Generalized Character theory on Cohomology

Master's Thesis
at Sorbonne Université

Théodore Guy Andreas DOUBINSKY
Advisor: Jesper Grodal

June 1, 2026

Contents

0.1 Introduction

It is void of exaggeration to say that representation theory has been central in the study of finite groups. In this theory one likes to study the (complex) representation ring $R(G)$, whose objects are virtual finite-dimensional representations (see [Ser67]). A *character* in representation theory can be understood as a map that to each complex representation $\rho : G \to Gl_n(\mathbb{C})$ sends $g \mapsto \chi_\rho(g) = \operatorname{tr} \rho(g)$ the trace of the matrix. This map does not depend on the choice of basis and is invariant over conjugation. Letting $Cl(G, \mathbb{C})$ denote the ring of functions $G \to \mathbb{C}$ that are conjugation-invariant (also called class functions), a well-known result in representation theory states that the irreducible representations form a $\mathbb{C}$-basis for $Cl(G, \mathbb{C})$ - in other words, we have an isomorphism $R(G) \otimes \mathbb{C} \xrightarrow{\sim} Cl(G, \mathbb{C})$.

As noted in [HKR92], one does not need all of $\mathbb{C}$ in the theorem above. We first of all notice that $\chi_\rho(g)$ is the sum of eigenvalues of $\rho(g)$, hence it lies in a cyclotomic extension of $\mathbb{Q}$. The isomorphism above then restricts to a map

$$R(G) \otimes L \xrightarrow{\sim} Cl(G, L), \tag{1}$$

where L denotes the union of all cyclotomic extensions. This gives a very nice Galois-theoretic interpretation of the theorem, which we will not detail here.

Another way to calculate the representation ring is with Artin's theorem. Given a subgroup $H \leq G$, restriction induces a homomorphism $R(G) \to R(H)$. The map

$$R(G) \to \lim_{A \in \mathcal{A}(G)} R(A),$$

where $\mathcal{A}(G)$ denotes abelian subgroups of G, is an isomorphism when the order of G is inverted.

A representation $\rho : G \to GL_n(\mathbb{C})$ can also be interpreted as a map $G \to U$ (the infinite unitary group). The associated map of classifying spaces $BG \to BU$ induces a vector bundle over BG and it follows that virtual representations are sent to virtual bundles and we get a morphism of rings

$$R(G) \to K^*(BG)$$

between the representation ring and the complex K-theory of the classifying space of G. A special case of the Atiyah-Segal completion theorem [AS69] gives an isomorphism

$$R(G)_{\widehat{I(G)}} \xrightarrow{\sim} K^*(BG),$$

where $R(G)_{\widehat{I(G)}}$ is the completion of $R(G)$ with respect to the augmentation ideal $I(G)$, the ideal consisting of degree zero virtual representations. This relation between representation theory and K-theory begs the natural question on whether some of the results found in representation theory echoes on in K-theory and possibly in the higher analogues of Morava K- and E- theory, $K(n)$ and $E(n)$ respectively.

In the year 2000, Hopkins, Kuhn and Ravenel published their long-awaited paper [HKR00], which gave generalizations of the results of representation theory introduced above. Their results were initially considered in the case of Morava K- and E-theories, but were generalized (to a varying degree) to so-called complex-oriented cohomology theories. These are more or less the generalized cohomology theories E^* for which $E^*(\mathbb{C}P^\infty) = E^*[[x]]$, and it is well-known that such theories have associated formal group laws (the precise definitions and relevant properties will be articulated in chapter 1). The authors use what they call *complex-oriented descent*, a generalization of the splitting principle that relates the complex oriented cohomology groups of a G-space X with the flag bundles related to equivariant vector bundles over X. Using descent Hopkins, Kuhn and Ravenel find a corresponding Artin's theorem on the following form. For any finite group G, finite G-CW complex X and complex-oriented cohomology theory E^*, there are natural maps

$$E^*(EG \times_G X) \to \lim_{A \in \mathcal{A}(G)} E^*(EG \times_A X) \to \int_{A \in \mathcal{A}(G)} E^*(BA \times X^A)$$

that become isomorphisms when tensored over $\mathbb{Z}[|G|^{-1}]$. This limit is taken over abelian subgroups of G where there is a morphism $A_1 \to A_2$ for each morphism $G/A_2 \to G/A_1$. In the case where $X = \{*\}$, the theorem gives that

$$E^*(BG) \to \lim_{A \in \mathcal{A}(G)} E^*(BA)$$

becomes an isomorphism when the order of G is inverted, which is the direct analogue of Artin's theorem found in representation theory.

The analogue of (1) primarily concerns the Morava E-theory, but applies to multiple cohomology theories. Specifically, we restrict ourselves to look at complex-oriented cohomology theories E^* wherein the coefficient ring is local and that the formal group law associated to the mod $\mathfrak{m}$ (the maximal ideal of E^*) reduction is a formal group law of height $n < \infty$. A large part of finding the cognate theorem is to identify what is meant by *class function* and *coefficient ring* in this new context. An observation on the $K(n)$-theoretic Euler characteristic gives that

$$\chi^G_{n,p} = \dim K(n)^{\text{even}}(BG) - \dim K(n)^{\text{odd}}(BG)$$

is equal to the number of orbits on $G_{n,p} = \operatorname{Hom}(\mathbb{Z}_p^n, G)$ with G-action by conjugation. Using this observation they find the analogues of

The goal of this master's thesis is to present a somewhat self-contained version of Hopkins, Kuhn and Ravenels paper. The memoir is divided into two chapters. In the first chapter we will include preliminary results needed for our main theorems. It starts off presenting some base results for equivariant topology, which is mostly included to help set the scene, and most of their results will not be used directly in later sections. Afterwards is presented a very loose exhibition on chromatic homotopy theory. The goal of this exhibition is to convincingly motivate the existence of Morava K- and E-theory and to justify the interest in calculating those groups. The reader might find themselves dissatisfied with the lack of proofs, that could not be included due to the thesis' 3 month time limit, but can hopefully find comfort in the material sourced, that is generally very well-written. The second chapter has as a main focus to treat Hopkins, Kuhn and Ravenel's paper [HKR00] and to present and prove its main theorems[1] The structure will follow their paper's quite closely but many supplementary proofs and additional details have been added to

0.2 Prerequisites

Although this memoir is trying to be somewhat self contained, the reader is of course assumed to have some familiarities with relevant mathematical fields. The reader is expected to be familiar with algebraic topology, specifically homotopy theory (for example [Hat02, Chapter 1 and 4]) as well as some baseline knowledge for stable homotopy theory.

There was originally planned an appendix for this thesis which would include the bases for G-equivariant topology. This has since been scrapped, but we do a lightning round of definitions in the conventions section below. The reader can otherwise consult the following buffet: [GM95], [MPC96], [Die79], [LMS] and Scwede's lecture notes [Sch18]. All of these with the exception of the last one risk being very outdated on the stable homotopy part, but are well-written and serve as a nice historical motivation.

0.3 Conventions

Many of our algebraic structures (e.g. cohomology theories, polynomial rings, ...) are canonically graded, and as a true mathematician cardinal sin of completely omitting these specifications unless of specific importance is comitted. It shall be assumed that morphisms between graded objects are graded morphisms.

We use the following notations for categories:

- **Top** is the category of (compactly generated weak Hausdorff) topological spaces.

[1] Although we omit the last main theorem in the paper, which analyzes for which groups $K(n)(BG)$ is concentrated in even degrees.

- Given a ring R, R-**Mod** is the category of left R-modules. Likewise, **Mod**-R is the category of right R-modules.
- For an abelian group A, A-**Mod** denotes the category of A-modules, i.e. $\mathbb{Z}[A]$-modules.

Let G be discrete a group. Writing $H \leq G$ means that H is a subgroup of G. A (left or right) G-space is a topological space equipped with continuous (resp. left or right) G-action. Unless specified we always work with left G-spaces. A G-map is a G-equivariant continuous map between G-spaces. Likewise a pointed G-space is a G-space with a basepoint fixed by the action and a pointed G-map is a basepoint-preserving G-map between pointed spaces. X/G is the orbit space and X^H is the fixed-point space for $H \leq G$.

A G-CW complex X is a CW-complex where all cells attached are on the form $G/H \times D^n$ with G-action induced by the obvious one on G/H. It follows that G-CW complexes have cellular action.

$EG \to BG$ is the universal G-principal bundle. Given a G-space X, the Borel construction $EG \times_G X$ is the product $EG \times G$ mod the relation $(t, x) \sim (tg, gx)$.

Overlapping notations

In this memoir, we will work with both character maps and Euler characteristics, both of which are historically always noted with the Greek letter chi. To chose one over the other would be like choosing a favorite child, which the author does not condone. We will therefore in an attempt to differentiate them note character maps as 'χ' and Euler characteristics as the more italicized 'χ', but of course context will prove more than enough to differentiate which object we are talking about.

0.4 Acknowledgements

This project would not have been possible without my advisor Jesper Grodal. I want to thank him for introducing me to this project, his support throughout and especially his patience with me. My regards also go to friends and family for moral support and an extra thank you to my math-friends with whom I have been able to correspond ideas with. Lastly, I would like to thank Jakob Kullberg for putting a roof over my head for the duration of the project, which has saved me the hell of finding affordable lodging in Copenhagen on such short notice.

Chapter 1

Preliminaries

1.1 An abridged introduction to chromatic homotopy theory

This section functions as a review of chromatic homotopy theory, which will give a baseline motivation as well as necessary results for the upcoming sections. It largely follows Ravenel's green book [Rav03] and Lurie's series of course notes [Lur10], but we will omit most proofs in the interest of compactness, perhaps to the reader's dismay. The language in this section may be a little outdated, and we will skip the hassle of doing a deep-dive into the (∞-)categorical context of spectra and the stable homotopy category. If the reader is interested in reading up on the modern language, the author recommends [Cno26] for an axiomatic and introduction or [BR20] for a more topological approach.

1.1.1 The bridge between stable homotopy and cohomology

The story of chromatic homotopy theory begins within stable homotopy theory and a somewhat fundamental theorem to this is the Brown representability theorem, which in some sense gives a correspondence between spectra and generalized cohomology theories. Before getting too far ahead of ourselves, let us state more precisely what this means.

Definition 1.1.1: A generalized (Eilenberg-Steenrod) unreduced cohomology theory is a sequence of contravariant functors $\mathcal{E}^i : CW_{\mathrm{Pairs}} \to \mathbf{Ab}$ satisfying the following axioms:

1. *Homotopy invariance:* If f and g are homotopic, then $f^* = g^*$.
2. *Exact sequence:* Any $i : A \subseteq X$ yields an exact sequence

$$\ldots \to \mathcal{E}^i(X, A) \to \mathcal{E}^i(X, \varnothing) \xrightarrow{i^*} \mathcal{E}^i(A) \to \mathcal{E}^{i+1}(X, A) \to \ldots$$

3. *Excision:* If $U \subseteq A \subseteq X$ with $\overline{U} \subseteq A^\circ$, the inclusion $(X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism $\mathcal{E}^i(X, A) \to \mathcal{E}^i(X \setminus U, A \setminus U)$

4. *Additivity:* The canonical map $\mathcal{E}^i(\coprod(X_j, A_j)) \xrightarrow{\sim} \prod \mathcal{E}^i(X_j, A_j)$ is an isomorphism.

Historically this definition had another axiom, often called the *dimension axiom*, where Eilenberg and Steenrod required that $\mathcal{E}^n(*, \varnothing) = 0$ when $n \neq 0$. It can be shown, e.g. in [Hat02, Thm. 4.59], that a cohomology theory (on CW complexes) that satisfies the axioms above and the dimension axiom is essentially determined by $G := \mathcal{E}^0(*, \varnothing)$, and in this case our cohomology theory $\mathcal{E}^*(-) \simeq H^*(-; G)$ is simply the singular cohomology with coefficients in G. The most standard examples for other generalized cohomology theories are topological (complex or real) K-theory and the cobordism cohomologies. Just as in the well-known singular case, a unreduced generalized cohomology theory $\mathcal{E}^*$ also yields a so-called reduced cohomology theory from pointed CW-complexes to abelian groups by setting $\tilde{\mathcal{E}}^*(X) := \mathcal{E}(X, *)$. Unreduced cohomology theories likewise have an axiomatic definition which we will omit here, since an unreduced theory corresponds to a reduced theory and vice versa.

Theorem 1.1.2: (Brown's representability theorem): A homotopy-invariant, contravariant functor $\tilde{h}$ from CW to **Set** is called a Brown functor if wedges get taken to products and given $X = A^\circ \cup B^\circ$, the map $\tilde{h}(X) \to \tilde{h}(A) \times_{\tilde{h}(A) \times \tilde{h}(B)} \tilde{h}(B)$ induced by the square

$$\begin{array}{ccc} h(X) & \longleftarrow & h(A) \\ \uparrow & & \uparrow \\ h(B) & \longleftarrow & h(A \cap B) \end{array}$$

is surjective. Any Brown functor is representable. Furthermore, any reduced cohomology theory, $\tilde{\mathcal{E}}^*$, is represented by a sequential spectrum, i.e. there exists a sequence of CW complexes E^i equipped with weak equivalences $E^i \to \Omega E^{i+1}$ such that

$$\mathcal{E}^i(X) = [X, E^i]_*$$

The proof is somewhat long and the original result can be found in [Bro62]. A more general, ∞-categorical result can be found in [Lur17]. Conversely, it is a nice exercise to show that a (sequential) spectrum represents a cohomology theory, and if said spectrum is multiplicative (defined later), then we have an obvious product structure on cohomology. The Brown representability theorem with this reciprocity also justifies the common abuse of notation where E^* both represents a cohomology theory and its related spectrum (although when strictly talking the spectrum, the star is often omitted), a convention which we will adhere to as well from here on out. A spectrum E also induces a homology theory on CW complexes by sending X to the (stable) homotopy groups

$$E_*(X) := \pi_*(E \wedge X)$$

with $(E \wedge X)_n = E_n \wedge X$. As remarked in [BR20, 5.1.6, 5.1.8, 6.5.7], Brown's representability says that the stable homotopy category is equivalent to cohomology theories on spectra. The dual, that is for homology, is false due to something called *phantom maps*, but due to Spanier-Whitehead duality we do have a Brown representability on homology for finite complexes.

1.1.2 Complex oriented cohomology theories and formal group laws

We restrict ourselves to multiplicative cohomology theories. By this we mean spectra E^* with a binary operation $E^* \wedge E^* \to E^*$ that is associative and unital up to homotopy. Here $\wedge$ denotes the smash product between spectra[1]. This operation takes a lot of work to define and is not done on sequential spectra. We will treat it here as something god-given that is to remain unquestioned.

When working with chromatic homotopy theory, we need to restrict our cohomology theories to satisfy a surjection criteria:

> **Definition 1.1.3:** A cohomology theory E is called *complex orientable* if the map $\tilde{E}^2(\mathbb{C}P^\infty) \to \tilde{E}^2(S^2) = E^0(*) = \pi_0 E$ induced by the canonical injection $S^2 \hookrightarrow \mathbb{C}P^\infty$ is surjective.

As stated in [Lur10, Lecture 4], we note that $\tilde{E}^2(S^2) = \pi_0 E$ has a canonical unit element $\bar{t}$, and a choice $t \in E^2(\mathbb{C}P^\infty)$ such that $t \mapsto \bar{t}$ is called a *complex orientation*.

To the readers utter shocklessness, a complex orientable cohomology together with a choice of complex orientation will be called *complex oriented*. Let E^* be a complex oriented cohomology theory and let $t \in E^2(\mathbb{C}P^\infty)$ be its complex orientation. It is fairly straightforward (see example 8 in [Lur10, Lecture 4]) to show that $E^*(\mathbb{C}P^\infty) \simeq \pi_*(E)[[t]]$ and $E^*(\mathbb{C}P^n) = \pi_*(E)[t]/(t)^{n+1}$. Likewise, $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = \pi_*(E)[[x, y]]$ (all of these facts can also be shown using the Atiyah-Hirzebruch spectral sequence, see [Ada74, Lemma 2.5]). Viewing $\mathbb{C}P^\infty$ as the projectivization of an infinite-dimensional complex vector space V, $\mathbb{C}P^\infty$ inherits a

[1] Also commonly noted as $\otimes$, since it induces a symmetric monoidal structure

commutative and associative multiplication $\mu : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$, which induces a map

$$\mu^* : E^*(\mathbb{C}P^\infty) \to E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = \pi_*(E)[[x, y]].$$

Let $F_E(x, y) = \mu^*(t)$ be the image of our complex orientation. F_E satisfies the criteria to be called a formal group law. As we will try to work towards justifying, the notion of formal groups and complex oriented cohomology theories are closely related.

Definition 1.1.4: A formal group law (FGL) over a commutative ring R is a formal series $F \in R[[X, Y]]$ satisfying:

1. **Identity:** $F(x, 0) = F(0, x) = x$.
2. **Associativity:** $F(x, F(y, z)) = F(F(x, y), z)$.
3. **Commutativity:** $F(x, y) = F(y, x)$.

A common notation of a formal group law is to write $x +_F y$ for $F(x, y)$.

A formal group law $F(x, y) = \sum c_{ij}x^iy^j$ is completely characterized by the coefficients c_{ij}, and the three properties in the definition above give sufficient and necessary relations between these coefficients (e.g. $c_{ij} = c_{ji}$ etc.) to form a formal group law.

Some examples of formal group laws are

1. The additive formal group law given by $x + y \in R[[x, y]]$.
2. The multiplicative formal group law, $x + y + xy \in R[[x, y]]$.

Let $S = \mathbb{Z}[\{c_{i,j}\}_{i,j}]$ and let I be the ideal in S generated by the relations described in the paragraph above. We form the Lazard ring $L = S/I$. One immediately sees that given any morphism $\phi : L \to R$ determines a formal group law $F(x, y) = \sum \phi(c_{i,j})X^iY^j$, and that this relation makes L universal in the sense that

$$\begin{aligned} \mathrm{Hom}(L, R) &\to \{\text{FGL's over } R\} \\ \phi &\mapsto \sum \phi(c_{i,j})X^iY^j \in R[[X, Y]] \end{aligned}$$

is bijective. Let $g(x) = \sum_{i=1}^{\infty} b_{i+1}x^i \in \mathbb{Z}[b_1, b_2, \ldots][[x]] =: M[[x]]$. The power series $g(g^{-1}(x) + g^{-1}(y)) \in M[[x, y]]$ is a formal group law over M and thus determines a map $\theta : L \to M$.

Theorem 1.1.5: (Lazard's Theorem) The map $\theta : L \to \mathbb{Z}[b_1, b_2, \ldots]$ is an isomorphism.

The proof of Lazard's theorem known to the author is rather calculatory and algebraic [Lur10, Lecture 3]. An important remark is that if we let $\deg c_{i,j} = 2(i+j-1)$ and $\deg b_i = 2i$ our isomorphism becomes a graded one.

One could ask why we call the definition above a formal *group* when we don't require inverses. Reassuringly, for purely formal reasons, all elements $x \in R$ has a *formal inverse* $i(x)$, which is a power series such that $F(x, i(x)) = 0$. A homomorphism of formal group laws between F and G is a power series $f \in R[[X]]$ such that $G(f(x), f(y)) = f(F(x, y))$. If f is invertible (i.e. if the coefficient of X is a unit) then f is an isomorphism. We say that f is a strict isomorphism if the coefficient of X is 1.

