the isomorphisms $\mathcal{F}(U) \xrightarrow{\tilde{g}} \mathcal{F}(U)$ define a G-action

$$\begin{aligned} \mu_U : G \times \Phi(\mathcal{F})(U) &\to \Phi(\mathcal{F})(U) \\ (g, s) &\mapsto \mathcal{F}(\tilde{g}^{-1})(s). \end{aligned} \tag{2.4.1}$$

It follows that the restriction maps are equivariant. Explicitly, let $\mu_U(g)$ denote the automorphism of $\Phi(\mathcal{F})(U)$ induced by $g \in G$, then

$$\begin{aligned} \mu_U(g) \circ \rho_{UV} &= \mathcal{F}(\tilde{g}^{-1}) \circ \mathcal{F}(\tilde{e}) = \mathcal{F}(\tilde{e} \circ \tilde{g}^{-1}) = \mathcal{F}(\tilde{g}^{-1} \circ \tilde{e}) \\ &= \mathcal{F}(\tilde{e}) \circ \mathcal{F}(\tilde{g}^{-1}) = \rho_{UV} \circ \mu_V(g). \end{aligned}$$

On morphisms, $\alpha : \mathcal{F} \to \mathcal{G}$, in $\mathrm{Shv}_G(Y, k)$ define

$$\Phi(\alpha) := \alpha \tag{2.4.2}$$

Since α is a natural transformation of G-sheaves it follows that the component maps are equivariant by definition of the action. Hence, we obtain a functor $\mathrm{Shv}_G(Y, k) \xrightarrow{\Phi} \mathrm{Shv}(Y, kG)$.

Now for a functor $\Psi : \mathrm{Shv}(Y, kG) \to \mathrm{Shv}_G(Y, k)$. Let $\mathcal{F} \in \mathrm{Shv}(Y, kG)$ be a sheaf. We must extend $\mathcal{F}$ from $\mathcal{U}(Y)$ to $\mathcal{U}_G(Y)$. Let μ_U denote the G-action map and $\mu_U(g)$ the corresponding automorphism for $\mathcal{F}(U)$. Define a functor $\Psi(\mathcal{F}) : \mathcal{U}_G(Y)^{op} \to \mathbf{Mod}(k)$ by $\Psi(\mathcal{F})(U) = \mathcal{F}(U)$ and for $\tilde{g} : U \to V$ the structure maps $\Psi(\mathcal{F})(\tilde{g})$ are the composition

$$\mathcal{F}(V) \xrightarrow{\rho_{VU}} \mathcal{F}(U) \xrightarrow{\mu_U(g^{-1})} \mathcal{F}(U). \tag{2.4.3}$$

This defines an extension of $\mathcal{F}$ to $\mathcal{U}_G(Y)$ since the restriction maps are equivariant. Explicitly, let $\tilde{g} : U \to V$ and $\tilde{h} : V \to W$, then

$$\Psi(\mathcal{F})(\tilde{h} \circ \tilde{g}) = \mu_U((hg)^{-1}) \circ \rho_{WU} = \mu_U(g^{-1}) \circ \rho_{VU} \circ \mu_V(h^{-1}) \circ \rho_{WV}.$$

For a natural transformation $\eta : \mathcal{F} \to \mathcal{G}$ of sheaves in $\mathrm{Shv}(Y, kG)$ we define

$$\Psi(\eta) = \eta. \tag{2.4.4}$$

Note that this defines a natural transformation of functors $\Psi(\mathcal{F}) \to \Psi(\mathcal{G})$ as the component maps, $\eta_U : \mathcal{F}(U) \to \mathcal{G}(U)$, are maps of kG-modules so equivariant.

We now construct the natural isomorphisms $\eta : \mathbf{1}_{\mathrm{Shv}(Y,kG)} \cong \Phi\Psi$ and $\varepsilon : \Psi\Phi \cong \mathbf{1}$. Let $\mathcal{F} \in \mathrm{Shv}(Y, kG)$ and define $\eta_{\mathcal{F}} : \mathcal{F} \to (\Phi\Psi)(\mathcal{F})$ by

$$\eta_{\mathcal{F}(U)} : \mathcal{F}(U) \xrightarrow{\text{identity}} (\Phi\Psi)(\mathcal{F})(U) = \mathcal{F}(U). \tag{2.4.5}$$

This is an isomorphism of k-modules so it remains only to check that $\eta_{\mathcal{F}(U)}$ is equivariant, that is, by definition of Ψ and Φ we need to show that for $g \in G$

$$\Psi(\mathcal{F})(\tilde{g}^{-1}) \circ \eta_{\mathcal{F}(U)} = \eta_{\mathcal{F}(U)} \circ \mu_U(g).$$

Indeed this holds as $\eta_{\mathcal{F}(U)}$ is the identity and by (2.4.3), $\Psi(\mathcal{F})(\tilde{g}^{-1}) = \mu_U(g)$. Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a map of sheaves, then the square

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\eta_{\mathcal{F}(U)}"] \arrow[d, "\alpha_U"'] & (\Phi\Psi)(\mathcal{F})(U) \arrow[d, "(\Phi\Psi)(\alpha)_U"] \\
\mathcal{G}(U) \arrow[r, "\eta_{\mathcal{G}(U)}"] & (\Phi\Psi)(\mathcal{G})(U)
\end{tikzcd}$$

commutes by definition of η and $(\Phi\Psi)(\alpha)_U$ so η is natural. Therefore, η is natural isomorphism $\mathbf{1} \to \Phi\Psi$.

For

$$\varepsilon : \Psi\Phi \to \mathbf{1}_{\mathrm{Shv}_G(Y,k)}$$

given $\mathcal{F} \in \mathrm{Shv}_G(Y, k)$ define $\varepsilon_{\mathcal{F}(U)}$ as

$$\varepsilon_{\mathcal{F}(U)} : (\Psi\Phi)(\mathcal{F})(U) \xrightarrow{\text{identity}} \mathcal{F}(U)$$

which is an isomorphism of k-modules. Let $\tilde{g} : V \to U$ be a morphism in $\mathcal{U}_G(Y)$, then the square

$$\begin{tikzcd} (\Psi\Phi)(\mathcal{F})(U) \arrow[r, "\varepsilon_{\mathcal{F}(U)}"] \arrow[d, "(\Psi\Phi)(\mathcal{F})(\tilde{g})"'] & \mathcal{F}(U) \arrow[d, "\mathcal{F}(\tilde{g})"] \\ (\Psi\Phi)(\mathcal{F})(V) \arrow[r, "\varepsilon_{\mathcal{F}(U)}"] & \mathcal{F}(U) \end{tikzcd}$$

commutes since

$$(\Psi\Phi)(\mathcal{F})(\tilde{g}) = \mu_U(g^{-1}) \circ \rho_{UV} = \mathcal{F}((\tilde{g}^{-1})^{-1}) \circ \mathcal{F}(\tilde{e}) = \mathcal{F}(\tilde{g}).$$

Finally, ε is natural since for a map $\alpha : \mathcal{F} \to \mathcal{G}$ of G-sheaves we have $(\Psi\Phi)(\alpha) = \alpha$ so

$$\begin{tikzcd} (\Psi\Phi)(\mathcal{F})(U) \arrow[r, "\varepsilon_{\mathcal{F}(U)}"] \arrow[d, "(\Psi\Phi)(\alpha)_U"'] & \mathcal{F}(U) \arrow[d, "\alpha_U"] \\ (\Psi\Phi)(\mathcal{G})(U) \arrow[r, "\varepsilon_{\mathcal{G}(U)}"] & \mathcal{G}(U) \end{tikzcd}$$

commutes. Therefore, $\varepsilon : \Psi\Phi \to \mathbf{1}_{\mathrm{Shv}_G(Y,k)}$ is a natural isomorphism and it follows that $\mathrm{Shv}_G(Y, k)$ and $\mathrm{Shv}(Y, kG)$ are equivalent for a space Y with trivial G-action. $\square$

Corollary 2.4.10. *For G a finite group there is an equivalence* $\mathrm{Shv}_G(*, k) \simeq \mathbf{Mod}(kG)$.

We conclude this section by discussing two other definitions of the category of equivariant sheaves. If the group G is a topological group, then these definitions may not agree with the earlier definition. However, in the case of a discrete group all three definitions agree. For this material our primary source is [BL06] while the material on simplicial spaces and sites comes from the stacks project [Sta21, Tag 09VI].

Let G be a group and X a G-space. Consider the simplicial space $X//G$ where $(X//G)_n = G^n \times X$ with face maps

$$d_i(g_1, \ldots, g_n, x) = \begin{cases} (g_2, \ldots, g_n, g_1^{-1}x), & i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n, x) & , \ 1 \leq i \leq n-1 \\ (g_1, \ldots, g_{n-1}, x), & i = n \end{cases} \tag{2.4.6}$$

and degeneracy maps

$$s_i(g_1, \dots, g_n, x) = (g_1, \dots, g_i, e, g_{i+1}, \dots, g_n, x).$$

where e denotes the identity element of G. Consider the truncation of this diagram to the following

$$\begin{tikzcd} G \times G \times X \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1"] \arrow[r, shift right=2, "d_2"'] & G \times X \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & X \arrow[l, "s_0"'] \end{tikzcd}$$

diagram. Observe that d_0 is the action map for X while d_n is projection.

Definition 2.4.11. A G-equivariant sheaf on X is a pair $(\mathcal{F}, \theta)$ where $\mathcal{F} \in \mathrm{Shv}(X)$ is a sheaf on X and θ is an isomorphism

$$\theta : d_0^*\mathcal{F} \xrightarrow{\cong} d_1^*\mathcal{F}$$

which satisfies the cocycle condition

$$d_0^*\theta \circ d_2^*\theta = d_1^*\theta = d_1^*\theta,\ \ s_0^*\theta = \mathbf{1}_{\mathcal{F}}.$$

A morphism of equivariant sheaves $f : (\mathcal{F}, \theta_{\mathcal{F}}) \to (\mathcal{G}, \theta_{\mathcal{G}})$ is a morphism $f : \mathcal{F} \to \mathcal{G}$ of sheaves such that

$$d_1^* f \circ \theta_{\mathcal{F}} = \theta_{\mathcal{G}} \circ d_0^* f.$$

We denote this category $\mathrm{Shv}_G(X)$.

Proposition 2.4.12. *If G is a discrete group, then the category $\mathrm{Shv}(\mathcal{U}_G(X))$ of G-sheaves on X and the category $\mathrm{Shv}_G(X)$ of G-equivariant sheaves on X are equivalent.*

The third and final definition of G-equivariant sheaf makes use of the entire simplicial space $X//G$ rather than just the truncation to degree two. However, this definition requires a bit more work. Recall, from Example 2.1.23 given a simplicial space $X_\bullet : \Delta^{op} \to \mathbf{Top}$ we may construct a site X^{Zar} for the simplicial space $X//G$. Then a sheaf $\mathcal{F}$ on X^{Zar} is given by:

(1) A collection $\{\mathcal{F}_n\}_{n \geq 0}$ with each $\mathcal{F}_n$ a sheaf on X_n.

