Higher-algebraic Picard invariants in modular representation theory

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Preface

Abstract

This thesis consists of three main parts, prefaced by a general introduction.

The first part is based on a paper joint with Richard Wong. We exhibit an ∞-categorical decomposition of the stable module category of a general finite group, and we show that, in the case of certain particularly simple finite p-groups, this decomposition can be interpreted as an instance of Galois descent. We then use this perspective to produce proof-of-concept calculations of the group of endotrivial modules for these p-groups.

In the second part, we move on to computations for more complicated groups. Of particular interest will be the case of the extraspecial groups, which have traditionally played a fundamental role in the theory of endotrivial modules. We analyse the Picard spectral sequence for the extraspecial groups and show that the E_2-page inherits a great deal of structure from a certain Tits building of isotropic subspaces with respect to a quadratic form.

In the third and final part, we move on to study the Dade group of endopermutation modules. We investigate how it can be realised as the Picard group of a certain ∞-category of genuine equivariant spectra. On our way, we produce a general framework for studying modules whose endomorphisms are trivial up to a specified subcategory of the representation category. This produces invariants that interpolate between the group of endotrivial modules and the Dade group, as well as other more exotic invariants that are of independent interest.

Resumé

Denne ph.d.-afhandling består af tre hoveddele samt en generel indledning.

Den første del er baseret på en artikel, som er skrevet i samarbejde med Richard Wong. Vi fremlægger en ∞-kategorisk dekomposition af stabile modul-∞-kategorier af en generel endelig gruppe, og vi viser, at i tilfælde af visse enkle endelige p-grupper, kan denne dekomposition fortolkes som et eksempel på 'Galois-nedstigning'. Vi bruger derefter dette perspektiv til at producere proof-of-concept-beregninger af gruppen af endotrivielle moduler til disse p-grupper.

I den anden del fortsætter vi med beregninger for mere komplicerede grupper. Af særlig interesse vil være tilfældet med de ekstraspeciale grupper, der traditionelt har spillet en grundlæggende rolle i teorien om endotrivielle moduler. Vi analyserer Picard-spektralsekvensen for de ekstraspeciale grupper og viser, at E_2-siden har en struktur, der nedarves fra en Tits-bygning af isotrope underrum med hensyn til en kvadratisk form.

I den tredje og sidste del går vi videre for at studere Dade-gruppen af endopermutationsmoduler. Vi undersøger, hvordan den kan realiseres som Picard-gruppen af en bestemt ∞-kategori af ægte ækvivariante spektre. Undervejs producerer vi en teoretisk ramme til undersøgelse af moduler, hvis endomorfismer er trivielle op til en specificeret underkategori af repræsentationskategorien. Dette producerer invarianter, der interpolerer mellem gruppen af endotrivielle moduler og Dade-gruppen, såvel som andre mere eksotiske invarianter, der er af uafhængig interesse.

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Contents

Chapter 0

Introduction

The origin of representation theory as a subject traces back to a now-famous correspondence between Ferdinand Georg Frobenius and Richard Dedekind, which took place in 1896. Dedekind proposed to Frobenius a conjecture on the factoring of a homogeneous polynomial arising as a certain determinant. In modern language, let G be a finite group with elements $g_1, \ldots, g_n$, and introduce a variable x_{g_i} for every element g_i. Define the $n \times n$ matrix X with entries $X_{ij} = x_{g_i g_j}$, and take its determinant, viewed as a complex homogeneous polynomial in n variables. What can we say about the splitting behaviour of this polynomial? In the case that G is abelian, Dedekind was able to prove that the determinant splits into linear factors, but he was unable to tackle the nonabelian case. Fascinated by the problem, Frobenius invented what is now known as character theory of finite groups to solve the factorisation problem in the general case, and reported his findings to Dedekind. After this, it wouldn't take long for Frobenius to undertake the first systematic study of the representation theory of finite groups.

With the advent of linear algebra, we have come to understand that a representation of a finite group G merely means a linear action of G onto a complex vector space V. Methods from linear algebra allow us to infer, with relative ease, various structural results about the representation theory of finite groups. Most notably, all representations split up uniquely into irreducible ones. As such, in order to fully understand the representation theory of a given group, the following two goals must be met.

- Classify all the irreducible representations.
- Devise an algorithm which finds the irreducible components of a given representation.

As it turns out, character theory leads us to both goals. It allows us to deduce that all irreducible representations may be found lying inside the regular representations. Moreover, characters exhibit orthogonality relations, which yield an efficient way of finding the irreducible components of a given reprsentations using only its character.

As the abstract concept of a field solidified, it would have become clear that representations can be made sense of for any base field. Whether the aforementioned structural results carry over, depends on the field. For algebraically closed fields of characteristic 0, essentially all results carry over verbatim. If you drop the assumption that your field is algebraically closed, representations sometimes struggle splitting up into their smallest pieces, but the general structural behaviour of representation theory remains unaffected. This is not the case, however, if you change your characteristic of your field.

If the characteristic of the field is coprime to the order of the group, the structural changes are manageable. We still have our semisimple splitting, and the consequent reduction to the aforementioned two goals. The prominent change occurs in character theory, where many important results, such as the orthogonality relations, simply break down. As such, we must devise a different way to find the irreducibles, and to decompose a given representation. Luckily, although some different methods are required, the representation theory tends to not behave all that much different from the characteristic-zero case.

All hell breaks loose, however, when when the characteristic of the field divides the order of the group. These so-called modular representations no longer decompose into irreducible components. Instead, we have to make do with a more crude decomposition, governed by the Krull–Schmidt theorem, which tells us that our representations split up uniquely into *indecomposable* summands. As such, to understand the modular representation theory of a group, our two goalposts are as follows.

- Classify all the indecomposable modular representations.
- Devise an algorithm which finds the indecomposable components of a given representation.

Since the decomposition is more crude, one may suspect that there may be many more indecomposable pieces to classify — and one's suspicion would be correct. In general, finding the indecomposable modular representations of a given group is an insurmountable task. Nonetheless, starting in the 1930s, Richard Brauer was the first to make serious progress towards both goals.

In an attempt to emulate the character theory of characteristic zero, he introduced what are now known as Brauer characters. When k is an algebraically closed field of characteristic p, there's a bijection between roots of unity in k and complex roots of unity of order coprime to p. Upon fixing such a bijection, the Brauer character of a characteristic-p representation assigns to each group element of order coprime to p the sum of complex roots of unity corresponding to the eigenvalues of that element in the given representation. The Brauer character of a representation is not quite as powerful as the classical character. For instance, representations are not determined uniquely by their Brauer characters. Nonetheless, the Brauer character manages to extract a great deal of structure from a given representation, and as such has become a powerful asset in the study of modular representations.

When defining the Brauer character, we specifically referred to elements of order coprime to p. What happens to the remaining elements? The answer is nothing. The Brauer character is defined only on those elements of G whose order is coprime to p. This defect becomes particularly striking when G is a p-group, since in that case, none of the elements of G (except the unit) have this property. As such, the modular representation theory of p-groups has no Brauer character theory at its disposal. Correspondingly, the modular representation theory is 'wild' — there is no hope of getting a good grasp on it in its entirety.

If we want to say something concrete about p-group representations over a field k of characteristic p, our best bet at this point is to forcibly simplify the world of modular representation theory to the point that it is no longer wild. Granted that we will never be able to understand modular representation up to isomorphism, we may still be able to understand them up to a more crude notion of equivalence. One particularly successful result of this line of thinking is the stable module category, which aims to classify modular representations 'up to projective summands'. Introduced in the 1970s, the stable module category was actively studied in the ensuing decade. Now, new developments in an entirely separate mathematical field have put it back to the forefront.

This mathematical field is called homotopy theory. Historically, homotopy theory arose as a branch of algebraic topology, and aimed to study spaces 'up to continuous deformation'. As time progressed, however, it became clear that the resulting 'homotopy-coherent' structures are in fact pervasive throughout mathematics; as such, homotopy theory soon grew out of its native environment into a field of its own. Ever since, it has proved itself to be spectacularly useful in many other areas of mathematics.

Homotopy-coherent structures have been found in various guises in representation theory, including in the stable module category. Post-hoc evidence of this was the existence of a triangulated structure on the stable module category. This triangulated structure can be enhanced to something homotopy-coherent, bringing the stable module category into the range of powerful homotopical machinery. Applying this machinery to produce new results is where the thesis you're currently reading ultimately fits in.

Let's highlight one such application. It had been understood for a long time that a great deal of representation-theoretic information of a group can be inferred from that of some or all of its subgroups. However, a global and quantitative statement was lacking, until the advent of homotopy theory, which has taught us that the representation category of a group admits a functorial, ∞-categorical decomposition into representation categories of subgroups. We will see in the first two chapters how this decomposition can lead to computatonal insights.

Chapter 1

Endotrivial modules for small p-groups via Galois descent

Abstract. We investigate the group of endotrivial modules for certain p-groups. Such groups were already been computed by Carlson–Thévenaz using the theory of support varieties; however, we provide novel homotopical proofs of their results for cyclic p-groups, the quaternion group of order 8, and for generalised quaternion groups using Galois descent and Picard spectral sequences.

This chapter is a modified version of [MW21], which has been written joint with Richard Wong. The contents of this chapter also appear in Wong's PhD thesis, cf. [Won21].

1.1 Overview

Throughout this paper, let G denote a finite group, and let k be a field of characteristic p, where p divides the order of G (i.e. the characteristic is modular). In this setting, one can study the representation theory of G over k. As p divides $|G|$, Maschke's theorem fails, which infamously implies that the structural phenomena of representation theory over modular characteristics are wildly different than the usual theory over other characteristics. Central to modular representation theory, then, is the study of the structural property of the category of kG-modules.

One particular instance of this is the problem of computing the group of endotrivial modules

$$T(G) := \{M \in \mathsf{Mod}^{\mathrm{fin}}(kG) : \mathrm{End}_k(M) \simeq k \oplus (\text{projective})\}.$$

That is, the (finite-dimensional) kG-modules M such that the endomorphism module decomposes as the direct sum of k, the trivial kG-module, and a projective kG-module. Notice that this forms a group under tensor product.

The group of endotrivial modules was first studied by Dade for the elementary abelian groups $(C_p)^n$ [Dad78], who regarded endotrivial modules as a stepping stone towards the study of the more general endopermutation modules. Endotrivial modules over p-groups were later classified in its entirety by Carlson and Thévenaz in [CT04] using purely representation-theoretic techniques, such as the theory of support varieties.

The group of endotrivial modules can be approached through homotopy theory — something we will make profound use of in this chapter. In Section 1.2, we realise the group of endotrivial modules as the Picard group of the stable module ∞-category. In fact, we obtain a Picard space, which admits a decomposition coming from a limit decomposition of the stable module ∞-category. This decomposition is then shown to be amenable to spectral sequence techniques.

In certain cases, the decomposition of the stable module ∞-category can be viewed through the lens of Galois theory. We take up this topic in Section 1.3 and we use a result of Rognes to give new proofs of the decomposition for cyclic p-groups and quaternion groups.

Finally, in Section 1.4 we evaluate the limit spectral sequences associated to the decomposition of the stable module category to explicitly compute the group of endotrivial modules for cyclic p-groups and generalised quaternion groups. Although these groups have already been computed, the method given here is entirely new. In particular, our approach allows for a new interpretation of the fact that the group of endotrivial modules over Q_8 depends on the arithmetic structure of the base field; we shall see that it arises naturally from a certain nonlinear differential in the limit spectral sequence. Furthermore, our approach to $T(Q_{2^n})$ is independent of the computation for $T(Q_8)$, whereas this was a crucial step in the classical approach.

1.2 Endotrivial modules and Picard spectra

Let G denote a finite group, and let k be a field of modular characteristic. As mentioned in Section 1.1 we define the group of endotrivial modules $T(G)$ as

$$T(G) := \left\{ M \in \mathsf{Mod}^{\text{fin}}(kG) : \text{End}_k(M) \simeq k \oplus (\text{projective}) \right\}.$$

Endotrivial modules form a group under tensor product. The group of endotrivial modules can be approached through homotopy theory, and the goal of this section is to illustrate how this can be done.

The failure of Maschke's theorem implies that not all kG-modules are projective. One can therefore additively localise the category of kG-modules $\mathsf{Mod}(kG)$ at the maps that factor through projective modules. The resulting localization is called the stable module category $\mathsf{StMod}(kG)$. It carries the structure of a tensor-triangulated category.

Given a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1}_{\mathcal{C}})$, one can study the Picard group $\text{Pic}(\mathcal{C})$ of $\otimes$-invertible object in $\mathcal{C}$. When taking $\mathcal{C}$ to be $\mathsf{StMod}(kG)$, we claim that the Picard group of $\mathcal{C}$ recovers the group of endotrivial modules. Certainly this has been known for a while, but we haven't been able to find a proof of this fact in the literature, so we digress for a moment to verify it.

Lemma 1.2.1. Two kG-modules M and N are equivalent in $\mathsf{StMod}(kG)$ if and only if there exist projective modules P and Q such that $M \oplus P \simeq N \oplus Q$.

Proof. If M and N are projectively equivalent, then the natural maps $f\colon M \hookrightarrow M \oplus P \xrightarrow{\sim} N \oplus Q \twoheadrightarrow N$ and $g\colon N \hookrightarrow N \oplus Q \xrightarrow{\sim} M \oplus P \twoheadrightarrow M$ form the desired equivalence, in that $g \circ f - \text{Id}$ factors through Q, and $f \circ g - \text{Id}$ factors through P.

Conversely, suppose we have maps $f\colon M \to N$ and $g\colon N \to M$ with

$$g \circ f - \mathrm{Id} = M \xrightarrow{\phi_1} P \xrightarrow{\phi_2} M$$
$$f \circ g - \mathrm{Id} = N \xrightarrow{\psi_1} Q \xrightarrow{\psi_2} N$$

Define $f'\colon M \to N \oplus P$ as (f, ϕ_1), and $g'\colon N \oplus P \to M$ as $g - \phi_2$. Then $g' \circ f' = \mathrm{Id}$, so that $M \oplus \operatorname{Ker} g' \simeq N \oplus P$. We're done if we show that $\operatorname{Ker} g'$ is projective. This follows once we verify that $f' \circ g' - \mathrm{Id}$ factors through a projective: indeed once we know this, we may observe that the map becomes $-\mathrm{Id}$ when restricted to $\operatorname{Ker} g'$ but after this restriction it of course still passes through this projective, so $\operatorname{Ker} g'$ becomes a summand thereof. So let's verify the claim. Simply observe that $f' \circ g' - \mathrm{Id}$ factors as

$$\begin{tikzcd}
N \oplus P \arrow[rr, "f' \circ g' - \mathrm{Id}"] \arrow[dr, swap, "(\psi_1, p, \phi_1 \circ g - \phi_1 \circ \phi_2)"] & & N \oplus P \\
& Q \oplus P \oplus P \arrow[ur, swap, "(\psi_2 - f \circ \phi_2, p_2 - p_1)"] &
\end{tikzcd}$$

and $Q \oplus P \oplus P$ is projective. ☐

From this lemma, we learn that a module M is $\otimes$-invertible if there exists a module N such that $M \otimes N \oplus Q \simeq k \oplus P$ for some projective modules P and Q. By applying the Krull–Schmidt theorem to this, we deduce two observations:

- The Q is not needed and we can simply write $M \otimes N \simeq k \oplus P$ for some projective module P;
- M and N split up as $M_0 \oplus (\mathrm{proj})$ and $N_0 \oplus (\mathrm{proj})$ where '(proj)' will henceforth be shorthand for 'some projective module which doesn't deserve its own symbol'.

Lemma 1.2.2. The Picard group of $\mathsf{StMod}(kG)$ is isomorphic to $T(G)$.

We generalise this result in Lemma 3.2.4, where we give a more systematic proof.

Proof. Suppose first that M is endotrivial. As M is finitely generated, we have $\mathrm{End}_k(M) \simeq M \otimes M^*$, and so M is $\otimes$-invertible with inverse M^*. Conversely, suppose M is a kG-module with $\otimes$-inverse N. By the discussion above we may write $M \otimes N \simeq k \oplus P$, and we have $M \simeq M_0 \oplus (\mathrm{proj})$ where M_0 is indecomposable.

Have a look at the commutative diagram

$$\begin{tikzcd}
M \otimes N \arrow[r, "\sim"] \arrow[d, swap, "1 \otimes f"] & k \oplus P \arrow[d, "\pi"] \\
M \otimes M^* \arrow[r, swap, "\mathrm{ev}"] & k
\end{tikzcd}$$

Here π is the projection map, ev the evaluation map on $M \otimes M^*$, and f is the map $N \to M^*$ sending n to $\phi\colon m \mapsto \pi(m \otimes n)$, where '$m \otimes n$' really refers to its isomorphic image in $k \oplus P$. As π admits a section, so does ev, which means k is a summand of $M \otimes M^*$. By tensoring with N we see that N is a summand of $M \otimes N \otimes M^*$. Now write

$$\begin{aligned} M \otimes N \otimes M^* &\simeq (k \oplus P) \otimes \big(M_0^* \oplus (\mathrm{proj})\big) \\ &\simeq M_0^* \oplus (\mathrm{proj}) \end{aligned}$$

This tells us that N is either projective or M_0^* plus something projective. In the former case, M was trivial and there was nothing to prove anyway, and in the latter case, we've found that $M \otimes M^* \oplus (\mathrm{proj}) \simeq k \oplus P$, and by the discussion above this lemma, this implies the desired result. $\square$

The stable module category is in fact the homotopy category of a stable symmetric monoidal ∞-category, which can be seen from the fact that $\mathsf{StMod}(kG)$ can be described as the Verdier quotient of the bounded derived category of kG-modules by the perfect complexes — a result first proved in [Ric89], and generalised in Section 3.C. This observation is what makes the study of endotrivial modules amenable to homotopical techniques.

To any symmetric monoidal ∞-category we can in fact associate a Picard *space* $\mathcal{P}\mathrm{ic}(\mathcal{C})$, defined as the ∞-groupoid underlying the full subcategory on the $\otimes$-invertible objects in $\mathcal{C}$. This is an enhancement of the classical Picard group.

Lemma 1.2.3. The homotopy groups of the Picard space are as follows:

$$\pi_t\,\mathcal{P}\mathrm{ic}(\mathcal{C}) \simeq \begin{cases} \mathrm{Pic}(\mathcal{C}) & \text{if } t = 0; \\ \pi_0(\Omega\,\mathcal{C})^\times & \text{if } t = 1; \\ \pi_{t-1}(\Omega\,\mathcal{C}) & \text{if } t \geq 2. \end{cases}$$

Here $\Omega\,\mathcal{C}$ is shorthand for the $\mathbb{E}_\infty$-ring $\mathrm{End}(\mathbb{1}_\mathcal{C})$ of endomorphisms of the $\otimes$-unit.

Proof sketch. Tensoring with a $\otimes$-invertible object tautologically describes an automorphism of $\mathcal{P}\mathrm{ic}(\mathcal{C})$. From this we observe that the Picard space decomposes as $\mathrm{Pic}(\mathcal{C}) \times B\,\mathrm{Aut}(\mathbb{1}_\mathcal{C})$. $\square$

Remark 1.2.4. In the literature the stable module ∞-category is often defined as the Ind-completion of the aforementioned Verdier quotient. For the purposes of finding the Picard group, the difference is irrelevant, as we will now show. A $\otimes$-invertible object in $\mathrm{Ind}(\mathcal{C})$ is clearly compact. By [Lur09, Lemma 5.4.2.4] the natural map $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})^\omega$ identifies with the idempotent completion of $\mathcal{C}$. Now $\mathsf{StMod}(kG)$ is idempotent-complete (but see Remark 3.5.1) hence the Picard group doesn't change upon passage to Ind-completion.

Associating a Picard space to a stable symmetric monoidal ∞-category is functorial under exact symmetric monoidal functors, and so we have a functor $\mathcal{P}\mathrm{ic}\colon \mathsf{Cat}^\otimes \to \mathcal{S}_*$. This functor commutes with limits, cf. [MS16, Proposition 2.2.3], which means that, whenever we have a limit decomposition of ∞-categories, we have a corresponding limit decomposition of Picard spaces.

In view of this, it is natural to recall the following. Whenever we have a diagram $\mathcal{F}\colon \mathcal{I}^{\mathrm{op}} \to \mathcal{S}_*$ of pointed spaces, there is a spectral sequence

$$E_2^{st} \simeq H^s(\mathcal{I}, \pi_t\,\mathcal{F}) \Rightarrow \pi_{t-s} \varprojlim_{\mathcal{I}} \mathcal{F}$$

whose E_2-page is given by the cohomology of the Ab-valued presheaf $\pi_t\,\mathcal{F}$ over the diagram $\mathcal{I}$. The spectral sequence dates back to the work of Bousfield–Kan, cf. [BK72]. Unfortunately, the Bousfield–Kan spectral sequence suffers from convergence issues and exhibits fringe phenomena, which makes it unreliable from a computational perspective.

We may circumvent this convergence issue by passing to spectra. More precisely, whenever we have a diagram $\mathcal{F}\colon \mathcal{I}^{\mathrm{op}} \to \mathsf{Sp}$ of *spectra*, there is a completely analogous spectral sequence (see [Lur12, Section 1.2.2]) but which does not exhibit fringe effects.

Now, if $\mathcal{C}$ is a symmetric monoidal stable ∞-category, then the Picard space is a grouplike $\mathbb{E}_\infty$-space, and can thus be viewed as a connective spectrum, which we call the Picard spectrum of $\mathcal{C}$, denoted $\mathfrak{pic}\,\mathcal{C}$. The functor $\mathfrak{pic}\colon \mathsf{Cat}_{\mathrm{st}}^{\otimes} \to \mathsf{Sp}_{\geq 0}$ commutes with limits as well, and so any limit decomposition of categories yields a limit decomposition of connective Picard spectra.