Given a formal group law F, a strict isomorphism f from F to the additive law is called a *logarithm for* F, and will commonly be noted $\log_F$. The following result can be found as theorem A2.1.6 in [Rav03].

Theorem 1.1.6: Let F be an FGL over R and let $f \in (R \otimes \mathbb{Q})[[x]]$ be given by

$$f(x) = \int_0^x F_2(t, 0)^{-1} \mathrm{d}t,$$

where $F_2(x, y) = \partial F / \partial y$. Then f is a logarithm for F, and thus F is isomorphic over $R \otimes \mathbb{Q}$ to the additive formal group law.

Proof: Note that F_2 is invertible since its constant term is one. We want to show that $F(x, y) = f^{-1}(f(x)+f(y))$, or equivalently that $w := f(F(x, y))-f(x)-f(y)$ is zero. Differentiating the associativity law $F(F(x, y), z) = F(x, F(y, z))$ with respect to z, we obtain

$$F_2(F(x, y), z) = F_2(x, F(y, z))F_2(y, z),$$

and in particular when $z = 0$:

$$F_2(F(x, y), 0) = F_2(x, y)F_2(y, 0). \tag{1.1}$$

By definition of w, we see that

$$\begin{aligned} \frac{\partial w}{\partial y} &= f'(F(x, y))F_2(x, y) - f'(y) \\ &= \frac{F(x, y)}{F_2(F(x, y), 0)} - \frac{1}{F_2(y, 0)} \end{aligned}$$

which by (1.1) is zero. Likewise, by symmetry, $\partial w / \partial x$ is zero, hence w can only have a constant term, which is zero since f and F have zero constant terms. □

Proposition 1.1.7: The power series $F_E(x, y)$ defined above for complex-oriented cohomology theories defines a formal group law over $\pi_*(E)$.

To get an idea as to why this is the case, remember that the H-space structure on $\mathbb{C}P^\infty$ ensures that

$$\begin{tikzcd} \mathbb{C}P^\infty \times \mathbb{C}P^\infty \arrow[r, "\mathrm{swap}"] \arrow[rr, bend left, "\mu"] & \mathbb{C}P^\infty \times \mathbb{C}P^\infty \arrow[r, "\mu"] & \mathbb{C}P^\infty \end{tikzcd}$$

commutes, which gives the commutativity criterion for F_E. A similar argument works for the other two criteria. □

As toy model, one can take the ordinary cohomology $H^*(-;\mathbb{Z}) = H\mathbb{Z}^*(-)$. The map $H\mathbb{Z}^*(\mathbb{C}P^\infty) \to H\mathbb{Z}^*(\mathbb{C}P^1)$ corresponds to the quotient $\mathbb{Z}[t] \to \mathbb{Z}[t]/t^2$ so complex orientability is immediate. The map μ can also be understood as the classifying map for the tensor of our universal bundle with itself, and thus (t being the Chern class of this bundle) we see that $\mu^*(t) = t_1 + t_2$, and the formal group law over $H\mathbb{Z}$ is simply the additive one.

Example 1.1.8: [Lur10, Lecture 5] or [Rav92, Appendix B.] Let $\mho_n$ denote the universal bundle over $BU(n)$. For every n, we define $MU(n)$ to be the Thom spectrum $\Sigma^{-n}\Sigma^\infty D(\mho_n)/B(\mho_n)$. There is a canonical map $MU(n-1) \to MU(n)$ induced by the inclusion $\mho_{n-1} \to \mho_{n-1} \oplus \mathbb{1} = \mho_n|_{BU(n-1)}$, which permits us to define the *complex bordism spectrum* as the colimit

$$MU = \varinjlim_{n \in \mathbb{N}} MU(n).$$

We will freely use the following facts about MU:

1. MU is multiplicative (induced by the direct sum of bundles)

2. MU is complex orientable with complex orientation determined by the map $\Sigma^{\infty-2}\mathbb{C}P^\infty \simeq MU(1) \to MU$.

3. MU is universal with respect to complex orientations. In other words, for any complex oriented cohomology theory E, we have a bijection

$$\{\text{Ring spectrum morphisms } MU \to E\} \leftrightarrow \{\text{Complex orientations on } E\}$$

Complex bordism is also universal for formal group laws.

Theorem 1.1.9: (Quillen) The map $L \to \pi_* MU$ that classifies the formal group law of MU is a graded isomorphism. Thus $\pi_* MU = \mathbb{Z}[b_1, b_2, \ldots]$ with $\deg b_i = 2i$.

Notation: Let F be a formal group law over R. We set $x +_F y := F(x, y)$. For any $x \in R$, set $[0]_F x := 0$ and we inductively for any positive integer $n \in \mathbb{N}$ define $[n]_F x := F(x, [n-1]_F x)$. Combining the two notations above, we immediately see that $[n]_F x = x +_F x +_F \ldots +_F x$ is simply iterating our addition over F. If $\log_F$ is a logarithm for F, our shorthand notations induce the formulas

$$\log_F(x +_F y) = \log_F(x) + \log_F(y), \quad \log_F([n]_F x) = n \log_F(x) \tag{1.2}$$

Proposition 1.1.10: Suppose $K \subseteq \mathbb{Q}$ is a subring of the rationals and R is a K-algebra. Let F be an FGL over R. For every $a \in K$, there is a power series $[a]_F x \in R[[x]]$ such that

1. For any $a \geq 0$, $[a]_F$ is defined as above, in other words there is no ambiguity on notation.
2. $[a_1 + a_2]_F x = [a_1]_F x +_F [a_2]_F x$
3. $[a_1 a_2]_F x = [a_1]_F([a_2]_F x)$.

These power series' will loosely be called formal group exponentials, and will also be noted simply $[a]x$ when the formal group law is implied.

Proof: Take $[-1]x := i(x)$. This with the points above induce a unique definition for all $a \in \mathbb{Z}$. Since $F(x, 0) = x$, it is not hard to verify that $[a]x = ax \mod x^2$. Thus if $a \in \mathbb{Z}$ is invertible in K, we can set $[a^{-1}]_F x := [a]_F^{-1} x$, and the rest follows from point 2 and 3. □

p-typical formal group laws

(Following [Rav03, Section A.2.1]) Suppose that $q \in R$ is an integer invertible in R. Let ζ be a primitive q'th root of unity on R. We can now define

$$f_q(x) := [q^{-1}] \left(\sum_{i=1}^{q}{}_{[F]} \zeta^i x \right).$$

Here, the notation $\sum_{i=1[F]}^{k} \alpha_i$ simply means $\alpha_1 +_F \ldots +_F \alpha_k$. Note that f_q is defined a power series over $R[\zeta]$, but as remarked in Ravenels book will have its coefficients in R.

Definition 1.1.11: If R is a $\mathbb{Z}_{(p)}$-algebra, we say that an FGL F over R is p-typical if $f_q = 0$ for any prime $q \neq p$.

When R is torsion-free, there is an alternative definition for p-typicality that perhaps gives more sense to the terminology. Namely any FGL F over a torsion free $\mathbb{Z}_{(p)}$-algebra R is called p-typical if the logarithm for F is on the form $\sum_{i\geq 0} \ell_i x^{p^i}$.

Theorem 1.1.12: (Cartier) Any FGL over a $\mathbb{Z}_{(p)}$-algebra R is canonically strictly isomorphic to a p-typical formal group law.

We also have a refinement of the classification theorem in the p-typical context:

Theorem 1.1.13: Let $L_p = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ with $\deg v_i = 2(p^i - 1)$. There is a canonical formal group law over L_p that is universal for p-typical FGLs, i.e. any p-typical formal group law over R is uniquely determined by a morphism $L_p \to R$. Furthermore, the map $L \otimes \mathbb{Z}_{(p)} \to L_p$ that characterizes this universal p-typical FGL is surjective, and L_p factors as a direct summand of $L \otimes \mathbb{Z}_{(p)}$.

The proof is again omitted here, and is somewhat similar to the original Lazard theorem. If the reader is interested, it can be found in [Rav03, Theorem A.2.1.25]. A consequence of the last statement of the theorem in particular states that

$$[p]_{L\otimes\mathbb{Z}_p} x \equiv v_i x^{p^i} \mod (p, b_1, b_2, \ldots, b_{p^i-1}),$$

where the formal exponentiation is taken in the canonical group law over $L \otimes \mathbb{Z}_p = \mathbb{Z}_p[b_1, b_2, \ldots]$.

Height of formal group laws.

Definition 1.1.14: [Rav03, Chapter A2.2] Let F be a formal group law over an $\mathbb{F}_p$-algebra R. F is said to have height n if the leading term of $[p]_F x$ is ax^{p^n}. If $[p]_F x = 0$ then we say that F has height ∞.

Just stating the definition without further comment would be disappointing, since we haven't justified that the leading term is always on this form. A lemma of [Rav03, A2.2.7] ensures us that any endomorphism f of formal groups is on the form $g(x^{p^n})$ with g a polynomial with non-zero coefficient on x. In particular the leading term of f is ax^{p^r}, hence height is defined for all formal group laws.

Proposition 1.1.15: The height of a formal group law is an isomorphism invariant.

This follows from the formula $f([p]_F x) = [p]_G f(x)$. □

Let R be an $\mathbb{F}_p$-algebra. We now define a series of formal group laws of height n.

Definition 1.1.16: Set $F_\infty(x, y) = x + y$. For any number $n \geq 1$ set F_n to be the p-typical formal group law defined by the morphism $L_p \to R$ that sends $v_n \mapsto 1$ and every other v_i to zero.

From the discussion after theorem 1.1.13, it follows that F_n has height n.

Theorem 1.1.17: Let F be an FGL of height n over a seperably closed field K of characteristic p with $n \leq \infty$. Then F is isomorphic to F_n.

1.1.3 Morava K-theory

It is hard to talk about spectra without mentioning the Morava K-theories. This class of spectra is universal in the sense that all multiplicative field spectra (i.e. those with all Künneth isomorphisms) are obtained from wedges of suspensions of these theories (a consequence of the main results of Devinatz-Hopkins-Smith nilpotence theorem [DHS88], see also [Lur10, Lecture 25]). To define these, we are going to need to know how to localize spectra.

Definition 1.1.18: Let X, Y, E be spectra.

1. X is called E-acyclic if the E-homology groups $E_*X := \pi_*(E \wedge X) = 0$.
2. Y is called E-local if for any E-acyclic spectrum X, $[X, Y]_{hSp} = 0$.

Definition 1.1.19: Let E, X be spectra. A *Bousfield localization* at E of X is a spectrum $L_E X$ together with an arrow $\eta : X \to L_E X$ satisfying the following conditions:

1. L_E is E-local.
2. The induced map on homology $E_*\eta$ is an isomorphism.
3. Any map $X \to Y$ with Y being E-local factors through $X \xrightarrow{\eta} L_E(X) \to Y$.

Theorem 1.1.20: (Bousfield) Such a localization exists and is functorial.

For more details on the above, see [BR20, Chapter 7] - particularly section 3.

Example 1.1.21: Let $p \in \mathbb{Z}$ be a prime. Let $\mathbb{S}\mathbb{Z}_{(p)}$ denote the Moore spectrum of the integers localized at p. For a spectrum X, its p-localization is defined to be $X_{(p)} = L_{\mathbb{S}\mathbb{Z}_{(p)}}X$. Extraordinarily, this localization is also given explicitly as the smash product $X \wedge \mathbb{S}_{(p)}$, the p-localization of the sphere spectrum, that can also be computed as the homotopy colimit

$$\operatorname{ho}\varinjlim(\mathbb{S} \xrightarrow{p_1} \mathbb{S} \xrightarrow{p_1 \cdot p_2} \mathbb{S} \xrightarrow{p_1 \cdot p_2 \cdot p_3} \dots),$$

where p_i is an enumeration of primes not equal to p.

Spectra E for which $L_E(-) = (-) \wedge L_E(\mathbb{S})$ are said to be smashing, and their localization have particularly nice properties. A consequence of this is that p-localization really acts like the p-localization in the algebraic sense: $\pi_*(X_{(p)}) = (\pi_*(X))_{(p)}$. This is not true for other types of localization (it is for example false for p-completion, which is a non-example of smashing spectra).

Looking at $MU_{(p)}$ we see that its coefficient ring becomes $\mathbb{Z}_{(p)}[t_1, t_2, \dots]$ with $\deg t_i = 2i$, which immediately refutes the naive idea that $MU_{(p)}$ could correspond to our universal p-typical Lazard ring as in theorem 1.1.13 - it however proves to be a useful starting point.

Theorem 1.1.22: There exists a spectrum BP that is a appears from a (multiplicative) retract $g : MU_{(p)} \to BP$ such that $\pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ that is universal for p-typical spectra (complex oriented whose associated FGL is p-typical) in the sense of theorem 1.1.13.

Sketch proof: [Rav03, Thm. 4.1.12] The goal is to construct an idempotent map $g : MU_{(p)} \to MU_{(p)}$ and set BP to be its image. This can be achieved with the help of two observations. 1) A complex orientation on a complex orientable spectrum E is equivalent to a map $MU \to E$ and 2) any map $MU_{(p)} \to MU_{(p)}$ (or $MU \to MU$) is completely determined by its behaviour on π_*. Thus, by universality of MU and complex orientations, a map $MU_{(p)} \to MU_{(p)}$ is equivalent to the choice of a power series over $\pi_*(MU_{(p)}) = \mathbb{Z}_{(p)}[c_1, c_2, \dots]$. One can then appropriately choose a power series f such that $f^2 = f$ (in fact this will be an isomorphism found in the proof of the Cartier theorem 1.1.12) and conclude from there.

We will now construct $K(n)$ by localizations and quotients on BP. Originally BP was constructed using Postnikov towers, and, as noted in the first section of *Nilpotence and stable homotopy theory II* [HS98], there is an easy way to localize and taking quotients on BP that doesn't need the entire arsenal of abstract nonsense to do.

Given a spectrum E that satisfies certain properties, with $R = \pi_* E$, and an element $x \in R$ that one could hope to kill, let $E/(x)$ be the spectrum defined as the cofiber

$$\Sigma^{\deg x} E \xrightarrow{x\cdot} E \to E/(x),$$

which we can appropriately extend to a notion of quotienting $E/(x_1, x_2, \ldots)$. Given a multiplicative subset $M \subseteq R$, we can hope that the homology theory

$$(E_{(M)})_*(-) := E_*(M) \otimes M^{-1}R$$

is represented by a spectrum $E_{(m)}$. Using Baas-Sullivan theory, one can conclude that our constructions above work for the Brown-Peterson spectrum BP (and its related quotients etc.) which lets us define the Morava K-theory as the one that corresponds to sending $\pi_* BP = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ to

$$\pi_* K(n) = \mathbb{Z}/p\mathbb{Z}[v_n, v_n^{-1}]$$

Every graded module over $\mathbb{Z}/p\mathbb{Z}[v_n, v_n^{-1}]$ is free, which is why $K(n)$ often by convention is called a *graded field*. By the discussions above, it follows that the formal group law associated to $K(n)$ is of height n.

Properties of $K(n)$ are as follows: [Rav92, Prop. 1.5.2]

1. $K(0)_*(X) = H(X; \mathbb{Q})$. In particular $K(0)_*(\mathrm{pt}) = \mathbb{Q}$.

2. $K(1)_*(X)$ is one of the $(p-1)$ isomorphic summands of mod p complex K-theory.

3. There is always a Künneth isomorphism

$$K(n)_*(X \times Y) \xrightarrow{\sim} K(n)_*(X) \otimes_{K(n)_*(\mathrm{pt})} K(n)_*(Y)$$

4. For a p-local finite CW-complex X (i.e. $\tilde{E}_*(X)$ is p-local for all homology theories) and n large enough,

$$\tilde{K}(n)_*(X) = K(n)(\mathrm{pt}) \otimes \tilde{H}_*(X; \mathbb{Z}/p\mathbb{Z}).$$

The Künneth isomorphism property makes cohomology groups $K(n)$ quite calculable compared to other cohomology theories, for example MU. We will in fact later on calculate $K(n)^*(BA)$ for all finite abelian groups.

1.1.4 Morava E-theory

The Morava E-theory has become somewhat of an umbrella term of many closely related cohomology theories. The construction we're interested in utilizes the Landweber exact functor theorem.

Theorem 1.1.23: (Landweber exact functor theorem): For all $0 \leq j \leq \infty$, let $I_j \subseteq \pi_*(BP)$ be the ideal $(p, v_1, \dots, v_{j-1})$ and let M be a BP-module that satisfies the following condition:

$(*)$ For every $1 \leq j < \infty$, the multiplication map

$$M \otimes_{\pi_* BP} \pi_*(BP)/I_j \xrightarrow{\cdot v_n} M \otimes_{\pi_* BP} \pi_*(BP)/I_j$$

is a monomorphism.

Then, the assignment

$$M_*(X) := M \otimes_{BP} BP_*(X)$$

is a homology theory.

For a proof, the reader is referred to Hopkin's course notes [Hop99] - the original references can be found in [Rav03, p. 115-116]. We will apply this theorem for the coefficient ring

$$E(n) = \mathbb{Z}_p[v_1, v_2, \dots v_n, v_{n-1}],$$

for which the condition $(*)$ is evidently true (multiplication by v_n is a monomorphism in $\pi_*(BP)$). This is also called the *Johnson-Wilson* spectrum.

To motivate the interest of $E(n)$, here are some facts about the spectrum [BR20, Section 7.4] (of which only the first one will prove of use to us).

1. $E(n)$ is a complex oriented multiplicative ring spectrum.
2. Localization with respect to $E(n)$ is smashing.
3. A spectrum is $E(n)$-local if and only if it is local with respect to $(K(0) \vee K(1) \vee \dots \vee K(n))$
4. $E(1)$ is a summand of p-localized complex K-theory.
5. Being $E(n)$-local implies being $E(n+1)$ local. It follows that there is a natural transformation of functors $L_{E(n+1)} \to L_{(E(n))}$.
6. If X is a p-local finite CW-spectrum, then

$$X \simeq \operatorname{ho}\lim(L_{E(0)}X \leftarrow L_{E(1)}X \dots)$$

A less-central fact that we *are* going to need is the following remark ([HKR00] and [BW89]).

Proposition 1.1.24: The Bousfield localization $L_{K(n)}E(n)$ of $E(n)$ with respect to $K(n)$ has coefficient ring $E(n)^*_{\widehat{I_n}}$, the completion of $E(n)^*$ with respect to I_n. We will denote this spectrum $\widehat{E}(n)$.

1.1.5 Hopf algebras and FGLs on cohomology

Hopf Algebras.

In the previous subsections we have introduced formal group laws and seen that complex oriented cohomology theories have a canonical formal group structure, but we are yet to see how the many properties introduced for FGL's can be applied to study our cohomology theories. The goal of this subsection is to take a small detour into Hopf algebras, which will permit us to properly weave the two aforementioned concepts more properly together. Results in this chapter are a mix of results who can be found in section 6.7 of [Wei94] and in the beginning of section 6 in [HKR00].