(2) A collection $\{\mathcal{F}(\varphi)\}_{\varphi \in \mathrm{Mor}(\Delta)}$ where $\mathcal{F}(\varphi)$ for $\varphi : [m] \to [n]$ is a map

$$\mathcal{F}(\varphi) : f_\varphi^* \mathcal{F}_m \to \mathcal{F}_n$$

satisfying the composition condition

$$\begin{tikzcd} f_{\psi\varphi}^* \mathcal{F}_l \arrow[rr, "\mathcal{F}(\psi \circ \varphi)"] \arrow[dr, "f_\varphi^* \mathcal{F}(\psi)"'] & & \mathcal{F}_n \\ & f_\varphi^* \mathcal{F}_m \arrow[ur, "\mathcal{F}(\varphi)"'] & \end{tikzcd}$$

for $\psi : [l] \to [m]$ and $\varphi : [m] \to [n]$ and where $f_\varphi : X_n \to X_m$ is the associated continuous map.

Definition 2.4.13. Let $X_\bullet : \Delta^{op} \to \mathbf{Top}$ be a simplicial space. A sheaf $\mathcal{F}$ on X^{Zar} is said to be *cartesian* if for all morphisms $\varphi : [m] \to [n]$ in Δ the structure maps $\mathcal{F}(\varphi) : f_\varphi^* \mathcal{F}_m \to \mathcal{F}_n$ are isomorphisms. Denote the full subcategory of $\mathrm{Shv}(X^{Zar})$ spanned by Cartesian sheaves by $\mathrm{Shv}_{eq}(X^{Zar})$.

Lemma 2.4.14 ([Sta21, Lemma 07TG])**.** *Let $d_i : X_n \to X_{n-1}$ denote the i^{th} face map. If $\mathcal{F}$ is a sheaf on X^{Zar} for $X_\bullet$ a simplicial space, then $\mathcal{F}$ is cartesian if and only if the structure maps $\mathcal{F}(\delta_i) : d_i^* \mathcal{F}_{n-1} \to \mathcal{F}_n$ are isomorphisms for all i and n.*

Proof. Recall, that Δ is generated by the coface and codegeneracy maps $\delta_i : [n-1] \to [n]$ and $\sigma_j : [n] \to [n-1]$. Hence, the only if direction is clear and we need only check that the structure maps $\mathcal{F}(\delta_i) : d_i^* \mathcal{F}_{n-1} \to \mathcal{F}_n$ and $\mathcal{F}(\sigma_j) : s_j^* \mathcal{F}_n \to \mathcal{F}_{n-1}$ are isomorphisms. Now since $\sigma_j \delta_i = \mathbf{1}_{[n]}$ for $i = j$ and $i = j + 1$ we get that that $\mathcal{F}(\sigma_j \circ \delta_j)$ is the identity. Therefore, if $\mathcal{F}(\delta_i)$ are all isomorphisms the result follows by the 2-of-3 for property for isomorphisms. □

Proposition 2.4.15. *Let X be a G-space and $X//G$ the corresponding simplicial space. The category $\mathrm{Shv}_G(X)$ of G-equivariant sheaves on X is equivalent to the category $\mathrm{Shv}_{eq}(X//G)$ of cartesian sheaves on $X//G$.*

Proof. See [Sta21, Lemma 0D7I, Lemma 0D7J] or [Del74, 6.1.2(b)]. □

2.5 Constructible Sheaves

We briefly discuss constructible sheaves which will be used to define the equivariant constructible derived category. Let X be a space and k a ring. Let $\mathrm{Shv}(\mathbf{C}, k)$ denote the category of sheaves of k-modules on a site $(\mathbf{C}, J)$ and $\mathrm{Shv}(X, k)$ the category of sheaves of k-modules on X.

Definition 2.5.1. A sheaf $\mathcal{F} \in \mathrm{Shv}(\mathbf{C}, k)$ is said to be *constant* if $\mathcal{F}$ is equivalent to the sheafification of a constant presheaf $P : U \mapsto A$.

Remark 2.5.2. Let $p : X \to *$ denote the map to the terminal object, then a sheaf $\mathcal{F}$ on X is constant if and only if $\mathcal{F}$ lies in the essential image of the inverse image functor p^*.

Let $j : U \subseteq X$ be a subspace, then the restriction of a sheaf $\mathcal{F}$ on X to U is $\mathcal{F}_{|U} := j^*\mathcal{F}$ the inverse image of the inclusion.

Definition 2.5.3. Let X be a topological space. A sheaf $\mathcal{F} \in \mathrm{Shv}(X, k)$ is said to be *locally constant* if for every $U \subseteq X$ open there is a cover $\{U_i \to U\}$ of U such that $\mathcal{F}_{|U_i}$ is a constant sheaf.

Definition 2.5.4. Let X be a compact Hausdorff space, then the *open cone* on X is defined to be the space

$$\mathring{c}X = X \times [0, 1)/(X \times \{0\}).$$

We define topologically stratified pseudomanifolds spaces inductively.

Definition 2.5.5. Let X be a topological space. Then:

- X is a 0-dimensional *topologically stratified space* if it is a countable set of points with the discrete topology.

- X is a n-dimensional *topological stratified pseudomanifold* if it is a paracompact Hausdorff space with a filtration

  $$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subseteq X_n = X$$

  by closed subspaces X_j such that the local normal triviality condition holds: For each $x \in S_J$ there is a neighborhood U of $x \in X$ and a compact topological stratified pseudomanifold L of dimension $n - j - 1$ with stratification

  $$\emptyset = L_{-1} \subset L_0 \subset L_1 \subset \cdots \subset L_{i-3} \subset L_{i-1} = L$$

and a homeomorphism

$$\phi : U \xrightarrow{\cong} \mathbb{R}^j \times \mathring{c}L$$

such that ϕ is a homeomorphism of $U \cap X_{i+j+1}$ onto $\mathbb{R}^j \times \mathring{c}L_i$ for $0 \leq i \leq n-j-1$ and

$$\phi : U \cap X_j \xrightarrow{\cong} \mathbb{R}^j \times \{x\}$$

is a homeomorphism.

If the filtration on X also satisfies the condition $X_{n-1} = X_{n-2}$ so $X \setminus X_{n-2}$ is dense in X, then X is said to be a *topologically stratified pseudomanifold.*

Example 2.5.6.
- Every manifold is a topologically stratified pseudomanifold.
- The wedge sum of two spheres is a topologically stratified pseudomanifold.
- All irreducible complex algebraic or analytic varieties may be considered as topological stratified pseudomanifolds.
- Complex algebraic varieties are topologically stratified spaces with a Whitney stratification.
- A *non-example* would be the open cone on three points.

Remark 2.5.7. Stratifications of spaces may be considered in much more general contexts as in [Lur, §A]. However, for our purposes the above definition is the one most commonly considered for constructible sheaves.

Definition 2.5.8. Let X and Y be stratified topological pseudomanifolds. A continuous map $f : X \to Y$ is stratified if

(1) For any connected component S of any stratum $Y_k \setminus Y_{k-1}$, then $f^{-1}(S)$ is a union of connected components of strata of X.

(2) For each y in a stratum $Y_i \setminus Y_{i-1}$ there is a neighborhood U of y in Y_i, a topologically stratified space

$$\emptyset = F_{-1} \subset \cdots \subset F_{k-1} \subset F_k = F$$

and a stratum preserving homeomorphism

$$F \times U \xrightarrow{\cong} f^{-1}(U).$$

The stratification on $F \times U$ is simply the product of U with the stratification of F.

Remark 2.5.9. For more on stratified spaces see [Bor09, Max19, Ban07, GM83].

Definition 2.5.10. Let X be a topological space with a stratification S with strata $\{S_i\}_{i \in I}$. We say that a sheaf $\mathcal{F}$ on X is *S-constructible* or just constructible if $\mathcal{F}_{|S_i}$ is a locally constant sheaf for all $i \in I$.

If $\mathcal{F} \in \mathrm{Shv}(X, R)$ is a sheaf of R-modules, then we will say that $\mathcal{F}$ is constructible if

(1) $\mathcal{F}_{|S_i}$ is a locally constant sheaf of R-modules

(2) the stalks $\mathcal{F}_x$ for all $x \in X$ are all finitely generated k-modules.

Remark 2.5.11. Let X be a stratified space with stratification S. Let k be a ring and $\mathrm{Shv}_c(X, k)$ denote the full subcategory spanned by constructible sheaves of k-modules. Then $\mathrm{Shv}_c(X, k)$ is a weak Serre subcategory of $\mathrm{Shv}(X, k)$ in the sense of [Sta21, Definition 02MO].

Definition 2.5.12. Let X be a G-space with a stratification S and $\mathcal{F} \in \mathrm{Shv}_G(X, k)$, then we say that $\mathcal{F}$ is *constructible* if $\mathcal{F}$ is constructible as a sheaf in $\mathrm{Shv}(X, k)$. Let $\mathrm{Shv}_{G,c}(X, k)$ denote the full subcategory of constructible sheaves.

Lemma 2.5.13. *The category $\mathrm{Shv}_{G,c}(X, k)$ is a weak Serre subcategory of $\mathrm{Shv}_G(X, k)$.*

Proof. A sheaf $\mathcal{F}$ is constructible by definition if it is constructible after applying the forgetful functor which is exact and the result follows. $\square$

2.6 ∞-Sheaves

We now turn to discussing some basic theory of sheaves on a site in the ∞-categorical setting. Here the main source is [Lur09b, Chapter 6]. The definitions and properties here are essentially the same as those of Section 2.1 with some slight modifications for the ∞-categorical setting.

Definition 2.6.1. Let $\mathcal{C}$ be an ∞-category. A *sieve* on $\mathcal{C}$ is a full subcategory $S \subseteq \mathcal{C}$ such that if $f : C \to C'$ is a morphism in $\mathcal{C}$ and $C' \in S$, then $C \in S$.

If $C \in \mathcal{C}$ is an object in $\mathcal{C}$, then a *sieve on C* is a sieve on the over ∞-category $\mathcal{C}_{/C}$.

Remark 2.6.2. In the above definition if $\mathcal{C} \simeq N(\mathbf{C})$ is the nerve of a 1-category, then this is precisely the same as the usual definition of a sieve on a 1-category. In fact something stronger is true. Namely, for any ∞-category there is a bijection between the sieves on the homotopy category $h(\mathcal{C}_{/C})$ and $(h\mathcal{C})_{/C}$. This is essentially due to the fact that a full subcategory is determined by its objects (See [Lur09b, Rem. 6.2.2.3]).