It is worth pointing out here that the limit spectral sequence lives in *nonconnective* spectra, whereas the Picard spectra are *connective* spectra. This is more than just a superficial difference, since the inclusion functor $\mathsf{Sp}_{\geq 0} \to \mathsf{Sp}$ does not commute with limits. However, a limit of connective spectra can be computed by taking a limit in the category of nonconnective spectra and then passing to the connective cover. Consequently, the discrepancy between a limit of connective spectra taken in $\mathsf{Sp}_{\geq 0}$ and taken in Sp is concentrated in negative degrees. In view of our main goal, which is to compute π_0 of Picard spectra, this discrepancy will never pose issues.

Let's summarise our findings into a lemma.

Lemma 1.2.5. Let $\mathcal{C}$ be a symmetric monoidal stable ∞-category, and suppose that $\mathcal{C}$ is realised as the limit over a diagram $\mathcal{I}^{\mathrm{op}} \to \mathsf{Cat}_{\mathrm{st}}^{\otimes}$. Then there is a spectral sequence

$$E_2^{st} \simeq H^s(\mathcal{I}, \pi_t\, \mathfrak{pic}\, \mathcal{C}_i) \Rightarrow \pi_{t-s} \varprojlim\nolimits_{\mathcal{I}} \mathfrak{pic}\, \mathcal{C}_i,$$

and for $* \geq 0$, $\pi_* \varprojlim_{\mathcal{I}} \mathfrak{pic}\, \mathcal{C}_i$ may be identified with $\pi_*\, \mathfrak{pic}\, \mathcal{C}$.

Remark 1.2.6. The Picard spectrum admits a delooping by the Brauer spectrum. Brauer spectra ought to admit descent as well — cf. for instance [Mat16, Prop. 3.45] — and so the (-1)-line for our limit spectral sequence of Picard spectra reveals information about the Brauer groups as well. We will find that the (-1)-line is often significantly more complicated than the nonnegative lines.

In view of Lemma 1.2.5, it would be beneficial to exhibit a limit decomposition for stable module categories, which is what we will now turn our attention to.

Let G be a finite group, and let $\mathcal{A}$ be a collection of subgroups of G satisfying the following properties.

- $\mathcal{A}$ is closed under finite intersections;
- $\mathcal{A}$ is closed under conjugation by elements of G;
- every elementary abelian p-subgroup of G is contained in a member of $\mathcal{A}$.

For any such collection $\mathcal{A}$, we define $\mathcal{O}_{\mathcal{A}}$ to be the full subcategory of the orbit category $\mathcal{O}_G$ spanned by the transitive G-sets with isotropy in $\mathcal{A}$, or in other words, the objects G/H for which H is in $\mathcal{A}$. We have the following result, which can be found in [Mat16, Corollary 9.16], and which is effectively a higher-categorical elaboration of Quillen's stratification theorem.

Theorem 1.2.7. If G, k and $\mathcal{A}$ are as above, then the stable module category of G decomposes as

$$\mathsf{StMod}(kG) \simeq \varprojlim\nolimits_{G/H \in \mathcal{O}_{\mathcal{A}}^{\mathrm{op}}} \mathsf{StMod}(kH).$$

The functoriality becomes apparent by noting that $\mathsf{StMod}(kH)$ can be identified as the category of module objects over the commutative algebra object $k(G/H)$ in $\mathsf{StMod}(kG)$.

As discussed in the previous subsection, any limit decomposition of stable symmetric monoidal ∞-categories yields a corresponding decomposition of Picard spectra

$$\mathfrak{pic}\,\mathsf{StMod}(kG) \simeq \varprojlim\nolimits_{G/H \in \mathcal{O}_{\mathcal{A}}^{\mathrm{op}}} \mathfrak{pic}\,\mathsf{StMod}(kH), \tag{1.2.8}$$

and hence a corresponding limit spectral sequence

$$E_2^{st} \simeq H^s\big(\mathcal{O}_{\mathcal{A}}, G/H \mapsto \pi_t\, \mathfrak{pic}\, \mathsf{StMod}(kH)\big) \Rightarrow \pi_{t-s}\, \mathfrak{pic}\, \mathsf{StMod}(kG). \tag{1.2.9}$$

To evaluate this spectral sequence, it is necessary to study both the objects and the differentials of this spectral sequence. To understand the differentials in the spectral sequence, we make use of the computational tools developed in [MS16, Part II]. We summarise their results here. We start with a symmetric monoidal stable ∞-category $\mathcal{C}$.

Lemma 1.2.10 ([MS16, Corollary 5.2.3]). For all $t \geq 2$, one has a functorial equivalence $\tau_{[t,2t-1]}\operatorname{Aut}(\mathbb{1}_{\mathcal{C}}) \simeq \tau_{[t,2t-1]}\mathbb{1}_{\mathcal{C}}$, where $\tau_{[\cdot,\cdot]}$ denotes the truncation functor with homotopy groups in the specified range. This induces a functorial equivalence $\tau_{[t+1,2t]}\,\mathfrak{pic}\,\mathcal{C} \simeq \Sigma\tau_{[t,2t-1]}\,\Omega\,\mathcal{C}$.

Recall that Ω was taken to be shorthand for the endomorphism spectrum of the $\otimes$-unit of $\mathcal{C}$. Now, Ω, too, commutes with limits, yielding a decomposition of $\Omega\,\mathsf{StMod}(kG)$ analogous to Eq. (1.2.8), and hence a limit spectral sequence analogous to Eq. (1.2.9). Lemma 1.2.10 then allows one to import differentials from the limit spectral sequence for Ω into the limit spectral sequence for Picard spectra.

Theorem 1.2.11 ([MS16, Comparison Tool 5.2.4]). Suppose we have a diagram $\mathcal{I}^{\mathrm{op}} \to \mathsf{Cat}_{\mathrm{st}}^{\otimes}$ of symmetric monoidal stable ∞-categories. Consider the limit spectral sequences

$$E_2^{st}(\mathfrak{pic}) = H^s\big(\mathcal{I}, \pi_t\,\mathfrak{pic}\,\mathcal{C}_i\big) \Rightarrow \pi_{t-s}\,\mathfrak{pic}\,\varprojlim_{\mathcal{I}} \mathcal{C}_i,$$
$$E_2^{st}(\Omega) = H^s\big(\mathcal{I}, \pi_t\,\Omega\,\mathcal{C}_i)\big) \Rightarrow \pi_{t-s}\,\Omega\,\varprojlim_{\mathcal{I}} \mathcal{C}_i.$$

Then we have an equality of differentials $d_r^{st}(\mathfrak{pic}) = d_r^{s,t-1}(\Omega)$ for all (s,t) such that either $t-s>0$ or $t \geq r+1$.

As it turns out, the spectral sequence for Ω is easier to understand, because the endomorphism spectra are $\mathbb{E}_\infty$-rings, which imbue the limit spectral sequence with a multiplicative structure. We will make frequent use of this advantage throughout our computations.

Let us now take a closer look at the endomorphism spectrum $\Omega\,\mathsf{StMod}(kG)$. This spectrum can be described explicitly, but before we are able to give the description, we need to recall some relevant definitions.

If X is a spectrum admitting a G-action, then we can capture the G-action in terms of a functor $BG \to \mathsf{Sp}$. We then associate to X its homotopy orbits X_{hG} and homotopy fixed points X^{hG}, defined as the colimit and limit, respectively, of the aforementioned functor. There is a norm map $X_{hG} \to X^{hG}$ whose cofibre is called the Tate construction, denoted X^{tG}.

As we've seen, any limit of spectra has an associated limit spectral sequence. Applied to X^{hG}, we obtain what is commonly called the homotopy fixed point spectral sequence (HFPSS). A dual spectral sequence, called the homotopy orbit spectral sequence, exists for X_{hG}, as does a four-quadrant spectral sequence for X^{tG}, called the Tate spectral sequence, which we'll encounter in Section 1.3.

If X is the Eilenberg–MacLane spectrum of a G-module M, then the homotopy groups of M^{tG} carry classical arithmetic information. To see this, note first that the homotopy groups of M_{hG} are given by

$$\pi_t\, M_{hG} = \begin{cases} H_t(G;M) & \text{if } t \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and that the homotopy groups of M^{hG} are given by

$$\pi_{-t} M^{hG} = \begin{cases} H^t(G; M) & \text{if } t \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\pi_0 M_{hG}$ is the classical set M_G of G-orbits while $\pi_0 M^{hG}$ is the set M^G of fixed points. The norm map $M_{hG} \to M^{hG}$ is necessarily zero on nonzero homotopy groups, while the map on π_0 is the classical norm map $N\colon M_{hG} \to M^{hG}$ sending an orbit $\{gm\}$ to the sum $\sum_g gm$. Through the long exact sequence of homotopy groups associated to the fibre sequence $M_{hG} \to M^{hG} \to M^{tG}$, one then infers that

$$\pi_t M^{tG} \simeq \widehat{H}^{-t}(G; M),$$

where $\widehat{H}$ denotes Tate cohomology, which is classically defined as

$$\widehat{H}^*(G; M) = \begin{cases} H^*(G; M) & \text{if } * \geq 1; \\ \operatorname{Coker} N & \text{if } * = 0; \\ \operatorname{Ker} N & \text{if } * = -1; \\ H_{-*-1}(G; M) & \text{if } * \leq -2. \end{cases}$$

If X is an $\mathbb{E}_\infty$-ring, then so is X^{tG}. In particular, if R is a classical ring with a G-action, then $\pi_* R^{tG}$ admits a cup product, which coincides with the ring structure on Tate cohomology. We refer the reader to Section 1.A for some explicit computations of Tate cohomology rings.

Lemma 1.2.12. There exists an equivalence of $\mathbb{E}_\infty$-rings

$$\Omega \operatorname{\mathsf{StMod}}(kG) \simeq k^{tG},$$

where the G-action on k is taken to be the trivial one.

On the level of homotopy groups, this is reflected by the classical fact that, in the triangulated stable module category,

$$\underline{\operatorname{Hom}}_{kG}(\Omega^t k, k) \simeq \widehat{H}^t(G; k).$$

Remark 1.2.13. If G is in fact a p-group, then $\mathsf{StMod}(kG)$ is in fact equivalent to $\mathsf{Mod}(k^{tG})$. We digress for a while to verify that this is the case.

Proof. By the Schwede–Shipley theorem ([Lur12, Theorem 7.1.2.1]), it suffices to show that the category has a compact generator, which we claim is k. Let us first see why k is nontrivial object in $\mathsf{Mod}^\omega_G(k)/\mathsf{Perf}(kG)$ to begin with. This is true so long as k isn't a perfect complex. Indeed this is the case: k is not compact on $\mathsf{Mod}^\omega_G(k)$, effectively because $H^{-*}(G; k) = \pi_* \operatorname{Map}_{\mathsf{Mod}^\omega_G(k)}(k, k)$ fails to be bounded in modular characteristic.

In general, if $\mathcal{C}$ is an ∞-category, then any object in $\mathcal{C}$ is compact in $\operatorname{Ind}(\mathcal{C})$ for formal reasons, so k defines a compact object. So then why does k generate $\mathsf{StMod}(kG)$? By definition, one must verify that, for any object M, if $\operatorname{Hom}(k, \Sigma^i M) = 0$ for all $i \in \mathbb{Z}$, then $M \simeq 0$. We may represent the $\Sigma^i M$ as kG-representations, so that the existence of a nonzero map $k \to \Sigma^i M$ may be identified with the existence of a fixed vector in the kG-representation $\Sigma^i M$. However, such nonzero maps always exist because kG-representations have a nonzero fixed vector. To see this, reduce to the case $k = \mathbb{F}_p$ by inspecting the underlying $\mathbb{F}_p$-vector space, and then apply a counting argument. This concludes the claim. $\square$

Let us go back to Eq. (1.2.9) again. In all examples of interest, we will take $\mathcal{A}$ to be a family of elementary abelian subgroups. In view of Lemma 1.2.3 and Lemma 1.2.12, the higher homotopy groups of $\mathfrak{pic}\,\mathsf{StMod}(kH)$ are well understood: they are given by Tate cohomology groups of elementary abelian groups. But what about the 0-th homotopy groups of $\mathfrak{pic}\,\mathsf{StMod}(kH)$, i.e. the Picard groups? The endotrivial modules of elementary abelian groups are understood via a result by a theorem of Dade, which states that the Picard group is necessarily generated by the suspension of the unit. We will take the computation of the Picard group for elementary abelians as a starting point, working our way up from there. Let's capture it as a lemma.

Lemma 1.2.14 ([Dad78])**.** The Picard group of the stable module category of elementary abelian groups is described as follows:

$$\pi_0\,\mathfrak{pic}\,\mathsf{StMod}\big(k(C_p)^n\big) \simeq \begin{cases} 0 & \text{if } p = 2 \text{ and } n = 1; \\ \mathbb{Z}/2\mathbb{Z} & \text{if } p \text{ is odd and } n = 1; \\ \mathbb{Z} & \text{if } n \geq 2. \end{cases}$$

Let us revisit what we know so far. We have our spectral sequence Eq. (1.2.9), and we can compare this spectral sequence functorially with an analogous spectral sequence for $\Omega\,\mathsf{StMod}(kG)$. The latter has a multiplicative structure, and the E_2-page can be described in terms of Tate cohomology groups. In fact, the spectral sequence may be recognised as a rather classical one. To see this, let's suppose we may take G to be a finite p-group with a single normal elementary abelian p-subgroup H. Then $\mathcal{O}_{\mathcal{A}} \simeq B(G/H)$, and the limit spectral sequence for $\Omega\,\mathsf{StMod}(kG)$ reads

$$E_2^{st} \simeq H^s\big(G/H; \widehat{H}^{-t}(H;k)\big) \Rightarrow \widehat{H}^{s-t}(G;k).$$

For nonpositive t, the spectral sequence is indeed isomorphic to the Hochschild–Serre spectral sequence associated to the extension $H \to G \to G/H$. This spectral sequence is sufficiently well studied that the differentials are known in all examples of interest. Via the Tate duality pairing (cf. Section 1.A) this allows us to deduce the differentials for positive t as well.

From a homotopical viewpoint, the comparison with the Hochschild–Serre spectral sequence can be seen by considering the natural map $k^{hH} \to k^{tH}$. The map is $W_G(H)$- i.e. G/H-equivariant and induces an isomorphism on nonnegative homotopy groups, so that their limit spectral sequence may be compared. Applied to k^{hH}, this is the homotopy fixed points spectral sequence, and it converges to the homotopy groups of $(k^{hH})^{hG/H}$, which is naturally isomorphic to k^{hG}.

Although a large swathe of differentials can now be understood using Theorem 1.2.11 and the comparison with the Hochschild–Serre spectral sequence, we will often find that there's a particular differential which strongly influences the development of the 0-line but which just barely falls outside the range of Theorem 1.2.11. For these differentials, we use an elegant formula of Mathew–Stojanoska. To match their statement with ours, let's assume that the diagram $\mathcal{I}$ consists of a single object so that the limit spectral sequences becomes an HFPSS.

Theorem 1.2.15 ([MS16, Theorem 6.1.1])**.** Let the notation be as in Theorem 1.2.11. Assume that $\mathcal{I}$ has a single object so that we may identify the limit spectral sequences with homotopy fixed point spectral sequences. Then we have the formula

$$d_r^{rr}(\mathfrak{pic})(x) = d_r^{r,r-1}(\Omega)(x) + x^2,$$

where the square refers to the multiplicative structure in the limit spectral sequence for Ω.

Remark 1.2.16. The classes of p-groups that are considered in this paper (cyclic p-groups and generalised quaternion groups) have a single elementary abelian p-subgroup. As a result, we can identify the limit spectral sequence with the homotopy fixed point spectral sequence and use Theorem 1.2.15. However, analogous methods apply to more complicated groups as well, as we'll see in Chapter 2.

1.3 Galois descent

Theorem 1.2.7 becomes especially simple when we may take the family $\mathcal{A}$ to consist of a single (necessarily normal) subgroup H. In such examples, the decomposition reduces to the much simpler

$$\mathsf{StMod}(kG) \simeq \mathsf{StMod}(kH)^{hG/H}. \tag{1.3.1}$$

Throughout this paper, we will consider the two families of p-groups where this phenomenon occurs:

- The cyclic p-groups C_{p^n}, and
- the generalised quaternion groups Q_{2^n}.

The cyclic p-groups obviously have a single elementary abelian subgroup H, which is isomorphic to C_p. As for the generalised quaternion groups, we recall that these may be defined e.g. algebraically as the groups

$$Q_{2^n} \simeq \langle \theta, \tau \mid \theta^{2^{n-1}} = \tau^4 = 1, \theta^{2^{n-2}} = \tau^2, \tau\theta\tau^{-1} = \theta^{-1} \rangle$$

The centre $H = Z(Q_{2^n})$ is the only nontrivial elementary abelian subgroup, being isomorphic to C_2, and the quotient is isomorphic to the Klein four group $(C_2)^2$ or the dihedral group $D_{2^{n-1}}$.

As it happens, we may re-interpret the decomposition as an instance of (faithful) Galois descent. The goal of this section is to give a new proof that Galois descent holds for $\mathsf{StMod}(kG)$, where G is any of the aforementioned groups. We will use a result of Rognes, along with a base-change argument, to prove that we have G/H-Galois extensions $k^{tG} \to k^{tH}$, and then proceed to prove that these Galois extensions are faithful, using a criterion involving the contractibility of a certain Tate construction.

We begin with the relevant definitions. Let $f\colon R \to S$ be a map of $\mathbb{E}_\infty$-ring spectra. We call it a G-Galois extension if there is a G-action on S such that the natural maps $R \to S^{hG}$ and $S \otimes_R S \to \operatorname{map}(G_+, S)$ are weak equivalences. We say that the Galois extension is faithful if S is faithful as an R-module. That is, if M is an R-module such that $S \otimes_R M$ is contractible, then so is M itself.

Whenever we have a faithful Galois extension, we have a good theory of descent called Galois descent:

Theorem 1.3.2. If $f\colon R \to S$ is a faithful G-Galois extension of $\mathbb{E}_\infty$-rings, then we have a natural equivalence of ∞-categories

$$\mathsf{Mod}(R) \simeq \mathsf{Mod}(S)^{hG}.$$

In the case of interest, this is of course in line with Eq. (1.3.1).

Remark 1.3.3. Lemma 1.2.14 is also known to have a proof using Galois descent, or rather, reverse Galois descent; see [Mat15]. Briefly, if A is an abelian p-group of p-rank n, then one can construct a fibre sequence of classifying spaces

$$B\mathbb{Z}^n \to BA \to B^2\mathbb{Z}^n \simeq B\mathbb{T}^n,$$

where $\mathbb{T}^n$ denotes the n-torus, which Mathew uses to prove that there exist faithful $\mathbb{T}^n$-Galois extensions of ring spectra

$$k^{h\mathbb{T}^n} \to k^{hA} \qquad \text{and} \qquad k^{t\mathbb{T}^n} \to k^{tA}.$$

In this case, it is the source rather than the target which is understood well, and Mathew proves conditions for an element in the Picard group of $k^{t\mathbb{T}^n}$ to descend to the Picard group of k^{tA}. This, along with a computation of $\mathrm{Pic}(k^{t\mathbb{T}^n})$, shows that $\mathrm{Pic}(k^{tA})$ is cyclic.

We start off by proving that we have Galois extensions $k^{hG} \to k^{hH}$ on homotopy fixed points — the analogous result on Tate fixed points will then follow from a base-change argument. Our main tool is the following result of Rognes.

Theorem 1.3.4 ([Rog08, Prop. 5.6.3])**.** Let Γ be a finite discrete group, and $P \to X$ a principal Γ-bundle. Suppose that X is path-connected and $\pi_1(X)$ acts nilpotently on $H_*(\Gamma; k)$. Then the map of cochain k-algebras $\mathrm{map}(X_+, k) \to \mathrm{map}(P_+, k)$ is a Γ-Galois extension.

Proof sketch. We sketch the idea of the proof. To see that $\mathrm{map}(X_+, k) \simeq \mathrm{map}(P_+, k)^{h\Gamma}$ follows from properties of principal Γ-bundles. Namely, we have that the (right) action of Γ on P induces a left Γ-action on $\mathrm{map}(P_+, k)$, which is compatible with the identification of X as the homotopy orbits of the action of Γ on P.

The interesting part is showing that

$$\mathrm{map}(P_+, k) \otimes_{\mathrm{map}(X_+,k)} \mathrm{map}(P_+, k) \simeq \mathrm{map}\big(\Gamma_+, \mathrm{map}(P_+, k)\big).$$

As a tensor product of ring spectra, the homotopy groups of the left-hand side may be computed using the Künneth spectral sequence

$$E^2_{s,t} \simeq \mathrm{Tor}^{\pi_*(A)}_{s,t}\big(\pi_*(B), \pi_*(B)\big) \Rightarrow \pi_{s+t}(B \otimes_A B).$$

Meanwhile, via the identification $\mathrm{map}\big(\Gamma_+, \mathrm{map}(P_+, k)\big) \simeq \mathrm{map}\big((\Gamma \times P)_+, k\big)$, the homotopy groups of the right-hand side can be computed using the Eilenberg–Moore spectral sequence,

$$E^2_{s,t} \simeq \mathrm{Tor}^{H^*(X;k)}_{s,t}\big(H^*(P;k), H^*(P;k)\big) \Rightarrow H^{-(s+t)}(P \times_X P; k).$$

Notice that the E_2-pages of these spectral sequences agree. Moreover, the filtrations are identified as well, as both can be viewed as being derived from the cobar construction. Therefore, if both spectral sequences converge strongly, then we obtain an equivalence between their targets, which is what we desire. Luckily, the Künneth spectral sequence is always strongly convergent, and a theorem of Shipley [Shi96] guarantees convergence of the Eilenberg–Moore spectral sequence if the hypotheses of Theorem 1.3.4 are satisfied. □

We apply Theorem 1.3.4 to the fibre sequence $G/H \to BH \to BG$, where G is a cyclic p-group or a generalised quaternion group, and H is its elementary abelian p-subgroup. Notice that BG is path-connected, and G, as a p-group, acts nilpotently on $H_*(G/H; k)$. We deduce that we have a G/H-Galois extension $\mathrm{map}(BG_+, k) \to \mathrm{map}(BH_+, k)$. Now since G acts trivially on k, these function spectra are naturally identified with homotopy fixed points, and the result follows.