Let k denote a field, which remains fixed for this section. Remember that a *coalgebra* over k is a k-module H equipped with a so-called coproduct morphism $\Delta : H \to H \otimes_k H$ and co-unit $\varepsilon : H \to k$, which verifies that the diagram

$$\begin{tikzcd}
H \arrow[rr, "\Delta"] \arrow[d, "\Delta"] \arrow[drr, "\mathbb{1}"] & & H \otimes H \arrow[d, "\varepsilon\otimes\mathbb{1}"] \\
H \otimes H \arrow[r, "\mathbb{1}\otimes\varepsilon"] & H \otimes k = H = k \otimes H &
\end{tikzcd}$$

commutes. Furthermore, we say that H is a coassociative coalgebra if $(\Delta \otimes \mathbb{1}) \circ \Delta = (\mathbb{1} \otimes \Delta) \circ \Delta$ (implicitly associating $k \otimes H$ with H). To no astonishment, a coalgebra is called cocommutative if $\Delta = \sigma \circ \Delta : H \to H \otimes H \to H \otimes H$, where σ denotes the classical braiding isomorphism.

Definition 1.1.25: We call H a *bialgebra* if it is an algebra and a coassociative and cocommutative coalgebra. A *Hopf algebra* H is a bialgebra equipped with an endomorphism of k-modules $s : H \to H$ (called the antipode) such that the following diagram commutes

$$\begin{tikzcd}
 & H \otimes H \arrow[rr, shift left, "s\otimes\mathbb{1}"] \arrow[rr, shift right, "\mathbb{1}\otimes s"'] & & H \otimes H \arrow[dr, "\text{multiplication}"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] & & k \arrow[rr, hook] & & H
\end{tikzcd}$$

Example 1.1.26:

1. For a group G, the space $k[G]$ can be given a canonical Hopf algebra structure. The coalgebra has coproduct induced by $\Delta : g \mapsto g \otimes g$ and its counit by $\varepsilon : g \mapsto 1$. The antipode is, as the name justifies, induced by $g \mapsto g^{-1}$. Verification that everything commutes how it should is as immediate as it could be.

2. Let A be a finite abelian group and R a k-algebra. The set of R-valued functions from A, noted R^A, form a Hopf algebra very similarly to the example above.

Formal group laws on cohomology theories.

To make this paper more readable, the author introduces a short-hand term for cohomology theories that are sufficiently *good* to satisfy some of the theorems seen later on in chapter 2. Let us suppose that E^* is a cohomology theory such that $\pi_* E$ is a complete local ring. If $p \in \mathfrak{m}$ is a prime, we see that the canonical morphism $L \to \pi_* E \to \pi_* E/\mathfrak{m}$ coming from the E^*-induced formal group law yields a formal group law over the a field of characteristic p. As a consequence, it makes sense to talk about the height of such a formal group law. We are now ready to restrict our cohomology theories.

Definition 1.1.27: Let E^* be a generalized cohomology theory. Suppose that

1. E^* is complex oriented

2. $\pi_* E$ is a complete local ring.

3. Let $\mathfrak{m} \subset \pi_* E$ denotes the maximal ideal. There is a prime $p \in \mathfrak{m}$ such that our mod $\mathfrak{m}$ reduction of our related FGL structure on E^* is of finite height $n < \infty$.

Then we will call E^* a cohomology theory of *finite residue height*, and n will be called the reduced height of E^*.

The cohomology theory $\widehat{E(n)}$ introduced in proposition 1.1.24 is an example of such a cohomology theory. A list of multiple other theories is given on page 6 of [HKR00].

Lemma 1.1.28: (Weierstrass preperation theorem) [Lan78, Theorems 5.2.1 and 5.2.2] Let R be a graded commutative ring which is complete with respect to the topology induced by powers of an ideal I. If a power series $\alpha(X) \in R[[X]]$ has $\alpha(X) \equiv uX^d \mod (I, X^{d+1})$ and u is a unit, then

1. Given any $f(X) \in R[[X]]$, there are unique $p(X) \in R[X]$ and $q(X) \in R[[X]]$ such that
$$f(X) = q(X)\alpha(X) + p(X)$$
and $\deg p < d$.

2. The ring $R[[X]]/\alpha(X)$ is free as an R-module with basis $1, X, \ldots, X^{d-1}$.

3. $\alpha(X) = \epsilon(X)g(X)$ with $\epsilon(X)$ a unit and $g(X)$ a monic polynomial with $\deg g = d$.

We will call d the *Weierstrass degree* of α.

Proof: For point 1, we will follow . Writing $\alpha(X) = \sum_{i=0}^{\infty} a_i X^i$, our hypotheses are equivalent to saying that $a_i \in I$ for $i \leq d$ and that there exists $b \in I$ such that $a_d - b = u$. We will restrict ourselves to the case when $a_d = u$. Let $h, t : R[[X]] \to R[[X]]$ be the head and tail end projections respectively, given by

$$h : \sum_{i=0}^{\infty} c_i X^i \mapsto \sum_{i=0}^{d-1} c_i X^i$$
$$t : \sum_{i=0}^{\infty} c_i X^i \mapsto \sum_{i=0}^{\infty} b_{n+i} X^i.$$

Notably, we see that solving the equation $f = q\alpha + p$ is equivalent to finding a solution to the equation $t(f) = t(q\alpha)$. We find that

$$\begin{aligned} t(f) &= t(q\alpha) \\ &= t(qh(\alpha) + qX^{n-1}t(\alpha)) \\ &= t(qh(\alpha)) + qt(\alpha) \end{aligned}$$

where the second inequality follows since $\alpha = h(\alpha) + X^{n-1}t(\alpha)$ and the third one since $t(X^{n-1}g) = g$ for any $g \in R[[X]]$. By assumption, $t(\alpha)$ has leading term a unit and is thus invertible in $R[[X]]$. Substituting $z = qt(\alpha)$, we are left to find a z that satisfies the equation

$$t(f) = t(zh(\alpha)t(\alpha)^{-1}) + z = \big(\mathbb{1}_{R[[X]]} + \psi\big)(z)$$

where ψ is the transformation that sends $g \mapsto t(h(\alpha)t(\alpha)^{-1}g)$. Since $h(\alpha) \in IR[[X]]$, it follows that ψ has codomain $IR[[X]]$. It follows by completeness that $\mathbb{1}+\psi$ is invertible with inverse $\frac{1}{\mathbb{1}+\psi} = \sum_{i=0}^{\infty}(-1)^i\psi$. Therefore $z = (\mathbb{1}_{R[[X]]}+\psi)^{-1}t(f)$ is a solution to our equation, which proves point 1. Point 2 immediately follows from point 1. For point 3 we factorize $X^d = qf + p$ with $q = \sum b_i X^i$, and we see that $b_0 a_n \equiv 1 \mod \mathfrak{m}$, b_0 is a unit and thus q is invertible. Now

$$f = q^{-1}(X^d - p),$$

which was to be shown. □

An almost-immediate consequence of this lemma is the first half of the following proposition.

Proposition 1.1.29: Let E^* be a cohomology theory of finite residue height n. Then

1. For any $r \geq 1$, $E^*[[x]]/([p^r]x)$ is a free as an E^*-algebra with basis $\{1, x, \dots x^{p^{rn}-1}\}$
2. There is a cocommutative coproduct on $E^*[[x]]/([p^r]x)$ induced by the formal group structure which, in turn, gives $E^*[[x]]/([p^r]x)$ the structure of a Hopf algebra.

Proof: Expanding the definitions of E^* being of height n mod $\mathfrak{m}$ exactly means that $[p](x) \equiv ux^{p^n} \mod (\mathfrak{m}, x^{p^n+1})$, with u a unit since E^* is local. It will likewise follow that $[p^r]x \equiv u_r x^{p^{rn}}$ where u_r is some power of u. The first point thus follows from lemma 1.1.28. To see the Hopf structure, we start off by noting that $E^*[[x]]/[p^r]x \otimes E^*[[y]]/[p^r]y \simeq E^*[[x, y]]/([p^r]x, [p^r]y)$. We induce our coproduct map $\mu^* : E^*[[t]]/[p^r]t \to E^*[[x, y]]/([p^r]x, [p^r]y)$ by our formal group structure given in proposition 1.1.7. Our antipode gets induced by $x \mapsto i(x)$ the formal inverse and counit is given by $x \mapsto 1$. The well-definedness stem from point 1 in this proposition. The rest follows analogously as in example 1.1.26. □

Our Hopf algebras have an induced inverse system (as Hopf algebras)

$$\dots \to E^*[[x]]/[p^{r+1}]x \to E^*[[x]]/[p^r]x \to \dots \tag{1.3}$$

where our maps are induced by $x \mapsto x$.

Definition 1.1.30: Let E^* be of finite reduced height and let R be a graded E^*-algebra. For $r \geq 0$, we set ${}_{p^r}F(R)$ to be the (abelian) group of algebra morphisms $E^*[[x]]/[p^r]x \to R$ with composition of $\psi, \varphi : E^*[[x]]/[p^r]x \to R$ given by the composition

$$E^*[[x]]/[p^r]x \to E^*[[x]]/[p^r]x \otimes E^*[[x]]/[p^r]x \xrightarrow{\psi\otimes\phi} R,$$

where the first map is our formal group law coproduct (see proposition 1.1.29).

The reader might Our inverse system (1.3) induces a sequence of groups[2]

$$\dots \to {}_{p^r}F(R) \to {}_{p^{r+1}}F(R) \to \dots$$

for which we define ${}_{p^\infty}F(R)$ to be the colimit.

[2]Group morphisms since the inverse system are morphisms of Hopf algebras

Theorem 1.1.31: Let K be a closed field of characteristic 0 that is also a E^*-algebra (E^* of finite reduced height). Then, ${}_{p^r}F(K)$ is isomorphic to $(\mathbb{Z}/p^r)^n$ and ${}_{p^\infty}F(K)$ is isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^n$

Proof: Note that our formal group law F_E over E^* directly induces a formal group law over K, which we will simply denote F. Elements of ${}_{p^r}F(K)$ are equivalent to roots of $[p^r]x$ in K. If $[p^r]x$ has p^{rn} roots of multiplicity 1, then ${}_{p^r}F(K)$ is a p-group of order p^{rn} whose orders all divide p^r, which would finish the first part of the proof. Let us thus prove this. By lemma 1.1.28 point 3, we can factorize $[p^r]x = \varepsilon(x)g(x)$ with ε a unit and g a polynomial of degree p^{rn}, which proves that $[p^r]x$ has p^{rn} roots in K with multiplicities counted. We only need to prove that these roots indeed are of multiplicity 1. It is enough to show that $[p^r]'x$ has no roots. Let $\log_F$ be a logarithm for F, see theorem 1.1.6. By assumptions on K we are guaranteed that $\log_F \in K[[x]]$. We can differentiate the following equation (see (1.2)) on both sides

$$\log_F([p^r]x) = p^r \log_F(x)$$

we obtain

$$[p^r]'(x) \log'_F([p^r]x) = p^r \log'_F(x).$$

Since $\log_F$ is a strict isomorphism of FGLs, our equation above gives $[p^r]'(x) = p^r \cdot u(x)$ with $u(x)$ a unit in $K[[x]]$, and $[p^r]'x$ has no roots in K. This proves the first part of the theorem. The second part follows from taking the colimit of our sequence. □

Chapter 2

Generalized characters

2.1 Complex Oriented descent

This section is dedicated to give necessary tools to prove the main theorems of [HKR00]. Since the framework here will be used for all other sections, it felt correct to present this section first.

Proposition 2.1.1: Let $\mathcal{C}$ and $\mathcal{A}$ be two categories and $F : \mathcal{C} \to \mathcal{C}$, $H : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ two functors. Suppose we are given natural transformations $p : F \Rightarrow \mathbb{1}_{\mathcal{C}}$ and $r : H \circ F^{\mathrm{op}} \Rightarrow H$ such that, for all $X \in \mathcal{C}$:

1. $p(X) \circ p(F(X)) = p(X) \circ F(p(X))$
2. $r(X) \circ H(p(X)) = \mathbb{1}_{H(X)}$.

Then, for all $X \in \mathcal{C}$, the following diagram is an equalizer:

$$H(X) \xrightarrow{H(p(X))} H(F(X)) \underset{H(F(p(X)))}{\overset{H(p(F(X)))}{\rightrightarrows}} H(F(F(X)))$$

Proof: The diagram is commutative by point 1 and we only need to justify the universality. Suppose $A \in \mathcal{C}$ with $\alpha : A \to H(F(X))$ such that the diagram

$$A \xrightarrow{\alpha} H(F(X)) \underset{H(F(p(X)))}{\overset{H(p(F(X)))}{\rightrightarrows}} H(F(F(X)))$$

commutes. Define $\beta = r(X) \circ \alpha : A \to H(X)$, and we would like that $\alpha = H(p(X)) \circ \beta$. However,

$$\begin{aligned}
H(p(x)) \circ \beta \quad & = H(p(X)) \circ r(X) \circ \alpha \\
(r \text{ is natural}) \quad & = r(F(X)) \circ H(F(p(X))) \circ \alpha \\
(\text{By definition of } \alpha) \quad & = r(F(X)) \circ H(p(F(X))) \circ \alpha \\
(\text{By point 2.}) \quad & = \alpha,
\end{aligned}$$

which was to be shown. □

2.1.1 Flag bundles

We start this subsection with a quick introduction / reminder of flag bundles. This mostly follows the structure found in [Die08, Chap. 19]. As we will not encounter other types of spaces in this text, we will assume all base spaces are (or have the homotopy type of) CW complexes. Recall that given a (complex) finite-dimensional vector bundle ξ with base space B and total space E we can define the *projective bundle*, $P(\xi)$, whose total space is the orbit space $E/\mathbb{C}^*$ with projection map induced by $p : E \to B$, which we will denote $\tilde{p}$.

It is easily verified that $P(\xi) \to B$ is a (topological) bundle with fibers $P(F_\xi)$, the projective space of the fibers. Viewing ξ as a principal $U(n)$-bundle (i.e. as a bundle $E \times_{U(n)} \mathbb{C}^n \to B$), we can form a pullback

$$\begin{tikzcd} E \times_H \mathbb{C}^{n-1} \cong E \times_{U(n)} (U(n) \times_H \mathbb{C}^n) \arrow[r] \arrow[d] & E \times_{U(n)} \mathbb{C}^n \arrow[d] \\ P(E) \arrow[r, "\tilde{p}"] & B. \end{tikzcd}$$

Where H denotes the subgroup $U(n-1) \times U(1)$ of $U(n)$. The readers already familiar with this concept probably knows the useful splitting principle associated to this construction, namely that $\tilde{p}^*\xi$ splits into the sum of a line bundle and an $(n-1)$ dimensional complex vector bundle. Repeating this process $(n-1)$ times, we get the so-called **flag bundle** $F(\xi)$ with a canonical map $F(\xi) \to B$, whose pullback splits into the sum of n line bundles. For an n-dimensional complex vector space V, let $F(V)$ denote the space of complete flags over V. It is not hard to see that $U(n)$ acts transitively on the elements of $F(V)$ and that the stabilizer of the standard (basis) flag is $T(n)$, the maximal torus of diagonal matrices. It follows that $F(V)$ can be interpreted as $U(n)/T(n)$ and furthermore that our flag bundle can be interpreted as the map

$$F(\xi) = E \times_{U(n)} U(n)/T(n) \simeq E/T(n) \to B.$$

Many of our later results only work for finite G-CW complexes, but the following proposition implies that we can still apply them for a flag bundle $F(\xi)$ over a finite complex.

Proposition 2.1.2: If ξ is an equivariant vector bundle over a finite G-CW complex then $F(\xi)$ has the homotopy type of a finite G-CW complex.

Sketch proof: Take the case when $X = G/H$ so that $F(\xi)$ is a compact, smooth G-manifold. As proved in [Ill74], a smooth compact G-manifold has the homotopy type of a finite G-CW complex. □

For any multiplicative cohomology theory, we have a generalized Leray-Hirsch theorem (see [Die08, Theorem 17.8.4])

Theorem 2.1.3: Let $p : E \to B$ be a topological fibre bundle with fibre F such that B and F are CW complexes and let h^* be an arbitrary multiplicative cohomology theory. Suppose there exists $c_1, \dots, c_n \in h^*(E)$ such that for each fiber F_{x_0} over x_0 the h^*-module $h^*(F)$ is free with basis $i^*(c_1), \dots, i^*(c_n)$. Then, $h^*(E)$ is a free $h^*(B)$-module with basis $c_1, \dots, c_n$. In other words every element in $h^*(E)$ is on the form $p^*(a_1)c_1 + \dots + p^*(a_n)c_n$ with $a_i \in h^*(B)$.

Remark 2.1.4: If E^* is a complex oriented multiplicative cohomology theory, then it satisfies the hypotheses of theorem 2.1.3 for any finite-dimensional complex vector bundle $\xi : E \to B$.

Sketch proof: The idea is to generalize Chern classes for complex oriented theories. One has (see [Lur10, Lecture 4]) that $E^*(BU(n)) = E^*[c_1, \dots, c_n]$. For general $\xi : E \to B$, the result follows from the classifying map $B \to BU(n)$ that is an isomorphism on the fibers (see [MS74]).

As a consequence of the remark and the theorem above, we have

Proposition 2.1.5: If ξ is an n-dimensional complex vector bundle over B, then $E^*(P(\chi))$ is free as a $E^*(B)$-module with basis $1, t_\xi, t_\xi^2, \dots, t_\xi^{n-1}$.

Proposition 2.1.6: Let ξ be a rank m complex vector bundle over B, and let $F(\xi)$ be its related flag bundle. For any complex-oriented cohomology theory E^*, we have

1. Let σ_i denote the i'th symmetric function and $c_i(\xi)$ the chern classes. Then $E^*(F(\xi)) = E^*(B)[x_1, \dots, x_m]/(\sigma_i(\{x_j\}) - c_i(\xi))$.

2. The $E^*(B)$-module $E^*(F(\xi))$ is free of rank $m!$

3. For any map $f : B' \to B$, the following map is an isomorphism:

$$E^*(B') \otimes_{E^*(B)} E^*(F(\xi)) \to E^*(F(f^*\xi)).$$

Proof: Part 1 follows from the proof of remark 2.1.4. Part 2 follows from iterating proposition 2.1.5 and part 3 follows from part 2 together with theorem 2.1.3. □

Applying proposition 2.1.6(3) to the map $F(\xi) \to B$ itself, we get the following result:

Corollary 2.1.7: We have a natural isomorphism

$$E^*(F(\xi)) \otimes_{E^*(B)} E^*(F(\xi)) \to E^*(F(\xi) \times_B F(\xi)).$$

□

Proposition 2.1.8: We have an equalizer on the form

$$E^*(B) \longrightarrow E^*(F(\xi)) \rightrightarrows E^*(F(\xi) \times_B F(\xi)).$$

Proof: The proof is reduced to verifying the hypotheses needed to apply proposition 2.1.1. Let $\mathcal{C}$ be the category of pairs (B, ξ) where ξ is a complex vector bundle over B. A morphism in $\mathcal{C}$ between (B, ξ) and (B', ξ') is a map $f : B \to B'$ such that $\xi \simeq f^*(\xi')$. Let $\mathcal{A} = E^*$-**Mod** be the category of E^*-modules. We set $F : \mathcal{C} \to \mathcal{C}$ to be the flag functor, i.e. $F(B, \xi) = (F(\xi), \rho^*(\xi))$ with $\rho : F(\xi) \to B$ being the induced projection. We also have the cohomology theory as our functor $H = E^*$. Let $p : F \Rightarrow \mathbb{1}$ be the natural transformation induced by ρ. By proposition 2.1.6, the composition

$$E^*(B) \xrightarrow{\rho^*} E^*(F(\xi)) \xrightarrow{\pi} E^*(F(\xi))(x_1, \ldots, x_m)$$

is an isomorphism. We can therefore define $r : E^* \circ F^{\mathrm{op}} \Rightarrow E^*$ as the transformation induced by $(\pi \circ \rho^*)^{-1} \circ \pi$. The two conditions in 2.1.1 are immediately verified, which concludes our proof. □

Proposition 2.1.9: Let G be a finite group and ξ an m-dimensional complex G-vector bundle over a G-space X. Let $Y \to B$ represent either

1. The map $EG \times_G F(\xi) \to EG \times_G X$, or,
2. The map $F(\xi)^A \to X^A$,

for any abelian subgroup $A \subseteq G$. Then,

(i) $E^*(Y)$, as a $E^*(B)$-module, is free of rank $m!$.