We now turn to defining Grothendieck topologies on an ∞-category $\mathcal{C}$. If $\mathcal{C}$ is an ∞-category with pullbacks, then we have projection functor $t : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ given by evaluation at 1. This is a biCartesian fibration. Let $f : C \to D$ be morpism in $\mathcal{C}$, then taking the pullback along $f : \Delta^1 \to \mathcal{C}$ and t gives a biCartesian fibration $\tilde{t} : P \to \Delta^1$ as Cartesian and coCartesian fibrations are closed under pullback by [Lur09b, Prop. 2.4.2.3]. This is precisely the data of a pair of adjoint functors

$$\mathcal{C}^{/C} \underset{f^*}{\overset{f_!}{\rightleftarrows}} \mathcal{C}^{/D}$$

where we may consider the functor $f_!$ as composition with f and f^* as pullback along f.

Now if $f : C \to D$ is a morphism in an ∞-category $\mathcal{C}$ with pullbacks. Then given a sieve S on $C \in \mathcal{C}$ we let f^*S denote the unique sieve on $D \in \mathcal{D}$ such that the full subcategory $f^*S \subseteq \mathcal{C}_{/D}$ and the sieve S determine the same sieve on $\mathcal{C}_{/f}$.

Definition 2.6.3. A *Grothendieck topology*, J, on $\mathcal{C}$ is for each $C \in \mathcal{C}$ a collection of sieves $J(C)$ on C which we call *covering sieves* such that

(1) (Maximality) The sieve $\mathcal{C}_{/C}$ is in $J(C)$.

(2) (Stability) if $f : D \to C$ is a morphism in $\mathcal{C}$ and $S \in J(C)$, then f^*S is in $J(D)$.

(3) (Transitivity) If $S \in J(C)$ and S' is any sieve on C such that for all $f : D \to C$ in S we have $f^*S' \in J(D)$, then S' is in $J(C)$.

Remark 2.6.4. In light of Remark 2.6.2 it follows that to give a Grothendieck topology on an ∞-category $\mathcal{C}$ is the same as giving a Grothendieck topology on the homotopy category $h\mathcal{C}$.

Definition 2.6.5. Let $\mathcal{C}$ be an ∞-category with a Grothendieck topology J. If $\mathcal{F} : \mathcal{C}^{op} \to \mathcal{S}$ is a presheaf on $\mathcal{C}$, then we say that $\mathcal{F}$ is a sheaf if for every $C \in \mathcal{C}$ and every covering sieve S of C the composition

$$S^{\triangleleft} \subseteq (\mathcal{C}_{/C})^{\triangleleft} \to \mathcal{C} \xrightarrow{\mathcal{F}^{op}} \mathcal{S}^{op}$$

is a colimit diagram in $\mathcal{S}^{op}$. We denote this category by $\mathrm{Shv}_J(\mathcal{C})$.

Remark 2.6.6. The notation for $\mathcal{S}$-valued sheaves and **Set**-valued sheaves is the same. Given our differing notation for 1-categories and ∞-categories it should be clear from context which we are referring to.

Remark 2.6.7. According to [Lur09b, Prop. 6.2.2.5] there is bijection between sieves on an object $C \in \mathcal{C}$ and monomorphisms into the representable functor $j(C)$ just as in the 1-categorical setting.

The category $\mathrm{Shv}_J(\mathcal{C})$ is a left exact localization of PSh($\mathcal{C}$ just as in the 1-categorical setting so we have adjoint functors

$$\begin{tikzcd} \mathrm{PSh}(\mathcal{C}) \arrow[r, shift left, "L"] & \mathrm{Shv}_J(\mathcal{C}) \arrow[l, shift left, "i"] \end{tikzcd}$$

where L is left exact. In particular, $\mathrm{Shv}_J(\mathcal{C})$ is an ∞-topos and presentable.

2.7 $\mathcal{C}$-valued ∞-Sheaves

Definition 2.7.1. Let $\mathcal{C}$ be a compactly generated ∞-category and $(\mathcal{T}, J)$ a (small) ∞-category with a Grothendieck topology and let $\mathcal{X} := \mathrm{Shv}_J(\mathcal{T})$ denote the corresponding ∞-topos.

(1) A functor $\mathcal{F} : \mathcal{T}^{op} \to \mathcal{C}$ is a $\mathcal{C}$-valued sheaf on $\mathcal{T}$ if for every $U \in \mathcal{T}$ and every covering sieve $S \subseteq \mathcal{T}_{/U}$ the composition

$$S^{\triangleleft} \subseteq (\mathcal{T}_{/U})^{\triangleleft} \to \mathcal{T} \xrightarrow{\mathcal{F}^{op}} \mathcal{C}^{op}$$

is a colimit diagram in $\mathcal{C}^{op}$. Let $\mathrm{Shv}(\mathcal{T}, \mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{T}^{op}, \mathcal{C})$ spanned by $\mathcal{C}$-valued sheaves.

(2) A $\mathcal{C}$-valued sheaf on $\mathcal{X}$ is a functor $F : \mathcal{X}^{op} \to \mathcal{C}$ which preserves small limits. Let $\mathrm{Shv}(\mathcal{X}, \mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C})$ spanned by such functors.

Informally, a functor $\mathcal{F} : \mathcal{T}^{op} \to \mathcal{C}$ is a $\mathcal{C}$-valued sheaf if for every $U \in \mathcal{T}$ and every covering sieve $S_{/U}$ of U the canonical map

$$\mathcal{F}(U) \to \varprojlim_{V \in S} \mathcal{F}(V)$$

is an equivalence in $\mathcal{C}$.

Remark 2.7.2. For ∞-categories $\mathcal{C}$ and $\mathcal{D}$ we let $\mathrm{Fun}^R(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by functors which admit left adjoints (are right adjoints). Similarly, $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ denotes the full subcategory of functors which admit right adjoints (are left adjoints). By [Lur09b, Prop. 5.2.6.2] there is a canonical equivalence

$$\mathrm{Fun}^R(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}^L(\mathcal{D}, \mathcal{C})^{op}.$$

In particular, we get an equivalence

$$\mathrm{Fun}^R(\mathcal{C}^{op}, \mathcal{D}) \simeq \mathrm{Fun}^R(\mathcal{D}^{op}, \mathcal{C})$$

using the fact that $\mathrm{Fun}^R(\mathcal{C}^{op}, \mathcal{D}) \simeq \mathrm{Fun}^L(\mathcal{C}, \mathcal{D}^{op})^{op}$.

Using the adjoint functor theorem [Lur09b, Cor. 5.5.2.9] and [Lur09b, Rem. 5.5.2.10] it follows that for $\mathcal{C}$ presentable we obtain an equivalence

$$\mathrm{Shv}(\mathcal{X}, \mathcal{C}) \simeq \mathrm{Fun}^R(\mathcal{X}^{op}, \mathcal{C}). \tag{2.7.1}$$

Indeed a functor $\mathcal{F} : \mathcal{X}^{op} \to \mathcal{C}$ which has a right adjoint preserves small limits and if $\mathcal{F} : \mathcal{X}^{op} \to \mathcal{C}$ preserve small limits, then $\mathcal{F}^{op} : \mathcal{X} \to \mathcal{C}^{op}$ preserves small colimits and so has a right adjoint, that is, lies in $\mathrm{Fun}^L(\mathcal{D}, \mathcal{C}^{op})$. From which it follows that $\mathcal{F} \in \mathrm{Fun}^R(\mathcal{D}^{op}, \mathcal{C})$ as

$$\mathrm{Fun}^L(\mathcal{X}, \mathcal{C}^{op})^{op} \simeq \mathrm{Fun}^R(\mathcal{C}^{op}, \mathcal{D}) \simeq \mathrm{Fun}^R(\mathcal{D}^{op}, \mathcal{C})$$

by the above discussion.

Since $\mathcal{C}$ is presentable it has all small limits. From this one is able to deduce that the two definitions in 2.7.1 are equivalent. This equivalence arises in the same way as for 1-categories. Let $\mathcal{X}$ denote the ∞-topos $\mathrm{Shv}_J(\mathcal{T})$. Let $j : \mathcal{T} \to \mathrm{PSh}(\mathcal{T})$ denote the Yoneda embedding and $L : \mathrm{PSh}(\mathcal{T}) \to \mathrm{Shv}_J(\mathcal{T})$ a left adjoint to the inclusion. Then the functor

$$\mathrm{Shv}(\mathcal{X}, \mathcal{C}) \xrightarrow{(L \circ j)^*} \mathrm{Shv}(\mathcal{T}, \mathcal{C})$$
$$\mathcal{F} \mapsto \mathcal{F} \circ L \circ j$$

given by precomposition with $L \circ j$ induces the desired equivalence. This is [Lur09a, Prop. 1.1.12].

Lemma 2.7.3. *Let $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0)$ be a compactly generated ∞-category and $j : \mathcal{C}_0^{op} \to \mathrm{Ind}(\mathcal{C}_0^{op})^{op}$ the Yoneda embedding. If $\mathcal{X}$ is an ∞-topos, then there is a canonical equivalence*

$$\mathrm{Shv}(\mathcal{X}, \mathrm{Ind}(\mathcal{C}_0)) \simeq \mathrm{Fun}^{lex}(\mathcal{C}_0^{op}, \mathcal{X}).$$

Proof. First, by [Lur09b, 5.3.5.10] and [Lur09b, 5.4.1.9] composition with the Yoneda embedding $j : \mathcal{C}_0^{op} \to \mathrm{Ind}(\mathcal{C}_0^{op})^{op}$ induces an equivalence

$$\mathrm{Fun}^R(\mathrm{Ind}(\mathcal{C}_0^{op})^{op}, \mathcal{X}) \xrightarrow{-\circ j} \mathrm{Fun}^{lex}(\mathcal{C}_0^{op}, \mathcal{X}).$$

Hence, we obtain

$$\mathrm{Fun}^R(\mathrm{Ind}(\mathcal{C}_0^{op})^{op}, \mathcal{X}) \simeq \mathrm{Fun}^R(\mathcal{X}^{op}, \mathrm{Ind}(\mathcal{C}_0)) = \mathrm{Shv}(\mathcal{X}, \mathcal{C}).$$

□

Now suppose we have an ∞-topos $\mathcal{X}$ and $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0)$ a compactly generated ∞-category. Then the above lemma gives equivalences

$$\mathrm{Shv}_J(\mathcal{T}, \mathcal{C}) \simeq \mathrm{Shv}(\mathcal{X}, \mathcal{C}) \simeq \mathrm{Fun}^{lex}(\mathcal{C}_0^{op}, \mathcal{X}).$$

Let $f = (f^*, f_*) : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi. Then we obtain a commuting diagram

$$\begin{tikzcd}
\mathrm{Shv}(\mathcal{X}, \mathcal{C}) \arrow[r, "-\circ f^*"] \arrow[d, "\simeq"'] & \mathrm{Shv}(\mathcal{Y}, \mathcal{C}) \arrow[d, "\simeq"] \\
\mathrm{Fun}^{lex}(\mathcal{C}_0^{op}, \mathcal{X}) \arrow[r, "f_*\circ -"] & \mathrm{Fun}^{lex}(\mathcal{C}_0^{op}, \mathcal{Y})
\end{tikzcd}$$

The functor $f_* \circ -$ given by postcomposition has a left adjoint given by postcomposition with f^*.