We proceed to use this to show that we have Galois extensions $k^{tG} \to k^{tH}$ for all aforementioned G and H. First, observe that given a Galois extension $R \to S$ and a map of ring spectra $R \to Q$, we can take the pushout along these maps to form the base-change $Q \to S \otimes_R Q$. The following result of Rognes provides conditions for this map to be a Galois extension.

Lemma 1.3.5 ([Rog08, Section 7.1]). *G-Galois extensions are stable under dualisable base change. Faithful G-Galois extensions are stable under arbitrary base change.*

From material that we discuss in the appendix, it turns out that in the cases we consider, one can identify the Tate constructions $k^{tG} \to k^{tH}$ as a (dualisable) base change of $k^{hG} \to k^{hH}$. Indeed, work of Greenlees (Theorem 1.A.4) allows us to view k^{tG} as a localization of k^{hG} away from the augmentation ideal I. This construction depends only on the radical of the ideal I. Now, since the groups C_p, C_{p^n}, and Q_{2^n} are Cohen–Macaulay, $\pi_* k^{hG}$ is free over a polynomial subalgebra $A \simeq k[x]$. The radical of the ideal (x) is the same as the radical of the augmentation ideal I, and so we obtain pushout diagrams

$$\begin{tikzcd} k^{hC_{p^n}} \arrow[r] \arrow[d] & k^{hC_p} \arrow[d] \\ k^{tC_{p^n}} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & k^{tC_p} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} k^{hQ_{2^n}} \arrow[r] \arrow[d] & k^{hC_2} \arrow[d] \\ k^{tQ_{2^n}} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & k^{tC_2} \end{tikzcd}$$

in the category $\mathsf{CAlg}(\mathsf{Sp})$ of ring spectra. In both cases the Tate constructions k^{tG} and k^{tH} are both formed by the same finite localizations (away from the ideal (x)). Since k^{tG} can be identified as a finite localization of k^{hG}, it is therefore dualisable. This yields the following theorem:

Theorem 1.3.6. *If G is either a cyclic group of prime power order or a generalised quaternion group, and H is its unique subgroup of order C_p, then the natural map $k^{tG} \to k^{tH}$ is a G/H-Galois extension of $\mathbb{E}_\infty$-rings.*

Remark 1.3.7. It is interesting to compare these results to Mathew's [Mat16, Thm. 9.17], where it's proved that the étale fundamental group of $\mathsf{StMod}(kG)$ identifies with the profinite completion of $\pi_1 |\mathcal{O}_{\mathcal{A}}|$, which of course simplifies to G/H if $\mathcal{A} = \langle H \rangle$. In particular, $k^{tG} \to k^{tH}$ can be viewed as the universal cover of k^{tG}.

Our next goal is to prove that the Galois extensions of Theorem 1.3.6 are faithful. We will repeatedly invoke the following criterion of Rognes to show that our Galois extensions are faithful.

Lemma 1.3.8 ([Rog08, Prop. 6.3.3]). *A G-Galois extension $f\colon R \to S$ is faithful if and only if the Tate construction S^{tG} is contractible.*

This is especially useful because of the existence of the multiplicative Tate spectral sequence: if X is a spectrum with a G-action for some group G, then there is a spectral sequence

$$E_2^{st} \simeq \widehat{H}^s(G; \pi_t X) \Rightarrow \pi_{t-s} X^{tG},$$

with differentials $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$, which lets us compute the homotopy groups of X^{tG} in terms of more readily accessible Tate cohomology groups. Moreover, we can leverage naturality and the cofibre sequence $X_{hG} \to X^{hG} \to X^{tG}$ to compare and import many differentials between the homotopy orbit spectral sequence, the homotopy fixed point spectral sequence, and the Tate spectral sequence.

Remark 1.3.9. We remark that it suffices to show that the extension on the level of homotopy fixed points $k^{hG} \to k^{hH}$ is faithful. Indeed, by Lemma 1.3.5, this implies that faithful Galois descent holds for $k^{tC_{p^n}} \to k^{tC_p}$ and $k^{tQ_{2^n}} \to k^{tC_2}$. However, the proof would proceed in exactly the same way (i.e. computing the Tate spectral sequence). Furthermore, we require the HFPSS calculations involving $k^{tC_{p^n}} \to k^{tC_p}$ and $k^{tQ_{2^n}} \to k^{tC_2}$ in our Picard spectral sequence calculations anyway, so we will not make this reduction.

Theorem 1.3.10. *The $C_{p^{n-1}}$-Galois extension $k^{tC_{p^n}} \to k^{tC_p}$ of $\mathbb{E}_\infty$-rings is faithful.*

Proof. Our goal is to compute the Tate spectral sequence

$$E_2^{st} \simeq \widehat{H}^s(C_{p^{n-1}}; \pi_t\, k^{tC_p}) \Rightarrow \pi_{t-s}(k^{tC_p})^{tC_{p^{n-1}}}$$

and show that the Tate spectrum $(k^{tC_p})^{tC_{p^{n-1}}}$ is contractible. To do so, we recall that the natural map $k^{hG} \to k^{tG}$ from homotopy fixed points to Tate fixed points allows us to import differentials from the HFPSS

$$E_2^{st} \simeq H^s\big(C_{p^{n-1}}; \pi_t\, k^{hC_p}\big) \Rightarrow \pi_{t-s}(k^{hC_p})^{hC_{p^{n-1}}}.$$

Note that $(k^{hC_p})^{hC_{p^{n-1}}} \simeq k^{hC_{p^n}}$. In fact, this spectral sequence can be identified with the Lyndon–Hochschild–Serre spectral sequence associated to the (central) extension $C_p \to C_{p^n} \to C_{p^{n-1}}$, which is well understood. We review this spectral sequence, distinguishing between the cases where $p = 2$ and p is odd.

If $p = 2$, then the E_2-page of the Hochschild–Serre spectral sequence, depicted in Fig. 1.1, is of the form

$$E_2^{st} \simeq H^s(C_{2^{n-1}}; k) \otimes H^{-t}(C_2; k) \simeq \begin{cases} k[x_1] \otimes k[t_1] & \text{if } n = 2; \\ k[x_2] \otimes \Lambda(x_1) \otimes k[t_1] & \text{if } n \geq 3. \end{cases}$$

Here, x_1 is of Adams degree $(-1, 1)$, x_2 is of degree $(-2, 2)$, and t_1 is of Adams degree $(-1, 0)$. A standard argument shows that $d_2(t_1)$ is nontrivial, whereas d_2 vanishes on the remaining generators for degree reasons. By multiplicativity, this determines the remaining differentials. The E_3-page has been illustrated in Fig. 1.2, and the spectral sequence collapses.

We can now leverage this information to the HFPSS computing $\pi_*\, k^{tC_{2^n}}$. Recall from Section 1.A that the ring $\pi_*\, k^{tC_2}$ is isomorphic to $k[t_1^{\pm 1}]$, so the E_2-page, illustrated in Fig. 1.3, is now given by

$$E_2^{s,t} \simeq H^s(C_{2^{n-1}}; \pi_t\, k^{tC_2}) \simeq \begin{cases} k[x_1] \otimes k[t_1^{\pm 1}] & \text{if } n = 2; \\ k[x_2] \otimes \Lambda(x_1) \otimes k[t_1^{\pm 1}] & \text{if } n \geq 3. \end{cases}$$

By multiplicativity, we can simply extend the differentials of Fig. 1.1 to negative powers of t_1 using the Leibniz rule. The E_3-page has been drawn out in Fig. 1.4, where it collapses.

We now use this information to compute the Tate spectral sequence. From Section 1.A we see that passing to Tate cohomology again amounts to inverting the relevant generators on cohomology (namely, x_1), and so we simply take the differentials of the HPFSS, and extend to negative s-degree by multiplicativity. The E_2-page has been drawn in Fig. 1.5. We see that every summand is now killed

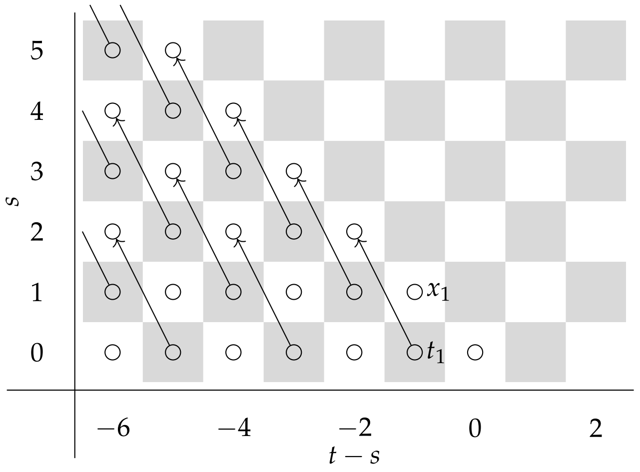


Figure 1.1: The Adams-graded E_2-page of the Hochschild–Serre spectral sequence associated to the extension $C_2 \to C_{2^n} \to C_{2^{n-1}}$. The circles denote a k-summand, and the nonzero differentials have been drawn.

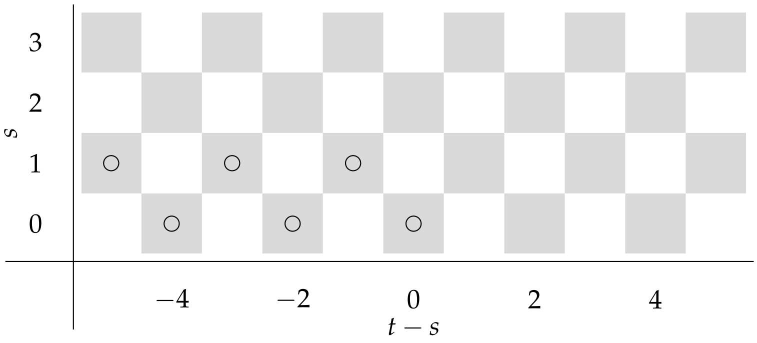


Figure 1.2: The E_3-page of the Hochschild–Serre spectral sequence associated to the extension $C_2 \to C_{2^n} \to C_{2^{n-1}}$. There are no remaining differentials, and the spectral sequence collapses.

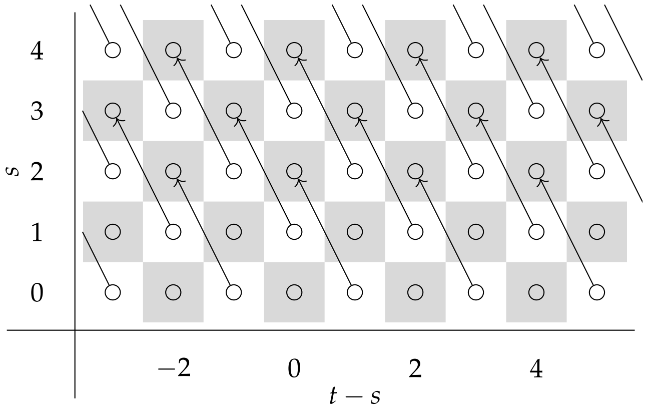


Figure 1.3: The Adams-graded E_2-page of the HFPSS computing the homotopy groups $\pi_* k^{tC_{2^n}}$ for $n \geq 2$. It is effectively just Fig. 1.1 extended to another quadrant.

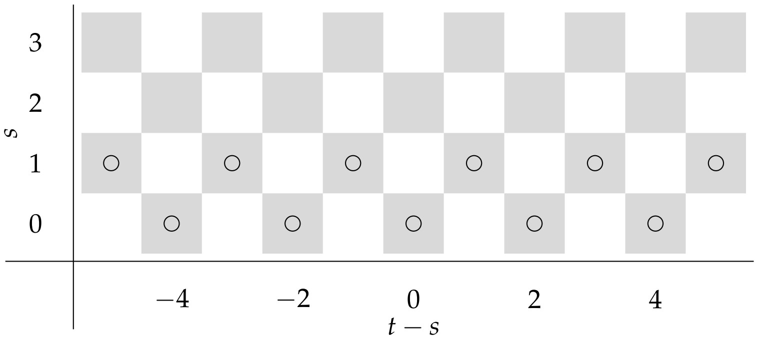


Figure 1.4: The E_3-page of the HFPSS computing $\pi_* k^{tC_{2^n}}$. There are no remaining differentials, and the spectral sequence collapses.

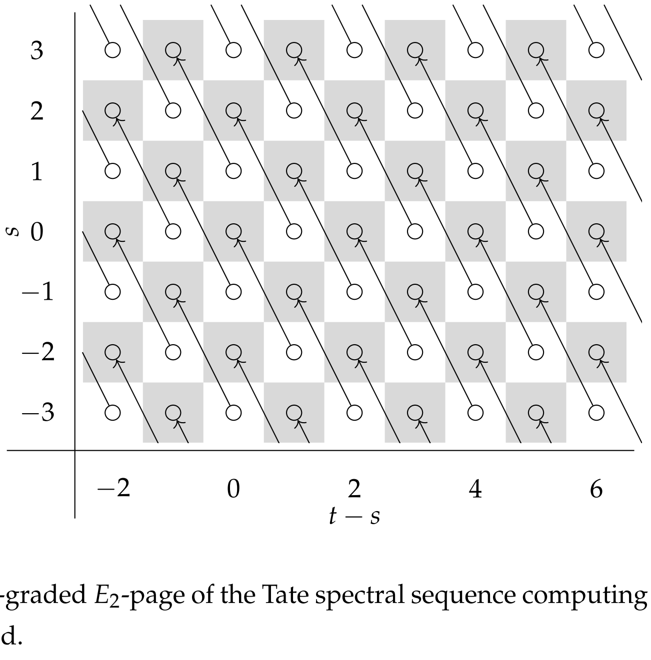


Figure 1.5: The Adams-graded E_2-page of the Tate spectral sequence computing the homotopy groups $\pi_*(k^{tC_p})^{tC_{p^{n-1}}}$ for p odd.

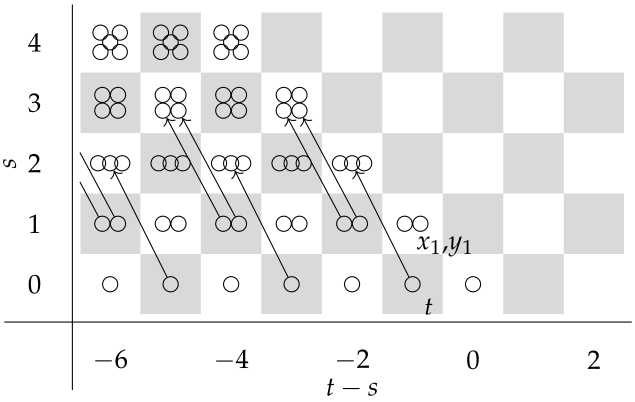


Figure 1.6: E_2-page of the Hochschild–Serre spectral sequence, or equivalently, the $(C_2)^2$-HFPSS, associated to the extension $C_2 \to Q_8 \to (C_2)^2$. To prevent cluttering, we have only illustrated the nonzero differentials for small s. Each circle represents a k-summand.

by a nontrivial differential. The E_3-page is therefore empty, and the Tate construction is contractible. By Lemma 1.3.8, we have therefore shown that $k^{tC_{2^n}} \to k^{tC_2}$ is a faithful Galois extension.

If p is odd, the proof technique is the same, though the multiplicative structure of the Hochschild–Serre spectral sequence changes. One now has

$$E_2^{st} \simeq H^s(C_{p^{n-1}}; k) \otimes H^{-t}(C_p; k) \simeq k[x_2] \otimes \Lambda(x_1) \otimes k[t_2] \otimes \Lambda(t_1),$$

with nontrivial differential $d_2(t_1) = x_2$. The E_2-page looks identical to Fig. 1.1, and the E_3-page to Fig. 1.2.

As before, by multiplicativity we can extend this to positive $(t - s)$-degree into the HFPSS for Tate spectra. Similarly, we can then further extend to negative s-degree into the Tate spectral sequence. The E_2-page of the latter looks identical to Fig. 1.5, and we conclude that the Tate construction is again contractible. Therefore, $k^{tC_{p^n}} \to k^{tC_p}$ is a faithful Galois extension. □

We now consider the case of the classical quaternion group Q_8. The reason we treat the case of Q_8 separately from the generalised quaternion case Q_{2^n} is because the group cohomology and Tate cohomology rings differ between these two cases, as do the resulting differentials.

Theorem 1.3.11. The $(C_2)^2$-Galois extension $k^{tQ_8} \to k^{tC_2}$ is faithful.

Proof. Our method is the same as in the cyclic p-group case: we first study the Hochschild–Serre spectral sequence associated to the extension $C_2 \to Q_8 \to (C_2)^2$, which can be identified with the HFPSS computing $\pi_* k^{hQ_8}$. We then leverage multiplicativity twice to compute the Tate spectral sequence

$$E_2^{st} \simeq \widehat{H}^s\big((C_2)^2; \pi_t k^{tC_2}\big) \Rightarrow \pi_{t-s}(k^{tC_2})^{t(C_2)^2}.$$

The Hochschild–Serre spectral sequence, regraded to match with the grading conventions of the HFPSS, has E_2-page of the form

$$E_2^{s,t} \simeq H^s\big((C_2)^2; k\big) \otimes \pi_t(k^{hC_2}) \simeq k[x_1, y_1] \otimes k[t_1].$$

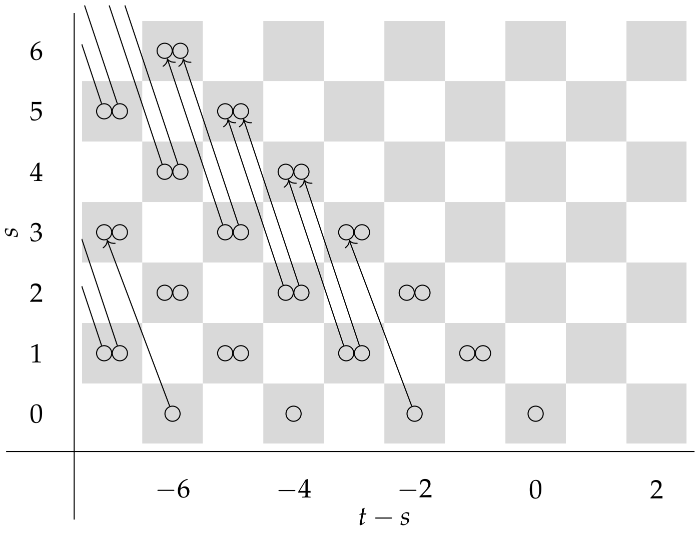


Figure 1.7: E_3-page of the Hochschild–Serre spectral sequence for $C_2 \to Q_8 \to (C_2)^2$.

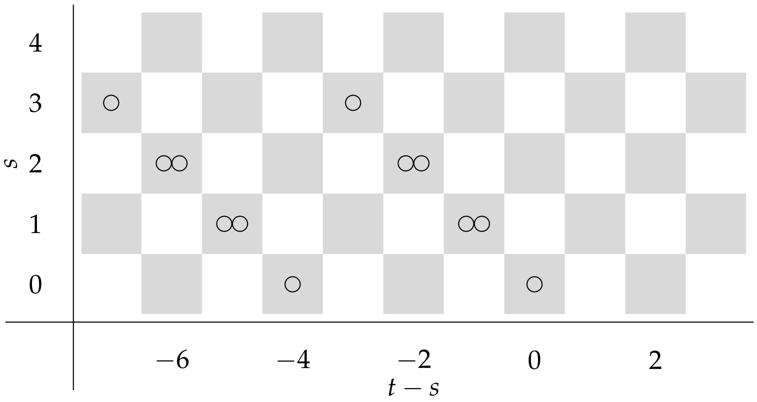


Figure 1.8: E_4-page of the Hochschild–Serre spectral sequence for $C_2 \to Q_8 \to (C_2)^2$. There are no remaining differentials, and the spectral sequence collapses.

where x_1 and y_1 are in Adams degree $(-1, 1)$ and t_1 is in Adams degree $(-1, 0)$. To understand the differentials, one can restrict to appropriate subgroups of Q_8, which yield natural maps of extensions. For example, one has a map of extensions

$$\begin{tikzcd}
1 \arrow[r] & C_2 \arrow[r] \arrow[d] & C_4 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & C_2 \arrow[r] & Q_8 \arrow[r] & C_2 \times C_2 \arrow[r] & 1
\end{tikzcd}$$

These extentions induce comparison maps of Hochschild–Serre spectral sequences for Q_8 and for C_4. The spectral sequence for C_4 had been outlined in the previous section, and we infer that $d_2(t_1) = x_1^2 + x_1y_1 + y_1^2$. The E_2-page has been drawn out in Fig. 1.6. By e.g. Kudo transgression one then finds that $d_3(t_1^2) = \mathrm{Sq}^1(d_2(t_1)) = x_1^2y_1 + x_1y_1^2$. The E_3-page and E_4-page have been drawn in Fig. 1.7 and Fig. 1.8, respectively. Observe that the spectral sequence collapses on the E_4-page.

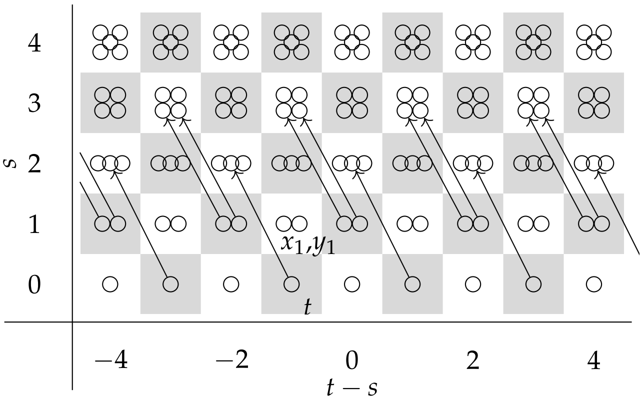


Figure 1.9: E_2-page of the $(C_2)^2$-HFPSS computing $\pi_* \, k^{tQ_8}$. It is obtained from Fig. 1.6 by inverting t.