(ii) We have an equalizer

$$E^*(B) \longrightarrow E^*(Y) \rightrightarrows E^*(Y \times_B Y)$$

Proof: The case $EG \times_G F(\xi) \to EG \times_G X$ is immediate by propositions 2.1.8 and 2.1.6. In the other case, we can assume without loss of generality that $G = A$

and $X = X^A$. In this case ξ is an A-equivariant bundle over a trivial A-space X. As proved as proposition 2.2 in [Seg68b], there is a natural decomposition

$$\xi = \bigoplus_{i=1}^{k} L_i \otimes \xi_i$$

with the sum running over irreducible representations L_i and the ξ_i are non-equivariant bundles over X of dimension $m_i := \dim \xi_i$. Since A is abelian, it follows that $\dim L_i = 1$ which yields $m_1 + \ldots + m_k = m$. Decomposing a complete flag $(v_1, v_2 \ldots, v_m) \in F(\xi)^A$, and reordering our vectors properly, we obtain a tuple of (complete) flags in $F(\xi_1) \times_X F(\xi_2) \times_X \ldots \times_X F(\xi_k)$. A reordering of this type is equivalent to a partition of m into k sets of respective cardinality m_i. Given such a partition and a tuple of flags in $F(\xi_1) \times_X F(\xi_2) \times_X \ldots \times_X F(\xi_k)$ likewise determines a unique flag in $F(\xi)^A$. In other words, we obtain

$$F(\xi)^A = \coprod F(\xi_1) \times_X F(\xi_2) \times_X \ldots \times_X F(\xi_k),$$

where the coproduct is taken over the $m!/(m_1! \ldots m_k!)$ partitions as described above. Using point 3 of proposition 2.1.6 on the diagram

$$\begin{tikzcd} F(\xi_1) \times_X F(\xi_2) \arrow[r] \arrow[d] & F(\xi_2) \arrow[d] \\ F(\xi_1) \arrow[r] & X, \end{tikzcd}$$

we obtain $E^*(F(\xi_1) \times_X F(\xi_2)) \simeq E^*(F(\xi_1)) \otimes_{E^*(X)} E^*(F(\xi_2))$, hence the left hand side is a free $E^*(X)$-module of rank $m_1!m_2!$. Applying the equivalent argument for our entire tuple, we see that $F(\xi)^A$ is a free module of rank

$$\frac{m!}{m_1! \ldots m_k!}(m_1! \ldots m_k!) = m!.$$

The equalizer diagram follows from the same arguments as the equalizer diagram in our first case. □

In the special case that E^* is the singular homology, proposition 2.1.9 in the second case implies

Corollary 2.1.10: Suppose ξ is an m-dimensional equivariant complex vector bundle with base space a finite G-CW complex X and $A \leq G$ is an abelian subgroup. Then $\chi(F(\xi)^A) = m!\xi(X^A)$.

□

2.2 A generalized Artin's theorem

The goal of this section is to prove the following theorem:

Theorem 2.2.1: Let E be a complex oriented cohomology theory, G a finite group and X a finite G-CW complex. There are natural maps

$$E^*(EG \times_G X) \to \lim_{A \in \mathcal{A}(G)} E^*(EG \times_A X) \to \int_{A \in \mathcal{A}(G)} E^*(BA \times X^A),$$

which become isomoprhisms when the order of G is inverted.

Here $\mathcal{A}(G)$ denotes the category of abelian subgroups of G with one morphism $H_1 \to H_2$ for each G-map $G/H_1 \to G/H_2$ (It follows that such a map induces $G \times_{H_1} X \to G \times_{H_2} X$ and $X^{H_1} \to X^{H_2}$ which defines our constructions above).

2.2.1 The Burnside ring

We start off by giving a very general general construction of groups from commutative monoids.

Definition 2.2.2: Let $(M, +)$ be a commutative (unital) monoid. We define the equivalence relation $\sim$ on $M \times M$ by

$(x_1, y_1) \sim (x_2, y_2) \iff$ There exists $k \in M$ such that $x_1 + y_2 + k = x_2 + y_1 + k$.

We set $\mathrm{Gro}(M) = M \times M/\sim$, called the **Grothendieck group**, with group structure $[x_1, y_1] + [x_2, y_2] = [x_1 + x_2, y_1 + y_2]$.

For example, we immediately see that $[x, y]^{-1} = [y, x]$, since $(1, 1) \sim (z, z)$ for any $z \in M$.

(Following Dieck, [Die79]) A finite G-set is a finite set S with left G-action. Obviously S decomposes into the disjoint union of its orbits, and each of these must necessarily be on the form G/H with $H \leq G$. The isomorphism classes of the building blocks G/H is in 1-to-1 correspondence with the conjugacy class of H. We form the semi-ring $A^+(G)$, whose elements are G-isomorphism classes of finite G-sets, addition is the disjoint union and multiplication is the Cartesian product.

Definition 2.2.3: The Burnside ring of a group G is the Grothendieck group $A(G) = \mathrm{Gro}(A^+(G))$.

It can directly be verified that this indeed forms a commutative ring structure with addition and multiplication directly inherited from the structure on $A^+(G)$.

To relate the Burnside ring to stable homotopy, we present a useful result which can be found in [Seg70]. For a finite dimensional real vector space V with linear G-action, let S^V denote its one-point compactification with we canonically point with ∞. We define the V-suspension on a pointed G-space X to be the smash product $\Sigma^V X = S^V \wedge X$. This lets us define the equivariant stable homotopy group between (pointed) G-spaces X and Y as

$$\{X, Y\}_G = \lim_{V} [\Sigma^V X, \Sigma^V Y]_G,$$

where

- $[-,-]_G$ is the equivariant pointed homotopy classes of G-maps.
- The limit is taken over the category of G-modules and embeddings which is ordered by $V \subseteq V'$ iff V is isomorphic to a sub-module of V'.

As with the non-equivariant case, this is indeed an abelian group.

We will use the following result without proof:

Proposition 2.2.4: There is an isomorhism of rings between the Burnside ring $A(G) \simeq \{S^0, S^0\}_G$ and the equivariant stable homotopy group between 0-spheres.

Let $\mathcal{C}_G$ be the category whose objects are finite G-CW complexes and $\mathrm{Hom}_{\mathcal{C}_G}(X, Y) = \{X, Y\}_G$. $A(G)$ gets to act on of $\mathcal{C}_G$. Furthermore, given any additive functor $F : \mathcal{C}_G \to \mathcal{D}$, $A(G)$ acts on F. This idea will be a central pillar of our proof, specifically when applied to the case where our functor is an equivariant cohomology theory.

Let us introduce a character theory on $A(G)$. For each subgroup $H \leq G$ we set $\chi_H : A(G) \to \mathbb{Z}$ with $\chi_H(S) = |S^H|$. Letting (H) denote the conjugacy class of H in G, we define $\phi = \prod_{(H)} \chi_H : A(G) \to \prod_{(H)} \mathbb{Z}$.

Lemma 2.2.5: [Die79, 1.2.2 and 1.2.3]
Let ϕ be as above.

1. ϕ is an injective ring homomorphism
2. When $|G|$ is inverted, ϕ is an isomorphism.

Proof: For point 1, we immediately see that $|S^H \amalg T^H| = |S^H| + |T^H|$ and $|(S \times T)^H| = |S^H||T^H|$ so the homomorphism part is direct, and injectivity is all that remains. Let $[S] \neq [0]$ with $\phi([S]) = 0$. As mentioned earlier, the cosets G/H form a basis of $A^+(G)$, and we can thus write $S = \sum_{(H)} \alpha_H G/H$. We introduce a partial order on subgroups of G by saying that $H \preccurlyeq K$ if H is conjugate in G to a subgroup of K. Let H be a maximal subgroup with respect to $\preccurlyeq$ such that $a_H \neq 0$. If $K \leq G$ such that $|(G/K)^H| \neq 0$. A quick calculation shows that when $(G/K)^H \neq \varnothing$ we necessarily have that $H \preccurlyeq K$, hence

$$0 = \phi_H(S) = a_H|G/H^H| = a_H|N_G(H)/H| \neq 0,$$

where $N_G(H)$ denotes the normalizer of H in G. This is a clear contradiction, which proves point 1. To prove point 2, we first need to pass to the rationalization

$$\begin{tikzcd} A(G) \arrow[r, "\phi"] \arrow[d] & \prod \mathbb{Z} \arrow[d] \\ A(G) \otimes \mathbb{Q} \arrow[r, "\phi_{\mathbb{Q}}"] & \prod \mathbb{Q}. \end{tikzcd}$$

We introduce the Weil group $W_G(H) = N_G(H)/H$ which we see has a free action on G/H by $wH.gH = gw^{-1}H$. In particular $W_G(H)$ acts freely on G/H^K, hence $|W_G(H)|$ divides $|G/H^K|$ and thus $\phi_{\mathbb{Q}}(G/H \otimes |W_G(H)|^{-1}) \in \prod \mathbb{Z}$. Writing x_H for $G/H \otimes |WH|^{-1}$ and letting $\epsilon_{(H)}$ be the canonical basis for $\prod_{(H)} \mathbb{Z}$, we see that $\phi_{\mathbb{Q}}(x_K) = \sum \alpha_{(H)}\epsilon_{(H)}$ has $\alpha_{(H)} = 1$ iff H and K are conjugate, and thus the x_H's form a $\mathbb{Z}$-basis for $\prod \mathbb{Z}$. Furthermore, the order of the cokernel of ϕ is $\prod_{(H)} |W_G(H)|$ and ϕ becomes an isomorphism when $|G|$ is inverted. $\square$

As a consequence of the proof, we can find a basis $e_i \in A(G)$ such that

$$\chi_K(e_H) = \begin{cases} 1, & \text{if } K \text{ and } H \text{ are conjugate in } G \\ 0 & \text{Otherwise.} \end{cases}$$

We thus obtain $\sum_{(H)} e_H = 1 \in A(G) \otimes \mathbb{Z}[|G|^{-1}]$. For any additive functor $F : \mathcal{C}_G \to \mathbb{Z}[|G|^{-1}]$-**Mod**, the splitting above induces a decomposition $F(X) = \prod_{(H)} e_H F(X)$.

2.2.2 The "key obervation"

The majority of the proof of the theorem relies on a proposition which [HKR00] rightfully calls the *key observation*, which becomes the bulk of the proof of the generalized Artin theorem.

The story, as it is told, begins with the following result of Adams, which can be seen as a generalization of lemma 2.2.5

> **Lemma 2.2.6:** (Adams, Unpublished) Let X be a finite G-CW complex and Y any space. The fixed point map
>
> $$f^G_{X,Y} : \{X, Y\}_G \otimes \mathbb{Z}[|G|^{-1}] \to \prod_{(H)} \{X^H, Y^H\}^{WH} \otimes \mathbb{Z}[|G|^{-1}] \tag{2.1}$$
>
> is an isomorphism.

Proof: We start off by proving the special case when $X = S^n$ and $Y = S^m$, both being equipped with trivial action. As noted in [Seg70, Proposition 2],there is an isomorphism

$$\{S^n, S^m\}_G = \bigoplus_{(H)} \pi^S_{n-m}(B(W_G(H))_+) \tag{2.2}$$

When $n = m$, proposition 2.2.4 together with lemma 2.2.5 gives that the map $f^G_{S^0,S^0}$ is indeed an isomorphism. When $m \neq n$, the right hand side of (2.2) is necessarily zero, since $BW_G(H)$ is an Eilenberg-MacLane space $K(W_G(H), 1)$. By inspection, the right hand side of (2.1) is as well zero as well, hence we have proved our special case. To generalize, we will utilize the following results of Adams:

$$\{G_+ \wedge_H X, Y\}_G \simeq \{X, Y\}_H \simeq \{X, G_+ \wedge_H Y\}_G \tag{2.3}$$

$$(G_+ \wedge_H X)^K = \coprod_{(K')} W_G(K')_+ \wedge_{W_H(K')} X^{K'} \tag{2.4}$$

We will now see how these arguments will let us conclude the lemma. From our special case together with (2.3) and (2.4) we see that $f^G_{G/H_+ \wedge S^n, S^m}$ is an isomorphism for all n, m, G and $H \leq G$ (since $G_+ \wedge_H S^n = G/H_+ \wedge S^n$). We now want to show that f^G_{X,S^m} is an isomorphism. Let us regard both sides of 2.1 as a functor in X with fixed $Y = S^m$, say $L(X)$ for the left-hand side and $R(X)$ for the right. Suppose that X is obtained from X' by attaching a cell $D^n \times G/H$. Taking the cofiber sequence we obtain a diagram:

$$\begin{array}{ccccccccc}
L(S^{n-1} \wedge G/H_+) & \leftarrow & L(X') & \leftarrow & L(X) & \leftarrow & L(\Sigma S^{n-1} \wedge G/H_+) & \leftarrow & L(\Sigma X') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
R(S^{n-1} \wedge G/H_+) & \leftarrow & R(X') & \leftarrow & R(X) & \leftarrow & R(\Sigma S^{n-1} \wedge G/H_+) & \leftarrow & R(\Sigma X').
\end{array}$$

The upper horizontal row is exact by standard results of homotopy theory. The bottom one is too once the following observations are taken into account:

1. Cofibers of G-maps $X \to Y \to Z$ are taken to cofibers of $W_G(H)$-maps $X^H \to Y^*H \to Z^H$.

2. Passage to $W_G(H)$ invariants is exact, since its order is invertible in $\mathbb{Z}[|G|^{-1}]$-**Mod**.

By assumption, all vertical arrows except the middle one are isomorphism, so one concludes with the 5-lemma. We similarly show that $f^G_{X,S^m \wedge G/H_+}$ and thus $f^G_{X,Y}$ is an isomorphism for any finite G-CW complex Y. Let Y is an arbitrary G-CW complex. Since X is finite and thus compact, any map $X \to Y$ must land in a fintie subcomplex $K \subseteq Y$, and we conclude by inspection that

$$\varinjlim_{K \in \mathcal{F}(Y)} f^G_{X,K} = f^G_{X,Y}$$

is an isomorphism. Here $\mathcal{F}(Y)$ simply means the category of finite subcomplexes of Y. This finishes our proof. □

The map above gives us an association $\mathbb{1}_X \overset{\sim}{\mapsto} \sum_{(H)} e_{H,X}$. Comparing this splitting with the splitting defined in the last subsection seems to suggest a redundancy. Indeed, the following lemma gives this exact relation.

Lemma 2.2.7: In the splitting above, $e_{H,X} = e_H \wedge \mathbb{1}_X$

Proof: Remember that the $e_H \in A(G) \otimes \mathbb{Z}[|G|^{-1}] = \{S^0, S^0\}_G \otimes \mathbb{Z}[|G|^{-1}]$ were exactly given such that the fixed point map was 1 in the K-component with K conjugate to H and 0 on all the other components. Sending $e_H \wedge \mathbb{1}_X$ through our correspondence (2.1) we obtain

$$\begin{aligned} e_H \wedge \mathbb{1}_X &\overset{\sim}{\mapsto} \sum_{(K)} (e_H \wedge \mathbb{1}_X)^K \\ &= \sum_{(K)} e_H^K \wedge \mathbb{1}_X^K \\ &= 1 \wedge \mathbb{1}_{X^H} = e_{H,K}. \end{aligned}$$

□

Applying this to the $A(G)$-induced splitting on additive functors we get the following.

Lemma 2.2.8: Let $\tilde{h} : \mathcal{C}_G^{\mathrm{op}} \to \mathbb{Z}[|G|^{-1}]-\mathrm{mod}$ be an additive functor and $f, g \in \{X, Y\}_G \otimes \mathbb{Z}[|G|^{-1}]$. If $f^H \simeq g^H$ then

$$e_H f^* = e_H g^* : e_H \tilde{h}(Z) \to e_H \tilde{h}(Y)$$

Lemma 2.2.9: For any additive functor $\tilde{h} : \mathcal{C}_G^{\mathrm{op}} \to \mathbb{Z}[|G|^{-}1]-\mathrm{mod}$ the projection-induced map $\pi \wedge \mathbb{1} : G/H_+ \wedge X \to X$ gives isomorphism

$$e_H \tilde{h}(X) \simeq e_H \tilde{h}(G/H_+ \wedge X)^{W_G(H)}.$$

Proof: We let $t \in \{S^0, G/H_+\}_G \otimes \mathbb{Z}[|G|^{-1}]$ be a map such that the induced map $t^H \in \{(S^0)^H, G/H_+^H\}_G \otimes \mathbb{Z}[|G|^{-1}] = \{S^0, W_G(H)_+\}_G \otimes \mathbb{Z}[|G|^{-1}]$ corresponds to the sum of the maps $1 \mapsto w$ where $1 \in S^0$ is the non-basepoint and $w \in W_G(H)$. We observe that $(\pi \circ t)^H$ is multiplication by $|W_G(H)|$ in $\{S^0, S^0\}_G \otimes \mathbb{Z}[|G|^{-1}]$ and that $(t \circ \pi)^H = \sum_{w \in W_G(H)} w$. After inverting $|G|$, these maps are isomorphisms and we conclude by lemma 2.2.8. □

Proposition 2.2.10: The natural G-map

$$G/H_+ \wedge X^H \to X$$

induced by $(gH, x) \mapsto gx$ yields and isomorphism

$$e_H\tilde{h}(X) \simeq e_H\tilde{h}(G/H_+ \wedge X^H)^{WH}$$

Proof: Our map above factorizes as

$$G/H_+ \wedge X^H \xrightarrow{\ell} G/H_+ \wedge X \xrightarrow{\pi\wedge 1} X$$

Where $\ell : (gH, x) \mapsto (gH, gx)$. Lemma 2.2.9 reduces our problem to showing the isomorphism for the first arrow. The fixed point map $\ell^H : W_G(H)_+ \wedge X^H \to W_G(H) \wedge X^H$ is simply $(wH, x) \mapsto (wH, wx)$, which is a homeormorphism. This is made clear once $G/H_+ \wedge X^H$ is equipped with right $W_G(H)$-action by $(gH, x).w = (gwH, w^{-1}x)$. □

2.2.3 Proof of Artin's theorem

Fix an embedding $G \subseteq U(m)$ (for m large enough) and we will let $F := F(U(m)) = U(m)/T(n)$ be the flag manifold (cf. page 25). Let $\mathfrak{C}_G$ denote the subcategory of $\mathbf{Top}^G$ of finite G-CW complexes. Here is the result of which the analogous Artin's theorem will follow.

Theorem 2.2.11: Suppose we have a functor $h : \mathfrak{C}_G^{op} \to \mathbb{Z}[|G|^{-1}]\mathrm{Mod}$ such that

1. $h(X) \to h(X \times F)$ is a monomorphism for all X
2. There exists an additive functor $\tilde{h} : \mathcal{C}_G^{op} \to \mathbb{Z}[|G|^{-1}]\mathrm{Mod}$ such that $\tilde{h}(X_+) = h(X)$ for all X.