We conclude by defining constant, locally constant, and constructible sheaves on a space. These may be defined in precisely the same way as in the 1-categorical case.

Definition 2.7.4. Let X be a topological space with a stratification $\{S_i\}$ and $\mathcal{C}$ a compactly generated ∞-category. Let $\mathcal{F} \in \mathrm{Shv}(X, \mathcal{C})$ be a sheaf, then:

(1) $\mathcal{F}$ is *constant* if $\mathcal{F}$ is equivalent to the sheafification of a constant presheaf. Equivalently, if $\mathcal{F}$ lies in the essential image of the pullback map $p : X \to *$.

(2) $\mathcal{F}$ is *locally constant* if there exists a cover $\{U_i\}$ of X such that $\mathcal{F}_{U_i}$ the restriction of $\mathcal{F}$ to each U_i is constant.

(3) $\mathcal{F}$ is *constructible* if for each strata S_i of X the restriction $\mathcal{F}_{|S_i}$ of $\mathcal{F}$ to each stratum is locally constant.

Let $\mathrm{Shv}_c(X, \mathcal{C})$ denote the full subcategory of constructible sheaves.

Lemma 2.7.5. *If X is a stratitifed space and $\mathcal{C}$ compactly generated, then there is an equivalence*

$$\mathrm{Shv}_c(X, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{lex}(\mathcal{C}_0^{op}, \mathrm{Shv}_c(X, \mathcal{S})).$$

Proof. The equivalence of Lemma 2.7.3 commutes with the pullback functor and the result follows. □

Definition 2.7.6. If $\mathcal{F} \in \mathrm{Shv}(X, \mathcal{C})$ and $\mathcal{C}$ is compactly generated, then we say that $\mathcal{F}$ is compact-valued if all the stalks $\mathcal{F}_x$ of $\mathcal{F}$ are valued in $\mathcal{C}_0$ the compact objects of $\mathcal{C}$. Let $\mathrm{Shv}_{cpt}(X, \mathcal{C}))$ denote the full subcategory of $\mathrm{Shv}(X, \mathcal{C})$ spanned by compact-valued sheaves.

2.8 Digression: The Derived Category of Sheaves

Suppose we have a site $\mathcal{T} = N(\mathbf{T})$ which is the nerve of a 1-category. Let R be a ring, then there are two ways we might consider the (unbounded) derived ∞-category of sheaves.

(1) We can consider $\mathcal{D}(\mathrm{Shv}(\mathbf{T}, R))$ the unbounded derived ∞-category of 1-sheaves of R-modules.

(2) We can consider $\mathrm{Shv}(\mathcal{T}, \mathcal{D}(R))$ the ∞-category of sheaves valued in the derived ∞-category of R-modules.

Note that the ∞-category $\mathcal{D}(R)$ is equivalently the ∞-category $Mod(R)$ of modules over the Eilenberg-Maclane ring spectrum associated to R by [Lur, Ex. 7.1.1.16]. In particular, by [Lur, Prop. 7.2.4.2] this is a compactly generated stable ∞-category and the subcategory of compact objects are precisely the perfect modules which correspond to complexes which are quasi-isomorphic to bounded chain complexes of finitely generated projective modules (See [Lur, Ex. 7.2.4.25]). Now the categories $\mathcal{D}(\mathrm{Shv}(\mathbf{T}, R))$ and $\mathrm{Shv}(\mathcal{T}, \mathcal{D}(R))$ are not necessarily equivalent. However, we have the following theorem from [Lur18], which we record here as [Lur18] is subject to change, which gives sufficient conditions for them to be equivalent.

Theorem 2.8.1 ([Lur18, Thm. 2.1.2.2]). *Let $(\mathcal{X}, \mathcal{O})$ be a spectrally ringed ∞-topos such that:*

(1) The structure sheaf $\mathcal{O}$ is discrete.

(2) For each object $X \in \mathcal{X}$ there is an effective epimorphism $U \to X$ where U is a discrete object of $\mathcal{X}$.

(3) The ∞-topos $\mathcal{X}$ is hypercomplete.

Then there is an canonical equivalence $\mathcal{D}(Mod(\mathcal{O})^{\heartsuit}) \simeq Mod(\mathcal{O})$.

Here $\mathcal{O}$ is a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$ and $Mod(\mathcal{O})^{\heartsuit}$ denotes the heart of a particular t-structure on $Mod(\mathcal{O})$.

The hypercomplete sheaves in an ∞-topos are precisely those which satisfy descent for all hypercovers. See [Lur09b, §6.5.2, 6.5.3]. A few facts about hypercomplete sheaves for our purposes are:

(1) The hypercomplete sheaves $\mathcal{X}^{hyp} \subseteq \mathcal{X}$ form an ∞-topos which is a left exact Bousfield localization of $\mathcal{X}$ [Lur09b, Cor. 6.5.3.13].

(2) The hypercomplete sheaves $\mathcal{X}^{hyp}$ contain all the n-truncated objects in $\mathcal{X}$ [Lur09b, Lemma 6.5.2.9].

(3) If $\mathcal{X} = \mathrm{Shv}(N(\mathbf{T})$ is an ∞-topos on a 1-site, then $\mathcal{X}^{hyp}$ is presented by the local model structure on the model category of simplicial presheaves on $\mathbf{T}$ [Lur09b, Prop. 6.5.2.14]. In particular, this is a Bousfield localization of the injective model structure on the model category of simplicial presheaves on $\mathbf{T}$ which present the ∞-category $\mathrm{PSh}(N(\mathbf{T}))$.

See also [Jar87] and [DHI04] for detailed treatments of these model structures.

Remark 2.8.2. In the case of $\mathrm{Shv}(\mathcal{T}, \mathcal{D}(R))$ where $\mathcal{T} = N(\mathbf{T})$, then we always have a map

$$\mathcal{D}(\mathrm{Shv}(\mathbf{T}, R)) \to \mathrm{Shv}(\mathcal{T}, \mathcal{D}(R))$$

taking a complex of sheaves to the sheaf valued in complexes and the theorem above may be viewed as saying this map is fully faithful and has essential image the hypercomplete sheaves.

We claim that the above theorem tells us that if $\mathcal{T} = N(\mathbf{T})$ is a Grothendieck site for a 1-category. Then we have an equivalence

$$\mathcal{D}(\mathrm{Shv}(\mathbf{T}, R)) \simeq \mathrm{Shv}^{hyp}(\mathcal{T}, \mathcal{D}(R))$$

where the right hand side denotes hypercomplete sheaves. Since R is a discrete $\mathbb{E}_\infty$-ring and we are considering $\mathcal{D}(R)$-valued sheaves on $\mathcal{X}^{hyp} = \mathrm{Shv}^{hyp}(\mathcal{T})$ the only condition that needs to be checked is (2). Thus, for every $X \in \mathcal{X}^{hyp}$ we need a discrete object $U \in \mathcal{X}^{hyp}$ along with a morphism $U \to X$ which is an effective epimorphism. Let $\tau_{\leq n} : \mathcal{X} \to \tau_{\leq n}\mathcal{X}$ denote the truncation functor [Lur09b, §5.5.6]. This functor is a localization functor in the sense that it has a fully faithful right adjoint. We note that the full subcategory $\tau_{\leq 0}\mathcal{X} \subseteq \mathcal{X}$ of discrete objects is precisely the 1-topos $\mathrm{Shv}(\mathbf{T})$, that is,

$$\tau_{\leq 0}\mathcal{X} \simeq N(\mathrm{Shv}(\mathbf{T})).$$

By [Lur09b, Prop. 7.2.1.14] a morphism $U \xrightarrow{\phi} X$ in an ∞-topos $\mathcal{X}$ is an effective epimorphism if and only if $\tau_{\leq 0}(\phi)$ is an effective epimorphism the 1-topos $h(\tau_{\leq 0}(\mathcal{X})) = \mathrm{Shv}(\mathbf{T})$. Furthermore, by [MM12, Thm. IV.7.8] we have that every epimorphism in a 1-topos is an effective epimorphism. It follows that for each $X \in \mathcal{X}^{hyp}$ we need only provide a map $U \xrightarrow{\phi} X$ with U discrete such that the $\tau_{\leq 0}(\phi)$ is an epimorphism in $\mathrm{Shv}(\mathbf{T})$. By [Lur09b, Prop. 5.5.6.16] left exact functors between ∞-categories with finite limits respect truncated objects. Hence, since $\mathcal{X}$ is a left exact localization of $\mathrm{PSh}(\mathcal{T})$ and $\mathcal{X}^{hyp}$ is a left exact localization of $\mathcal{X}$ it follows that the result holds if it holds in $\mathrm{PSh}(\mathcal{T})$. The result follows.

As a consequence of the above discussion. Let X be a topological space and R a ring. Let $Mod(R)$ denote the ∞-category of module spectra over the Eilenberg-Maclane spectrum for R. As constructible sheaves are hypercomplete by [Lur, Prop. A.5.9] we get that the subcategory

$$\mathrm{Shv}_c(X, \mathcal{D}(R)) \subseteq \mathrm{Shv}^{hyp}(X, \mathcal{D}(R))$$

corresponds to the full subcategory $\mathcal{D}_c(\mathrm{Shv}(X,R)) \subseteq \mathcal{D}(\mathrm{Shv}(X,R))$ of sheaves which are homologically constructible. Similarly, the full subcategory of constructible compact-valued sheaves

$$\mathrm{Shv}_{c,cpt}(X, \mathcal{D}(R)) \subseteq \mathrm{Shv}_c(X, \mathcal{D}(R))$$

corresponds to the subcategory $\mathcal{D}_{c,cpt}(\mathrm{Shv}(X,R)) \subseteq \mathcal{D}_{cpt}(\mathrm{Shv}(X,R))$ of sheaves with constructible homology and perfect stalk complex.

For additional discussion of this section see [mat] and [Jan20, §2.4, Appendix B].

2.9 Equivariant ∞-Sheaves

We conclude the discussion of sheaves by discussing equivariant ∞-sheaves and the equivariant derived category.