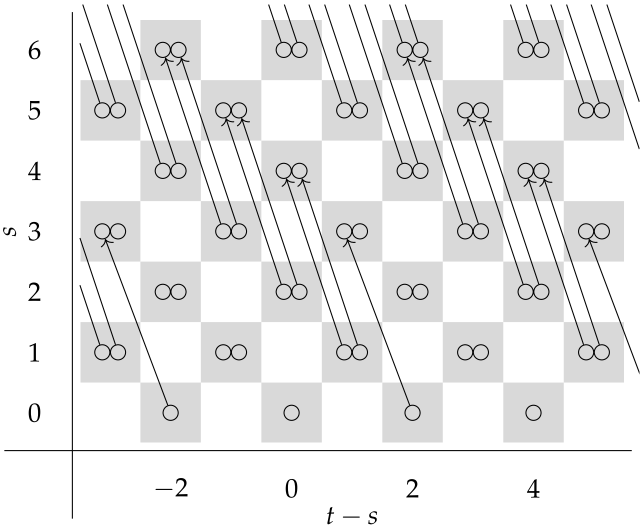


Figure 1.10: E_3-page of the $(C_2)^2$-HFPSS computing $\pi_* \, k^{tQ_8}$.

We can now again compute $\pi_* \, k^{tQ_8}$ via the HFPSS. Since the ring structure of $\pi_* \, k^{tC_2}$ is given simply by $k[t_1^{\pm 1}]$, we may again import all differentials from the Hochschild–Serre spectral sequence and extend using multiplicativity. The E_2-page is illustrated in Fig. 1.9. It develops in the expected way: the E_3-page is outlined in Fig. 1.10, and the E_4-page in Fig. 1.11.

We further extend to negative s-degree so as to obtain the Tate spectral sequence

$$E_2^{st} \simeq \widehat{H}^{-s}\big((C_2)^2, \pi_{t-s} \, k^{tC_2}\big) \Rightarrow \pi_{t-s}(k^{tC_2})^{t(C_2)^2}.$$

Here, some care must be taken in extending to the Tate spectral sequence, as the Tate cohomology ring of $(C_2)^2$ isn't just given by a naive Laurent polynomial ring. As computed in Section 1.A, the multiplicative structure in *nonnegative* degree is identified with that of the cohomology ring. But in negative degrees, we have the following. There is a distinguished element α in $\widehat{H}^{-1}\big((C_2)^2; k\big)$, and the cup product yields a perfect pairing $\widehat{H}^r\big((C_2)^2; k\big) \otimes \widehat{H}^{-r-1}\big((C_2)^2; k\big) \to \widehat{H}^{-1}\big((C_2)^2; k\big) \simeq \langle \alpha \rangle$. The remaining cup products, in particular all products of negative-degree elements, are zero. In view of

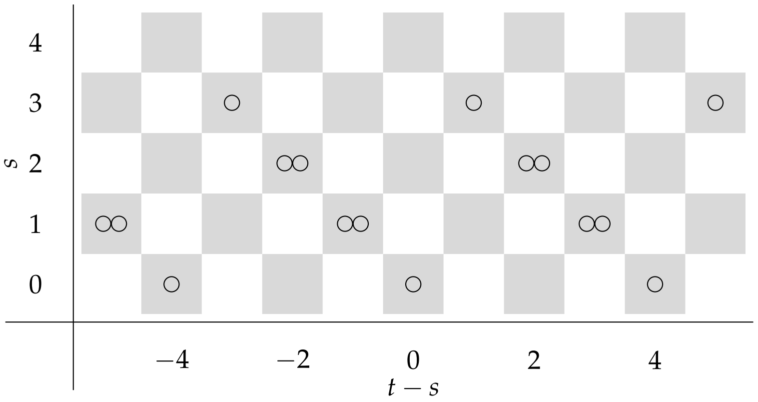


Figure 1.11: E_4-page of the $(C_2)^2$-HFPSS computing $\pi_* \, k^{tQ_8}$. Compare with Fig. 1.8.

the perfect pairing, we denote the negative-degree classes by $\alpha x_1^{-a} y_1^{-b}$, though it is not a cup product of α by some element $x_1^{-a} y_1^{-b}$.

It is thanks to the pairing that we can extend the differentials to negative s-degree. For instance, we have

$$\begin{aligned} d_2(\alpha x_1^{-a} y_1^{-b} \otimes t_1^{-1}) &= d_2(t_1^{-1}) \cdot \alpha x_1^{-a} y_1^{-b} \\ &= \alpha x_1^{2-a} x_2^{-b} + \alpha x_1^{1-a} x_2^{1-b} + \alpha x_1^{-a} x_2^{2-b} \end{aligned}$$

We've drawn the E_2-page on Fig. 1.12. The E_3-page and E_4-page of the Tate spectral sequence have been drawn in Fig. 1.13 and Fig. 1.14.

In the HFPSS, the spectral sequence collapses at the E_4-page for degree reasons, but in the Tate spectral sequence, there's room for nontrivial d_4-differentials. We claim that these differentials are indeed nontrivial. We begin with the following square of cofibre sequences.

$$\begin{tikzcd}
(k^{tC_2})_{h(C_2)^2} \arrow[r] & (k^{tC_2})^{h(C_2)^2} \arrow[r] & (k^{tC_2})^{t(C_2)^2} \\
(k^{hC_2})_{h(C_2)^2} \arrow[r] \arrow[u] & (k^{hC_2})^{h(C_2)^2} \arrow[r] \arrow[u] & (k^{hC_2})^{t(C_2)^2} \arrow[u] \\
(k_{hC_2})_{h(C_2)^2} \arrow[r] \arrow[u] & (k_{hC_2})^{h(C_2)^2} \arrow[r] \arrow[u] & (k_{hC_2})^{t(C_2)^2} \arrow[u]
\end{tikzcd}$$

We can identify the middle term as k^{hQ_8} and the bottom left term as k_{hQ_8}. Moreover, thanks to Theorem 1.3.6 we can identify the top middle term with k^{tQ_8}. This simplifies the diagram to

$$\begin{tikzcd}
(k^{tC_2})_{h(C_2)^2} \arrow[r] & k^{tQ_8} \arrow[r] & (k^{tC_2})^{t(C_2)^2} \\
(k^{hC_2})_{h(C_2)^2} \arrow[r] \arrow[u] & k^{hQ_8} \arrow[r] \arrow[u] & (k^{hC_2})^{t(C_2)^2} \arrow[u] \\
k_{hQ_8} \arrow[r] \arrow[u] & k_{hQ_8} \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

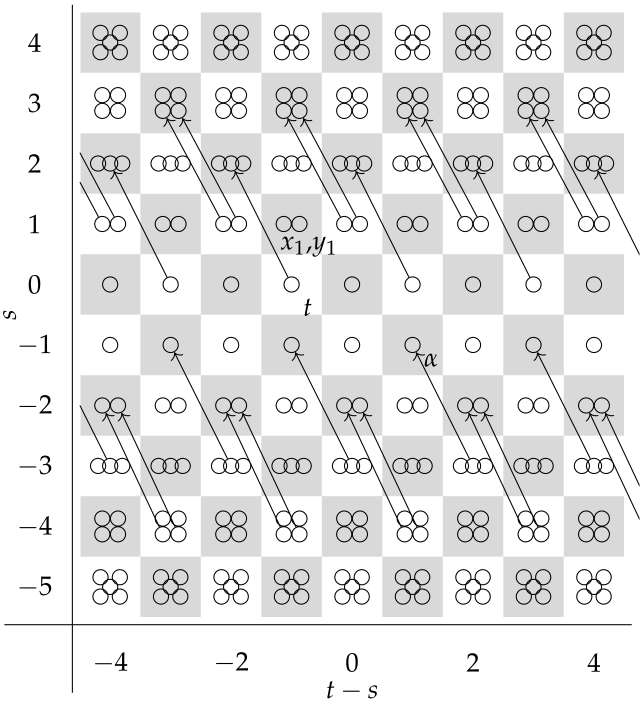


Figure 1.12: E_2-page of the Tate spectral sequence computing the homotopy groups $\pi_*(k^{tC_2})^{t(C_2)^2}$. To prevent cluttering, only the nonzero differentials for small s are drawn.

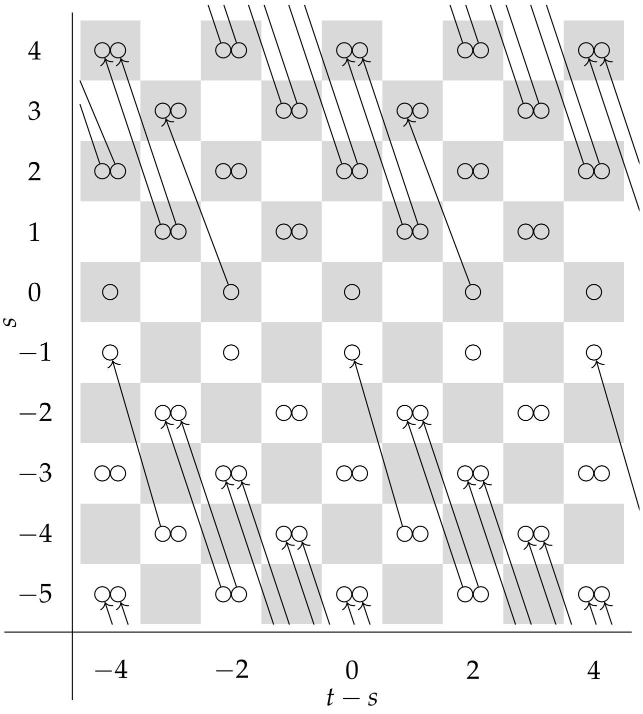


Figure 1.13: E_3-page of the Tate spectral sequence computing the homotopy groups $\pi_*\left((k^{tC_2})^{t(C_2)^2}\right)$. All nonzero differentials have been illustrated.

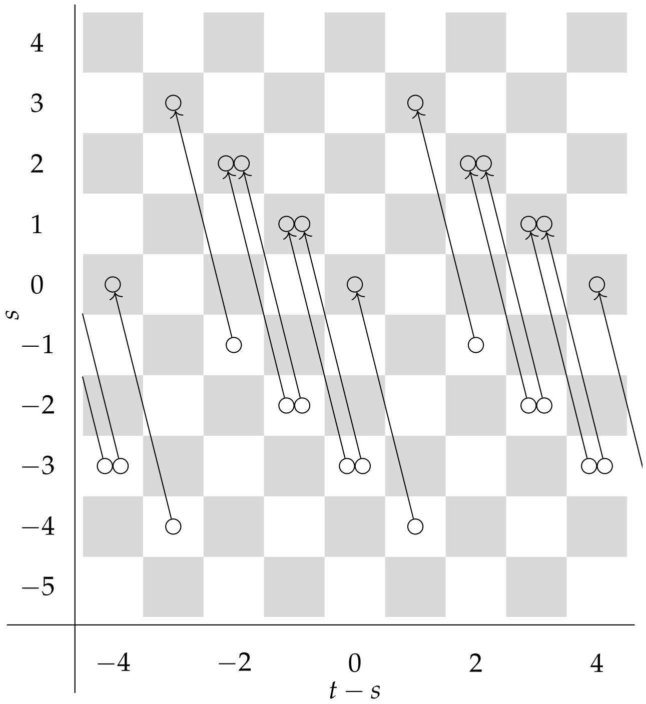


Figure 1.14: E_4-page of the Tate spectral sequence computing the homotopy groups $\pi_*(k^{tC_2})^{t(C_2)^2}$. These nontrivial differentials do not come from the HFPSS.

This forces the map $(k^{hC_2})^{t(C_2)^2} \to (k^{tC_2})^{t(C_2)^2}$ to be an isomorphism. Now, to both we may functorially asssociate a Tate spectral sequence. The E_4-page of $(k^{tC_2})^{t(C_2)^2}$ has been illustrated in Fig. 1.14, and that of $(k^{hC_2})^{t(C_2)^2}$ is the same but truncated so as to live in t-degree ≤ 0. The comparison map of spectral sequences is the obvious one. This comparison forces the d_4-differentials in Fig. 1.14 for $t - s > 1$ to be nontrivial, and by multiplicativity, this nontriviality propagates to negative $(t-s)$-degree. Thus, the E_∞-page is empty and the Tate construction $(k^{tC_2})^{t(C_2)^2}$ is contractible. ☐

Remark 1.3.12. This argument also implies that $k^{hQ_8} \to k^{hC_2}$ is a faithful Galois extension. Also, by the commutative squares above, we may observe the equivalences of spectra $(k^{hC_2})_{h(C_2)^2} \simeq (k^{hC_2})^{h(C_2)^2}$ and $(k^{tC_2})_{h(C_2)^2} \simeq (k^{tC_2})^{h(C_2)^2}$.

Finally, we consider the generalised quaternion groups beyond Q_8.

Theorem 1.3.13. The Q_{2^n}/C_2-Galois extension $k^{tQ_{2^n}} \to k^{tC_2}$ is faithful.

Proof. Our method is the same as in the previous cases. In fact, the associated spectral sequence diagrams look exactly the same as in the case of Q_8; the only difference is that the multiplicative structure changes.

We first study the Hochschild–Serre spectral sequence associated to the extension $C_2 \to Q_{2^n} \to D_{2^{n-1}}$, and then we extend this spectral sequence to produce the four-quadrant Tate spectral sequence

$$E_2^{st} \simeq \widehat{H}^s\big(D_{2^{n-1}}; \pi_t\, k^{tC_2}\big) \Rightarrow \pi_{t-s}(k^{tC_2})^{tD_{2^{n-1}}}.$$

For all $n \geq 4$, the cohomology ring $H^*(D_{2^{n-1}}; k)$ is given by $k[x_1, y_1, z_2]/(x_1 y_1)$, where $|x_1| = |y_1| = 1$ and $|z_2| = 2$. Moreover, $\mathrm{Sq}^1(z_2) = (x_1 + y_1)z_2$. It is convenient to set $u_1 = x_1 + y_1$ and write the cohomology ring as $k[x_1, u_1, z_2]/(u_1 x_1 + x_1^2)$.

Since C_2 is central in Q_{2^n}, the E_2-page of the Hochschild–Serre spectral sequence has the form

$$E_2^{st} \simeq H^s(D_{2^{n-1}}; k) \otimes \pi_t(k^{hC_2}) \simeq k[x_1, u_1, z_2]/(u_1 x_1 + x_1^2) \otimes k[t_1].$$

We have a nontrivial d_2-differential $d_2(t_1) = u_1^2 + z_2$, as can be computed by restricting to appropriate subgroups of Q_{2^n}, and by Kudo transgression, we have $d_3(t_1^2) = u_1 z_2$. These again spawn all the other differentials via the Leibniz rule. Although the multiplicative generators are different, the E_2-, E_3-, and E_4-page look exactly the same as those for Q_8 — cf. Fig. 1.6, Fig. 1.7, and Fig. 1.8.

We extend the spectral sequence using multiplicativity to the HFPSS computing $\pi_* k^{tQ_{2^n}}$. Again, since $\pi_* k^{tC_2}$ is simply $k[t_1^{\pm 1}]$, we can extend without much issue. The pages are again identical to Q_8, and are illustrated in Fig. 1.9, Fig. 1.10, and Fig. 1.11. We then further extend to the Tate spectral sequence. As in the Q_8 case, some care must be taken when extending, because the multiplicative structure of $\widehat{H}^*(D_{2^{n-1}}; k)$ is nontrivial. As shown in Section 1.A, the Tate cohomology ring is the usual cohomology ring in positive degrees, and there's again a perfect pairing onto $\widehat{H}^{-1}(D_{2^{n-1}}; k) \simeq \langle \alpha \rangle$, and we use the perfect pairing to extend the differentials to negative s-degree. The E_2- and E_3-page look the same as in Fig. 1.12 and Fig. 1.13.

For the same reason as in Q_8, there is room for nontrivial differentials on the E_4-page of the Tate spectral sequence. The proof that they are indeed nontrivial is exactly the same: one has the square of cofibre sequences

$$\begin{tikzcd}
(k^{tC_2})_{hD_{2^{n-1}}} \arrow[r] & k^{tQ_{2^n}} \arrow[r] & (k^{tC_2})^{tD_{2^{n-1}}} \\
(k^{hC_2})_{hD_{2^{n-1}}} \arrow[r] \arrow[u] & k^{hQ_{2^n}} \arrow[r] \arrow[u] & (k^{hC_2})^{tD_{2^{n-1}}} \arrow[u] \\
k_{hQ_{2^n}} \arrow[r] \arrow[u] & k_{hQ_{2^n}} \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

which implies that the map $(k^{hC_2})^{tD_{2^{n-1}}} \to (k^{tC_2})^{tD_{2^{n-1}}}$ is an equivalence. This forces the nontriviality of some d_4-differentials, and the nontriviality of all other differentials then follows by multiplicativity. Thus the Tate construction is again contractible. ☐

1.4 Computation of endotrivial modules

In this chapter, we will evaluate the limit spectral sequence for the Picard spectrum to compute the group of endotrivial modules for the cyclic p-groups and generalised quaternion groups. We use the comparison tool of Theorem 1.2.11 to compare most differentials to the limit spectral sequence for Ω.

For the groups that we consider, the limit decomposition can be re-interpreted as an instance of Galois descent. Accordingly, the limit spectral sequence for Ω is a familiar object. Indeed, by Lemma 1.2.12, $\Omega\,\mathsf{StMod}(kG)$ is simply k^{tG}, and the limit spectral sequence is simply an extension of the Hochschild–Serre spectral sequence to two quadrants. We have already evaluated this spectral

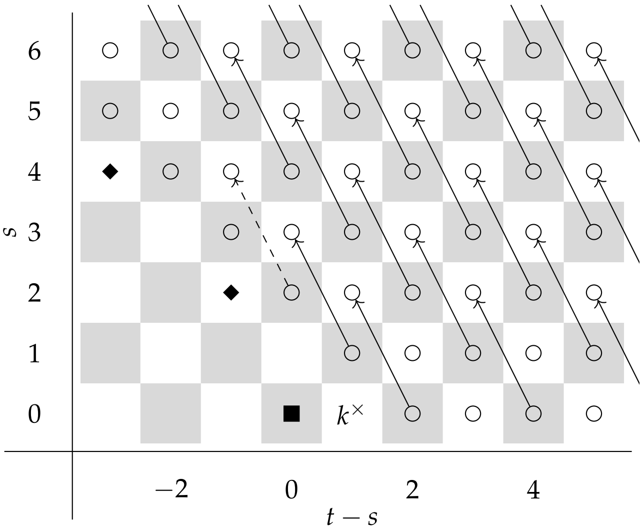


Figure 1.15: E_2-page of the limit spectral sequence for the Picard spectrum of $\mathsf{StMod}(kC_{p^n})$. The circles denote a k-summand again. The black square is 0 if $p = 2$ and $\mathbb{Z}/2\mathbb{Z}$ if p is odd. The black diamond is the group $k^\times/(k^\times)^{p^{n-1}}$. The known nonzero differentials have been illustrated. The dashed differential is of special interest, as it falls within the range of Theorem 1.2.15.

sequence in Section 1.3, under the guise of an HFPSS computing $\pi_*\, k^{tG}$. In view of this, we will see that most of the work which remains will be to compute some unstable differentials.

In this section, our aim is to compute the Picard group of $\mathsf{StMod}(kC_{p^n})$. The limit spectral sequence of Eq. (1.2.9) reads

$$E_2^{st} \simeq H^s(C_{p^{n-1}}; \pi_t\, \mathfrak{pic}\, \mathsf{StMod}(kC_p)) \Rightarrow \pi_{t-s}\, \mathfrak{pic}\, \mathsf{StMod}(kC_{p^n}).$$

Because the groups involved are abelian, all conjugation actions are trivial, hence so is the action of $C_{p^{n-1}}$ on the π_t. The E_2-page has been sketched in Fig. 1.15. Let's take a look at the differentials, distinguishing between the cases $p = 2$ and odd p.

We start with the case $p = 2$. Differentials in the stable range may be compared with the differentials of the multiplicative spectral sequence

$$E_2^{st} \simeq H^s(C_{p^{n-1}}; \pi_t\, \Omega\, \mathsf{StMod}(kC_p)) \Rightarrow \pi_{t-s}\, \Omega\, \mathsf{StMod}(kC_{p^n})$$

using Theorem 1.2.11. We have evaluated this spectral sequence in the proof of Theorem 1.3.10.

The relevant differentials in the unstable range are d_2^{01} and d_2^{22}. The former is zero, because $k^\times$ has no 2-torsion. (In addition, we know that the 1-line should have a surviving $k^\times$ anyhow.) The latter may be understood via Theorem 1.2.15. The corresponding differential $d_2^{21}(\Omega)$ of the spectral sequence for $\Omega\, \mathsf{StMod}(kC_{2^n})$ was the linear map $\langle t_1^{-1}x_1^2\rangle \to \langle t_1^{-2}x_1^4\rangle$ sending $t_1^{-1}x_1^2$ to $t_1^{-2}x_1^4$. Consequently, Theorem 1.2.15 tells us d_2^{22} in the limit spectral sequence for Picard spectra is the map sending a scalar α in k to $\alpha + \alpha^2$. The kernel of this map is given by the elements α such that $\alpha + \alpha^2 = \alpha(\alpha + 1) = 0$, of which there's only two, namely 0 and 1. Therefore the kernel is $\mathbb{Z}/2\mathbb{Z}$, irrespective of the underlying field k.

The E_3-page is now summarised in Fig. 1.16. It's easily seen that, from the 0-line upward, no nontrivial differentials can exist, and we deduce the following.