Then for any G-CW complex Finite?, every map

$$h(X) \to \lim_{A\in\mathcal{A}(G)} h(G \times_A X) \to \int_{A\in\mathcal{A}(G)} h(G/A \times X^A)$$

is an iso.

Proof of the Artin-type theorem 2.2.1, assuming theorem 2.2.11: Setting $h(X) = E^*(EG \times_G X) \otimes \mathbb{Z}[|G|^{-1}]$ and $\tilde{h} = \tilde{E}^*(EG_+ \wedge_G X) \times \mathbb{Z}[|G|^{-1}]$, condition 1. of the theorem above is satisfied using proposition 2.1.9 and the second one by the classical reduced-unreduced cohomology correspondence. the result immediately follows from our theorem.

Proof of theorem 2.2.11: The proof will be split off into two parts. In the first part we will use our key observation to obtain an isomorphism $h(X) \to \prod_{(A)} e_A h(G/A \times X^A)^{W_G(A)}$ with the product taken over conjugacy classes of abelian subgroups. The second part will then apply the method of finding equalizers from proposition 2.1.1 to conclude.

By the second assumption and proposition 2.2.10, there is a composition of isomorphisms

$$h(X) \to \prod_{(H)} e_H h(X) \to \prod_{(H)} e_H h(G/H \times X^H)^{W_G(H)},$$

and we thus get a commutative diagram

$$\begin{array}{ccc} h(X) & \xrightarrow{\quad\sim\quad} & \prod_{(H)} e_H h(G/H \times X^H)^{W_G(H)} \\ \downarrow & & \downarrow \\ h(X \times F) & \xrightarrow{\sim} & \prod_{(H)} e_H h(G/H \times X^H \times F^H)^{W_G(H)} \end{array}.$$

The leftmost map is a monomorphism (by assumption), so the rightmost must also be on each component. Since $F^H = \varnothing$ for every non-abelian H, the bottom right component of our diagram above must restrict to the product over abelian groups. The right arrow being a component-wise monomorphism means that the top isomorphism reduces to the desired

$$h(X) \to \prod_{\substack{(A) \\ A \text{ abelian}}} e_A h(G/A \times X^A)^{WA}. \tag{2.5}$$

We will now set up the conditions for proposition 2.1.1 to be used twice. In the first case, we set $\mathcal{C} = \mathfrak{C}_G$ and $\mathcal{H} = h$. Let $\mathcal{F}_1 : X \mapsto \coprod_{A \in \mathcal{A}(G)} G \times_A X$ and $\mathcal{F}_2(X) : X \mapsto \coprod_{A \in \mathcal{A}(G)} G/A \times X^A$. The maps

$$\begin{array}{cc} G/A \times X^A \to G \times_A X, & G \times_A X \to X \\ (gA, x) \mapsto (g, g^{-1}x) & (g, x) \mapsto gx \end{array}$$

induces maps $p_1 : \mathcal{F}_1(X) \to X$ and $p_2 : \mathcal{F}_2(X) \xrightarrow{q} \mathcal{F}_1(X) \to X$. The retractions r_1 and r_2 are obtained from the following commutative diagram

$$\begin{array}{ccc} h(X) & \xrightarrow{\quad\sim\quad} & \prod_{(A)} e_A h(G/A \times X^A)^{W_G(A)} \\ \big\downarrow{\scriptstyle h(p_1)} & & \uparrow \\ \prod_{A \in \mathcal{A}(G)} h(G \times_A X) & \xrightarrow{h(q)} & \prod_{A \in \mathcal{A}(G)} h(G/A \times X^A) \end{array}$$

We can now apply proposition 2.1.1 twice to obtain the equalizers

$$h(X) \longrightarrow h\Big(\coprod_{A\in\mathcal{A}(G)} X \times_A G\Big) \rightrightarrows h\Big(F_1\Big(\coprod_{A\in\mathcal{A}(G)} X \times_A G\Big)\Big)$$

and

$$h(X) \longrightarrow h\Big(\coprod_{A\in\mathcal{A}(G)} G/A \times X^A\Big) \rightrightarrows h\Big(F_2\Big(\coprod_{A\in\mathcal{A}(G)} G/A \times X^A\Big)\Big)$$

Calculating the equalizers above, we obtain $\lim_{\mathcal{A}(G)} h(G\times_A X)$ and $\int_{\mathcal{A}(G)} h(G/A\times X^A)$ respectively, which was to be shown. $\square$

2.3 The K(n)-theoretic Euler Characteristic

A calculation (analogous to section 2.4.1) shows that the Morava K-theory $K(n)^*(BA)$ is concentrated in even degrees for all finite abelian groups A. It was conjectured by Hopkins, Kuhn and Ravenel that this result holds for all (finite) groups G. Although the conjecture has been refuted, the study of it gave a principal observation that will be a good motivation for the main results of section 2.4.3. The observation in question is a relation between the $K(n)$-theoretic Euler characteristic

$$\chi^G_{n,p} = \dim K(n)^{\text{even}}(BG) - \dim K(n)^{\text{odd}}(BG)$$

and the number of G-orbits of a certain group $G_{n,p} \subseteq G^n$ of commuting p-elements acted upon by conjugation. We dedicate this section to outline this relationship.

2.3.1 Additive functors and their bases

Definition 2.3.1: A function χ : {Finite G-CW complexes} $\to M$, where M is an Abelian group is called **additive** if the following properties are verified:

1. $\chi(X) = \chi(Y)$ if X and Y are homotopy equivalent through G-homotopies.
2. $\chi(X\cup Y) = \chi(X) + \chi(Y) - \chi(X\cap Y)$
3. $\chi(\varnothing) = 0$.

We can nicely characterize additive functions with the following lemma.

Lemma 2.3.2: The cannonical map $S \to [S] \in A(G)$ that sends finite G-sets to its class in the Burnside ring induces a universal additive function $\chi_{\text{univ}} : \mathcal{C}_G \to A(G)$ in the sense that the following map is a bijection:

$$\begin{aligned}\mathrm{Hom}(A(G), M) &\to \{\text{additive functions with values in } M\}\\ f &\mapsto f\circ\chi_{\text{univ}}\end{aligned}$$

Proof: We define $\chi_{\text{univ}}(X)$ inductively. When the dimension $\dim X = 0$, X is a finite G-set and we can set $\chi_{\text{univ}}(X) = [X]$. Now, suppose that X is obtained by attaching some $D^n \times G/H$ to Y for $n \geq 1$ and $H \leq G$ a subgroup. By induction, we can suppose that $\chi_{\text{univ}}(Y)$ is well-defined, and we see that

$$\begin{aligned}\chi_{\text{univ}}(X) &= \chi_{\text{univ}}(Y \cup (D^n \times G/H)) \\ &= \chi_{\text{univ}}(Y) + \chi_{\text{univ}}((D^n) \times G/H) - \chi_{\text{univ}}(S^{n-1} \times G/H) \\ &= \chi_{\text{univ}}(Y) + \chi_{\text{univ}}(G/H) - \chi_{\text{univ}}(S^{n-1} \times G/H),\end{aligned}$$

and we are left with no choice for the value of $\chi_{\text{univ}}(X)$ (again using the induction assumption for $S^{n-1} \times G/H$)[1] It follows that χ_{univ} is well-defined. By the same calculation as above, we can even conclude that *every additive function is completely determined by its values on the sets of the form G/H for $H \leq G$*. Given an additive function $\chi : \mathcal{C}_G \to M$, define

$$\begin{aligned} f : \{\text{Finite } G\text{-sets}\} &\to M \\ S &\mapsto \chi(S). \end{aligned}$$

It is immediate that $f(G/H)$ only depends on the conjugacy class of H, and that it factors through as $\{\text{Finite } G\text{-sets}\} \xrightarrow{\chi_{\text{univ}}} A(G) \to M$, which proves surjectivity. Injectivity is immediate. □

Let us fix $M = \mathbb{Z}[|G|^{-1}]$ for the remainder of this section. We introduce two additive functions, which we will soon see form bases for all additive functions.

Definition 2.3.3: For every subgroup $K \subseteq G$ set $\chi_K(X) = \chi(X^K)$, where χ denotes the classical Euler characteristic. We also define μ_K by downwards induction as follows: Set $\mu_G = \chi_G$. If $K \leq G$ is a subgroup such that $\mu_{K'}$ is well-defined for every $K \lneq K'$, we set

$$\mu_K = \chi_K - \sum_{K' \gneq K} \mu_{K'},$$

in other words, the $\{\mu_K\}$ are the additive functions such that

$$\chi_K = \sum_{K' \geq K} \mu_{k'}$$

for every $K \leq G$.

And, as promised,

[1] If one does not like this argument, $\chi_{\text{univ}}(S^{n-1} \times G/H)$ can also be calculated inductively on n by covering the sphere with 2 extended hemispheres. It follows that $\chi(S^{n-1} \times G/H)$ fluctuates between 0 and $2\chi_{\text{univ}}(G/H)$ depending on the parity of $n-1$.

Lemma 2.3.4: If $\mathcal{K} \subseteq \{K : K \subseteq G\}$ is a system of representatives of conjugacy classes of subgroups of G, the sets $\{\mu_K : K \in \mathcal{K}\}$ and $\{\chi_K : K \in \mathcal{K}\}$ form two bases for additive functions with values in $\mathbb{Z}[|G|^{-1}]$.

Proof: By lemma 2.3.2, for χ_K to form a basis it suffices to show that the maps $A(G) \to \mathbb{Z}[|G|^{-1}]$ defined by $S \mapsto |S^K|$ form a basis for $\mathrm{Hom}(A(G), \mathbb{Z}[|G|^{-1}])$. But this is immediately implied by proposition 2.2.5. By its definition, μ_k must also form a basis. □

It will be useful later on to describe additive functions χ in terms of the basis μ_k.

Proposition 2.3.5: If χ is any additive function, then

$$\begin{aligned}\chi &= \sum_{(H)} \frac{1}{|W_G(H)|}\chi(G/H)\mu_H \\ &= \frac{1}{|G|}\sum_{H} |H|\chi(G/H)\mu_H\end{aligned}$$

Proof: Since we can write $\chi = \sum \alpha_K \chi_K$, it suffices by linearity of our formula in the proposition to prove the case $\chi = \chi_K$. Remember from eariler that $W_G(H)$ acts freely on G/H^K. We remark that this action has one orbit for each group $H' \sim H$ that contains K and in particular

$$\chi_K(G/H) = |(G/H)^K| = \sum_{H' \geq K,\, H' \sim H} |W_G(H)|,$$

which lets us conclude that

$$\begin{aligned}\sum_{(H)} \frac{1}{|W_G(H)|}|(G/H)^K|\mu_H &= \sum_{(H)} \sum_{\substack{H' \geq K, \\ H' \sim H}} \mu_H \\ &= \sum_{(H)} \frac{|G|}{|N_G(H)|}\mu_H \\ &= \sum_{K \leq H} \mu_H \\ &= \chi_K.\end{aligned}$$

The third equality follows since $|G|/|N_G(H)|$ is equal to the number of subgroups H' conjugate to H. □

Complex oriented additive functions

Inspired by corollary 2.1.10 and proposition 2.1.9, we see that a G-equivariant $K(n)$-theoretic Euler characteristic of a flag bundle should be $n!$ times the characteristic of its base space (we will define the precise notions and show this in the next subsection). We can therefore restrict our results above to a more specific case.

Definition 2.3.6: An additive function χ is called **complex oriented** if given any n-dimensional G-equivariant complex vector bundle ξ over X, we have

$$\chi(F(\xi)) = n!\chi(X).$$

This gives a refinement of lemma 2.3.4:

Theorem 2.3.7: An additive function χ is complex oriented iff its decomposition into the basis χ_K is on the form

$$\chi = \sum_{i=0}^{n} a_{A_i}\chi_{A_i},$$

with every A_i abelian.

Proof: By corollary 2.1.10, we know that χ_A is complex oriented. It immediately follows that a linear combination of these has to be complex oriented as well. For the converse, let χ be a complex oriented additive function, which we can decompose $\chi = \sum_{(H)} a_H\chi_H$. Let V be a faithful representation of G (e.g. one could choose a faithful $\mathbb{C}[G]$-module), and let n denote its dimension.

We remind the reader that any stabilizer group of $F(V)$ is abelian; for any flag $(V_1, \ldots, V_n) \in F(V)$, we see that $g \in G$ is in the stabilizer if and only if for any $v_1, \ldots, v_n$ that represents our flag we have $gv_i = \lambda_i v_i$ with $\lambda_i \in \mathbb{C}^*$. In particular, any stabilizer subgroup is abelian. Let $H \leq G$ be a subgroup such that $F(V)^H$ is non-empty, say $x \in F(V)^H$. This immediately implies that $H \leq \mathrm{stab}_G(x)$ is a subgroup of an abelian group, hence abelian itself. By contraposition, we see that $\chi_H((F(V))) = 0$ for any non-abelian subgroup $H \leq G$. We calculate

$$\begin{aligned}
n!\sum_{(H)} a_H\chi_H(X) &= n!\chi(X) \\
&= n!\chi(X \times F(V)) \\
&= \sum_{H} b_H\chi_H(X \times F(V)) \\
&= \sum_{H} b_H\chi_H(X)\chi_H F(V)) \\
&= \sum_{A \text{ Abelian}} b_A\chi_A(X)\chi_A(F(V)) \\
&= n! \sum_{A \text{ Abelian}} b_A\chi_A(X).
\end{aligned}$$

Here, the fourth equality implicitly uses that χ_H is also multiplicative (in the obvious way) and the sixth equality uses one that χ_A is complex orientable, $V \simeq *$ and thus $\chi_A(V) = 1$. By linear independence of χ_H, we can conclude that the only non-zero a_H have H abelian. $\square$

Definition 2.3.8: Just like when we defined μ_K in the non-complex oriented case, we get a complex oriented version $\mu_A^{\mathbb{C}}$ for every abelian $A \subseteq G$ by a similar downwards induction such that

$$\sum_{\substack{A \subseteq B, \\ B \text{ Abelian}}} \mu_B^{\mathbb{C}} = \chi_A$$

We specifically define the Möbius function $\mu_{\mathcal{A}(G)}(A) := \mu_A(\text{pt.})$

We can refine proposition 2.3.5 to the complex oriented case:

Proposition 2.3.9: For any complex oriented additive function, we have

$$\chi = \frac{1}{|G|} \sum_A |A| \chi(G/A) \mu_A^{\mathbb{C}}$$

Proof: In the proof of 2.3.7, we saw that if Y is a G-space with only abelian stabilizers, then $\chi_H(Y) = 0$ for any non-abelian H. It follows directly by downwards induction that $\mu_H = 0$ when H is non-abelian and $\mu_H = \mu_H^{\mathbb{C}}$ when H is abelian. Applying 2.3.5 to $X \times F(V)$, the rest of the proof follows exactly as in 2.3.7. $\square$

2.3.2 Presentation and proof of the theorem

We fix a prime p and let $K(n)$ denote the n'th Morava K-theory at p. For any finite G-CW complex $X \in \mathcal{C}_G$, set

$$\chi_{n,p}^G(X) = \dim K(n)^{\text{even}}(EG \times_G X) - \dim K(n)^{\text{odd}}(EG \times_G X),$$

i.e. $\chi_{n,p}^G$ is the equivariant $K(n)$-theoretic Euler characteristic We immediately remark that $\chi_{n,p}^G(\{*\}) = \chi_{n,p}^G$, the characteristic introduced in the beginning of this chapter. $\chi_{n,p}^G$ is immediately additive and the complex orientability will follow from the following result.

Lemma 2.3.10: Suppose X is a finite G-CW complex. Then

1. $K(n)^*(EG \times_G X)$ is finite dimensional vector space over $K(n)^*$.

2. $\chi_{n,p}^G$ is a complex oriented additive function.

3. For any abelian subgroup $A \subseteq G$ with p-Sylow subgroup $A_{(p)}$, we have

$$\chi^G_{n,p}(G/A) = |A_{(p)}|^n$$

Proof: Point 2 immediately follows from point 1 together with proposition 2.1.9. By the same proposition, we see that if $K(n)(F(V) \times X)$ is a finite dimensional over $K(n)^*$, then $K(n)(X)$ must be as well. We can thus replace X with $F(V) \times X$ which is a finite G-CW complex by proposition 2.1.2. Since stabilizers $\mathrm{stab}_G(\mathbf{v}, x) = \mathrm{stab}_G(\mathbf{v}) \cap \mathrm{stab}_G(x)$ are abelian, we can assume that $X \times F(V)$ is obtained from attaching cells on the form $G/A \times D^k$ with A abelian.

Note that $K(n)^*(X^{(m)}, X^{(m-1)}) = K^*(\vee_\alpha S^n) = \prod K(n)^{*-m}(*)$ is a free $K^*(*)$-module with one basis element for each m-cell of Y. By the long exact sequence of the pair $(X^{(m)}, X^{(m-1)})$, we inductively conclude that statement 1 holds true if and only if it holds true for $X = G/A$ with A abelian. But then $K(n)^*(EG \times_G G/A) = K(n)^*(BA)$, and point 1 follows from a calculation of this group, which we will omit since its deduction follows from the same arguments as in section 2.4.1[2] - see in particular corollary 2.4.4. Point 3 follows from the same result, using the fact that the cohomology groups are concentrated in even degrees. □

Combining proposition 2.3.9 with lemma 2.3.10 makes the following theorem:

Theorem 2.3.11: For any finite G-CW complex X,

$$\chi^G_{n,p}(X) = \frac{1}{|G|} \sum_A |A||A_{(p)}|^n \mu^{\mathbb{C}}_A(X)$$

□

The goal of the rest of this section is to interpret the right hand side of the theorem above. Let $G_{n,p}$ denote the subset of n-tuples $(g_1, \dots, g_n)$ in G that satisfy

1. Every $g_i g_j = g_j g_i$, in other words all the elements in the tuple commute.
2. Every g_i is of order p^{m_i} for some $m_i \geq 0$.

We equip this set with a G-action by conjugation, i.e.

$$g(g_1, \dots, g_n) = (gg_1g^{-1}, \dots, gg_ng^{-1})$$

There is a straightforward identification $G_{n,p} \simeq \mathrm{Hom}(\mathbb{Z}^n_p, G)$ in the following sense. Suppose are given a morphism $\phi : \mathbb{Z}^n_p \to G$. Let $g_i = \phi(0, \dots, 0, 1, 0, \dots, 0) = \phi(e_i)$

[2] In section 2.4 we restrict ourselves to complete and local Cohomology theories, but since $K(n)^*$ is a field the required results follow immediately.

be the image of the i'th unit. We claim that $(g_1, \dots, g_n)$ is an element of $G_{n,p}$. First of all $\phi(e_j)\phi(e_i) = \phi(e_i)\phi(e_j)$. We now show g_i is of order p^{m_i}. Let ℓ_i denote the order of g_i. Let ϕ_i be the composition $\mathbb{Z}_p \to \mathbb{Z}_p^n \xrightarrow{\phi} G$, where the first map is the inclusion to the i'th coordinate. We see that $\ker \phi_i = (\ell_i)$, and let $k = v(\ell_i)$ be the p-adic valuation of ℓ_i. We will show that $p^k \in (\ell_i)$, which in turn implies that $\ell_i = p^k$ which would finish our proof. Let $(b_i)_{i\geq 0}$ be the p-adic expansion of ℓ_i. We see that

1. $b_k\ell_i$ has leading term 1.