Definition 2.9.1. Let G be a discrete group and X a G-space. The ∞-category of *G-sheaves* or *G-equivariant sheaves* is the ∞-category

$$\mathrm{Shv}_G(X) := \mathrm{Shv}(N(\mathcal{U}_G(X)), \mathcal{S}).$$

Given an equivariant map $f : X \to Y$ we obtain a map

$$f^{-1} : N(\mathcal{U}_G(Y)) \to N(\mathcal{U}_G(X))$$

as in the 1-categorical case. This induces a functor of ∞-categories

$$\mathrm{PSh}_G(X) \xrightarrow{(f^{-1})^*} \mathrm{PSh}_G(Y)$$

by precomposition with f^{-1}. Using [Lur09b, Prop. 4.3.3.7] we obtain left and right Kan extensions of $(f^{-1})^*$

$$\begin{tikzcd} \mathrm{PSh}_G(X) \arrow[r, "(f^{-1})^*"] & \mathrm{PSh}_G(Y). \arrow[l, bend right, "(f^{-1})_!"'] \arrow[l, bend left, "(f^{-1})_*"] \end{tikzcd}$$

Now since a Grothendieck topology on $\mathcal{C}$ is the same as a Grothendieck topology on $h\mathcal{C}$ we get that $f^{-1} : N\mathcal{U}_G(Y) \to N\mathcal{U}_G(X)$ respects the Grothendieck topology. It follows that if $\mathcal{F} \in \mathrm{PSh}_G(X)$ is already a sheaf, then $(f^{-1})^*\mathcal{F}$ lies in $\mathrm{Shv}_G(Y)$. Hence, we obtain a pushforward functor.

$$f_* : \mathrm{Shv}(X) \to \mathrm{Shv}(Y).$$

For the inverse image functor $f^* : \mathrm{Shv}_G(Y) \to \mathrm{Shv}_G(X)$ observe that the composition

$$\mathrm{Shv}_G(Y) \subseteq \mathrm{PSh}_G(Y) \xrightarrow{(f^{-1})_!} \mathrm{PSh}_G(X) \xrightarrow{\mathbf{a}_X} \mathrm{Shv}_G(X)$$

is a left adjoint to f_* by precisely the same argument as in the 1-categorical case. Thus, we have a pair of adjoint functors

$$\begin{tikzcd} \mathrm{Shv}_G(X) \arrow[r, shift right, "f_*"'] & \mathrm{Shv}_G(Y) \arrow[l, shift right, "f^*"'] \end{tikzcd}$$

which define a geometric morphism $f = (f^*, f_*) : \mathrm{Shv}_G(X) \to \mathrm{Shv}_G(Y)$ provided f^* is left exact.

Unfortunatley, the author is unsure how to show that the functor $f^* : \mathrm{Shv}_G(Y) \to \mathrm{Shv}_G(X)$ is left exact in the ∞-categorical setting for similar reasons to why we were not able to show the left Kan extension is left exact in the 1-categorical case. A potential approach would be to somehow use [Lur09b, Prop. 6.2.3.20] which characterizes the inverse image functors of geometric morphisms.

Definition 2.9.2. Let G be a discrete group and X a G-space. Let $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0)$ be a compactly generated ∞-category. The ∞-category of *G-sheaves or G-equivariant $\mathcal{C}$-valued sheaves on X* is the ∞-category

$$\mathrm{Shv}_G(X, \mathcal{C}) := \mathrm{Shv}(N(\mathcal{U}_G(X)), \mathcal{C}).$$

Or equivalently either of the categories $\mathrm{Fun}^R(\mathrm{Shv}_G(X)^{op}, \mathcal{C})$ or $\mathrm{Fun}^{lex}(\mathcal{C}_0^{op}, \mathrm{Shv}_G(X))$ by Definition 2.7.1 or Lemma 2.7.3.

2.10 The Equivariant Derived Category

We conclude by defining the G-equivariant derived category for a finite group G over a commutative ring k. We then discuss the alternative definitions which must be used in the case the group G is not finite following [BL06]. In this setting we assume all spaces are sufficiently nice. Specifically, we assume X is Hausdorff and locally homeomorphic to a pseudomanifold of bounded dimension. Note that such a space is locally compact of finite cohomological dimension and is also locally completely paracompact. A space is said to be locally completely paracompact if every point has an open neighborhood all of whose open subsets are paracompact.

Definition 2.10.1. Let G be a finite group and X a G-space. The equivariant derived category is the unbounded derived stable ∞-category of the G-equivariant sheaves on X, denoted

$$\mathcal{D}_G(X, k) := \mathcal{D}(\mathrm{Shv}_G(X, k)).$$

If X has a stratification, then the constructible equivariant derived ∞-category is

$$\mathcal{D}_{G,c}(X, k) := \mathcal{D}_c(\mathrm{Shv}_G(X))$$

the full subcategory spanned by complexes which are homologically constructible.

Remark 2.10.2. Following the discussion in the digression this is equivalently the ∞-category $\mathrm{Shv}_G^{hyp}(X, \mathcal{D}(k))$ of hypercomplete sheaves valued in the derived ∞-category of k and the constructible sheaves in $\mathrm{Shv}_G^{hyp}(X, \mathcal{D}(k))$ correspond to the homologically constructible sheaves.

By [BL06, §8] given a continuous equivariant map we may simply define the functors

$$f_* : \mathcal{D}_G^+(X, k) \to \mathcal{D}_G^+(X, k)$$
$$f^* : \mathcal{D}_G^b(X, k) \to \mathcal{D}_G^b(X, k)$$

as the usual derived functors. Furthermore, as an equivariant sheaf is constructible if it is constructible after applying the forgetful functor it follows that these functors descend to functors of constructible sheaves. In this context we are not entirely sure how to define the functors $f^!$.

As we are unable to define the six operations for the definition of the equivariant derived ∞-category above. We give the definition of the equivariant derived 1-category in the case of a non-finite group for which the six operations are defined. For the six operations themselves we simply list their properties which will be used in Section 3. For more details on these constructions see [BL06]. In particular sections 2.1, 2.7, 2.8, and 3.

We still consider a finite group G and G-space X. Let EG be a the total space for the classifying space BG of G. Let $P = EG \times X$ and $\overline{P} = (EG \times X)/G$ be the quotient by the G-action. We have a diagram

$$X \xleftarrow{p} P \xrightarrow{q} \overline{P}$$

where p is the projection to X and q is the quotient map. Let $D^b(X)$ denote the bounded derived category of sheaves of k-modules. Define a category $D^b_G(X, P)$ with objects triples $(\mathcal{F}_X, \overline{\mathcal{F}}, \beta)$ such that $\mathcal{F}_X \in D^b(X)$, $\overline{\mathcal{F}} \in D^b(\overline{P})$, and $\beta : p^*(\mathcal{F}_X) \cong q^*(\overline{F})$ is an isomorphism in $D^b(P)$. Note that there is an obvious forgetful functor sending an object $(\mathcal{F}_X, \overline{\mathcal{F}}, \beta)$ to $\mathcal{F}_X \in \mathcal{D}^b(X)$.

Definition 2.10.3. The equivariant derived category is the category

$$D^b_G(X) := D^b(X, P).$$

The equivariant constructible derived category $D^b_{G,c}(X)$ is the full subcategory of $D^b_G(X)$ spanned by $\mathcal{F} = (\mathcal{F}_X, \overline{\mathcal{F}}, \beta)$ such that $\mathcal{F}_X$ is homologically constructible in $D^b(X)$.

Remark 2.10.4. The definition above works for any group G with a sufficiently nice space EG. For a precise statement see [BL06, Def. 1.9.1].

Remark 2.10.5. The category above may also be defined as a fibered category. The definition as a fibered category is used to define the t-structure, triangulated structure, and the six operations. See [BL06, §2.4].

We now turn to the equivariant six operations for $D^b_G(X)$.

V1. Let $\mathcal{F}, \mathcal{G} \in D^b_G(X)$, then there exist objects $\mathcal{F} \otimes \mathcal{G}$ and $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ in $D^b_G(X)$ induced by bifunctors $- \otimes -$ and $\mathrm{Hom}(-,-)$ such that

$$\mathrm{Hom}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \cong \mathrm{Hom}(\mathcal{F}, \mathrm{Hom}(\mathcal{G}, \mathcal{H}))$$

V2. For $f : X \to Y$ an equivariant map there are functors

$$f_*, f_! : D^b_G(X) \to D^b_G(Y)$$
$$f^*, f^! : D^b_G(Y) \to D^b_G(X)$$

such that:

V2.1. The assignment is functorial, that is, for $f : X \to Y$ and $g : Y \to Z$ we have natural isomorphisms

$$(fg)^* \cong g^* f^*, \ (fg)^! \cong g^! f^!$$
$$(fg)_* \cong f_* g_*, \ (fg)_! \cong f_! g_!.$$

V2.2. f^* is naturally left adjoint to f_* and $f_!$ is naturally left adjoint to $f^!$.

V3. For $\mathcal{F}, \mathcal{G}, \mathcal{H} \in D^b_G(X)$ there are natural functorial isomorphisms

$$\mathrm{Hom}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \cong \mathrm{Hom}(\mathcal{F}, \mathrm{Hom}(\mathcal{G}, \mathcal{H}))$$
$$f^*(\mathcal{F} \otimes \mathcal{G}) \cong f^*(\mathcal{F}) \otimes f^*(\mathcal{G}).$$

V4. There is a canonical map $f_! \to f_*$ which is an isomorphism when f is proper.

V5. For $Y \subset X$ a closed G-subspace, $U = X \setminus Y$ let $i : Y \hookrightarrow X$ and $j : U \hookrightarrow X$ denote the inclusions, then for $\mathcal{F} \in D^b_G(X)$ we have exact triangles

$$i_! i^!(\mathcal{F}) \to \mathcal{F} \to j_* j^*(\mathcal{F})$$
$$j_! j^!(\mathcal{F}) \to \mathcal{F} \to i_* i^*(\mathcal{F})$$

functorial in $\mathcal{F}$ and compatible with the forgetful functor to $D^b(X)$.

V6. (Equivariant Verdier Duality) There is an object $\omega_{G,X} \in D^b_G(X)$ which defines a dualizing functor $\mathbb{D}_{G,X} := \mathrm{Hom}(-, \omega_{G,X})$ on $D^b_G(X)$ where the functor $\mathbb{D}_{G,X}$ is such that:

V6.1. There is a canonical map

$$\mathcal{F} \to \mathbb{D}(\mathbb{D}(\mathcal{F})))$$

in $D^b_G(X)$.

V6.2. For a G-equivariant map $f : X \to Y$ there are canonical isomorphisms

$$\mathbb{D} f_! \cong f_* \mathbb{D}$$
$$f^! \mathbb{D} \cong \mathbb{D} f^*.$$

V6.3. The functor $\mathbb{D}_{G,X}$ commutes with the forgetul functor to $D^b(X)$. More precisely if $\mathbb{D}_X$ denotes the Verdier duality functor for $D^b(X)$, then the square

$$\begin{array}{ccc} D^b_G(X) & \xrightarrow{\mathbb{D}_{G,X}} & D^b_G(X) \\ \big\downarrow & & \big\downarrow \\ D^b(X) & \xrightarrow{\mathbb{D}_X} & D^b(X) \end{array}$$

commutes where the vertical maps are the forgetful functor.