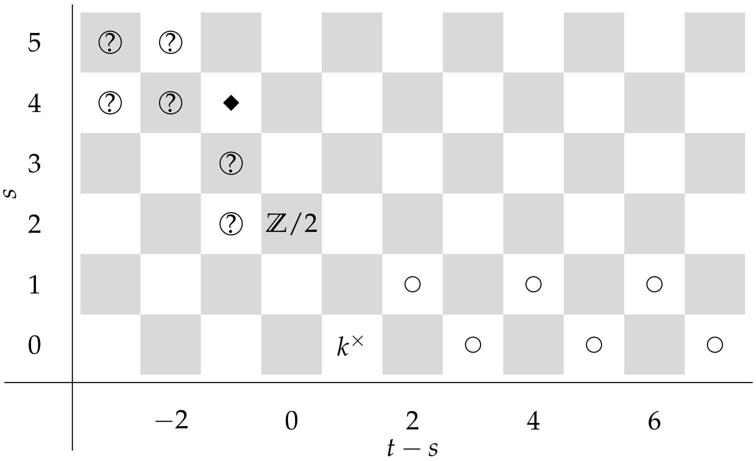


Figure 1.16: E_3-page of the limit spectral sequence for $\mathsf{StMod}(kC_{2^n})$. Notice that the 0-line has only one nonzero group remaining. The black diamond is the group E_3^{43}, which is the quotient of k by the subgroup of those c for which the equation $x^2 + x + c$ has a root in k. Classes indicated by a question mark have unknown value, as they fall outside the range where we can understand the differentials.

Theorem 1.4.1. For all fields k of characteristic 2, and all $n \geq 2$, the Picard group of $\mathsf{StMod}(kC_{2^n})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Remark 1.4.2. The appearance of $k^\times/(k^\times)^{p^{n-1}}$ on the (-1)-line of the E_2-page is somewhat curious. If k is perfect, then this group vanishes; in contrast, if say $k = \mathbb{F}_p(x)$, then x represents a nontrivial element in the quotient. In view of Remark 1.2.6, would it be fair to conjecture whether perfect-ness of k influences the Brauer group of $k^{tC_{p^n}}$?

We now turn to the case where p is odd. Several minor differences arise.

- The Picard group of $\mathsf{StMod}(kC_p)$ is $\mathbb{Z}/2\mathbb{Z}$ rather than 0 if the prime p is odd.
- As we already observed in Section 1.3, the cup product structure on $H^*(C_{p^{n-1}};k)$ is different.
- The squaring operation in Theorem 1.2.15 dies in the context of odd characteristic, which alters the outcome of the Adams-graded $(0,2)$-position of the spectral sequence.

The second point causes the odd-prime analogue of the E_2-page of the Hochschild–Serre spectral sequence to have different multiplicative generators, but as we found in Section 1.3, both the E_2-page and E_3-pages of the Hochschild–Serre spectral sequence look exactly the same as the $p = 2$ case.

To compute the Picard spectral sequence for p odd in Fig. 1.15, we can again import differentials in the stable range. It remains to study the unstable differentials. As before, d_2^{01} is necessarily trivial. d_2^{00} is trivial as well, because $k^\times/(k^\times)^{p^{n-1}}$ has no 2-torsion, and so the $\mathbb{Z}/2\mathbb{Z}$ in E_2^{00} survives. The differential d_2^{22} is again governed by Theorem 1.2.15. Since we're in odd characteristic, the squaring operation vanishes, and the differential d_2^{22} is identified with the corresponding differential $d_2^{21}(\Omega)$ of Fig. 1.3, which is seen to be an isomorphism $k \to k$, and hence E_3^{22} is 0 rather than $\mathbb{Z}/2\mathbb{Z}$.

The E_3-page is summarised in Fig. 1.17. As before, there are no more nontrivial differentials that can alter the outcome, and we deduce the following result.

Theorem 1.4.3. For all fields k of odd characteristic p and all $n \geq 2$, the Picard group of $\mathsf{StMod}(kC_{p^n})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

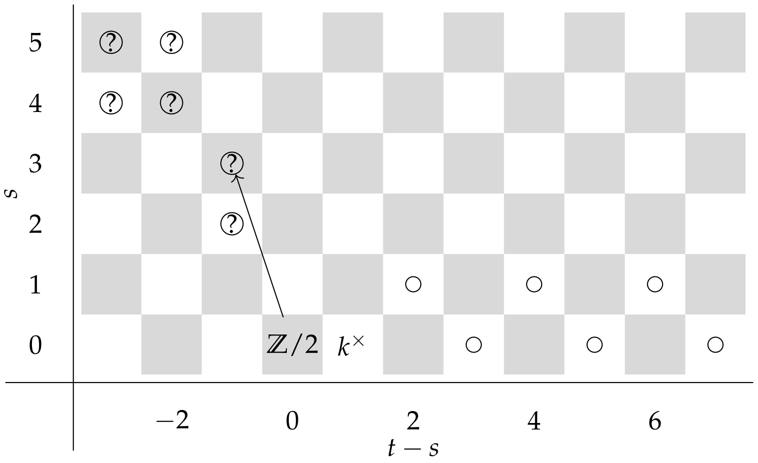


Figure 1.17: E_3-page of the limit spectral sequence for $\mathsf{StMod}(kC_{p^n})$ for p odd. Compare with Fig. 1.16. The only differential with possibly nontrivial domain and codomain is $d_3^{0,0}$, but this differential must be 0, as $E_3^{3,2}$, arising as a subgroup of $E_2^{2,2} \simeq k$, has no 2-torsion.

Remark 1.4.4. For varying n, the Tate cohomology rings of C_{p^n} are isomorphic (with the exception of the case $p = 2$ and $n = 1$). Nonetheless, the Tate fixed points $k^{tC_{p^n}}$ are inequivalent, as they can be distinguished via e.g. Sq^1. Can distinct p-groups have equivalent Tate constructions?

We will now turn our attention to Q_8, and calculate the Picard spectral sequence

$$E_2^{st} = H^s\big((C_2)^2; \pi_t \,\mathfrak{pic}\, \mathsf{StMod}(kC_2)\big) \Rightarrow \pi_{t-s} \,\mathfrak{pic}\, \mathsf{StMod}(kQ_8).$$

The E_2-page has been illustrated in Fig. 1.18. The terms for $t \geq 2$ are cohomology groups, which we have also encountered in Theorem 1.3.11. As for $t = 1$, we notice that the crucial term $E_2^{11} = H^1\big((C_2)^2; k^\times\big)$ is zero; indeed, there are no nontrivial maps $(C_2)^2 \to k^\times$ because $k^\times$ never has any 2-torsion.

Using Theorem 1.2.11, the differentials in the stable range may be directly imported from the HFPSS computing $\pi_* \, k^{tQ_8}$. The E_2-page, illustrated in Fig. 1.9, was given by

$$E_2^{st} \simeq k[x_1, y_1] \otimes k[t_1],$$

with $d_2(t_1) = x_1^2 + x_1y_1 + y_1^2$ and $d_3(t_1^2) = x_1^2y_1 + x_1y_1^2$.

There is an unstable differential $d_2^{22}(\mathfrak{pic})$, which by Theorem 1.2.15 we may compare to $d_2^{21}(\Omega)$. The differential $d_2^{21}(\Omega)$ of the Hochschild–Serre spectral sequence is a k-linear map $k^3 \to k^5$ defined by

$$d_2^{21}(\Omega)\colon \begin{cases} t_1^{-1}x_1^2 & \mapsto t_1^{-2}(x_1^2 + x_1y_1 + y_1^2)x_1^2 \\ t_1^{-1}x_1y_1 & \mapsto t_1^{-2}(x_1^2 + x_1y_1 + y_1^2)x_1y_1 \\ t_1^{-1}y_1^2 & \mapsto t_1^{-2}(x_1^2 + x_1y_1 + y_1^2)y_1^2 \end{cases}$$

The resulting differential $d_2^{22}(\mathfrak{pic})$ may now be computed by hand. It has been described diagramatically in Table 1.19. We easily see that there's only one possible nonzero element in the kernel, namely $x_1^2 + x_1y_1 + y_1^2$, and hence the kernel is $\mathbb{Z}/2\mathbb{Z}$ regardless of the field k.

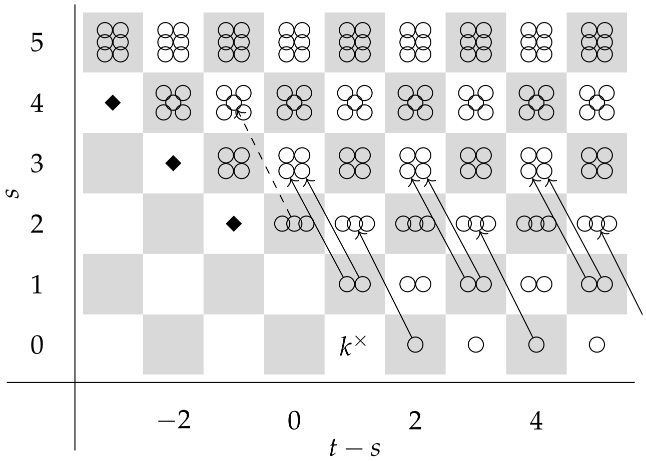


Figure 1.18: E_2-page of the limit spectral sequence computing the Picard group of $\mathsf{StMod}(kQ_8)$. The nontrivial differential has been illustrated. The group indicated by the black diamond is trivial if k is a perfect field. Known nonzero diagonals have been illustrated only for small s to prevent cluttering. The dashed diagonal is governed by Theorem 1.2.15.

		$t_1^{-2}x_1^4$	$t_1^{-2}x_1^3y_1$	$t_1^{-2}x_1^2y_1^2$	$t_1^{-2}x_1y_1^3$	$t_1^{-2}y_1^4$
$\lambda t_1^{-1}x_1^2$	$\mapsto$	$\lambda + \lambda^2$	λ	λ		
$\mu t_1^{-1}x_1y_1$	$\mapsto$		μ	$\mu + \mu^2$	μ	
$\nu t_1^{-1}y_1^2$	$\mapsto$			ν	ν	$\nu + \nu^2$

Table 1.19: Behaviour of $d_2^{22}(\mathfrak{pic})$ in the limit spectral sequence for Q_8. Here λ and μ denote a scalar in k. Notice that the differential is not k-linear, although it is $\mathbb{F}_2$-linear.

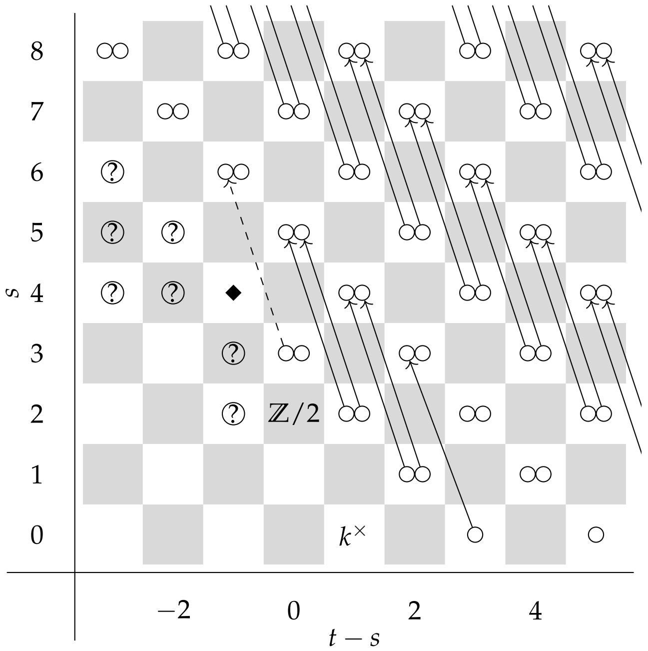


Figure 1.20: E_3-page of the limit spectral sequence computing the Picard group of $\mathsf{StMod}(kQ_8)$. The nontrivial differentials have been illustrated. The group illustrated by the black diamond is the cokernel of the nonlinear map described by Table 1.19.

On the E_3-page, which has been illustrated in Fig. 1.20, a similar situation arises: the stable differentials are imported from Fig. 1.10, but there's a possibly nontrivial unstable differential $d_3^{33}(\mathfrak{pic})$. The corresponding differential $d_3^{32}(\Omega)$ from the Hochschild–Serre spectral sequence is the k-linear map

$$d_3^{32}(\Omega)\colon \begin{cases} t_1^{-2}[x_1y_1^2] & \mapsto t_1^{-4}([x_1^3y_1^3]+[x_1^2y_1^4]) \\ t_1^{-2}[x_1^2y_1] & \mapsto t_1^{-4}([x_1^4y_1^2]+[x_1^3y_1^3]) \end{cases}$$

Using this, we readily compute to find that the behavior of the $d_3^{33}(\mathfrak{pic})$ differential is described by

$$d_3^{33}(\mathfrak{pic})\colon \begin{cases} \lambda t_1^{-2}[x_1y_1^2] & \mapsto \lambda t_1^{-4}[x_1^2y_1^4]+\lambda^2 t_1^{-4}[x_1^4y_1^2] \\ \mu t_1^{-2}[x_1^2y_1] & \mapsto \mu^2 t_1^{-4}[x_1^2y_1^4]+\mu t_1^{-4}[x_1^4y_1^2] \end{cases}$$

Elements in the kernel of this differential correspond to pairs (λ,μ) such that $\lambda+\mu^2=0$ and $\lambda^2+\mu=0$. Since k is of characteristic 2, this corresponds to pairs (λ,λ^2) such that $\lambda+\lambda^4=0$. Clearly, there are trivial solutions $\lambda=0$ and $\lambda=1$, but if k contains a primitive cube root of unity ω, then we may also take $\lambda=\omega$ and $\lambda=\omega^2$. We thus find that

$$\operatorname{Ker} d_3^{33} \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & \text{if } k \text{ has a third root of unity;} \\ \mathbb{Z}/2\mathbb{Z} & \text{otherwise.} \end{cases}$$

We're now ready to write out the E_4-page of the limit spectral sequence. A portion of it has been illustrated in Fig. 1.21. The spectral sequence collapses — at least in the relevant range $t-s\geq 0$ — and we find that the line $t-s=0$ depends on the structure of the field. If k has a third root of unity,

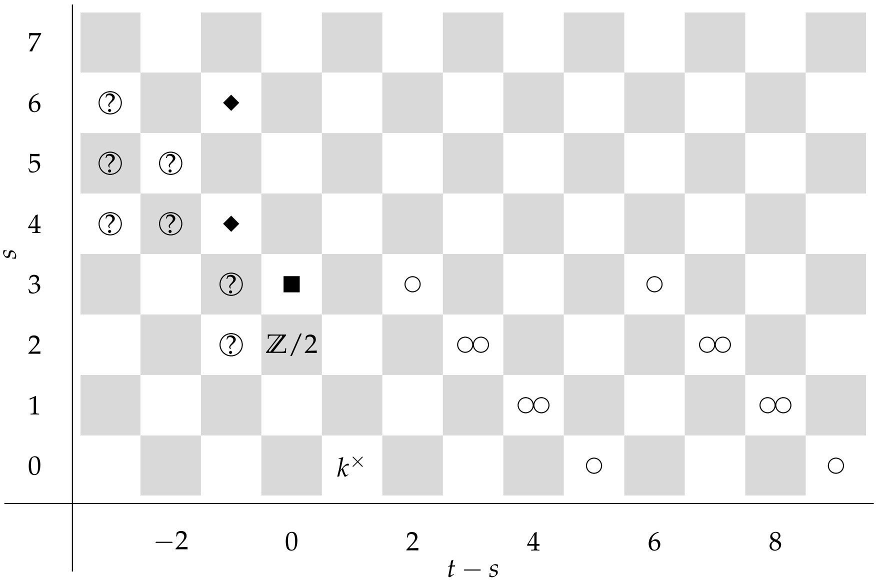


Figure 1.21: E_4-page of the limit spectral sequence for computing the Picard group of $\mathsf{StMod}(kQ_8)$. There are no remaining differentials, and the spectral sequence collapses. The black square is E_3^{33}, and it depends on the structure of the field k. It is either $\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$. The groups illustrated by the black diamonds are the cokernels of the nonlinear maps d_2^{22} and d_3^{33}.

then there's a copy of $\mathbb{Z}/2\mathbb{Z}$ and a copy of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ on the 0-line, while if k does not have a third root of unity, there are two surviving copies of $\mathbb{Z}/2\mathbb{Z}$.

In both cases, there's room for nontrivial extension problems. Nonetheless it's easy to overcome these problems: The 4-fold periodicity of the homotopy groups of k^{tQ_8} implies that the unit is an element of order 4 in the Picard group. The only groups with the indicated extensions and an element of order 4 are $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, hence we deduce the following result.

Theorem 1.4.5. Let k a field of characteristic 2. Then

$$\operatorname{Pic}\left(\mathsf{StMod}(kQ_8)\right) \simeq \begin{cases} \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } k \text{ has a primitive cube root of unity;} \\ \mathbb{Z}/4\mathbb{Z} & \text{otherwise.} \end{cases}$$

Remark 1.4.6. The exotic generator of the Picard group of $\mathsf{StMod}(kQ_8)$ has a known explicit description as a G-representation. Following [CT00], we find that it is captured by the associations

$$i \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad j \mapsto \begin{pmatrix} 1 & 0 & 0 \\ \omega & 1 & 0 \\ 0 & \omega^2 & 1 \end{pmatrix},$$

where ω denotes a principal cube root of unity. It would be interesting to have a homotopical construction of this object.

Remark 1.4.7. This method of computing the group of endotrivial modules differs dramatically from the work of Carlson, Thévenaz and others, who used representation-theoretic techniques (namely, the theory of support varieties). In the case of the quaternion group, they explicitly construct the endotrivial modules above, and prove that no other endotrivial modules exist.

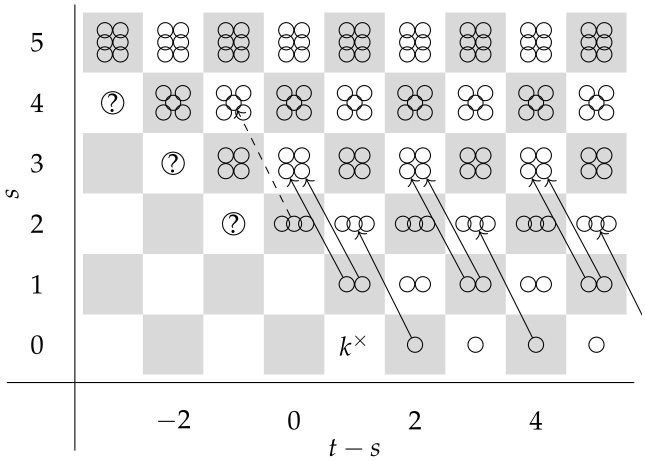


Figure 1.22: E_2-page of the limit spectral sequence for computing the Picard group of $\mathsf{StMod}(kQ_{2^n})$. The nontrivial differential has been illustrated. Terms indicated by a question mark are the higher cohomology of $D_{2^{n-1}}$ with coefficients in $k^\times$, and are likely well behaved when k is perfect. The dashed diagonal is governed by Theorem 1.2.15.

Finally, we consider the Picard spectral sequence

$$E_2^{st} = H^s\left(D_{2^{n-1}}; \pi_t\, \mathfrak{pic}\, \mathsf{StMod}(kC_2)\right) \Rightarrow \pi_{t-s}\, \mathfrak{pic}\, \mathsf{StMod}(kQ_{2^n}).$$

The E_2-page, illustrated on Fig. 1.22, looks effectively the same as that of Q_8, and indeed is computed in the same way. The differentials in the stable range are imported from the associated HFPSS

$$\begin{aligned} E_2^{st} &\simeq H^s(D_{2^{n-1}}; k) \otimes \pi_t(k^{tC_2}) \\ &\simeq k[x_1, u_1, z_2]/(u_1x_1 + x_1^2) \otimes k[t_1^{\pm 1}] \Rightarrow \pi_*\, k^{tQ_{2^n}} \end{aligned}$$

which we computed in the proof of Theorem 1.3.13. We found that $d_2(t_1) = u_1^2 + z_2$ and $d_3(t_1^2) = u_1z_2$, and that the pages looked identical to the analogous spectral sequences for Q_8, which were illustrated in Fig. 1.9 and Fig. 1.10.

The crucial unstable differential is again $d_2^{22}(\mathfrak{pic})$, which we compute through Theorem 1.2.15. The differential $d_2^{21}(\Omega)$ is the k-linear map $k^3 \to k^5$ defined by

$$d_2^{21}(\Omega)\colon \begin{cases} t_1^{-1}u_1^2 & \mapsto t_1^{-2}(u_1^2 + z_2)u^2 \\ t_1^{-1}z_2 & \mapsto t_1^{-2}(u_1^2 + z_2)z_2 \\ t_1^{-1}u_1x_1 & \mapsto t_1^{-2}(u_1^2 + z_2)u_1x_1 \end{cases}$$

We use this to compute d_2^{22} by hand; the result has been indicated in Table 1.23. The only nonzero element in the kernel is $t_1^{-1}(u_1^2 + z_2)$, so the kernel is $\mathbb{Z}/2\mathbb{Z}$ regardless of the field k. This brings us to the E_3-page, illustrated in Fig. 1.24.