2. We can, as when constructing inverses in $\mathbb{Z}_p$, kill all the higher terms of $b_k\ell_i$ (by adding linear combinations of ℓ_i^{-1}),

which yields the bijective correspondence we looked for (the inverse identification is equivalent). We will now work towards better understanding $\mathrm{Hom}(\mathbb{Z}_p^n, G)$ with respect to our G-action above.

Lemma 2.3.12:

$$\frac{1}{|G|}|\mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}_p^n, G)| = |\mathrm{Hom}(\mathbb{Z}_p^n, G)/G|$$

Proof: For any $k \in \mathbb{Z}$, let $r : \mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}_p^n) \to \mathrm{Hom}(\mathbb{Z}_p^n)$ be the restriction map $\alpha \mapsto \alpha|_{\{0\}\times\mathbb{Z}_p^n}$. We can identify the fiber $r^{-1}(\beta)$ with the centralizer group $C_G(\mathrm{Im}\,\beta)$ the following way. Any map $\alpha : \mathbb{Z} \to \mathbb{Z}_p^n$ such that $\alpha(0, x) = \beta(x)$ is uniquely determined by β and its value $\alpha(1, 0)$. Furthermore, since $\beta(x)\alpha(1, 0) = \alpha(1, x) = \alpha(1, 0)\beta(x)$ for all x, this element is necessarily (and sufficiently) an element of our centralizer of (the image of) β. We can therefore calculate:

$$\begin{aligned}
|\mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}_p^n)| &= \sum_{\beta:\mathbb{Z}_p^n\to G} |C_G(\mathrm{Im}\,\beta)| \\
&= |G| \sum_{\beta:\mathbb{Z}_p^n\to G} \frac{|C_G(\mathrm{Im}\,\beta)|}{|G|} \\
&= |G||\mathrm{Hom}(\mathbb{Z}_p^n, G)/G|
\end{aligned}$$

where the last equality follows from the G-isomorphism

$$\mathrm{Hom}(\mathbb{Z}_n^p, G) = \coprod_{\beta\in\mathrm{Hom}(\mathbb{Z}_n^p, G)/G} G/C_G(\mathrm{Im}\,\beta)$$

□

Lemma 2.3.13:

$$|\mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}_p^n, G)| = \sum_{A \text{ Abelian}} |\mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}_p^n, A)| \mu_{\mathcal{A}(G)}(A)$$

Proof: Any $\alpha \in \mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}_p^n, G)$ has abelian image, being the image of an abelian group, hence

$$\begin{aligned} |\mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}_p^n, G)| &= \sum_{\alpha \in \mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}_p^n, G)} 1 \\ &= \sum_{\alpha \in \mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}_p^n, G)} \sum_{\mathrm{im}(\alpha) \subseteq A} \mu_{\mathcal{A}(G)}(A) \\ &= \sum_{A \text{ Abelian}} \sum_{\substack{\alpha \in \mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}_p^n, G) \\ \mathrm{im}\, \alpha \leq A}} \mu_{\mathcal{A}(G)}(A) \\ &= \sum_{A \text{ Abelian}} |\mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}_p^n, A)| \mu_{\mathcal{A}(G)}(A) \end{aligned}$$

□

Corollary 2.3.14:

$$|\mathrm{Hom}(\mathbb{Z}_p^n, G)/G| = \frac{1}{|G|} \sum_{A \text{ Abelian}} |A||A_{(p)}|^n \mu_{\mathcal{A}(G)}(A).$$

Proof: This is a consequence of the two preceding lemmas and that $|\mathrm{Hom}(\mathbb{Z}_p^n \times \mathbb{Z}, A)| = |A||A_{(p)}|^n$ for any abelian A. □

Combining this with theorem 2.3.11 we get the following interpretation of the Euler characteristic.

Theorem 2.3.15: The Morava K-theory Euler characteristic $\chi_{n,p}^G$ is equal to the number of G-orbits in $G_{n,p}$.

□

Some calculations using the theorems

Example 2.3.16: Let us apply theorem 2.3.15 to the symmetric group $G = \mathrm{Sym}_m$. If $p > m$, none of our elements $\sigma \in G$ have order p^r, hence $(G)_{n,p}$ has only one orbit.

If $m = p$ and $n = 1$ the set $G_{1,p}$ has $1 + (p-1)!$ elements (the neutral element and all p-cycles). But all p-cycles are conjugate to each other and thus $G_{1,p}$ has

2 orbits. Let us also calculate the same case but when $n = 2$. First we see that the number of elements in our G-set is

$$|G_{2,p}| = [1 + (p-1)!] + [(p-1)!p] = 1 + (p+1) \cdot (p-1)!,$$

where the first square bracket counts elements on the form (e, σ) and the second one (σ, σ') with σ a p-cycle. Note that in the latter case two p-cycles commute if and only if $\sigma' = \sigma^\ell$ for some ℓ (an orbit-stabilizer theorem argument for the conjugation action). Our number of orbits can be calculated using Burnside's lemma. To calculate $|G_{2,p}^\sigma|$ for $\sigma \in \mathrm{Sym}_p$, note that

1. The element $(1, 1)$ is stabilized by all elements.

2. Elements on the form (σ, σ') with at least one p-cycle (by symmetry we can assume σ to be non-trivial). are stabilized by the $|G_\sigma| = |G|/|G \cdot \sigma| = p$ elements of the cyclic group generated by σ. Since $\sigma' \in \langle \sigma \rangle$ these elements indeed fix our entire tuple.

To conclude, we obtain

$$\begin{aligned} |G| \cdot |G_{2,p}/G| &= \sum_{x \in G_{2,p}} |G_x| \\ &= |G| + p \cdot (p+1) \cdot (p-1)! \\ &= (p+2)p!, \end{aligned}$$

and thus our K-theoretic Euler characteristic is $p + 2$.

The same idea works for general n. One derives from the formula $|G_{n,p}| = |G_{n-1,p}| + (p-1)!p^{n-1}$ that $|G_{n,p}| = 1 + (p-1)! \sum_{i=0}^{n-1} p^i$. Then

$$|G_{n,p}/G| = 1 + \sum_{i=0}^{n-1} p^i.$$

In an expository article of the same results ([HKR92]), Hopkins, Kuhn and Ravenel used theorem 2.3.11 to calculate the $K(n)$ Euler characteristic of Sym_4 and Sym_3 for all n and obtained that $\chi_{n,3}(\mathrm{Sym}_4) = \chi_{n,3}(\mathrm{Sym}_3) = (3^n + 1)/2$, which is another formula for the same result above.

Example 2.3.17: Let us apply theorem 2.3.11 to the dihedral group $G = D_{2mp^r}$ of order $2mp^r$ and its action on the polygon P of mp^r points (with no 2-cell). We can assume that m and p are coprime. Our group G has a cyclic subgroup C_{mp^r} of order mp^r, corresponding to our rotations, and mp^r cyclic subgroups $C_2^{(i)}$ of order

2, corresponding to our mp^r reflections. Since C_{mp^r} acts freely on P, we obtain $\mu^{\mathbb{C}}_{C_{mp^r}}(P) = 0$.

Case 1: mp^r is odd (so $p \neq 2$). In this case, the abovementioned groups together with the subgroups of C_{mp^r} are the only abelian subgroups of D_{2mp^r}. For any rotation subgroup $0 \lneq A \leq C_{mp^r}$, we still obtain $\mu^{\mathbb{C}}_A(P) = 0$. For any reflection subgroup $C_2^{(i)}$, we have $\mu^{\mathbb{C}}_A(P) = 2$. For the trivial subgroup, we have

$$\mu^{\mathbb{C}}_0(P) = \chi_0(P) - \sum_{i=1}^{mp^r} \mu^{\mathbb{C}}_{C_2^{(i)}}(P) = 0 - 2mp^r.$$

We obtain

$$\chi^G_{n,p}(P) = \frac{1}{2mp^r}(4mp^r - 2mp^r) = 1.$$

Case 2: $p \neq 2$, m is even. This case differs from our first one since each reflection group $C_2^{(i)}$ is contained in a Klein 4-group $K^{(i)}$ (generated by the reflection and the 180 degree rotation). Since these groups all contain a rotation and are maximal abelian subgroups, we have $\mu^{\mathbb{C}}_{K^{(i)}}(P) = 0$, and our calculation does not change.

Case 3: $p = 2$. In this case the biggest change is that every $C_2^{(i)}$ has a non-trivial Sylow 2-group, hence

$$\chi^G_{n,2}(P) = \frac{1}{2^{r+1}m}\left[2^r m(2 \cdot 2 \cdot 2^n) - 2^{r+1}m\right] = (2^{n+1} - 1).$$

2.4 Generalized characters

Remember from the introduction that we for finite groups G had an isomorphism

$$L \otimes R(G) \to Cl(G; L)$$

which we would like to generalize in some way to some kind of isomorphism

$$\mathfrak{L} \otimes E^*(BG) \to \mathfrak{Cl}(G; \mathfrak{L})$$

for some undefined coefficient ring $\mathfrak{L}$ and class function object $\mathfrak{Cl}$. In light of the previous section, the notion of *class functions* should intuitively reflect maps that are invariant over some $G_{n,p}$-conjugation, which will act as a motivation for our definitions. For the rest of this memoir, we will fix E^* to be a cohomology theory of finite residue height n.

We will in this section both need the fixed-point sets as well as function spaces, which traditionally are denoted in the same way. To avoid confusion (although we will give context clues), the set of functions $X \to Y$ will be denoted $Y^{\leftarrow X}$.

2.4.1 Calculating $E^*(BA)$

Lemma 2.4.1: (Special thanks to Ben Steffan) Let $m \in \mathbb{Z}$ be an integer and E^* be of finite residue height. Then,

$$E^*(B\mathbb{Z}/m\mathbb{Z}) \simeq E[[x]]/[m]x.$$

Proof: We apply the Eilenberg-Maclane functor $K(-, 2)$ to the short exact sequence $0 \to \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ to obtain a sequence

$$\mathbb{C}P^\infty \to \mathbb{C}P^\infty \to K(\mathbb{Z}/m\mathbb{Z}, 2)$$

Going twice to the left in the fiber sequence yields a sequence

$$S^1 \to B\mathbb{Z}/m\mathbb{Z} \to \mathbb{C}P^\infty,$$

so $B\mathbb{Z}/m\mathbb{Z}$ is a homotopy pullback of the homotopy commutative diagram

$$\begin{tikzcd} B\mathbb{Z}/m\mathbb{Z} \arrow[r] \arrow[d] & * \arrow[d] \\ \mathbb{C}P^\infty \arrow[r] & \mathbb{C}P^\infty. \end{tikzcd}$$

To calculate the homotopy pullback, we only need to replace the right map by a fibration. We can for example use the universal S^1-bundle $S^\infty \to \mathbb{C}P^\infty$, which in turn implies that $B\mathbb{Z}/m\mathbb{Z} \to \mathbb{C}P^\infty$ is a principal $S^1 = U(1)$-bundle. It is thus the sphere bundle related to a complex line bundle $K' \to \mathbb{C}P^\infty$. Let T denote the Thom space of our bundle, i.e. T is the mapping cone

$$B\mathbb{Z}/m\mathbb{Z} \to \mathbb{C}P^\infty \to T.$$

We now get a homotopy commutative diagram

$$\begin{tikzcd} B\mathbb{Z}/m\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{R}^\infty \arrow[d] \\ \mathbb{C}P^\infty \arrow[r, "\cdot m"] \arrow[d] & \mathbb{C}P^\infty \arrow[d] \\ T \arrow[r] & \mathbb{C}P^\infty, \end{tikzcd}$$

for which $\mathbb{C}P^\infty$ is the Thom space of our universal S^1-bundle. It follows from bundle theory (here we follow the notions of [Die08, Sections 17.8 and 17.9]) that $x \in E^*(\mathbb{C}P^\infty) = E^*[[x]]$ is a Thom class, and, by naturality of Thom and Euler classes, our Gysin sequence for the vertical maps on the left become

$$\ldots \to E^*[[x]] \xrightarrow{\cdot[m]x} E^*[[x]] \to E^*B\mathbb{Z}/m\mathbb{Z} \to \ldots$$

By the Weierstrass preparation theorem 1.1.28 $[m]x$ cannot be a zero divisor. The lemma then follows. □.

There is a nice interpretation of the generator $x \in E^*[[x]]/[m]x$ above. The map $B\mathbb{Z}/m\mathbb{Z} \to \mathbb{C}P^\infty$ corresponds to a line bundle over $B\mathbb{Z}/m\mathbb{Z}$ and thus to a map $\chi : \mathbb{Z}/m\mathbb{Z} \to U(1) = S^1$. Introducing the notation $G^* = \mathrm{Hom}_{\mathrm{Grp}}(G, S^1)$ for the character ring, we deduce that x can be interpreted as the first chern class of the (bundle associated to the) map $B\chi \in (B\mathbb{Z}/m\mathbb{Z}^*)$. Also notice that χ must be a generator of $(\mathbb{Z}/m\mathbb{Z})^*$, and that a choice of generator χ' in $(\mathbb{Z}/m\mathbb{Z})^*$ induces a generating class $x \in E^*(B\mathbb{Z}/m\mathbb{Z})$.

Corollary 2.4.2: If $m = sp^r$ with $p \not| s$, then $E^*(B\mathbb{Z}/m\mathbb{Z})$ is free of rank p^{rn}.

Proof: Combine lemma 2.4.1 with proposition 1.1.29. □

What is particularly nice about being finitely generated is that the following Künneth isomorphism trivializes the calculations for $E^*(BA)$.

Lemma 2.4.3: Suppose that $E^*(Y)$ is finitely generated free module over E^*. Then, for any X, the map

$$E^*(Y) \otimes_{E^*} E^*(X) \to E^*(Y \times X) \tag{2.6}$$

is an isomorphism.

Proof: First remark that the lemma true when X is a point. We consider both sides of the equation (2.6) to be functors in X - we will call left hand side functor L^* and the right one R^*. Given a (homotopy) pushout diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

by assumptions on Y, we obtain Mayer-Vietoris sequences[3]

$$L^{n-1}(B \cap C, A) \to L^n(D, A) \to L^n(B, A) \oplus L^n(C, A) \to L^n(B \cap C, A)$$

[3] Here we implicitly think of D as being the double mapping cylinder

that map naturally to a Mayer-Vietoris sequence for R^*. Wedges (infinite included) get taken to products, and we can then conclude from the case where X is a point that our transformation of functors $L^*(X) \to R^*(X)$ is an isomorphism for all X (as in lemma 2.2.10). □

Corollary 2.4.4: For any finite abelian group A with p-Sylow subgroup $A_{(p)}$. Then $E^*(BA)$ is free of rank $|A_{(p)}|^n$ as an E^*-module. In particular, we have a Künneth isomorphism

$$E^*(BA) \otimes E^*(X) \to E^*(BA \times X)$$

Proof: Write $A = \bigoplus_i \mathbb{Z}/m_i\mathbb{Z}$. Corollary 2.4.2 implies that we can use our Künneth formula, lemma 2.4.3 which gives $E^*(BA) = \bigotimes_i E^*(B\mathbb{Z}/m_i\mathbb{Z})$. □

To finish off this section, the interpretation of Chern classes of the character ring remarked earlier on inspires a nice interpretation for $E^*(BA)$.

Proposition 2.4.5: Suppose we are given a finite abelian group A and an E^*-algebra R. The natural transformation

$$\begin{aligned} \mathrm{Hom}_{E^*-alg}(E^*(BA), R) &\to \mathrm{Hom}(A^*, {}_{p^\infty}F(R)) \\ f &\mapsto [\chi \mapsto f \circ \mathbf{c}_1(\chi)] \end{aligned}$$

is an isomorphism. Here $\mathbf{c}_1(\chi)$ denotes the endomorphism on ${}_{p^\infty}F(R)$ that sends x to $c_1(\chi)$, the first Chern class of the associated bundle over BA.

Proof: Since finite products of groups is taken to finite sums, we can by 2.4.2 assume that $A = \mathbb{Z}/p\mathbb{Z}^r$ and ${}_{p^r}F(R) = \mathrm{Hom}_{E^*}(E^*(BA), R)$. Fix a choice of generator $\chi_0 \in A^*$ and associate x to be its Chern class, such that $E^*(BA) = E^*[[x]]/[p^r]x$. We obtain an inverse

$$\begin{aligned} \mathrm{Hom}(A^*, {}_{p^r}F(R)) &\to \mathrm{Hom}(E^*(BA), R) \\ g &\mapsto g(\chi) \end{aligned}$$

□

2.4.2 The coefficient ring

For a finite group G, an element $\alpha \in G_{n,p} = \mathrm{Hom}(\mathbb{Z}_p^n, G)$ must factor through $\mathbb{Z}_p^n \to \mathbb{Z}/p^r\mathbb{Z}^n \to G$ for a fixed r large enough - it suffices to chose r such that all p-elements of G has order $\leq p^r$. For the rest of this section, we will let $\Lambda_r = (\mathbb{Z}/p^r\mathbb{Z})^n$. Then $\Lambda = \lim \Lambda_r = \mathbb{Z}_p^n$ and $\Lambda^* = (\mathbb{Q}_p/\mathbb{Z}_p)$.

Remember from definition 1.1.30 that ${}_{p^r}F$ is defined as the group of Hopf algebra morphisms $E^*[[x]]/[p^r]x \to R$.

Proposition 2.4.6: Let R be an E^*-algebra. There is a bijection between group homomorphisms $A \to {}_{p^r}F(R)$ and Hopf algebra morphisms $R[[x]]/[p^r]x \to R^{\leftarrow A}$.

Proof: Given $\phi : A \to {}_{p^r}F(R)$ we associate $\tilde{\phi} : R[[x]]/[p^r]x \to R^{\leftarrow A}$ defined by $x \mapsto (a \mapsto \phi(a)(x))$, which, though being defined only on E^*, canonically extends to R. This is a Hopf algebra morphism since $\phi(ab)(x) = \phi(a)(x) +_E \phi(b)(x)$ by assumption. This assignment has a clear inverse, which completes the proof. $\square$

We will now try to better understand this correspondence. To a morphism $\phi : \Lambda_r^* \to {}_{p^r}F(R)$ we let

$$\Delta = \prod_{\alpha_i \neq \alpha_j} (\phi(\alpha_i) - \phi(\alpha_j))$$

denote its discriminant.