If X is a G-space which is a stratified pseudomanifold, then we have the following:

(C1) The full subcategory $D^b_{G,c}(X) \subseteq D^b_G(X)$ of constructible sheaves is preserved by $\otimes$, Hom, and $\mathbb{D}_{G,X}$.

(C2) The canonical morphism $\mathcal{F} \to \mathbb{D}(\mathbb{D}(\mathcal{F}))$ is an isomorphism for $\mathcal{F} \in D^b_{G,c}(X)$

(C3) If $f : X \to Y$ is a stratified G-equivariant map of stratified pseudomanifolds with G-action, then f^*, $f^!$, f_*, and $f_!$ all preserve constructibility.

Remark 2.10.6. As observed in [BL06, §2.7] the equivariant derived category above is the 2-limit of the diagram

$$D^b(X) \xrightarrow{p^*} D^b(P) \xleftarrow{q^*} D^b(\overline{P}).$$

This suggests defining the equivariant derived ∞-category as a limit of the diagram

$$\mathcal{D}^b(X) \xrightarrow{p^*} \mathcal{D}^b(P) \xleftarrow{q^*} \mathcal{D}^b(\overline{P})$$

in $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ the ∞-category of stable ∞-categories. We note that following [BL06, §2.7] on can instead take a 2-limit over a larger diagram to obtain the equivariant derived category. One would modify the limit in $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ in a corresponding fashion. We say more on this in Section 4.

3 Smith-Treumann Theory for Sheaves

This section is essentially just material in [Tre19, §4] with a few of our own remarks.

3.1 The Smith Operation

This is essentially [Tre19, §4]. Let $G = \mathbb{Z}/p$ be the cyclic group of order p and k a field of characteristic p. Let kG denote the corresponding group algebra. Let X be a G-space and let $i : X^G \hookrightarrow X$ denote the inclusion of the fixed points. We assume that all spaces are sufficiently nice to define constructible sheaves, that is, X should be a stratified pseudomanifold.

Let Y be a G-space with trivial G-action and let $D^b_c(Y, kG)$ denote the full subcategory of $D^b(Y, kG)$ spanned by complexes with constructible homology. Let $\mathrm{Perf}(Y, kG)$ denote the full subcategory of $D^b(Y, kG)$ spanned by sheaves of kG-modules whose stalk complexes are perfect, that is, the stalk complexes are finitely generated (bounded) complexes of projective kG-modules. Note that kG is a local ring and so these are simply complexes of finitely generated free kG-modules. Let $\mathrm{Perf}(Y, k^{tG})$ denote the Verdier quotient $D^b_c(Y, kG)/\mathrm{Perf}(Y, kG)$ and let $Q : D^b_c(Y, kG) \to \mathrm{Perf}(Y, k^{tG})$ denote the Verdier quotient map.

Remark 3.1.1. If $Y = *$ is a point, then this is precisely the stable module category $\mathbf{stmod}(kG)$ by Theorem 1.3.3. We would like to think of the Verdier quotient $D^b_c(Y, kG)/\mathrm{Perf}(Y, kG)$ as an appropriate ∞-category of sheaves valued in the stable module ∞-category of kG or equivalently, by Theorem 1.4.7 the category of modules over the Tate spectrum k^{tG}.

Recall, if G acts trivially on a space Y, then we have and equivalence

$$\mathrm{Shv}_G(Y, k) \simeq \mathrm{Shv}(Y, kG).$$

which induces an equivalence $D^b_G(Y, k) \simeq D^b(Y, kG)$. Let X be a G-space and let $i : X^G \to X$ denote the inclusion of the fixed points.

Definition 3.1.2 ([Tre19, Def. 4.2])**.** The *Smith operation* is the composition

$$\mathbf{Psm} : D^b_G(X, k) \xrightarrow{i^*} D^b_G(X^G, k) \simeq D^b(X^G, kG) \xrightarrow{Q} \mathrm{Perf}(X^G, k^{tG}).$$

3.2 Six Operations for The Smith Operation

We now show that $\mathrm{Perf}(Y, k^{tG})$ obtains a symmetric monoidal structure, duality functor, and functors f_*, f^*, $f_!$, and $f^!$ for a map $f : Y \to Y'$ of spaces with trivial action.

Lemma 3.2.1. *Let X be a finite dimensional space with free G action, then the global sections functor*

$$\Gamma_G : D^b_{G,c}(X,k) \to D^b_{G,c}(*,k) = D^b(\mathbf{mod}(kG))$$

is valued in $\mathrm{Perf}(kG)$.

Proof. Note that the category $D^b_{G,c}(X,k)$ is generated (as a triangulated category) by the constant sheaves on G-invariant closed subsets. Hence, let Y be a G-invariant closed subset of X with a G-invariant triangulation. The global sections Γ_G are quasi-isomorphic to the simplicial cochain complex with coefficients in k of the triangulation along with the natural G-action. As X is finite dimensional this is a bounded complex. As G acts freely on Y it acts freely on the i-simplices and so this complex is a bounded complex of free kG-modules and we are done. $\square$

Let $i : X^G \hookrightarrow X$ denote the inclusion, then this is a proper map. In particular we get that $i_* \cong i_!$ and i_* is fully faithful. It follows that the unit $\mathbf{1} \xrightarrow{\cong} i^! i_*$ is an isomorphism so we get an isomorphism $i^! i_* i^* \cong i^*$. Hence, applying $i^!$ to the unit map $\mathbf{1} \to i_* i^*$ there is a natural map $i^! \to i^*$.

Lemma 3.2.2 ([Tre19, Thm. 4.1])**.** *Let X be a G-space and let $i : X^G \hookrightarrow X$ the inclusion. The cone on the natural map $i^! \to i^*$ lies in* $\mathrm{Perf}(X^G, kG)$.

Remark 3.2.3. Lemma 3.2.2 implies that the Smith operation $\mathbf{Psm}$ may equivalently be defined as the composition

$$D^b_G(X,k) \xrightarrow{i^!} D^b_G(X^G,k) \simeq D^b(X^G, kG) \xrightarrow{Q} \mathrm{Perf}(X^G, k^{tG}).$$

The G-equivariant Verdier duality operation from V6. is defined using a fibered category definition of $D^b_G(X)$. For a space Y with trivial action under the equivalence

$$D^b_G(Y,k) \simeq D^b(\mathrm{Shv}_G(Y,k)) \simeq D^b(\mathrm{Shv}(Y,kG))$$

that the G-equivariant Verdier duality operation becomes like the k-linear dual operation $\mathrm{Hom}_k(-,k)$ for $\mathbf{stmod}(kG)$. In essence the duality operation becomes

$$\mathrm{Hom}(-, p^! C_{BG})$$

where C_{BG} denotes the constant sheaf of k-vector spaces on the classifying space BG of G concentrated in degree 0 and

$$p : (EG \times X)/G \to EG/G = BG$$

is induced by the projection from $EG \times X \to EG$. It follows in particular, that on stalks it becomes the k-linear duality operation on $\mathbf{mod}(kG)$. It follows that since $\mathcal{F} \in \mathrm{Perf}(Y, kG)$ if and only if each stalk lies in $\mathrm{Perf}(kG)$ that the duality operation descends to an operation on $\mathrm{Perf}(X^G, k^{tG})$ which we denote $\mathbb{D}_{k^{tG}}$. Similarly, a symmetric monoidal structure may be constructed on $D^b_c(Y, kG)$ by taking the tensor product over k.

Let $f : Y \to Y'$ be a morphism of spaces with trivial G-action. Then for the functors f^*, f_*, $f^!$, $f_!$ to descend to functors on $\mathrm{Perf}(Y, k^{tG})$ and $\mathrm{Perf}(Y', k^{tG})$ it is sufficient to check on f^* and $f_!$ as Verdier duality preserves perfect sheaves. Let $\mathcal{F}' \in D^b_c(Y', kG)$, then the stalk of $f^*\mathcal{F}'$ at $y \in Y'$ is equivalently the stalk of $\mathcal{F}'$ at $f(y)$. Thus, since f^* preserves stalks it follows that it descends to a functor $f^* : \mathrm{Perf}(Y', k^{tG}) \to \mathrm{Perf}(Y, k^{tG})$. For the functor $f_!$ the result follows from the following lemma.

Lemma 3.2.4 ([Tre19, Prop. 4.3])**.** *If $\mathcal{F}$ lies in $\mathrm{Perf}(Y, kG)$, then $f_!\mathcal{F}$ lies in $\mathrm{Perf}(Y', kG)$.*

Proposition 3.2.5 ([Tre19, Thm. 4.2])**.** *Let X and Y be G-spaces. Let $f : X \to Y$ be a G-equivariant map. Then the square*

$$\begin{tikzcd} D^b_{G,c}(Y,k) \arrow[r, "f^*"] \arrow[d, "\mathbf{Psm}_Y"'] & D^b_{G,c}(X,k) \arrow[d, "\mathbf{Psm}_X"] \\ \mathrm{Shv}(Y^G, k^{tG}) \arrow[r, "f^*_{X^G}"] & \mathrm{Perf}(X^G, k^{tG}) \end{tikzcd}$$

commutes up to natural isomorphism.

Proof. Let $i_X : X^G \hookrightarrow X$ and $i_Y : Y^G \hookrightarrow Y$ be the inclusions and $f_{X^G} : X^G \to Y^G$ denote the restriction of f to X^G. Clearly $f \circ i_X = i_Y \circ f_{X^G}$ so there is a natural isomorphism $i_X^* \circ f^* \cong f^*_{X^G} \circ i_Y^*$ of functors

$$D^b_{G,c}(Y, k) \to D^b_{G,c}(X^G, k)$$

which induces a natural isomorphism $\mathbf{Psm}_Y \circ f^*_{X^G}$ and $f^* \circ \mathbf{Psm}_X$ of functors

$$D^b_{G,c}(Y, k) \to \mathrm{Perf}(X^G, k^{tG}).$$

$\square$

Proposition 3.2.6 ([Tre19, Thm. 4.3]). *Let X be a G-space. The square*

$$\begin{tikzcd} D^b_{G,c}(X,k) \arrow[r, "\mathbb{D}_{X,G}"] \arrow[d, "\mathbf{Psm}"'] & D^b_{G,c}(X,k) \arrow[d, "\mathbf{Psm}"] \\ \mathrm{Perf}(X^G, k^{tG}) \arrow[r, "\mathbb{D}_{k^{tG}}"] & \mathrm{Perf}(X^G, k^{tG}) \end{tikzcd}$$

commutes up to natural isomorphism.