On the E_3-page there's again an unstable differential, $d_3^{33}(\mathfrak{pic})$. In the analysis of the HFPSS, the differential $d_3^{32}(\Omega)$ was determined to be the map $k^2 \to k^2$ defined by

$$d_3^{32}(\Omega)\colon \begin{cases} t_1^{-2}[u_1z_2] & \mapsto t_1^{-4}[u_1^2z_2^2] \\ t_1^{-2}[z_2x_1] & \mapsto t_1^{-4}[u_1z_2^2x_1] \end{cases}$$

		$t_1^{-2}u^4$	$t_1^{-2}u_1^2z_2$	$t_1^{-2}z_2^2$	$t_1^{-2}u_1^3x_1$	$t_1^{-2}u_1z_2x_1$
$\lambda t_1^{-1}u_1^2$	$\mapsto$	$\lambda+\lambda^2$	λ			
$\lambda t_1^{-1}z_2$	$\mapsto$		λ	$\lambda+\lambda^2$		
$\lambda t_1^{-1}u_1x_1$	$\mapsto$				$\lambda+\lambda^2$	λ

Table 1.23: Behaviour of $d_2^{22}(\mathfrak{pic})$ in the limit spectral sequence for Q_{2^n}. Here λ denotes a scalar in k. Compare with Table 1.19.

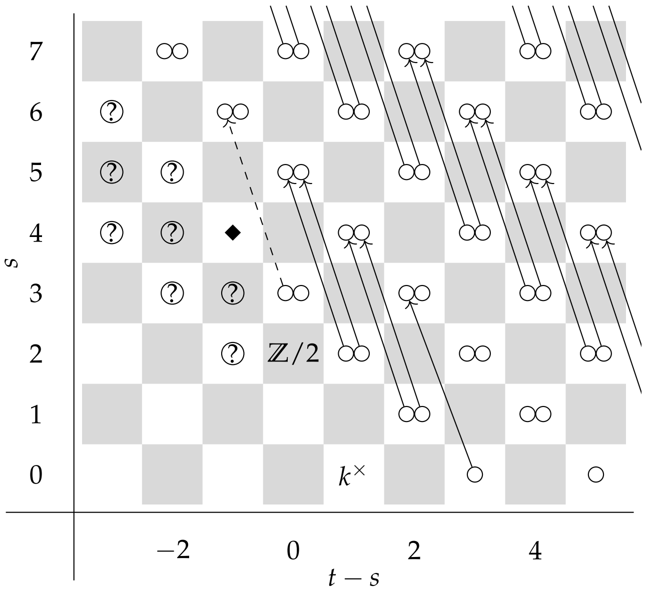


Figure 1.24: E_3-page of the limit spectral sequence for computing the Picard group of $\mathsf{StMod}(kQ_{2^n})$. The nontrivial differentials have been illustrated. The group illustrated by the black diamond is the cokernel of the nonlinear map described by Table 1.23.

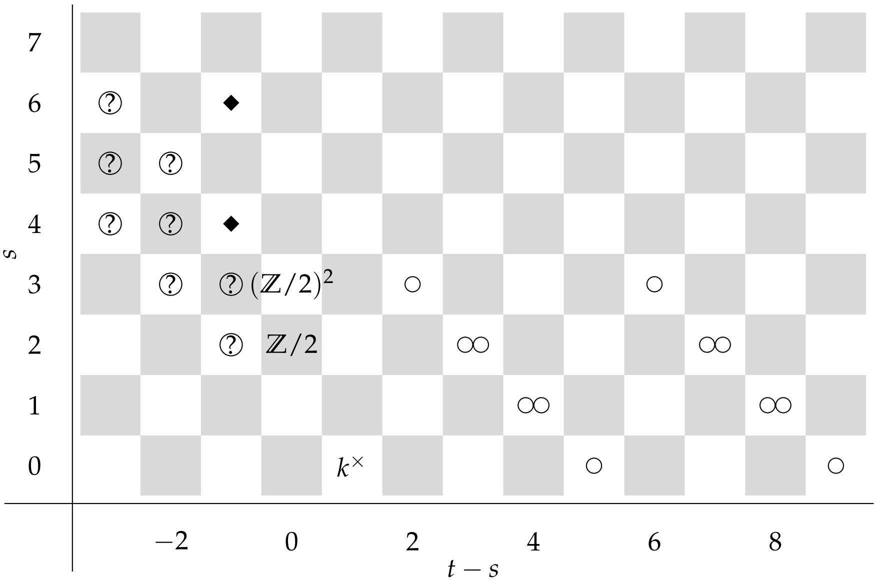


Figure 1.25: E_4-page of the limit spectral sequence for computing the Picard group of $\mathsf{StMod}(kQ_{2^n})$. There are no remaining differentials, and the spectral sequence collapses. The groups illustrated by the black diamonds are the cokernels of the nonlinear maps d_2^{22} and d_3^{33}.

Using Theorem 1.2.15 again, we compute $d_3^{33}(\mathfrak{pic})$ by hand again to find that

$$d_3^{33}(\mathfrak{pic})\colon \begin{cases} \lambda t_1^{-2}[u_1 z_2] & \mapsto \lambda t_1^{-4}[u_1^2 z_2^2] + \lambda t_1^{-4}[u_1 z_2^2 x_1] \\ \mu t_1^{-2}[z_2 x_1] & \mapsto \mu^2 t_1^{-4}[u_1^2 z_2^2] + \mu^2 t_1^{-4}[u_1 z_2^2 x_1] \end{cases}$$

We see that for an element to be in the kernel of d_3^{33}, we need $\lambda + \lambda^2$ and $\mu + \mu^2$ to be 0. Therefore, both λ and μ can be either 0 or 1, so that the kernel is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$, irrespective of k.

The relevant portion of E_4-page is now in Fig. 1.25. There are no further differentials which may influence the line $t - s = 0$. As in the case of Q_8, there's room for a nontrivial extension problem, which is resolved by observing the 4-fold periodicity of the Tate cohomology of Q_{2^n}. We thus conclude the following result.

Theorem 1.4.8. The Picard group of $\mathsf{StMod}(kQ_{2^n})$, where $n \geq 4$, is given by $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ for all fields k of characteristic 2.

Remark 1.4.9. To compute the group of endotrivial modules of the generalised quaternion groups, Carlson–Thévenaz rely on the computation for Q_8. More precisely, they prove the following result: Let G be a non-cyclic p-group, and let $\mathcal{E}$ denote the family of subgroups H such that H is an extraspecial 2-group that is not isomorphic to D_8, or an almost extraspecial 2-group, or an elementary abelian group of rank 2. Then the restriction map

$$\mathrm{Res}\colon \operatorname{Pic}\mathsf{StMod}(kG) \to \prod_{H\in\mathcal{E}} \operatorname{Pic}\mathsf{StMod}(kH)$$

is injective. They then apply it to Q_{2^n}: noting that Q_8 naturally sits in Q_{2^n} as a subgroup, they study this restriction map to explicitly construct the endotrivial modules for Q_{2^n}. In contrast, with our method, the computations for the generalised quaternion groups are independent of the computations for Q_8.

1.A Tate cohomology of Cohen–Macaulay groups

The goal of this section is to describe the multiplicative structure of the Tate cohomology groups $\widehat{H}^*(G;k)$ for a class of p-groups G that we call Cohen–Macaulay groups. We will specifically be interested in the case that G is elementary abelian or a dihedral 2-group. The methods presented in this section are certainly not new or original, but are necessary for our spectral sequence calculations.

Recall that a Noetherian local ring R is called Cohen–Macaulay if its depth and Krull dimension coincide. Upon globalising, a Noetherian ring R is called Cohen–Macaulay if its localisations are Cohen–Macaulay. We shall say that a p-group G is Cohen–Macaulay if its cohomology ring $H^*(G;k)$ is a Cohen–Macaulay ring.

Let G be a Cohen–Macaulay group. By Noether normalisation, there exists a graded polynomial subring $A = k[x_1, \dots, x_n]$ of $R = H^*(G;k)$ such that R is finitely generated over A. By Hironaka's criterion, also known as miracle flatness [Vak17, Thm. 26.2.10], R is flat, and hence free, over A. The converse holds as well: if we can realise R as a free module over a polynomial ring $k[x_1, \dots, x_n]$, then G is Cohen–Macaulay.

Example 1.A.1. The elementary abelian p-groups and the dihedral 2-groups are Cohen–Macaulay. This can be seen simply by inspecting their cohomology rings, which we had computed in Section 1.4,

Remark 1.A.2. The depth and dimension of $H^*(G;k)$ are closely related to the structure of the elementary abelian p-subgroups of G. For instance, a famous theorem of Quillen says that the Krull dimension of $H^*(G;k)$ is equal to the maximal p-rank of any elementary abelian p-subgroup of G, and a theorem of Duflot says that the depth of $H^*(G;k)$ is greater than or equal to the largest p-rank of any *central* elementary abelian p-subgroup. This need not imply that the p-rank and the central p-rank coincide, even when G is Cohen–Macaulay: the dihedral 2-groups have p-rank 2 but central p-rank 1.

To study the Tate cohomology rings of Cohen–Macaulay groups, we will interpret the Tate fixed points k^{tG} as a Čech spectrum, which allows us to do computations using the so-called Čech cohomology spectral sequence. By the Cohen–Macaulay condition, this spectral sequence will then drastically simplify.

We start off with some definitions. Let R be an $\mathbb{E}_\infty$-ring spectrum. Given $x_1 \in \pi_* R$, we define the Koszul spectrum $K(x_1)$ as the fibre of the inclusion map $R \to R[x_1^{-1}]$. If $I = (x_1, \dots, x_n)$ in $\pi_* R$ is a finitely generated ideal, then we define the Koszul spectrum $K(I)$ as a tensor product $K(x_1) \otimes_R \cdots \otimes_R K(x_n)$. Up to homotopy, this construction depends only on the radical of the ideal I. We also define the Čech spectrum $R[I^{-1}]$ to be the cofibre $K(I) \to R \to R[I^{-1}]$.

Remark 1.A.3. We may regard $R[I^{-1}]$ as the localisation away from the ideal I. Note that if $I = (x_1)$ is principal, then the Čech spectrum $R[I^{-1}]$ is precisely $R[x_1^{-1}]$. However, for an arbitrary finitely generated ideal $I = (x_1, \dots, x_n)$, $R[I^{-1}]$ is generally not the same as the localization at a multiplicatively closed subset of $\pi_* R$, cf. [GM95, Thm. 5.1].

The Tate fixed points k^{tG} of a p-group G may be identified with a Čech spectrum, thanks to the following theorem of Greenlees.

Theorem 1.A.4 ([Gre95, Thm. 4.1])**.** Let G be a p-group acting trivially on the Eilenberg–MacLane spectrum k. Let $R = k^{hG}$ be the homotopy fixed points, so that $\pi_{-*}(R) \simeq H^*(G;k)$. Define I to be the augmentation ideal $I = \ker(H^*(G;k) \to k)$. Then there is a homotopy equivalence between $R[I^{-1}]$ and k^{tG}.

For a Čech spectrum $R[I^{-1}]$ there exists a Čech cohomology spectral sequence

$$E_2 \simeq \check{C}H_I^{-s,-t}(\pi_* R) \Rightarrow \pi_{s+t} R[I^{-1}],$$

which allows us to compute the homotopy groups of $R[I^{-1}]$ using methods of commutative algebra. Together with Theorem 1.A.4 this produces a spectral sequence computing the Tate cohomology, as well as the cup product, of a p-group G.

If $H^*(G;k)$ is Cohen–Macaulay, then by miracle flatness it is free over a polynomial subalgebra $A \subseteq H^*(G;k)$, say $A = k[\zeta_1, \dots, \zeta_n]$. Note that the radical of the ideal $J = (\zeta_1, \dots, \zeta_n)$ is the ideal I of elements in positive degrees. Since the Čech spectrum of an ideal depends only on the radical, we may identify k^{tG} with $R[J^{-1}]$. Moreover, since R is free over A, it suffices to study the Čech cohomology spectral sequence

$$E_2 \simeq \check{C}H^*_{(\zeta_1,\dots,\zeta_n)}\big(k[\zeta_1, \dots, \zeta_n]\big) \Rightarrow \pi_* A[J^{-1}].$$

It is easy to calculate the E_2-page of this spectral sequence, as we have induced long exact sequences relating Čech cohomology to local cohomology coming from the fibre sequence $K(I) \to R \to R[I^{-1}]$:

Lemma 1.A.5 ([GM95])**.** For an R-module M, we have an exact sequence

$$0 \longrightarrow H_I^0(M) \longrightarrow M \longrightarrow \check{C}H_I^0(M) \longrightarrow H_I^1(M) \longrightarrow 0$$

and an isomorphism

$$H_I^s(M) \simeq \check{C}H_I^{s-1}(M) \qquad \text{for all } s \geq 1.$$

Since the $(\zeta_1, \dots, \zeta_n)$ form a regular sequence in $A = k[\zeta_1, \dots, \zeta_n]$, the local cohomology groups $H_J^j(A)$ vanish for all j away from 0 and $n = \dim A$. Therefore the E_2-page of our spectral sequence is concentrated in the two rows $s = 0$ and $s = n-1$, where we have $\check{C}H_J^0(A) \simeq A$, and $\check{C}H_J^{n-1}(A) \simeq k[x_1^{-1}, \dots, x_n^{-1}]$. Moreover, the multiplication in the spectral sequence allows us to recover the multiplication structure on $\pi_* A[J^{-1}]$. Finally, to compute the cohomology ring of $R[I^{-1}]$, notice that $\pi_* k^{tG}$ is free over $\pi_* A[J^{-1}]$ so we may tensor up the spectral sequence to $\pi_* k^{tG}$ without exactness issues.

Now let's see these principles in action, starting with the elementary abelian groups. Recall that

$$H^*\big((C_p)^n;k\big) \simeq \begin{cases} k[x_1, \dots, x_n] & \text{if } p = 2; \\ k[x_1, \dots, x_n] \otimes \Lambda(y_1, \dots, y_n) & \text{if } p \text{ is odd.} \end{cases}$$

If $p = 2$ then the generators have degree 1, while if p is odd then $|x_i| = 2$ and $|y_i| = 1$. Notice that the cohomology ring is free over the polynomial ring $A = k[x_1, \dots, x_n]$; in fact the cohomology ring A identifies with $\pi_* k^{h\mathbb{T}^n}$, and the Čech cohomology spectral sequence may be expressed as

$$E_2 \simeq \check{C}H_I^*(\pi_* k^{h\mathbb{T}^n}) \Rightarrow \pi_* k^{t\mathbb{T}^n}.$$

As before, the E_2-page of this spectral sequence is concentrated in two rows, at $s = 0$ and $s = n-1$. In these rows, we have $\check{C}H_I^0(\pi_* k^{h\mathbb{T}^n}) \simeq \pi_* k^{h\mathbb{T}^n}$ and $\check{C}H_I^{n-1}(\pi_* k^{h\mathbb{T}^n}) \simeq k[x_1^{-1}, \dots, x_n^{-1}]$, shifted in t-degree by n for $p = 2$ (Fig. 1.26), or $2n$ for p odd (Fig. 1.27).

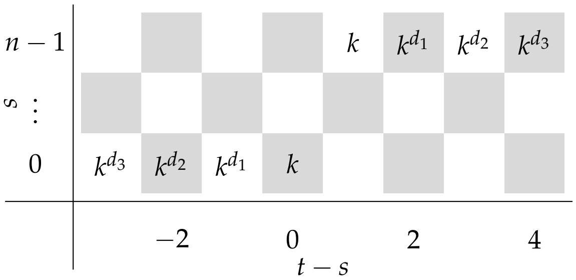


Figure 1.26: The Adams-graded E_2-page of the Čech cohomology spectral sequence computing $\pi_* k^{t\mathbb{T}^n}$, with $p = 2$. Here $d_i = \dim \pi_i k^{h\mathbb{T}^n} = \binom{n-1+i}{n-1}$.

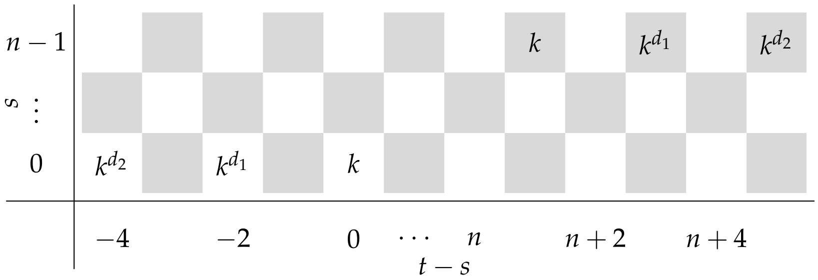


Figure 1.27: The Adams-graded E_2-page of the Čech cohomology spectral sequence computing $\pi_* k^{t\mathbb{T}^n}$, with p odd. Here $d_i = \dim \pi_{2i} k^{h\mathbb{T}^n} = \binom{n-1+i}{n-1}$.

The spectral sequence collapses for degree reasons. Note that, unless $p = 2$ and $n = 1$, there is nowhere for the product of two elements in positive degree to land, and so all products must be zero. This determines the multiplicative structure on $\pi_* k^{t\mathbb{T}^n}$.

In turn, this allows us to infer what the multiplication in $\widehat{H}^*\big((C_p)^n; k\big)$ should be. For $p = 2$, there is nothing to do, since the group cohomology ring is isomorphic to the polynomial algebra $\pi_{-*} k^{h\mathbb{T}^n}$. For concreteness, we illustrate the cup product structure of the Tate cohomology of $(C_2)^2$ in Fig. 1.28. For p odd, we need to tensor the Čech cohomology spectral sequence computing $\pi_* k^{t\mathbb{T}^n}$ with the exterior algebra $\Lambda(y_1, \ldots, y_n)$ where y_i is represented in Adams degree $(-1, 0)$ so as to obtain the multiplication in $\widehat{H}^*\big((C_p)^n; k\big)$.

Now let's turn our attention to the dihedral 2-groups. Recall that for $n \geq 3$, $H^*(D_{2^n}; k) \simeq k[x_1, x_2, z]/(x_1 x_2)$, where $|x_i| = 1$ and $|z| = 2$. Moreover, $\mathrm{Sq}^1(z) = (x_1 + x_2)z$. Writing $u = x_1 + x_2$, we observe that $H^*(D_{2^n}; k)$ is Cohen–Macaulay with ideal $I = (u, z)$. We take A to be $k[u, z]$, and we therefore obtain the Čech cohomology spectral sequence as illustrated in Fig. 1.29. Notice that the spectral sequence has no differentials. As such, the multiplication in $\pi_* A[I^{-1}]$ may be described as follows. In negative degrees, multiplication in $\pi_* A[I^{-1}]$ is the same as multiplication in $k[u, z]$. In positive degrees, we have a class α which generates the algebra $\check{C}H^{n-1}_J(A) \simeq k[u^{-1}, z^{-1}]$. One has $\alpha \cup u = \alpha \cup z = 0$ and $\alpha \cup \alpha = 0$.

To calculate the multiplicative structure of $\widehat{H}^*(D_{2^n}; k)$, we tensor the spectral sequence with the exterior algebra $\Lambda(x_1)$, where x_1 is represented in Adams degree $(-1, 0)$. The resulting spectral sequence has been illustrated in Fig. 1.30.

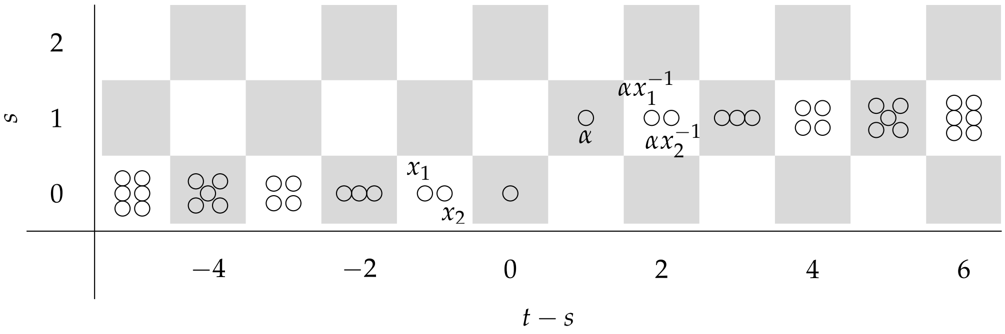


Figure 1.28: The Adams-graded E_2-page of the Čech cohomology spectral sequence computing $\pi_* \, k^{t(C_2)^2}$. All multiplicative generators have been labelled explicitly.

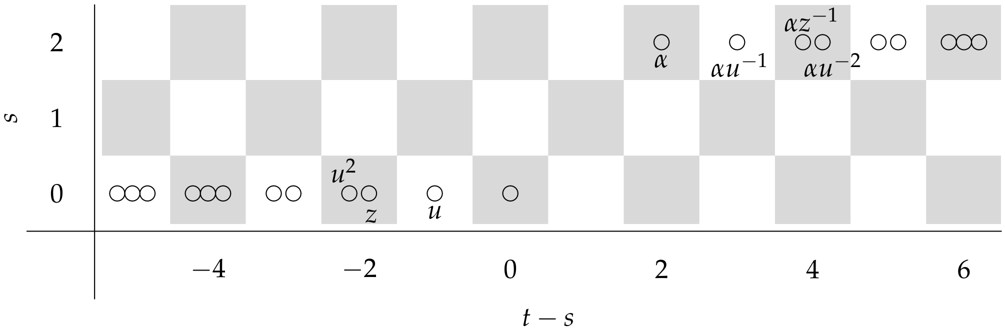


Figure 1.29: The Adams-graded E_2-page of the Čech cohomology spectral sequence computing $\pi_* \, A[I^{-1}]$, where $A \simeq k[u,z]$. There are no differentials and the spectral sequence collapses.

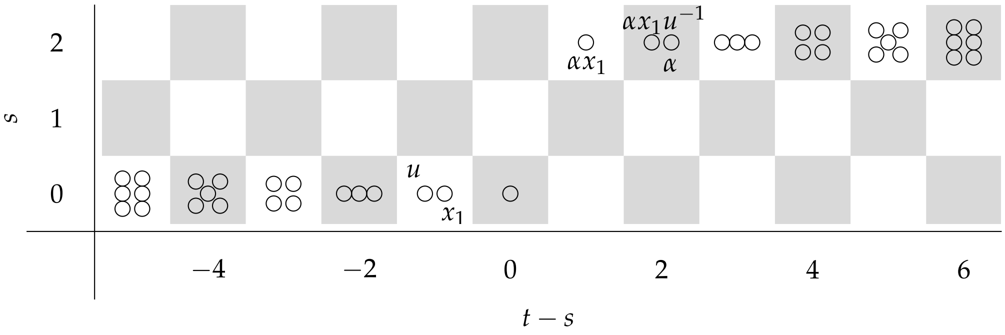


Figure 1.30: The Adams-graded E_2-page of the Čech cohomology spectral sequence computing $\pi_* \left(k^{tD_{2^n}}\right)$. There are no differentials again.