Lemma 2.4.7: Suppose $\phi : \Lambda_r^* \to {}_{p^r}F(R)$ is a homomorphism and let $\tilde{\phi} : R[[x]]/[p^r]x \to R^{\leftarrow \Lambda_r^*}$ be the related morphism as in proposition 2.4.6. If $\sim$ denotes the relation $a \sim b$ iff they are equal mod a unit, then

1. $\Delta \sim \prod_{0 \neq \alpha \in \Lambda_r^*} \phi(\alpha)^{|\Lambda_r^*|}$
2. If $\tilde{\phi}$ is a monomorphism, then $\prod_{0 \neq \alpha \in \Lambda_r^*} \phi(\alpha) \sim p^r$. In particular $\Delta \sim p^{rp^{rn}}$

Proof: Remember our notations $x +_F y := F_E(x, y)$, $x -_F y = x +_F i(y)$ with $i(y)$ being the formal inverse in our formal group law as in chapter 1. The relation $x -_F y = (x - y)v(x, y)$ with $v(x, y) \in E^*[[x, y]]^\times$ gives $x -_F y \sim x - y$. We now obtain

$$\begin{aligned}
\Delta &= \prod_{\substack{\alpha \neq \beta, \\ \alpha, \beta \neq 0}} (\phi(\alpha) - \phi(\beta)) \\
&\sim \prod_{\alpha, \beta} \phi(\alpha - \beta) \\
&= \prod_{\alpha - \beta = \gamma} \prod_{\gamma \neq 0} \phi(\gamma) = \prod_{\alpha} \phi(\alpha)^{|\Lambda_r^*|},
\end{aligned}$$

which proves the first statement. For the second, use point 3 of the Weiestrass preperation theorem 1.1.28 to factorize $[p^r]x = g(x)\epsilon(x)$ with g a monic polynomial of degree p^{rn} and $\epsilon(x) \in E^*[[x]]$ a unit. If $\tilde{\phi}$ is a monomorphism, $g(x) = \prod_{\alpha \in \Lambda_r^*}(x - \phi(\alpha))$. Looking at the x-coefficient gives

$$\prod_{\alpha \neq 0} \phi(\alpha) = \epsilon(0)p^r.$$

For the right hand side, one can see the proof of 1.1.31, which gives $[p^r]'(x) = p^r u(x)$ with u a unit. This finishes our proof. $\square$

Proposition 2.4.8: Let $\phi : \Lambda_r^* \to {}_{p^r}F(R)$ be a group morphism with associated morphism $\tilde{\phi} : R[[x]]/[p^r]x \to R^{\leftarrow \Lambda_r^*}$. Then

1. $\tilde{\phi}$ is a monomorphism if and only if $\phi(a)$ is not a zero divisor for all $0 \neq a \in \Lambda_r^*$.

2. $\tilde{\phi}$ is an isomorphism if and only if $\phi(a)$ is a unit for all $0 \neq a \in \Lambda_r^*$.

Proof: We equip $R[[x]]/[p^r]x$ with the R-basis given by powers of x and $R^{\Lambda_r^*}$ the basis of "Dirac functions" δ_a. For example, the map $a \mapsto \phi(a)(X)$ decomposes into $\sum_{a\in\Lambda_r^*} \phi(a)(X)\delta_a$. In the chosen bases, it follows that $\tilde{\phi}$ be written as the matrix

$$\begin{pmatrix} 1 & 1 & 1 & \dots \\ \phi(a_1)(x) & \phi(a_2)(x) & \phi(a_3)(x) & \dots \\ \phi(a_1)(x^2) & \phi(a_2)(x^2) & \phi(a_3)(x^2) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(known as the Vandermonde matrix). Its determinant is known to be Δ, hence the proposition follows from point 1 of lemma 2.4.7. $\square$

Definition/Proposition 2.4.9: We let $\mathcal{L}_r$ be the functor that takes an E^* algebra R and sends it to the set of morphisms $\phi : \Lambda_r \to {}_{p^r}F(R)$ that satisfies point 2 in proposition 2.4.8. This functor is representable by an E^*-algebra $L_r(E^*)$, which is finite and faithfully flat over $p^{-1}E^*$.

Proof: By proposition 2.4.5, the identity morphism on $E^*(B\Lambda_r)$ corresponds to a morphism $\phi_{\text{univ}} : \Lambda_r^* \to {}_{p^r}F(E^*(B\Lambda_r))$ which is universal in the sense that any map $\Lambda_r^* \to {}_{p^r}F(R)$ factors through ϕ_{univ}. Let M be a multiplicative subset of $E^*(B\Lambda_r)$ generated by (the images of) $\phi_{\text{univ}}(a)$ with $a \neq 0$, and we define $L_r(E^*) = M^{-1}E^*(B\Lambda_r)$. A morphism $L_r(E^*) \to R$ corresponds to a morphism $\phi : \Lambda_r^* \to {}_{p^r}F(E^*(B\Lambda_r)) \to {}_{p^r}F(R)$ that has each $\phi(a)$ a unit. The converse statement is true by similar arguments. It follows that L_r represents $\mathcal{L}_r$.

Since $L_r(E^*)$ is a direct limit of flat E^*-modules, it is itself a flat module over E^*. Let $\overline{E^*(B\Lambda_r)}$ be the image of $E^*(B\Lambda_r)$ in $L_r(E^*)$. By proposition 2.4.8(1), the map $\overline{E^*(B\Lambda_r)}[[x]]/[p^r]x \to (\overline{E^*(B\Lambda_r)})^{\Lambda_r^*}$ associated to $\overline{\phi_{univ}}$ is a monomorphism and lemma 2.4.7(2) says that, up to a unit, the determinant of $\overline{\phi_{univ}}$ is $p^{rp^{rn}}$. Consequentially, $p^{-1}\overline{E^*(B\Lambda_r)} = L_r(E^*)$ so $L_r(E^*)$ is flat and finite over $p^{-1}E^*$, and faithful flatness is what remains to be shown. Remember that this is equivalent to every homogeneous prime ideal $\mathfrak{p} \in p^{-1}E^*$ extending to a prime ideal $\mathfrak{q} \in L_r(E^*)$. Let $\mathfrak{p}$ be such an ideal and choose a closed, graded field K and morphism $f : p^{-1}E^* \to K$ with kernel $\mathfrak{p}$. By theorem 1.1.31, we have an isomorphism $\mu : \Lambda^* \to L$, and our correspondence from proposition 2.4.5 makes μ induce a map $E^*(BA) \to K$, that extends to a $p^{-1}E^*$-algebra morphism $L_r(E^*) \to K$ with kernel a prime ideal $\mathfrak{q}$ that extends $\mathfrak{p}$. □

The action of $\mathrm{Aut}(\Lambda_r)$ on $\mathcal{L}_r$ induces an action on $L_r(E)$.

Proposition 2.4.10: The ring of invariants of $\mathrm{Aut}(\Lambda_r)$ on $L_r(E^*)$ is $p^{-1}E^*$.

Proof: It is enough to find a faithfully flat $p^{-1}E^*$-algebra R such that the ring of invariants on $R \otimes_{E^*} L_r(E^*)$ is R. First observe that the ring $R \otimes_{E^*} L_r(E^*)$ represents the functor that sends R-algebras T to the set of Hopf T-algebra isomorphisms

$$T[[x]]/[p^r]x \to T^{\leftarrow \Lambda_r^*}.$$

Choosing $R = L_r(E^*)$ itself (equipped with no action), we see that $L_r(E^*)[[x]]/[p^r]x \simeq L_r(E^*)^{\Lambda_r^*}$ itself (cf. 2.4.5, 2.4.6 and 2.4.8), hence $L_r(E^*) \otimes_{E^*} L_r(E^*)$ represents the functor that sends $L_r(E^*)$-algebras T to the set of Hopf algebra automorphisms on $T^{\Lambda_r^*}$. By the next lemma, the same functor is represented by $L_r(E^*)^{\leftarrow \mathrm{Aut}(\Lambda_r)}$. So $L_r(E^*) \otimes L_r(E^*) \approx L_r(E^*)^{\leftarrow \mathrm{Aut}(\Lambda_r)}$, but the latter has $\mathrm{Aut}(\Lambda_r)$-fixed points all the constant functions, i.e. $L_r(E^*)$ itself, which was to be shown. □

Lemma 2.4.11: For a ring R and A a finite abelian group, the functor that takes R-algebras

$$T \mapsto \{\text{Hopf algebra automorphisms } T^{\leftarrow A} \to T^{\leftarrow A}\}$$

is represented by $R^{\leftarrow \mathrm{Aut}(A)}$.

Proof: We first find a representative for the functor that, for fixed finite sets B_1, B_2, assigns an R-algebra T to the set of T-algebra morphisms $T^{\leftarrow B_1} \to T^{\leftarrow B_2}$. We claim that $R^{\leftarrow \mathrm{Hom}_{\mathrm{Set}}(B_1,B_2)}$ is this representative. Indeed, to a map $\mu : R^{\leftarrow \mathrm{Hom}_{\mathrm{Set}}(B_1,B_2)} \to T$ we can associate

$$R^{\leftarrow B_1} \xrightarrow{\mathrm{ev}} R^{\leftarrow B_2 \times \mathrm{Hom}_{\mathrm{Set}}(B_2,B_1)} = R^{\leftarrow \mathrm{Hom}_{\mathrm{Set}}(B_2,B_1)^{\leftarrow B_2}} \xrightarrow{\mu} T^{\leftarrow B_2},$$

which extends T-linearly to a map $T^{\leftarrow B_1} \to T^{\leftarrow B_2}$. We leave it to the reader to check that (1.) this association is natural and (2.) that the correspondence is isomorphic when B_1 is a point. Taking disjoint unions of points decomposes our formulae above into products, which proves that $R^{\leftarrow \mathrm{Hom}_{\mathrm{Set}}(B_1,B_2)}$ is indeed our representative. The lemma then follows from restricting the above accordingly. □

We are now ready to define our coefficient ring for our generalized characters. The obvious restriction of a map $\Lambda_r^* \to {}_{p^r}F(R)$ to $\Lambda_{r-1}^* \to {}_{p^{r-1}}F(R)$ induces a morphism $L_{r-1}(E^*) \to L_r(E^*)$, and we can take the colimit

$$L(E^*) = \underset{\overrightarrow{r}}{\mathrm{colim}}\, L_r(E^*).$$

It follows, since each term $L_{r-1}(E^*) \to L_r(E^*)$ is $\mathrm{Aut}(\Lambda)$-equivariant (by factoring through $\mathrm{Aut}(\Lambda) \to \mathrm{Aut}(\Lambda_r)$ for each r) that we get an action of $\mathrm{Aut}(\Lambda)$ on $L(E^*)$. The following properties of $L(E^*)$ are immediately deduced.

Corollary 2.4.12:

1. $L(E^*)$ represents the functor that sends an E^*-algebra R to the set of isomorphisms a.
2. $L(E^*)$ is faithfully flat over E^*.
3. The invariant ring $L(E^*)^{\mathrm{Aut}(\Lambda)} = p^{-1}E^*$.

□

2.4.3 Generalized characters

In this subsection, X will denote a finite G-CW complex and r an integer large enough so that all p-torsion of G divides p^r. It follows that $\mathrm{Hom}(\Lambda_r, G) = G_{n,p}$ (see page 41). Let

$$\begin{aligned}\mathrm{Fix}_{n,p}(G,X) &:= \coprod_{\alpha \in \mathrm{Hom}(\Lambda_r, G)} X^{\mathrm{Im}(\alpha)} \\ &= \{(\alpha, x) \in \mathrm{Hom}(\Lambda_r, G) \times X : \text{ for all } \lambda \in \Lambda_r,\ \alpha(\lambda)x = x\},\end{aligned}$$

which has an obvious left $G \times \mathrm{Aut}(\Lambda_r)$-action with G acting diagonally. The fixed-point sets have the following properties [Kuh89, Prop. 6.2]

Remark 2.4.13:

1. $\mathrm{Fix}_{n,p}(G, \mathrm{pt}) = \mathrm{Hom}(\Lambda_r, G)$
2. $\mathrm{Fix}_{n,p}(G, -)$ preserves G-pushouts.
3. We have a natural G-homeomorphism

$$G \times_H \mathrm{Fix}_{n,p}(H, X) = \mathrm{Fix}_{n,p}(G, G \times_H X).$$

Their proofs are as straightforward as writing them out.

A morphism $\alpha : \Lambda_r \to G$, gives a map of Λ_r-spaces $E\Lambda_r \to EG$ and therefore a map $\tilde{\alpha} : B\Lambda_r \times X^{\mathrm{Im}(\alpha)} \to EG \times_G X$. We then obtain, with the help of our Künneth isomorphism 2.4.3,

$$\tilde{\alpha}^* : E^*(EG \times_G X) \to E^*(B\Lambda_r) \otimes_{E^*} E^*(X^{\mathrm{Im}(\alpha)}) \to L_r(E^*) \otimes_{E^*} E^*(X^{\mathrm{Im}(\alpha)}).$$

This will give our character map $\chi^G_{n,p}$.

Definition 2.4.14: Taking the product over $\alpha \in \mathrm{Hom}(\Lambda_r, G)$ gives a map

$$\chi^G_{n,p} : E^*(EG \times_G X) \to L_r(E^*) \otimes_{E^*} E^*(\mathrm{Fix}_{n,p}(G, X)).$$

We will now define an action of $G \times \mathrm{Aut}(\Lambda_r)$ on $L_r(E^*) \otimes_{E^*} E^*(\mathrm{Fix}_{n,p}(G, X))$ as follows. G has its canonical action on $\mathrm{Fix}_{n,p}(G, X)$ and we have the diagonal action of $\mathrm{Aut}(\Lambda_r)$ on the tensor product.

Lemma 2.4.15: $\mathrm{Im}\,\chi_{n,p} \subseteq (L_r(E^*) \otimes_{E^*} E^*(\mathrm{Fix}_{n,p}(G, X)))^{G \times \mathrm{Aut}(\Lambda_r)}$.

Proof: We will show that each group individually acts trivially on the image - this will be completely independent of E^*. First let $\alpha \in \mathrm{Hom}(\Lambda, G)$ and $\phi \in \mathrm{Aut}(\Lambda)$. By inspection, the following diagram commutes

$$\begin{tikzcd}
B\Lambda_r \times X^{\mathrm{im}\alpha\circ\phi} \arrow[r, "\phi\times\mathbb{1}"] \arrow[d, "\alpha\circ\phi"] & B\Lambda_r \times X^{\mathrm{im}\alpha} \arrow[d, "\alpha"] \\
EG \times_G X \arrow[r, equal] & EG \times_G X.
\end{tikzcd}$$

For G-invariance, let $c_g : EG \times_G X \to EG \times_G X$ be the morphism induced by $(k, x) \mapsto (gkg^{-1}, gx) \in G \times X$. We have the commutative diagram

$$\begin{tikzcd}
B\Lambda_r \times X^{\mathrm{im}\alpha} \arrow[r, "\mathbb{1}\times g\cdot"] \arrow[d] & B\Lambda_r \times X^{g(\mathrm{im}\alpha)g^{-1}} \arrow[d] \\
EG \times_G X \arrow[r, "c_g"] & EG \times_G X.
\end{tikzcd}$$

Admitting that c_g is homotopic to the identity [Seg68a, Proposition 2.1], the lemma follows. □

Set $Cl_{n,p}(G, X; L(E^*)) := L_r(E^*) \otimes_{E^*} E^*(\mathrm{Fix}_{n,p}(G, X))^G$. The above lemma implies that we have a character map $\chi^G_{n,p} : E^*(EG \times_G X) \to Cl_{n,p}(G, X; L(E^*))$.

Example 2.4.16: Let us calculate the ring of class maps $Cl_{n,p}(A,*;L(E^*))$ for A a finite abelian group, which prove useful later. Here we obtain

$$\begin{aligned} Cl_{n,p}(A,*;L(E^*)) &= L(E^*)\otimes_E E^*(\mathrm{Fix}_{n,p}(A,*))^A \\ &= L(E^*)\otimes_E E^*(\mathrm{Hom}(\Lambda_r,A)) \\ &= \bigoplus_{\mathrm{Hom}(\Lambda,A)} L(E^*) \\ &= L(E^*)^{\leftarrow \mathrm{Hom}(\Lambda,A)}. \end{aligned}$$

We are now ready to prove the main theorem.

Theorem 2.4.17: The algebra $L(E)$ is faithfully flat over $p^{-1}E^*$, and the invariants $L(E^*)^{\mathrm{Aut}(\mathbb{Z}_p^n)} = p^{-1}E^*$. The character map $\chi_{n,p}^G$ induces isomorphisms

$$L(E^*)\otimes_{E^*} E^*(EG\times_G X)\simeq Cl_{n,p}(G,X;L(E^*))$$

and

$$p^{-1}E^*(EG\times_G X)\simeq Cl_{n,p}(G,X;L(E))^{\mathrm{Aut}(\Lambda_r)}$$

In particular, when X is a point we obtain

$$L(E^*)\otimes_{E^*} E^*(BG)\simeq Cl_{n,p}(G;L(E^*)),$$

which is the exact generalization that we hinted at in the introduction. The proof of this theorem will be a categorical axiomatic approach.

Definition 2.4.18: Let W^* be a contravariant functor from the category $\mathcal{GC}$ of pairs (G,X) (G a finite group, X a finite G-CW complex) to the category of graded abelian groups. We introduce the following properties on W:

1. **Mayer-Vietoris**: Any pushout D of $B\leftarrow A\rightarrow C$ in $\mathcal{GC}$ gives a long exact sequence
$$\ldots\to W^n(D)\to W^n(B)\oplus W^n(C)\to W^n(A)\xrightarrow{\delta} W^{n+1}(D)\to\ldots$$
2. **Induction**: If $H\leq G$ and X is a H-space then the obvious map $(H,X)\to(G,G\times_H X)$ induces isomorphism
$$W^*(G,G\times_H X)\xrightarrow{\sim} W^*(H,X).$$
3. **Descent**: If F is the flag bundle of an equivariant vector bundle over X then there is an equalizer (coming from the obvious maps)
$$W^*(G,X)\longrightarrow W^*(G,F)\rightrightarrows W^*(G,F\times_X F).$$

As we have seen multiple times in this project, natural transformations between these functors are isomorphisms if they are isomorphisms in the simplest cases. There is an intuitive argument for this; since these functors are all cohomology-esque (we had in fact for earlier cases actual cohomology theories), which is also why the proofs are analogous to the result for cohomology [Hat02, 3.17].[4]

Lemma 2.4.19: Let $\eta : W \to Z$ be a natural transformation between functors satisfying the properties above such that η commutes with the connecting morphism δ in the Mayer-Vietoris sequence and is homotopy invariant. If η is an isomorphism for all pairs of abelian groups A and the point then η is an isomorphism for all finite groups G and finite G-CW complexes.

Proof: Let G be a finite group and X a G-space. By descent we can replace X with $F(V) \times X$ with $F(V)$ the flag manifold of a faithful finite dimensional representation. We thus only have to show isomorphisms for G-CW complexes with cells on the form $G/A \times D^n$ with $A \leq G$ abelian. Suppose for induction on the dimension on $X = X^{(m)}$ that our lemma holds true for G-CW complexes of dimension less than m and suppose X is obtained from Y by attaching one cell. Applying the Mayer-Vietoris sequence to our attachment pushout yields

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & W^n(X) & \longrightarrow & W^n(Y) \oplus W^n(H) & \longrightarrow & W^n(K) & \xrightarrow{\delta} & W^{n+1}(X) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & Z^n(X) & \longrightarrow & Z^n(Y) \oplus Z^n(H) & \longrightarrow & Z^n(K) & \xrightarrow{\delta} & Z^{n+1}(X) & \longrightarrow & \dots
\end{array}$$

where $K = S^{m-1} \times G/A$ and $H = D^m \times G/A$. By induction assumption the second vertical arrow is an isomorphism and we have reduced our case to when $X = D^n \times G/A$. By homotopy invariance, this is equivalent to when $X = G/A$ and by induction $\eta_{(G,G/A)}$ is an isomorphism if and only if $\eta_{(A,\mathrm{pt})}$ is, which was our hypothesis. □

Proof of theorem 2.4.17: We would like to use lemma 2.4.19 on $\chi_{n,p}$. Let us verify the axioms for

$$\begin{aligned}
W_G(X) &= L(E^*) \otimes_{E^*} E^*(EG \times_G X) \\
Z_G(X) &= Cl_{n,p}(G, X; L(E^*)) = L(E^*) \otimes E^*(\mathrm{Fix}_{n,p}(G, X))^G
\end{aligned}$$

The sequence of the Mayer-Vietoris property comes from the classical Mayer-Vietoris sequence on E^*. Exactness follows from flatness of $L(E^*)$ and the fact that passage to G-invariants is exact (see further below). Descent follows in the same way with

[4] The reason we don't simply refer to this theorem every time and instead proving it each time is because it would take just as much reasoning to show that our functors indeed are cohomology theories.

help from complex oriented descent, proposition 2.1.9. Induction follows from the homotopy equivalence

$$EG \times_G (G \times_H X) \simeq EH \times_H X$$

and the homeomorphism

$$G \times_H \mathrm{Fix}_{n,p}(H, X) = \mathrm{Fix}_{n,p}(G, G \times_H X)$$

from remark 2.4.13.