Proof. We want to show there is a natural isomorphism $\mathbf{Psm} \circ \mathbb{D}_X \cong \mathbb{D}_{k^{tG}} \circ \mathbf{Psm}$. To show this it is sufficient to have a natural map

$$\mathbb{D}_{X^G} i^* \mathcal{F} \to i^* \mathbb{D}_X \mathcal{F}$$

in $D^b_{G,c}(X^G, k)$ which becomes an isomorphism in $\mathrm{Perf}(X^G, k^{tG})$. Consider the natural isomorphism of functors

$$\mathbb{D}_{X^G} i^* \xrightarrow{\cong} i^! \mathbb{D}_X$$

which exists by V6.2.. We have a natural map $i^! \to i^*$ which gives natural map $i^! \mathbb{D}_X \to i^* \mathbb{D}_X$. Thus, we obtain a natural map

$$\mathbb{D}_{X^G} i^* \mathcal{F} \xrightarrow{\cong} i^! \mathbb{D}_X \mathcal{F} \to i^* \mathbb{D}_X \mathcal{F}$$

which we claim becomes an isomorphism in $\mathrm{Perf}(X^G, k^{tG})$. As the map $\mathbb{D}_{X^G} i^* \mathcal{F} \xrightarrow{\cong} i^! \mathbb{D}_X \mathcal{F}$ is already an isomorphism it is sufficient that $i^! \mathbb{D}_X \mathcal{F} \to i^* \mathbb{D}_X \mathcal{F}$ becomes an isomorphism which holds if the cone lies in $\mathrm{Perf}(X^G, kG)$. This holds by Lemma 3.2.2 and we are done. □

Proposition 3.2.7 ([Tre19, Thm. 4.4]). *Let X and Y be G-spaces. Let $f : X \to Y$ be G-equivariant. The squares*

$$\begin{tikzcd} D^b_{G,c}(X,k) \arrow[r, "f_!"] \arrow[d, "\mathbf{Psm}"'] & D^b_{G,c}(Y,k) \arrow[d, "\mathbf{Psm}"] \\ \mathrm{Perf}(X^G, k^{tG}) \arrow[r, "(f_{|X^G})_!"] & \mathrm{Shv}(Y^G, k^{tG}) \end{tikzcd} \qquad \begin{tikzcd} D^b_{G,c}(X,k) \arrow[r, "f_*"] \arrow[d, "\mathbf{Psm}"'] & D^b_{G,c}(Y,k) \arrow[d, "\mathbf{Psm}"] \\ \mathrm{Perf}(X^G, k^{tG}) \arrow[r, "(f_{|X^G})_*"] & \mathrm{Shv}(Y^G, k^{tG}) \end{tikzcd}$$

commute up to natural isomorphism.

Proof. First, by V6.2. we have a natural isomorphism $\mathbb{D} \circ f_! \circ \mathbb{D} \cong f_*$ so it is sufficient to consider the case of $f_!$. Let $i_X : X^G \hookrightarrow X$ and $i_Y : Y^G \hookrightarrow Y$ denote the inclusions. Note that these maps are proper so by V4. the canonical map $(i_X)_! \to (i_X)_*$ is an isomorphism. In particular $(i_X)_!$ is right adjoint to i_X^* and similarly for i_Y. Let $f_{X^G} : X^G \to Y^G$ denote the restriction of f to X^G, then $f \circ i_X = i_Y \circ f_{X^G}$. Thus, by V2.1. $(-)_!$ is functorial so we obtain a sequence of natural isomorphisms

$$i_Y^* f_! (i_X)_! i_X^* \cong i_Y^* (f i_X) i_X^* = i_Y^* (i_Y f_{X^G}) i_X^* \cong i_Y^* (i_Y)_! (f_{X^G})_! i_X^*.$$

Since i_X^* is left adjoint to $(i_X)_!$, then we obtain a natural transformation

$$i_Y^* f_! = i_Y^* f_! \mathbf{1} \to i_Y^* f_! (i_X)_! i_X^*.$$

Similarly, we have a natural transformation

$$i_Y^* (i_Y)_! (f_{X^G})_! i_X^* \to \mathbf{1} (f_{X^G})_! i_X^* = (f_{X^G})_! i_X^*.$$

Stringing these together we obtain a natural transformation

$$i_Y^* f_! \xrightarrow{\alpha} (f_{X^G})_! i_X^*$$

of functors $D^b_{G,c}(X,k) \to D^b_{G,c}(Y^G,k)$. It follows we obtain an induced natural transformation

$$\mathbf{Psm}_Y \circ f_! \xrightarrow{\tilde{\alpha}} (f_{k^tG})_! \circ \mathbf{Psm}_X.$$

We claim that this is a natural isomorphism for which it is sufficient to show that the cone on

$$i_Y^* f_! \mathcal{F} \xrightarrow{\alpha_{\mathcal{F}}} (f_{X^G})_! i_X^* \mathcal{F}$$

lies in $\mathrm{Perf}(Y^G, kG)$. Since this may be checked on stalks we may assume without loss of generality that Y is a point. Thus, we are reduced to showing that

$$D^b_{G,c}(X,k) \xrightarrow{p_!} D^b_{G,c}(*,k) \simeq D^b(\mathbf{mod}(kG))$$

is valued in $\mathrm{Perf}(kG)$. However, as in Lemma 3.2.1 we may further reduce to the case of a constant sheaf on a closed G-invariant subset Z since these sheaves generate $D^b_{G,c}(X,k)$. As in Lemma 3.2.1 we may further reduce to the case of $\mathcal{F}$ a constant sheaf on a closed G-invariant subset Z since these sheaves generate $D^b_{G,c}(X,k)$. The result follows by a similar argument to Lemma 3.2.1 since for $\mathcal{F}$ the constant sheaf $p_!(\mathcal{F})$ is quasi-isomorphic to the simplicial cohomology with compact support of Z with coefficients in $\mathcal{F}$ with the obvious G-action. ◻

4 Remarks

This final section contains conjectural statements we had hoped to be able to show, at least in part, in the course of this project. Due to time constraints and several wrong turns we were not able to show much of any of these statements. We believe that these statements are either known or at least easily proven (or disproven) by experts. Some may even follow directly from the 1-categorical counterparts.

Following Remark 2.10.6. Let G be a topological group and M an ∞-acyclic locally connected free G-space (See [BL06, §1.9] for terminology). Let X be a G-space and $\mathcal{D}(X)$ denote the unbounded derived ∞-category of sheaves of k-modules for some ring k. Let $P = M \times X$ and $\overline{P}$ be the quotient. We have the diagram

$$X \xleftarrow{p} P \xrightarrow{q} \overline{P}$$

where p is the projection and q the quotient map. We obtain a corresponding diagram

$$\mathcal{D}(X) \xrightarrow{p} \mathcal{D}(P) \xleftarrow{q} \mathcal{D}(\overline{P}) \tag{4.0.1}$$

of presentable stable ∞-categories of sheaves of k-modules. Let $\mathcal{D}_G(X)$ denote the limit of this diagram in the category $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ of stable ∞-categories. Similarly, we may consider the pullback of

$$\mathcal{D}^b(X) \xrightarrow{p} \mathcal{D}^b(P) \xleftarrow{q} \mathcal{D}^b(\overline{P})$$

which we denote $\mathcal{D}^b_G(X)$.

Conjecture 4.0.1. (1) The stable ∞-categories $\mathcal{D}_G(X)$ and $\mathcal{D}^b_G(X)$ admit t-structures.

(2) The stable ∞-category $\mathcal{D}^b_G(X)$ may be viewed as the bounded part of the t-structure on $\mathcal{D}_G(X)$.

(3) The homotopy category of $\mathcal{D}^b_G(X)$ is the equivariant derived 1-category $D^b_G(X)$.

(4) There are obvious forgetful functors to $\mathcal{D}(X)$ and $\mathcal{D}^b(X)$ the derived ∞-categories of sheaves of k-modules.

(5) The categories $\mathcal{D}_G(X)$ and $\mathcal{D}^b_G(X)$ support a full six functor formalism as described in Section 2.10.

(6) In [BL06, §2.6] for a subgroup $H \subseteq G$ restriction, induction, and quotient (for normal subgroup) functors are defined. Such functors should also exist and satisfy similar properties.

Recalling from Section 2.8 for a site $(\mathbf{C}, J)$ we have an equivalence

$$\mathcal{D}(\mathrm{Shv}(\mathbf{C}, k)) \xrightarrow{\simeq} \mathrm{Shv}^{hyp}(N(\mathbf{C}), \mathcal{D}(k))$$

between the derived ∞-category of sheaves and the hypercomplete sheaves with values in the derived ∞-category. Using this equivalence we have that the limit over the diagram

$$\mathcal{D}(X) \xrightarrow{p^*} \mathcal{D}(P) \xleftarrow{q^*} \mathcal{D}(\overline{P})$$

which we may instead consider as

$$\mathrm{Shv}^{hyp}(X, \mathcal{D}(k)) \xrightarrow{p^*} \mathrm{Shv}^{hyp}(P, \mathcal{D}(k)) \xleftarrow{q^*} \mathrm{Shv}^{hyp}(\overline{P}, \mathcal{D}(k)).$$

Furthermore, if the space is stratified, then a constructible sheaf $\mathrm{Shv}_c^{hyp}(X, \mathcal{D}(k))$ may be identified with a complex of homologically constructible sheaves. Let $\mathrm{Shv}_G^{hyp}(X, \mathcal{D}(k))$ denote the corresponding limit and $\mathrm{Shv}_{G,c}^{hyp}(X, \mathcal{D}(k))$ in the case of constructible sheaves. Let $\mathrm{Shv}^{hyp}(X, \mathcal{D}(kG))$ be the ∞-category of hypercomplete sheaves and $\mathrm{Shv}_c^{hyp}(X, \mathcal{D}(kG))$ the constructible sheaves. Let $\mathrm{Perf}(X, kG)$ denote the full subcategory of sheaves which are compact-valued, that is, whose stalks are perfect complexes of kG-modules. We may then consider the Verdier quotient $\mathrm{Shv}_c^{hyp}(X, \mathcal{D}(kG))/\,\mathrm{Perf}(X, kG)$ which we denote $\mathrm{Perf}(X, k^{tG})$.

Conjecture 4.0.2. There is a variant of the Smith operation

$$\mathbf{Psm} : \mathrm{Shv}_{G,c}^{hyp}(X, \mathcal{D}(k)) \xrightarrow{i^*} \mathrm{Shv}_{G,c}^{hyp}(X, \mathcal{D}(k)) \simeq \mathrm{Shv}_c^{hyp}(X, \mathcal{D}(kG)) \to \mathrm{Perf}(X, k^{tG}).$$

Moreover, we believe it may be possible to realize the Verdier quotient $\mathrm{Shv}_c^{hyp}(X, \mathcal{D}(kG))/\,\mathrm{Perf}(X, kG)$ as a subcategory of $\mathrm{Shv}(X, Mod(k^{tG}))$ the ∞-category of sheaves of modules over the Tate spectrum.