Chapter 2

On endotrivial modules of extraspecial p-groups

Abstract. We perform homotopical computations of the group of endotrivial modules for general finite groups. Our emphasis will lie on the extraspecial groups, which have traditionally played a fundamental role in the theory of endotrivial modules. We analyse the Picard spectral sequence for the extraspecial groups and show that the E_2-page inherits a great deal of structure from a certain Tits building of isotropic subspaces with respect to a quadratic form.

2.1 Extraspecial p-groups

The goal of this section is to take a closer look at the nature of extraspecial p-groups, particularly their subgroup structure. We might as well start with their definition: A p-group G is called extraspecial if its centre $Z(G)$ is cyclic of order p, and the quotient $G/Z(G)$ is a nontrivial elementary abelian p-group. For instance, if $p = 2$ then the smallest extraspecial groups are Q_8 and D_8.

Lemma 2.1.1. *In an extraspecial p-group G, one has $[G, G] = Z(G)$.*

Proof. Since $G/Z(G)$ is abelian, $[G, G]$ is contained within $Z(G)$. As $Z(G)$ is cyclic of prime order, either $[G, G]$ is trivial or it's all of $Z(G)$. It cannot be trivial, because then all of G is abelian and hence $G/Z(G)$ is trivial, which was assumed not to be the case. ☐

We shall regard $Z(G)$ as the underlying group of $\mathbb{F}_p$, and $G/Z(G)$ as the underlying group of an $\mathbb{F}_p$-vector space V. We nonetheless choose to stick with multiplicative notation.

The structure of G is captured by a bilinear form $B\colon V \times V \to \mathbb{F}_p$, which we define as follows. Take a pair $(\overline{x}, \overline{y}) \in V \times V$, lift it to a pair (x, y) in $G \times G$, and let $B(\overline{x}, \overline{y})$ be the commutator $[x, y]$, which by Lemma 2.1.1 lives in $\mathbb{F}_p$.

Lemma 2.1.2. *B is a well-defined map. It is a nondegenerate alternating (and hence skew-symmetric) bilinear form on V.*

From this lemma, general linear algebra forces the rank of V to be even, from which we infer that every extraspecial p-group has order p^{1+2n} for some $n \geq 1$.

Proof. We first prove that B is well defined. Let $x, y \in G$ and let $z \in Z$. We show that $[xz, y] = [x, y]$. The proof that $[x, yz] = [x, y]$ is similar. For all $x, y, z \in G$ for all groups G, one has the identity $[xz, y] = [x, y]^z[z, y]$, but by Lemma 2.1.1, $[G, G]$ is abelian and therefore the identity simplifies to

$$[xz, y] = [x, y][z, y] \qquad \text{for all } x, y, z \in G.$$

Clearly if z is in the centre then $[z, y] = e$ and we're done.

The identity above immediately shows that B is bilinear. It is alternating for obvious reasons. To prove nondegeneracy, simply observe the fact that $[x, g] = 0$ for all $g \in G$ means precisely that x is in the centre. □

To understand the Picard group of the stable module category $\mathsf{StMod}(kG)$ of an extraspecial p-group, we would like to apply Theorem 1.2.7 for a family of subgroups $\mathcal{A}$. As the full subgroup structure of an extraspecial group is complicated, it is important to make a clever choice for the family $\mathcal{A}$. We choose to take $\mathcal{A}$ to be the collection of elementary abelian p-subgroups of G containing $Z(G)$. This family is closed under intersection and conjugation, and every elementary abelian p-subgroup P of G is contained within a p-subgroup that also contains $Z(G)$, namely the internal direct product $P\,Z(G)$.

We proceed to investigate the structure of $\mathcal{A}$. Generally speeaking, subgroups of G containing $Z(G)$ are in correspondence with subgroups of $G/Z(G)$, and the *abelian* subgroups containing $Z(G)$ correspond to those subgroups of $G/Z(G)$ on which the bilinear form B vanishes. To recognise which of these lift to an *elementary* abelian subgroup of G, it helps to define an additional structure map $Q\colon V \to \mathbb{F}_p$ as follows. Start with an element $\overline{x} \in V$, choose a lift to an element $x \in G$, and let $Q(\overline{x})$ be x^p. The elementary abelian subgroups of G in $\mathcal{A}$ are then precisely those subgroups of $G/Z(G)$ on which both B and Q vanish identically.

Lemma 2.1.3. The map Q is well defined. If $p = 2$ then it is a quadratic form, while if p is odd, it is $\mathbb{F}_p$-linear.

Proof. Take $x \in G$ and $z \in Z(G)$. We wish to show that $x^p = (xz)^p$. Since z lives in the centre, we can move all the z's in the expression to the right, so that $(xz)^p = x^p z^p$. Now since $Z(G)$ is cyclic of order p, z^p must be trivial, and hence we're done.

To prove that Q is quadratic or bilinear, we shall evaluate $Q(xy)$ for $x, y \in G$. Let us verify the identity

$$(xy)^n = x^n y^n [y, x]^{n(n-1)/2},$$

which in fact holds for any nilpotent group G of nilpotence class at most 2. To prove this identity, write

$$\begin{aligned} y^{-n}x^{-n}(xy)^n &= y^{-n}x^{-n+1}y(xy)^{n-1} \\ &= y^{-n}x^{-n+2}y^2(xy)^{n-2}[y, x] \\ &= y^{-n}x^{-n+3}y^3(xy)^{n-3}[y, x]^3 \\ &= \cdots = y^{-n}x^{-n+n}y^n(xy)^{n-n}[y, x]^{1+2+\cdots+(n-1)} \\ &= [y, x]^{n(n-1)/2} \end{aligned}$$

In each step, we move $[y, x]$ to the far right, which we're allowed to do, because $[G, G]$ is assumed to be contained in $Z(G)$.

Apply the identity to the case where $n = p$. If $p = 2$, then

$$Q(xy) = Q(x)\,Q(y)\,B(x,y).$$

The bilinear form $B(x,y)$ was alternating and hence antisymmetric. In characteristic 2, we may as well say that $B(x,y)$ is symmetric. Consequently, Q is a nondegenerate quadratic form on V. On the other hand, if p is odd, then $p(p-1)/2$ is a multiple of p, and hence $[y,x]^{n(n-1)/2}$ becomes trivial, from which we infer that

$$Q(xy) = Q(x)\,Q(y),$$

as desired. ☐

Remark 2.1.4. Alternatively, extraspecial groups may be recognised as certain central extensions of the elementary abelian group C_p^n by C_p. Central extensions are classified by the second cohomology, and the structure maps B and Q that we uncovered can be expressed in terms of 2-cocycles.

We have managed to classify the family $\mathcal{A}$ entirely in terms of linear algebra. Let's now make things a bit more concrete. We will henceforth focus on the case $p = 2$, reducing the case of odd primes to a remark at the end of the section.

If $p = 2$, we have the identity $B(x,y) = Q(xy)Q(x)^{-1}Q(y)^{-1}$, and so if Q vanishes uniformly on a subspace of V, then so does B. As such, the subgroups in $\mathcal{A}$ are in one-to-one correspondence with the so-called totally isotropic subspaces of V with respect to the nondegenerate quadratic bilinear form Q. It is a well-known fact that, for every each even dimension $2n$, there are only two such quadratic forms $\mathbb{F}_2^{2n} \to \mathbb{F}_2$ up to isomorphism, distinguished by their Arf invariants. We may define them explicitly as

$$\begin{aligned} Q_+(x) &= x_1x_2 + \cdots + x_{2n-1}x_{2n}, \\ Q_-(x) &= x_1x_2 + \cdots + x_{2n-1}x_{2n} + x_1 + x_2. \end{aligned}$$

This gives us a way to list all objects in $\mathcal{O}_{\mathcal{A}}$. At least in principle, that is — a computer-aided computation indicates that the number of totally isotropic subspaces increases exponentially in n. As such, it would be infeasible to brute-force compute resolutions.

Although the orbit category $\mathcal{O}_{\mathcal{A}}$ has many objects, its morphisms are nicely structured, and this will be of great help to us. Observe that, is a normal subgroup of G with quotient $\overline{G}$, and H and K are subgroups of G containing N, then

$$\mathrm{Hom}_G(G/H, G/K) \simeq \mathrm{Hom}_{\overline{G}}\left(\overline{G}/\overline{H}, \overline{G}/\overline{K}\right);$$

in particular, applied to the case where G is extraspecial and $N = Z(G)$, we see that the morphisms in $\mathcal{O}_{\mathcal{A}}$ can be computed by passing to the orbit category of the quotient $G/Z(G)$. This quotient is abelian. For a general abelian group A with subgroups H_1 and H_2, one has

$$\mathrm{Hom}_A(A/H_1, A/H_2) \simeq \begin{cases} A/H_2 & \text{if } H_1 \subseteq H_2; \\ \varnothing & \text{otherwise.} \end{cases}$$

We thus find the following down-to-earth description of $\mathcal{O}_{\mathcal{A}}$ for an extraspecial 2-group G. Write Q for the quadratic form on the $\mathbb{F}_2$-vector space $V = G/Z(G)$. Then $\mathcal{O}_{\mathcal{A}}$ has as objects all totally isotropic subspaces W of V, and

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{A}}}(W', W) = \begin{cases} V/W & \text{if } W' \subseteq W; \\ \varnothing & \text{otherwise.} \end{cases}$$

Let T be the category whose objects are all totally isotropic subspaces W and V, and where now

$$\mathrm{Hom}_T(W', W) = \begin{cases} * & \text{if } W' \subseteq W; \\ \varnothing & \text{otherwise.} \end{cases}$$

We remark that T may be regarded as a sort of Tits building for the subspace V equipped with the quadratic form Q. Now, there's a functor $F\colon \mathcal{O}_{\mathcal{A}} \to T$ sending every object W to itself, and as such, by the results of Section 2.A, there's a Grothendieck spectral sequence

$$E_2^{pq} = H^p\big(T; W \mapsto H^q(F \downarrow W; \mathcal{F})\big) \Rightarrow H^{p+q}(\mathcal{O}_{\mathcal{A}}; \mathcal{F}).$$

Fixing a subspace W, we now consider the comma category $F \downarrow W$. We observe that this category consists of all subspaces U of W, with

$$\mathrm{Hom}_{F\downarrow W}(U', U) = \begin{cases} V/U & \text{if } U' \subseteq U; \\ \varnothing & \text{otherwise.} \end{cases}$$

Notice that for all such U and U', there is a direct sum decomposition $V/U = (V/W) \oplus (W/U)$: the natural quotient map $V/U \to V/W$ induces a further functor $G_W\colon (F \downarrow W) \to B(V/W)$, whose unique comma category now admits an initial object, namely W, so that the derived limits over any presheaf on this comma category are trivial. Consequently, the Grothendieck spectral sequence applied to G_W degenerates and we conclude that

$$H^*(F \downarrow T, \mathcal{F}) \simeq H^*\big(V/W; \mathcal{F}(W)\big),$$

where the V/W-action on $\mathcal{F}(W)$ are defined by the restriction maps of the presheaf $\mathcal{F}$. The functoriality in T is the expected one: if $W' \to W$ is a map in T, then the map $H^*\big(V/W; \mathcal{F}(W)\big) \to H^*\big(V/W'; \mathcal{F}(W')\big)$ may be specified in terms of the quotient map $V/W' \to V/W$ and the map $W' \to W$ in $\mathcal{O}_{\mathcal{A}}$ corresponding to $0 \in V/W$. In summary, the spectral sequence for computing derived limits over $\mathcal{O}_{\mathcal{A}}$ simplifies to become

$$E_2^{pq} = H^p\big(T; W \mapsto H^q(V/W; \mathcal{F}(W))\big) \Rightarrow H^{p+q}(\mathcal{O}_{\mathcal{A}}; \mathcal{F}).$$

Remark 2.1.5. Our category T is acted on in an obvious way by the subgroup $\mathrm{O}_{2n}^{\pm}(\mathbb{F}_2)$ of the general linear group of $V = G/Z(G) \simeq \mathbb{F}_2^{2n}$ consisting of those linear maps which preserve the quadratic form $Q_{\pm}$. This action induces an action on $\mathsf{StMod}(kG)$ (as well as on $D^b(kG)$, cf. [Mat16, Prop. 9.13]). It would be interesting to know if there is a representation-theoretic meaning of this action.

In principle, our spectral sequence need not be of much help. An inductive argument involving the Grothendieck spectral sequence shows that if P is a poset with a chain $c_0 < \cdots < c_n$ of length n, then $H^*(P; \mathcal{F})$ has the potential to be nonzero for $* = 0, \ldots, n$. In particular, the E_2-page of the spectral sequence above can have numerous nonzero vertical lines, and exhibit nontrivial differentials. Fortunately for us, we will see that when $\mathcal{F} = \pi_t\, \mathfrak{pic}\, \mathsf{StMod}(k[\,\cdot\,])$, the cohomologies simplify quite drastically.

Let's first look at the case $t = 0$. By virtue of Lemma 1.2.14, $\mathcal{F}(W)$ has value $\mathbb{Z}$ unless $W = 0$, in which case the value becomes 0. The functoriality of $\mathcal{F}$ is easy to determine: the generator of the Picard group is given by the suspension of the unit, and every morphism in $\mathcal{F}$ comes from an underlying functor of stable module categories which is symmetric monoidal and exact by design, so for every morphism $W' \to W$, the induced map $\mathcal{F}(W) \to \mathcal{F}(W')$ is the identity — unless of course $W' = 0$ and $W \neq 0$. In view of this, we can describe the cohomology groups as $H^s(\mathcal{O}_{\mathcal{A}\setminus 0}(G); \mathbb{Z})$. Only the case $s = 0$ is relevant to the development of the limit spectral sequence; in that case, computing the group is a matter of counting the path components of $\mathcal{O}_{\mathcal{A}\setminus 0}(G)$. In view of our Grothendieck spectral sequence, we can equivalently choose to count the path components of $T \setminus 0$.

Remark 2.1.6. At a first glance, an obvious source of path components is maximal 1-dimensional totally isotropic subspaces, corresponding to maximal elementary abelian groups of rank 2, taken up to G-conjugacy. Maximal rank-2 elementary abelians are known to play a distinguished role in the study of torsionfree endotrivial modules — and we'll come back to this point in Section 2.3.

We will skip the case $t = 1$, instead treating it on an ad hoc basis in the case of D_8 in the next section.

Consider the case $t \geq 2$. Again something fortunate happens in this case: if $W' \to W$ is a map in T, corresponding to an inclusion $H \hookrightarrow K$ of (necessarily normal) subgroups of G, then the induced map $H^*(V/W; \mathcal{F}) \to H^*(V/W'; \mathcal{F})$ is *always* 0. This is because the associated restriction map on $\mathcal{F}$ amounts to taking the transfer homomorphism on homology with respect to the inclusion $H \hookrightarrow K$. As all groups are p-groups, the index of H in K is a power of p, and consequently the transfer map vanishes. The functoriality in T is therefore trivial, and our Grothendieck spectral sequence easily lets us find that

$$H^s(\mathcal{O}_{\mathcal{A}}; \mathcal{F}) \simeq \left(\bigoplus_P H^s\big(G/P; \widehat{H}^{1-t}(P;k)\big) \right) \oplus H^{s-1}\big(G/Z(G); \widehat{H}^{1-t}(Z(G);k)\big), \qquad (2.1.7)$$

where P ranges over the *maximal* elementary abelian subgroups of G. The G/P-action on $\widehat{H}^{1-t}(P;k)$ is induced by conjugation, and is usually nontrivial. We'll refer to all but the last component as the noncentral component, and to the last component as the central component.

Remark 2.1.8. The trivial functoriality in T is nothing specific to extraspecial p-groups; we merely used the fact that G is a p-group to argue that the subgroup inclusion have index a power of p, which is enough to conclude that the transfer maps vanish.

As before, by Theorem 1.2.11 there is a natural isomorphism between the cohomology groups

$$H^s\big(\mathcal{O}_{\mathcal{A}}; \pi_t \,\mathfrak{pic}\,\mathsf{StMod}(k[\,\cdot\,])\big) \simeq H^s\big(\mathcal{O}_{\mathcal{A}}; \pi_{t-1}\,\Omega\,\mathsf{StMod}(k[\,\cdot\,])\big)$$

inducing comparison maps between their respective descent spectral sequences, and the $\mathbb{E}_\infty$-structure of the endomorphism ring of the unit endows the spectral sequence corresponding to the right-hand side with a cup product structure.

We are led to consider the product structure of

$$E_2^{st} = H^s\big(\mathcal{O}_{\mathcal{A}}; \pi_t\,\Omega\,\mathsf{StMod}(k[\,\cdot\,])\big).$$

In view of Eq. (2.1.7), it is tempting to suspect that the products somehow distribute over the summands in an obvious way. This is not the case, however. For one, the decomposition in Eq. (2.1.7) fails to hold for negative homotopy groups of $\Omega\,\mathsf{StMod}\big(k[\,\cdot\,]\big)$ since the functoriality in T is a (nontrivial) map on cohomology rather than a (trivial) transfer map.

Partial information about the cup product structures may be gleaned from various comparison maps. Consider, for instance, the inclusion $\iota\colon \mathcal{M} \hookrightarrow \mathcal{O}_{\mathcal{A}}$ of the full subcategory $\mathcal{M}$ spanned by the maximal objects. For any presheaf $\mathcal{F}$, there is a natural map $\iota^*\colon H^*\big(\mathcal{O}_{\mathcal{A}};\mathcal{F}\big) \to H^*\big(\mathcal{M};\mathcal{F}\circ\iota\big)$. Now take $\mathcal{F}$ to be $\pi_t\,\Omega\,\mathsf{StMod}\big(k[\,\cdot\,]\big)$ for varying t. Then ι^* preserves cup products. For $t \geq 2$, ι^* is the projection onto the noncentral component of Eq. (2.1.7). This lets us conclude that, when restricting attention to the noncentral component, the cup product can effectively be computed on the separate summands. However, we crucially cannot exclude the possibility of products of elements in the noncentral component to land in the central component.

Similarly we may consider the functor $\pi\colon \mathcal{O}_{\mathcal{A}} \to \mathcal{Z}$ where $\mathcal{Z}$ is the subcategory consisting of the same objects as $\mathcal{O}_{\mathcal{A}}$ but with the endomorphisms of the maximal objects taken out. Derived limits over $\mathcal{Z}$ are described by $G/Z(G)$-cohomology shifted in degree. Owing to this degree shift, cup products of derived limits over $\mathcal{Z}$ are trivial, and by mapping to the derived limit over $\mathcal{O}_{\mathcal{A}}$, we infer that cup products on the central component are trivial.

We can use the same comparison maps $\iota\colon \mathcal{M} \hookrightarrow \mathcal{O}_{\mathcal{A}}$ and $\pi\colon \mathcal{O}_{\mathcal{A}} \to \mathcal{Z}$ to infer partial information about the differentials of the limit spectral sequence. We learn that, upon restricting to a given summand in Eq. (2.1.7), the differentials can be understood in the expected manner: take the Hochschild–Serre spectral sequences

$$E_2^{st} = H^s\big(G/P; H^t(P;k)\big) \Rightarrow H^{s+t}(G;k)$$

for maximal abelian P, and

$$H^s\big(G/Z(G); H^t(Z(G);k)\big) \Rightarrow H^{s+t}(G;k)$$

with $Z(G)$ the centre of G, and proceed to reflect in the t-axis using Tate duality pairings to understand the differentials for the negative Tate cohomology groups. As before, however, we cannot rule out that the differentials out of the noncentral components may have nontrivial values in the central component.

Remark 2.1.9. Some care must be taken to ensure that we can really extend the spectral sequence in a natural way to negative Tate cohomology groups. The Tate cohomology groups for these spectral sequences are $\widehat{H}^*(C_2^n;k)$, which by Section 1.A has numerous zero divisors in negative degrees. This is in contrast to the ring structure on $\widehat{H}^*(C_p;k)$ — the only Tate cohomology ring we had considered thus far. Nonetheless, this will not cause any problems.

Remark 2.1.10. We briefly consider the case where p is odd. By Lemma 2.1.3, Q is now an $\mathbb{F}_p$-linear map. Up to isomorphism, it is either zero or nonzero. We can choose a basis of V with respect to which B is the symplectic map

$$B(x,y) = (x_1y_{n+1} + \cdots + x_ny_{2n}) - (x_{n+1}y_1 + \cdots + x_{2n}y_n),$$

and Q is one of the two maps

$$\begin{aligned} Q_+(x) &= x_1 + \cdots + x_{2n} \\ Q_-(x) &= 0. \end{aligned}$$

The elementary abelian subgroups of G containing $Z(G)$ are now classified by those subspaces on which both B and Q vanish. As before, we take a functor to a 'Tits building' and run a Grothendieck spectral sequence. The subsequent methods and results are similar to the case $p = 2$.