We now need to show that $\chi^A_{n,p}(pt)$ is an isomorphism on all finite abelian groups A. Since all domains and codomains transform products of abelian groups to tensor products we may only consider the case when $A = \mathbb{Z}/p^r\mathbb{Z}$. We can reinterpret $A = \mathrm{Hom}(S^1, \mathbb{Z}/p^r\mathbb{Z})$, which canonically lets us identify $\mathrm{Hom}(\Lambda, A) \simeq \Lambda^*_r$ and $E^*(BA) = E[[x]]/([p^r]x)$ by lemma 2.4.1. We then obtain

$$\begin{array}{ccc} L(E^*) \otimes E^*(BG) & \xrightarrow{\chi^G_{n,p}} & L(E^*)^{\mathrm{Hom}(\Lambda, A)} \\ \| & & \| \\ L(E^*)[[x]]/([p^r]x) & & L(E^*)^{\Lambda^*_r}, \end{array}$$

and $\chi^G_{n,p}$ identifies with the isomorphism $L_r(E^*)[[x]]/[p^r]x \to L_r(E^*)^{\Lambda^*_r}$ that is obtained from the canonical $E^*(BA) \to L_r(E^*)$ (in the same way as in the proof of proposition 2.4.10). $\square$

Let us verify the claim in the proof that passage to G-invariants is exact. Specifically, let us check that if

$$A \xrightarrow{\mu} B \xrightarrow{\nu} C$$

is an exact sequence in the category of R-modules with G-action such that $|G|$ is invertible in R, then the passage to G-invariants

$$A^G \xrightarrow{\mu^G} B^G \xrightarrow{\nu^G} C^G$$

is exact. $\mathrm{im}\mu^G \subseteq \ker \nu^G$ immediately, and if $b \in \ker \nu^G$ and we choose $a \in A$ such that $\mu(a) = b$, then $|G|^{-1} \sum_{g \in G} ga \in A^G$ yields the element we look for.

2.4.4 An induction theorem

The Transfer map

Suppose that $p : T \to X$ is an n-sheeted covering space. We will in this section introduce a map $E^*(X) \to E^*(T)$ called the transfer map and study some properties regarding this. We will follow chapter 4 of Adams' *Infinite Loop spaces* [Ada78].

Adams in turn credits most of these ideas to Roush's PhD thesis [Rou72]. The transfer map has a very direct interpretation in the case of singular cohomology: a singular simplex $\sigma : \Delta^n \to X$ can be lifted to a chain $\sum_i \tilde{\sigma}_i \in C_n(T)$ running over all simplices such that $p \circ \tilde{\sigma}_i = \sigma$. We call the assignment above the transfer $\mathrm{Tr} : C_n(X) \to C_n(T)$, and one immediately sees that it commutes with the differential and thus passes to homology - and similarly to cohomology. This explicit idea does unfortunately not work for general cohomology theories, and we therefore need a more general construction.

Instead of finding our transfer map directly on cohomology, we will instead for each relative, finite[5] covering $(X, A) \to (Y, B)$ find a map of spectra $\mathrm{tr} : \Sigma^\infty(Y/B) \to \Sigma^\infty(X/A)$. The construction is purely geometric and we only present a sketch of the ideas here. Let us start in the case where X and Y are finite complexes. For n large enough, X embeds itself into $Y \times (I^n)^\circ$ such that our covering factors as $X \to Y \times (I^n)^\circ \to Y$, the last map being the obvious one. We can extend this embedding to small cubical tubular neighborhood $X \times I^n$ to obtain a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "p"'] & X \times I^n \arrow[r, "\epsilon"] \arrow[d] & Y \times I^n \arrow[dl, "\pi"] \\ & Y. & \end{tikzcd}$$

Collapsing appropriately outside the tubular neighborhood gives a map $\Sigma^n Y/B \to \Sigma^n X/A$. This map is, up to suspending enough, independent[6] of choice of ϵ and n, which provides the map $\Sigma^\infty Y/B \to \Sigma^\infty X/A$ which we wanted.

Let us state some properties of these transfer maps, which becomes the flavor of which will be useful in the following sections. Let us start by giving a concise list of the properties in question.

Proposition 2.4.20: The transfer maps defined above satisfy the following properties.

1. **Functoriality:** The transfer map corresponding to the identity is the identity and is functorial on compositions of coverings $X \to Y \to Z$.

2. **Naturality**: Given a map of covering spaces, i.e. a commutative diagram

$$\begin{tikzcd} (T, U) \arrow[r, "\alpha"] \arrow[d, "p"] & (T', U') \arrow[d, "p'"] \\ (X, Y) \arrow[r, "\beta"] & (X', Y'), \end{tikzcd}$$

[5] The number of sheets here is allowed to vary on each path component

[6] In the homotopical sense.

gives a (homotopy) commutative diagram

$$\begin{tikzcd} \Sigma^\infty T/U \arrow[r, "\Sigma^\infty \alpha"] & \Sigma^\infty T'/U' \\ \Sigma^\infty X/Y \arrow[u, "\mathrm{Tr}"] \arrow[r, "\Sigma^\infty \beta"] & \Sigma^\infty X'/Y'. \arrow[u, "\mathrm{Tr}"'] \end{tikzcd}$$

3. **Additivity**: Given two coverings $(T_1, U_1) \xrightarrow{p_1} (X, Y) \xleftarrow{p_2} (T_2, U_2)$ then the transfer map of $p_1 \amalg p_2$ is the coproduct of the transfer maps of the other two coverings.

4. **Multiplicativity**: Given coverings $p_1 : (T, U) \to (X, Y)$ and $p_2 : (T', U') \to (X', Y')$, the transfer map related to $p_1 \prod p_2$ is identified with the smash of our other 2 transfers.

Most of these follow directly from the definition. Additivity follows from naturality on the double fiber diagram

$$\begin{tikzcd} T_1 \arrow[r] \arrow[d] & T_1 \amalg T_2 \arrow[d] & T_2 \arrow[l] \arrow[d] \\ X \arrow[r, equal] & X \arrow[r, equal] & X \end{tikzcd}$$

Multiplicativity is however quite hard since it requires working on smash products of spectra and the reader is referred to [Rou72] (Adams only does it in the finite-dimensional case). Perhaps a modern reference does it elegantly, but such has not been found by the author.

The proposition above has a canonical interpretation for cohomology. We will for simplicity later state it in full as a corollary.

Corollary 2.4.21: The transfer maps on cohomology satisfies the following properties.

1. **Naturality**: Transfer maps convert a fiber diagram into a commutative square of cohomology groups.

2. **Identity**: The transfer map of the identity $X \to X$ is the identity $E^*(X) \to E^*(X)$.

3. **Additivity**: Given two coverings $T_1 \to X \leftarrow T_2$, the following triangle commutes

$$\begin{tikzcd} E^*(T_1 \amalg T_2) \arrow[rr, equal] \arrow[dr, "\mathrm{Tr}"'] & & E^*(T_1) \oplus E^*(T_2) \arrow[dl, "\mathrm{Tr} \oplus \mathrm{Tr}"] \\ & E^*(X) & \end{tikzcd}$$

4. **Multiplicativity**: The map $E^*(T) \to E^*(X)$ is a morphism of $E^*(X)$-modules.

For the last part we of course implicitly use that E^* is a multiplicative spectrum. □

The induction theorem

Let $C \leq A$ be two finite abelian groups. Remember that this induces a covering space $BC \to BA$, and we thus find a transfer map $E^*(BC) \xrightarrow{\mathrm{Tr}} E^*(BA)$.

Proposition 2.4.22: If $A \lneq \Lambda_r$ the composition $E^*(BA) \xrightarrow{\mathrm{Tr}} E^*(B\Lambda_r) \to L_r(E^*)$ is zero.

Proof: Let μ be non-zero element of the kernel of $\Lambda_r^* \to A^*$. Remember from defintion 2.4.9 that $L_r(E^*) = S^{-1}E^*(B\Lambda_r)$ is localized with respect to the image of a morphism $\phi : \Lambda_r^* \to E^*(B\Lambda_r)$. Let $x = \phi(\mu)$ and, by design, we see that the restriction of x to $E^*(BA)$ (i.e. the image of $i^* : E^*(B\Lambda_r) \to E^*(BA)$) is zero. It then follows that for any $y \in E^*(BA)$ we have

$$x\,\mathrm{Tr}(y) = \mathrm{Tr}(i^*(x)y) = 0,$$

since our transfer map is a map of modules. Thus every $\mathrm{Tr}(y)$ is a zero divisor, but since $x = \phi(\mu)$, it is a unit and $\mathrm{Tr}(y)$ necessarily gets eliminated in $L_r(E^*)$. □

Lemma 2.4.23: Let S be a finite Λ_r-space and $S^{\Lambda_r} = \varnothing$ and Y a space with has trivial Λ_r-action. The composition

$$E^*(E\Lambda_r \times_{\Lambda_r} (S \times Y)) \xrightarrow{\mathrm{Tr}} E^*(B\Lambda_r \times Y) \to L_r(E^*) \otimes_{E^*} E^*(Y)$$

is zero.

Proof: By additivity of the transfer map, S can be assumed to be on the form $S = \Lambda_r/A$ with A a proper subgroup. The homotopy equivalence $EG \times_G (G \times_H Y) = EH \times_H Y$ gives in our case that $E\Lambda_r \times_{\Lambda_r} (\Lambda_r/A \times Y) = E\Lambda_r \times_{\Lambda_r} (\Lambda_r/A \times Y) = BA \times Y$ (this is also where the transfer map comes from). This together with the Künneth formula 2.4.3 means that our composition above is on the form

$$E^*(BA) \otimes E^*(Y) \to E^*(B\Lambda_r) \otimes E^*(Y) \to L_r(E^*) \otimes E^*(Y).$$

Additivity and multiplicativity of the transfer map then gives that this composition simply is the map of proposition 2.4.22 tensored with $E^*(Y)$. □

Theorem 2.4.24: Let $H \leq G$ be a subgroup, X a G-space and $x \in E^*(EH \times_H X)$. Then:

$$\chi^G_{n,p}(\alpha)(\mathrm{Tr}(x)) = \sum_{gH \in (G/H)^{\mathrm{Im}\,\alpha}} \chi^H_{n,p}(g.\alpha)(x).$$

Proof: Let H, G, X and x be as in the statement of the theorem and $\alpha : \Lambda \to G$ a morphism. Chose r large enough so that α factors through Λ_r. The map $B\Lambda_r \times X^{\mathrm{im}\alpha} \to EG \times_G X$ factors and fits into a composition of pullback diagrams

$$\begin{tikzcd}
E\Lambda_r \times_{\Lambda_r} (G/H \times X^{\mathrm{im}\alpha}) \arrow[r] \arrow[d] & E\Lambda_r \times_{\Lambda_r} (G \times_H X) \arrow[r] \arrow[d] & \underbrace{EG \times_G (G \times_H X)}_{=EH \times_H X} \arrow[d] \\
B\Lambda_r \times X^{\mathrm{im}\alpha} \arrow[r] & E\Lambda_R \times_{\Lambda_r} X \arrow[r] & EG \times_G X.
\end{tikzcd}$$

By naturality of the transfer, calculating the $\chi^G_{n,p}(\alpha)(\mathrm{Tr}(x))$ is the same as going through the composition

$$\begin{aligned} E^*(EH \times_H X) &\to E^*(E\Lambda_r \times_{\Lambda_r} (G/H \times X^{\mathrm{im}\alpha})) \\ &\xrightarrow{\mathrm{Tr}} E^*(B\Lambda_r \times X^{\mathrm{im}\alpha}) \to L_r(E^*) \otimes E^*(X^{\mathrm{im}\alpha}). \end{aligned} \tag{2.7}$$

We shall now look closer at the second term of this composition. The space $E\Lambda_r \times_{\Lambda_r} (G/H \times X^{\mathrm{im}\alpha})$ can be decomposed by looking at the $gH \in G/H$. If $gH \in (G/H)^{\mathrm{im}\alpha}$, then $(y_1, (gH, x)) \sim (\lambda y_1, (gH, x))$ for any $y_1, y_2 \in E\Lambda_r$ and $\lambda \in \Lambda_r$, hence gH generates a summand on the form $B\Lambda_r \times \{gH\} \times X^{\mathrm{im}\alpha}$. Let S denote the set of elements in gH that aren't stabilized by α. It follows that

$$E\Lambda_r \times_{\Lambda_r} (G/H \times X^{\mathrm{im}\alpha}) = \left(\amalg_{gH \in (G/H)^{\mathrm{im}\alpha}} B\Lambda_r \times \{gH\} \times X^{\mathrm{im}\alpha}\right) \coprod \left(E\Lambda_r \times_{\Lambda_r} (S \times X^{\mathrm{im}\alpha})\right)$$

On the latter component of this disjoint union, the composition in (2.7) is zero by lemma 2.4.23. On the former component, and specifically on each gH-summand, x is sent through (2.7) to $\chi^G_{n,p}(g.\alpha)(x)$. The theorem then follows from additivity of the transfer. □

Bibliography

[Ada74] J.F. Adams. *Stable Homotopy and Generalised Homology*. University of Chicago Press, 1974.

[Ada78] J.F. Adams. *Infinite Loop Spaces: Hermann Weyl Lectures, The Institute for Advanced Study*. Annals of Mathematics Studies. Princeton University Press, 1978.

[AS69] M. F. Atiyah and G. B. Segal. Equivariant K-theory and completion. *Journal of Differential Geometry*, 3(1-2):1 – 18, 1969.

[BR20] D. Barnes and C. Roitzheim. *Foundations of Stable Homotopy Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University press, 2020.

[Bro62] Edgar Brown. Cohomology theories. *ANNALS OF MATHEMATICS, Vol. 75, No. 3, May, 1962*, 1962.

[BW89] Andrew Baker and Urs Würgler. Liftings of formal groups and the artinian completion of bp. *Mathematical Proceedings of the Cambridge Philosophical Society*, 106(3):511–530, 1989.

[CF06] P. E. Conner and E. E. Floyd. *The Relation of Cobordism to K-Theories*. Springer Berlin, Heidelberg, 2006.

[Cno26] Baastian Cnossen. Stable homotopy theory and higher algebra (in progress). https://drive.google.com/file/d/1ivHDIqclbg2hxmUEMTqmj2TnsAHQxVg9/view, 2026.

[DHS88] Ethan S. Devinatz, Michael J. Hopkins, and Jeffrey H. Smith. Nilpotence and stable homotopy theory i. *Annals of Mathematics*, 128(2):207–241, 1988.

[Die79] Tammo Tom Dieck. *Transformation Groups and Representation Theory*. Lecture Notes in Mathematics. Springer Berlin, Heidelberg, 1979.

[Die08] T. Dieck. *Algebraic Topology*. EMS textbooks in mathematics. European Mathematical Society, 2008.

[GM95] J.P.C. Greenlees and J.P. May. Chapter 8 - equivariant stable homotopy theory. In I.M. JAMES, editor, *Handbook of Algebraic Topology*, pages 277–323. North-Holland, Amsterdam, 1995.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[HKR92] Michael J. Hopkins, Nicholas J. Kuhn, and Douglas C. Ravenel. Morava k-theories of classifying spaces and generalized characters for finite groups. In Jaume Aguade, Manuel Castellet, and Frederick Ronald Cohen, editors, *Algebraic Topology Homotopy and Group Cohomology*, pages 186–209, Berlin, Heidelberg, 1992. Springer Berlin Heidelberg.

[HKR00] Michael J. Hopkins, Nicholas J. Kuhn, and Douglas C. Ravenel. Generalized group characters and complex oriented cohomology theories. *J. Amer. Math. Soc. 13 (2000), 553-594*, 2000.

[Hop99] Michael Hopkins. Complex oriented cohomology theories and the language of stacks course notes for 18.917, taught by mike hopkins. 1999.

[HS98] Michael J. Hopkins and Jeffrey H. Smith. Nilpotence and stable homotopy theory ii. *Annals of Mathematics*, 148(1):1–49, 1998.

[Ill74] Sören Illman. Whitehead torsion and group actions. *Annales Academiae Scientiarum Fennicae. Series A I*, 1975, 01 1974.

[Kuh89] Nicholas J. Kuhn. *Character rings in algebraic topology*, page 111–126. London Mathematical Society Lecture Note Series. Cambridge University Press, 1989.

[Lan78] Serge Lang. *Cyclotomic Fields*. Graduate texts in mathematics ; 59. Springer International Publishing., Cham, 1978.

[LMS] L. Gaunce Lewis, J. Peter May, and Mark Steinberger. *Equivariant Stable Homotopy Theory*. Lecture notes in mathematics. Springer Berlin, Heidelberg.

[Lur10] Jacob Lurie. Lecture notes on chromatic homotopy theory. `https://www.math.ias.edu/~lurie/252xnotes`, 2010.

[Lur17] Jacob Lurie. Higher algebra. 2017.

[MPC96] J.P. May, R.J. Piacenza, and M. Cole. *Equivariant Homotopy and Cohomology Theory: Dedicated to the Memory of Robert J. Piacenza*. Regional conference series in mathematics. American Mathematical Society, 1996.

[MS74] J.W. Milnor and J.D. Stasheff. *Characteristic Classes*. Annals of mathematics studies. Princeton University Press, 1974.

[Rav92] D.C. Ravenel. *Nilpotence and Periodicity in Stable Homotopy Theory*. Annals of Mathematics Studies. Princeton University Press, 1992.

[Rav03] D.C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres*. AMS Chelsea Publishing Series. AMS Chelsea Pub., 2003.

[Rou72] Fred W. Roush. Transfer in generalized cohomology theories. 1972.

[RW80] D. Ravenel and W. Wilson. The morava k-theories of eilenberg-maclane spaces and the conner-floyd conjecture. *American journal of mathematics*, 1980.

[Sch18] Stefan Schwede. Lectures on equivariant stable homotopy theory. 2018.

[Seg68a] Graeme Segal. Classifying spaces and spectral sequences. *Publications mathématiques de l'IHÉS*, 1968.

[Seg68b] Graeme Segal. Equivariant k-theory. *Publications mathématiques de l'IHÉS*, 1968.

[Seg70] G. B. Segal. Equivariant stable homotopy theory. *Actes Congrès intern. Math. tome 2*, pages 59 – 63, 1970.

[Ser67] J.P. Serre. *Représentations linéaires des groupes finis*. Collection Méthodes. Hermann, 1967.

[Wei94] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994.