Finally, we conclude with some statements about equivariant sheaves of spaces. In [Lur09b, 6.3] ∞-categories $\mathcal{LTOP}$ and $\mathcal{RTOP}$ of ∞-topoi are defined. The ∞-category $\mathcal{LTOP}$ has objects ∞-topoi and functors $f^* : \mathcal{X} \to \mathcal{Y}$ if and only if f^* preserves small colimits and finite limits. The ∞-category

$\mathcal{RTOP}$ has objects ∞-topoi and functors $f_* : \mathcal{X} \to \mathcal{Y}$ between ∞-topoi if and only if f_* has a left adjoint which is left exact. In other words $\mathcal{LTOP}$ consists of the left adjoints of geometric morphisms of ∞-topoi while $\mathcal{RTOP}$ has the right adjoints to geometric morphisms. Let G be a finite group and X a G-space we may consider it as a functor $BG \xrightarrow{A} \mathcal{RTOP}$ where BG is the nerve of the category with a single object an morphisms G (cf. [ES21]). This functor sends the object to $\mathrm{Shv}(X)$ and each $g \in G$ to $g^* : \mathrm{Shv}(X) \to \mathrm{Shv}(X)$.

Conjecture 4.0.3. There is an equivalence of ∞-topoi

$$\mathrm{Shv}(N(\mathcal{U}_G(X))) \simeq \varprojlim_{BG} A =: \mathcal{X}^{BG}.$$

Remark 4.0.4. If the group G is not discrete, then we may resolve the space using the simplicial space $X//G$ from 2.4.6. The corresponding limit should give the correct notion of G-equivariant ∞-sheaves on X for a topological group.

A Model Categories

In this appendix we fix some terminology for model categories following [Rie14] and [MP11].

Definition A.0.1. Let $\mathbf{M}$ be a category and $i : A \to B$ and $f : X \to Y$ be morphisms in $\mathbf{M}$. A *lifting problem* for i and f is a commutative square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "v"'] \arrow[ur, dashed, "\tilde{f}"] & Y. \end{tikzcd}$$

A *lift* is a map $\tilde{f} : B \to X$ making the triangles commute. If any lifting problem between i and f has a solution we say that i has the *left lifting property* with respect to f (LLP) and f has the *right lifting property* with respect to i (RLP). In either equivalent case we denote this $i \boxslash f$.

Definition A.0.2. If I is a collection of morphisms in a category $\mathbf{M}$ write $I^{\boxslash}$ for the collection of morphisms which have the *right lifting property* against each $i \in I$. Similarly, write ${}^{\boxslash}I$ for the collection of morphisms which have the *left lifting property* again I.

Definition A.0.3 (Weak Factorization System)**.** Let $\mathbf{M}$ be a category and $(\mathcal{L}, \mathcal{R})$ a pair of morphism classes in $\mathbf{M}$. We say that $(\mathcal{L}, \mathcal{R})$ is a *weak factorization system* if

1. (factorization) every morphism $f : M \to N$ in $\mathbf{M}$ may be factored as a morphism in $\mathcal{L}$ followed by a morphism in $\mathcal{R}$,
2. (lifting) $\mathcal{L} \boxslash \mathcal{R}$, and
3. (closure) $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$.

If there is a set J such that $\mathcal{R} = J^{\boxslash}$ and $\mathcal{L} = {}^{\boxslash}J^{\boxslash}$, then the weak factorization system is said to be *cofibrantly generated.*

Remark A.0.4. If the first two axioms hold, then the closure axiom may be replaced by the requirement that $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts ([Rie14, Lemma 11.2.3]).

Let $\sigma_i : [1] \to [2]$ denote the map in Δ which misses the i^{th} element.

Definition A.0.5. Let $\mathbf{M}$ be a category. A *functorial factorization* on $\mathbf{M}$ is a section of the precomposition wiht σ_1 functor

$$\begin{aligned} \mathrm{Fun}([2], \mathbf{M}) &\xrightarrow{(\sigma_1)^*} \mathrm{Fun}([1], \mathbf{M}) \\ ([2] \xrightarrow{F} \mathbf{M}) &\mapsto ([1] \xrightarrow{F \circ \sigma_1} \mathbf{M}). \end{aligned}$$

Given two objects $A, B \in \mathbf{M}^{[1]}$ and a morphism $(u, v) : A \to B$ (a commutative square)

$$\begin{tikzcd} A_0 \arrow[r, "u"] \arrow[d, "f"'] & B_0 \arrow[d, "g"] \\ A_1 \arrow[r, "v"'] & B_1, \end{tikzcd}$$

then a functorial factorization produces a pair of commutative squares fitting into the diagram

$$\begin{tikzcd} A_0 \arrow[r, "u"] \arrow[d, "Lf"] \arrow[dd, bend right=60, dashed, "f"'] & B_0 \arrow[d, "Lg"'] \arrow[dd, bend left=60, dashed, "g"] \\ Ef \arrow[r, "E(u,v)"'] \arrow[d, "Rf"] & Eg \arrow[d, "Rg"'] \\ A_1 \arrow[r, "v"'] & B_1. \end{tikzcd}$$

Definition A.0.6. Let $\mathcal{M}$ be category. A *model structure* on $\mathcal{M}$ is a triple $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ of collections of morphisms in $\mathbf{M}$ called *weak equivalences*, *cofibrations*, and *fibrations*, such that

(1) $\mathcal{W}$ satisfies the 2-of-3 property,

(2) $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is a weak factorization system,

(3) $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is a weak factorization system.

Call the collection $\mathcal{F} \cap \mathcal{W}$ *trivial fibrations* and the collection $\mathcal{C} \cap \mathcal{W}$ *trivial cofibrations.*

Definition A.0.7. The homotopy category of a model category $\mathbf{M}$ is the localization $h\mathbf{M} := \mathbf{M}[\mathcal{W}^{-1}]$ at the class of weak equivalences.

Theorem A.0.8 (The Small Object Argument [MP11,Rie14, Prop. 15.1.11, Thm 12.2.2])**.** *If I is a small set of maps in a category $\mathcal{M}$, then there exists a functorial factorization which makes $({}^{\boxslash}(I^{\boxslash}), I^{\boxslash})$ a weak factorization system.*

B Verdier Quotients

We briefly record some results on Verdier quotients of stable ∞-categories for reference. This material can be found in [BGT13, Section. 5] and [Mat16, Section 2].

Following [Lur, 1.1.4] we let $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ denote the ∞-category of stable ∞-categories and exact functors. This is a presentable ∞-category.

Definition B.0.1. Let $\mathcal{C}$ be a stable ∞-category and $\mathcal{D} \subseteq \mathcal{C}$ a stable subcategory. The *Verdier quotient* $\mathcal{C}/\mathcal{D}$ is the cofiber (pushout)

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ 0 \arrow[r] & \mathcal{C}/\mathcal{D} \end{tikzcd}$$

in $\mathrm{Cat}_\infty^{\mathrm{Ex}}$.

The above construction gives a universal property. Namely, for $\mathcal{E}$ an ∞-category in $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ to give an exact functor $\mathcal{C}/\mathcal{D} \to \mathcal{E}$ is the same as giving an exact functor $\mathcal{C} \xrightarrow{F} \mathcal{E}$ such that $F(D) \simeq 0$ for all $D \in \mathcal{D}$.

Recall, that for $\mathcal{C}$ an ∞-category and W a collection of morphisms, then the localization $\mathcal{C}[W^{-1}]$ has the universal property that any functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(f)$ is an isomorphism for all $f \in W$, then F factors (uniquely) through $\mathcal{C}[W^{-1}]$. We note that this construction is compatible with Ind-completion in the sense that the canonical map $\mathrm{Ind}(\mathcal{C}[W^{-1}]) \to \mathrm{Ind}(\mathcal{C})[S^{-1}]$ is an equivalence where S is the image of W under the Yoneda embedding $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$.

Proposition B.0.2. *Let $i : \mathcal{D} \hookrightarrow \mathcal{C}$ denote the inclusion of a stable subcategory into a stable ∞-category $\mathcal{C}$ and S the collection of morphisms f in $\mathcal{C}$ such that* $\mathrm{cofib}(f) \in \mathcal{D}$. *Then there is a canonical equivalence $\mathcal{C}[S^{-1}] \to \mathcal{C}/\mathcal{D}$.*

Proof. Observe the canonical map $\mathcal{C} \to \mathcal{C}/\mathcal{D}$ sends each element f of S to an equivalence. This follows as every map of S has cofiber in $\mathcal{D}$ so $F(f)$ has trivial cofiber and is therefore an equivalence in $\mathcal{C}/\mathcal{D}$. Hence, by the universal property of $\mathcal{C}[S^{-1}]$ we obtain a factorization

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr] & & \mathcal{C}/\mathcal{D}. \\
& \mathcal{C}[S^{-1}] \arrow[ur, "\tilde{F}"'] &
\end{tikzcd}$$

We claim that $\tilde{F}$ is the desired canonical equivalence. There is a commutative diagram

$$\begin{tikzcd}
\mathcal{C}[S^{-1}] \arrow[r, "\tilde{F}"] \arrow[d, hook] & \mathcal{C}/\mathcal{D} \arrow[d, hook] \\
\mathrm{Ind}(\mathcal{C}[S^{-1}] \arrow[r, "\simeq"] & \mathrm{Ind}(\mathcal{C}/\mathcal{D})
\end{tikzcd}$$

with the vertical maps the Yoneda embedding. The bottom horizontal map is the map induced by $\tilde{F}$ and is an equivalence by the discussion preceding the proposition as well as [BGT13, Prop. 5.7, 5.13]. It follows $\tilde{F}$ is fully faithful and $\tilde{F}$ is essentially surjective as F is essentially surjective and factors through $\tilde{F}$. $\square$

Proposition B.0.3 ([BGT13, 5.10, 5.11])**.** *A functor $F : \mathcal{C} \to \mathcal{D}$ of stable ∞-categories is an equivalence, if and only if $h\mathcal{C} \xrightarrow{hF} h\mathcal{D}$ is an equivalence.*

In other words we may check equivalence of stable ∞-categories at the level of triangulated categories.

Proposition B.0.4. *Let* $(\mathcal{C}, \otimes, \mathbf{1})$ *be a stable symmetric monoidal* ∞*-category and* $i : \mathcal{D} \hookrightarrow \mathcal{C}$ *the inclusion of a stable subcategory such that if* $X \in \mathcal{C}$ *and* $Y \in \mathcal{D}$*, then* $X \otimes Y \in \mathcal{D}$*. Then* $\mathcal{C}/\mathcal{D}$ *is naturally a symmetric monoidal* ∞*-category.*

Proof. See [Mat16, 2.16]. □

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