2.2 The case of D_8

Now let's apply the generalities developed in the previous section to D_8. The subgroup structure of D_8 is rather complicated:

$$\begin{tikzcd}
 & & D_8 & & \\
 & \langle r^2 s, s\rangle \arrow[ur] & \langle r\rangle \arrow[u] & \langle rs, r^3 s\rangle \arrow[ul] & \\
\langle r^2 s\rangle \arrow[ur] & \langle s\rangle \arrow[u] & \langle r^2\rangle \arrow[ul] \arrow[u] \arrow[ur] & \langle r^3 s\rangle \arrow[u] & \langle rs\rangle \arrow[ul] \\
 & & \{e\} \arrow[ull] \arrow[ul] \arrow[u] \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

Apart from $\langle r\rangle$, all nontrivial subgroups are elementary abelian. But it luckily suffices to take $\mathcal{A}$ to be the collection of subgroups containing the centre $Z(D_8) = \langle r^2\rangle$. The resulting orbit category is

$$\mathcal{O}_{\mathcal{A}} = \begin{tikzcd}
\langle r^2 s, s\rangle \arrow[loop above, "\{e,r\}"] & & \langle rs, r^3 s\rangle \arrow[loop above, "\{e,r\}"'] \\
 & \langle r^2\rangle \arrow[ul, shift left, "e"] \arrow[ul, shift right, "r"'] \arrow[ur, shift right, "e"'] \arrow[ur, shift left, "r"] \arrow[loop below, "\{e,r,s,rs\}"] &
\end{tikzcd}$$

As expected from the previous section, this is precisely the category of totally isotropic subspaces W of $V = D_8/Z(D_8) \simeq \mathbb{F}_2^2$ with respect to the quadratic form $Q_+(x) = x_1x_2$, with $\mathrm{Hom}(W', W)$ given by either V/W or $\varnothing$.

Remark 2.2.1. The case of Q_8 fits within this pattern as well. V is 2-dimensional again, but this time, one has $Q(x) = x_1x_2 + x_1 + x_2$, which has just a single totally isotropic subspace, namely 0, corresponding to the unique elementary abelian subgroup $\{\pm 1\}$ of Q_8. It may therefore be regarded as a 'degenerate' case which might as well be treated separately the way we did.

We consider the functor $F\colon \mathcal{O}_{\mathcal{A}} \to T$ as before, where T is now the category

$$T = \begin{tikzcd}
\langle r^2 s, s\rangle & & \langle rs, r^3 s\rangle \\
 & \langle r^2\rangle \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

We have a spectral sequence

$$E_2^{pq} = H^p\Big(T; W \mapsto H^q(V/W; \mathcal{F}(W))\Big) \Rightarrow H^{p+q}(\mathcal{O}_{\mathcal{A}}; \mathcal{F})$$

for computing cohomology over our orbit category in terms of cohomology over T. This spectral sequence collapses, as its only vertical lines are on $q = 0$ and $q = 1$, which may be computed to be the kernel and cokernel of the difference $\mathcal{F}(\langle r^2 s, s\rangle) \oplus \mathcal{F}(\langle rs, r^3 s\rangle) \to \mathcal{F}(\langle r^2\rangle)$.

At this point, we have our recipe to understand the terms in the E_2-page of the spectral sequence. We take $\mathcal{F}$ to be $\pi_t\, \mathfrak{pic}\, \mathsf{StMod}\big(k[\,\cdot\,]\big)$. If $t = 0$, then Lemma 1.2.14 tells us that $\mathcal{F}$ sends $\langle r^2, s\rangle$ and $\langle rs, r^2\rangle$ to $\mathbb{Z}$, and $\langle r^2\rangle$ to 0. The functoriality in the endomorphisms is clear: the underlying functors on the level of stable module categories are exact and hence in particular preserve suspension. As the Picard groups are generated by the suspension of the unit, all endomorphisms in $\mathcal{O}_{\mathcal{A}}$ induce identity maps on $\mathcal{F}(\,\cdot\,)$. Consequently,

$$H^*\big(\mathcal{O}_{\mathcal{A}}; \mathcal{F}\big) \simeq H^*(C_2; \mathbb{Z}) \oplus H^*(C_2; \mathbb{Z}),$$

whose terms are easily computed by hand.

If $t = 1$, then $\mathcal{F}$ sends all objects to the group $k^\times$, and all morphisms to the identity map. This yields the following algebraic description of the cohomology of $\mathcal{F}$. For $i = 1, 2$ we let ι_i^* be the map $H^*(C_2 \times C_2; k^\times) \to H^*(C_2; k^\times)$ induced by the inclusion maps into the two components. Then

$$H^*\big(\mathcal{O}_{\mathcal{A}}; \mathcal{F}\big) \simeq \mathrm{Ker}(\iota_1^* - \iota_2^*) \oplus \mathrm{Coker}(\iota_1^{*-1} - \iota_2^{*-1}).$$

In general, the terms are rather complicated. As the coefficient group need not be well behaved, one needs to invoke the general Künneth formula to express $H^*\big((C_2)^2; k^\times\big)$ in terms of $H^*(C_2; k^\times)$, which causes Tor_1-terms to appear. However, we point out that $H^*\big(\mathcal{O}_{\mathcal{A}}; \mathcal{F}\big)$ is always $k^\times$ for $* = 0$, and 0 for $* = 1$. In addition, if k is assumed to be perfect, then the situation simplifies drastically: all the nontrivial cohomology groups $k^\times/(k^\times)^2$ vanish, and as a result all nonzero cohomologies of $\mathcal{F}$ are 0.

For $t \geq 2$, we recall that the trivial functoriality in T induced a decomposition of the cohomology groups over $\mathcal{O}_{\mathcal{A}}$, which in our case reads as

$$\begin{aligned} H^s\big(\mathcal{O}_{\mathcal{A}}; \pi_t\, \mathfrak{pic}\, \mathsf{StMod}(k[\,\cdot\,])\big) \simeq{}& H^s\big(D_8/\langle r^2 s, s\rangle; \widehat{H}^{1-t}(\langle r^2 s, s\rangle)\big) \\ &\oplus H^s\big(D_8/\langle rs, r^3 s\rangle; \widehat{H}^{1-t}(\langle rs, r^3 s\rangle)\big) \\ &\oplus H^{s-1}\big(D_8/\langle r^2\rangle; \widehat{H}^{1-t}(\langle r^2\rangle)\big). \end{aligned} \tag{2.2.2}$$

We remark that the action of the quotient group on the Tate cohomology is nontrivial in the first two cases. Indeed, note that conjugation by a nontrivial representative (in both cases, r) acts nontrivially on the subgroups $\langle r^2 s, s\rangle$ and $\langle rs, r^3 s\rangle$; this induces nontrivial functors on the stable module categories and hence on the homotopy groups of the Picard spectra. In both cases, the action is abstractly isomorphic to the nontrivial C_2-action on $C_2 \times C_2$ obtained by permuting the terms.

To understand differentials, we consider the functors from $\mathcal{M}$ and to $\mathcal{Z}$ the way we outlined in the previous section. Upon restricting to the three summands in Eq. (2.2.2), the differentials in the stable realm are governed by the three Hochschild–Serre spectral sequences

$$\begin{aligned} E_2^{st} &= H^s\big(D_8/\langle r^2 s, s\rangle; H^t(\langle r^2 s, s\rangle)\big) \Rightarrow H^{s+t}(D_8; k), \\ E_2^{st} &= H^s\big(D_8/\langle rs, r^3 s\rangle; H^t(\langle rs, r^3 s\rangle)\big) \Rightarrow H^{s+t}(D_8; k), \\ E_2^{st} &= H^s\big(D_8/\langle r^2\rangle; H^t(\langle r^2\rangle; k)\big) \Rightarrow H^{s+t}(D_8; k). \end{aligned}$$

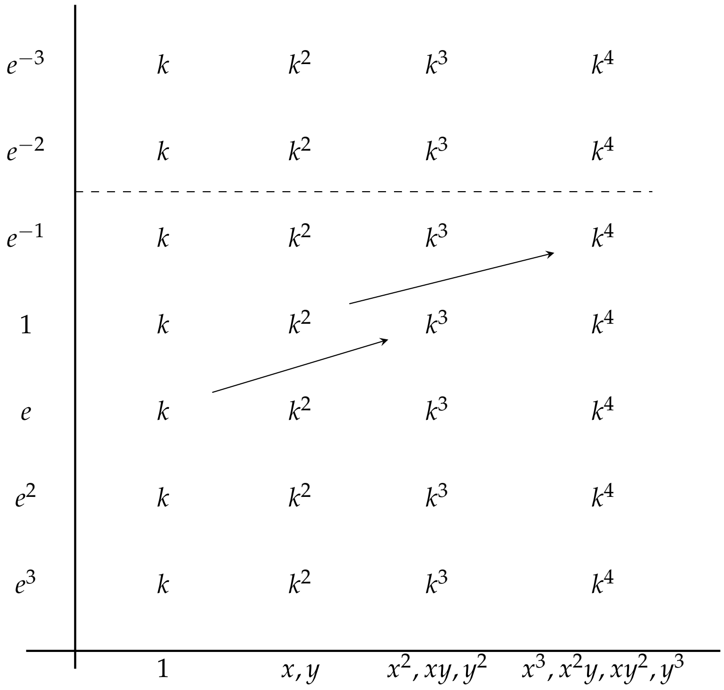


Figure 2.1: Serre-graded (!) E_2-page of the spectral sequence for $\Omega\,\mathsf{StMod}(kD_8)$ induced by the central extension. On the first quadrant, it may also be regarded as the Hochschild–Serre spectral sequence associated to the same extension.

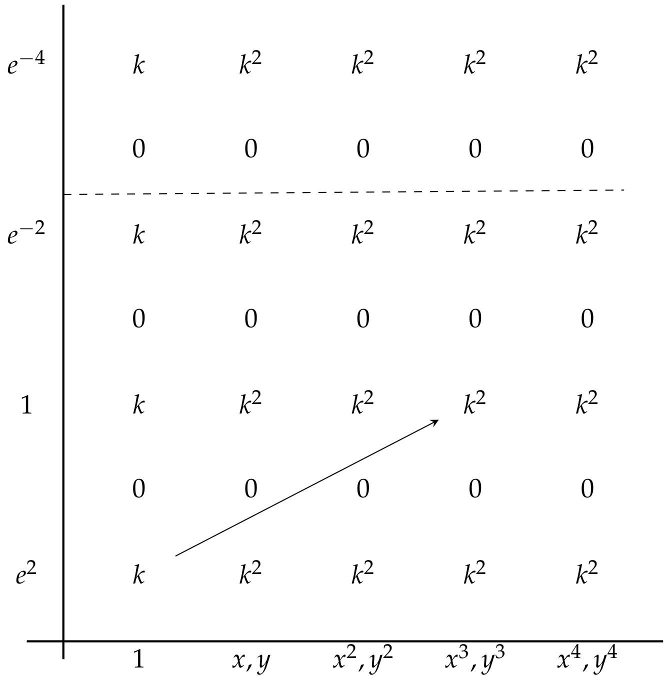
The Hochschild–Serre spectral sequences for D_8 are well known. The first spectral sequence has its E_2-page drawn out on Fig. 2.1. One has $d_2(x) = d_2(y) = 0$, and through comparison with simpler spectral sequences, or computation of transgression, one learns that $d_2(e) = xy$. The E_3-page is sketched in Fig. 2.2, where it collapses.

The other two spectral sequences, coming from the noncentral extensions, are isomorphic to each other, and their E_2-page is sketched on Fig. 2.3. This spectral sequence collapses already on the E_2-page.

We immediately notice something peculiar: On each diagonal line of Fig. 2.3, infinitely many terms survive. For the results to be consistent, this effectively forces differentials from the central extension to kill most of the terms occurring in the two spectral sequences of the noncentral extensions. To this end, we make the following conjecture:

Conjecture 2.2.3. The d_2-differential of the limit spectral sequence for $\mathfrak{pic}\,\mathsf{StMod}(kD_8)$ sends the two generators $u \in H^0\big(D_8/P; \widehat{H}^{-1}(P;k)\big)$ corresponding to the two noncentral extensions of D_8 (cf. Fig. 2.3) to the generators $e^{-2}x$ and $e^{-2}y$ in $H^1\big(D_8/Z(D_8); \widehat{H}^{-2}(Z(D_8);k)\big)$ (cf. Fig. 2.1). The same differential sends the generators $\overline{x^n y^n} \in H^0\big(D_8/P; \widehat{H}^{-2n-1}(P;k)\big)$ of the noncentral component to $e^{-2n-2}x$ and $e^{-2n-2}y$.

Let's assemble what we know so far, and what the conjecture implies. In Fig. 2.4 we have illustrated the E_2-page of the limit spectral sequence of D_8. We have illustrated with blue circles which



e^{-4}	k	k^2	k^2	k^2	k^2
	0	0	0	0	0
e^{-2}	k	k^2	k^2	k^2	k^2
	0	0	0	0	0
1	k	k^2	k^2	k^2	k^2
	0	0	0	0	0
e^2	k	k^2	k^2	k^2	k^2
	1	x, y	x^2, y^2	x^3, y^3	x^4, y^4

Figure 2.2: E_3-page of the spectral sequence for $\Omega\,\mathsf{StMod}(kD_8)$ induced by the central extension. There are no remaining differentials, and $E_3 = E_\infty$.

summands get killed by virtue of nontrivial differentials of the two noncentral E_2-spectral sequences, and we have illustrated with red circles which summands get killed owing to Conjecture 2.2.3. Some additional care must be taken with the unstable differential d_2^{22}: specifically, we must expect the squaring operation to be trivial in order for things to work out. In Fig. 2.5 we have written out the E_3-page of the limit spectral sequence, assuming our hypotheses are correct.

It seems reasonable that there is an ad hoc explanation of the behaviour of the differentials, for instance by arguing that we know what the positive lines should converge to and ruling out other candidate behaviours on a case-by-case basis. Any such method will likely fail to work for groups beyond D_8, owing to the massive increase in complexity. For this reason we shall refrain from even trying.

Rather, we believe that a first step to finding the differentials should be a systematic computation of the cup product structure on the E_2-page of the Ω-spectral sequence — one which naturally lends itself to a generalisation to larger extraspecial groups. It's reasonable, though not guaranteed, that an understanding of the cup product structure will be sufficient to infer Conjecture 2.2.3.

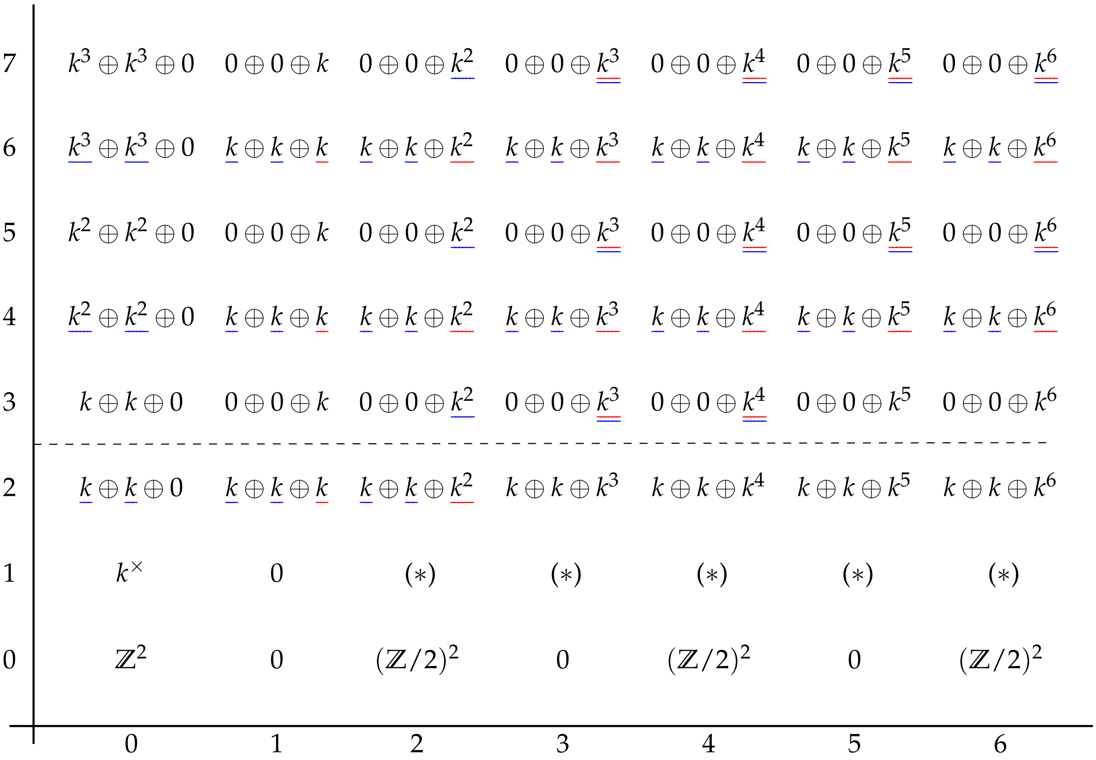
In any case, upon inspecting Fig. 2.5 we arrive at the following result.

Corollary 2.2.4. Assuming Conjecture 2.2.3, the Picard group of $\mathsf{StMod}(kD_8)$ is $\mathbb{Z} \oplus \mathbb{Z}$.

Remark 2.2.5. There is room for nontrivial differentials out of $E_2^{00} = \mathbb{Z} \oplus \mathbb{Z}$. Indeed the targets are completely torsion, and so the kernel of any nontrivial differential would remain abstractly isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. It can be deduced whether such nontrivial differentials occur, and how often:

3	$\langle \overline{x^2+y^2}, \overline{xy} \rangle$	$\langle \overline{xy}e \rangle$	$\langle \overline{xy}e^2 \rangle$	$\langle \overline{xy}e^3 \rangle$	$\langle \overline{xy}e^4 \rangle$	$\langle \overline{xy}e^5 \rangle$
2	$\langle \overline{x+y} \rangle$	0	0	0	0	0
1	$\langle u \rangle$	$\langle ue \rangle$	$\langle ue^2 \rangle$	$\langle ue^3 \rangle$	$\langle ue^4 \rangle$	$\langle ue^5 \rangle$
0	$\langle 1 \rangle$	$\langle e \rangle$	$\langle e^2 \rangle$	$\langle e^3 \rangle$	$\langle e^4 \rangle$	$\langle e^5 \rangle$
-1	$\langle x+y \rangle$	0	0	0	0	0
-2	$\langle x^2+y^2, xy \rangle$	$\langle xye \rangle$	$\langle xye^2 \rangle$	$\langle xye^3 \rangle$	$\langle xye^4 \rangle$	$\langle xye^5 \rangle$
-3	$\langle x^3+y^3, x^2y+xy^2 \rangle$	0	0	0	0	0
-4	$\langle x^4+y^4, x^3y+xy^3, x^2y^2 \rangle$	$\langle x^2y^2e \rangle$	$\langle x^2y^2e \rangle$	$\langle x^2y^2e \rangle$	$\langle x^2y^2e \rangle$	$\langle x^2y^2e \rangle$
	0	1	2	3	4	5

Figure 2.3: E_2-page of the spectral sequence for $\Omega\,\mathsf{StMod}(kD_8)$ associated to the two noncentral extensions. The overline indicates the Tate dual element, e.g. $\overline{xy} \cup xy = u$. The Hochschild–Serre spectral sequence on the lower half has no nontrivial differentials, and by multiplicativity neither does the top half.



7	$k^3 \oplus k^3 \oplus 0$	$0 \oplus 0 \oplus k$	$0 \oplus 0 \oplus \textcolor{blue}{\underline{k^2}}$	$0 \oplus 0 \oplus \textcolor{blue}{\underline{\textcolor{red}{\underline{k^3}}}}$	$0 \oplus 0 \oplus \textcolor{blue}{\underline{\textcolor{red}{\underline{k^4}}}}$	$0 \oplus 0 \oplus \textcolor{blue}{\underline{\textcolor{red}{\underline{k^5}}}}$	$0 \oplus 0 \oplus \textcolor{blue}{\underline{\textcolor{red}{\underline{k^6}}}}$
6	$\textcolor{blue}{\underline{k^3}} \oplus \textcolor{blue}{\underline{k^3}} \oplus 0$	$\textcolor{blue}{\underline{k}} \oplus \textcolor{blue}{\underline{k}} \oplus \textcolor{red}{\underline{k}}$	$\textcolor{blue}{\underline{k}} \oplus \textcolor{blue}{\underline{k}} \oplus \textcolor{red}{\underline{k^2}}$	$\textcolor{blue}{\underline{k}} \oplus \textcolor{blue}{\underline{k}} \oplus \textcolor{red}{\underline{k^3}}$	$\textcolor{blue}{\underline{k}} \oplus \textcolor{blue}{\underline{k}} \oplus \textcolor{red}{\underline{k^4}}$	$\textcolor{blue}{\underline{k}} \oplus \textcolor{blue}{\underline{k}} \oplus \textcolor{red}{\underline{k^5}}$	$\textcolor{blue}{\underline{k}} \oplus \textcolor{blue}{\underline{k}} \oplus \textcolor{red}{\underline{k^6}}$
5	$k^2 \oplus k^2 \oplus 0$	$0 \oplus 0 \oplus k$	$0 \oplus 0 \oplus \textcolor{blue}{\underline{k^2}}$	$0 \oplus 0 \oplus \textcolor{blue}{\underline{\textcolor{red}{\underline{k^3}}}}$	$0 \oplus 0 \oplus \textcolor{blue}{\underline{\textcolor{red}{\underline{k^4}}}}$	$0 \oplus 0 \oplus \textcolor{blue}{\underline{\textcolor{red}{\underline{k^5}}}}$	$0 \oplus 0 \oplus \textcolor{blue}{\underline{\textcolor{red}{\underline{k^6}}}}$
4	$\textcolor{blue}{\underline{k^2}} \oplus \textcolor{blue}{\underline{k^2}} \oplus 0$	$\textcolor{blue}{\underline{k}} \oplus \textcolor{blue}{\underline{k}} \oplus \textcolor{red}{\underline{k}}$	$\textcolor{blue}{\underline{k}} \oplus \textcolor{blue}{\underline{k}} \oplus \textcolor{red}{\underline{k^2}}$	$\textcolor{blue}{\underline{k}} \oplus \textcolor{blue}{\underline{k}} \oplus \textcolor{red}{\underline{k^3}}$	$\textcolor{blue}{\underline{k}} \oplus \textcolor{blue}{\underline{k}} \oplus \textcolor{red}{\underline{k^4}}$	$\textcolor{blue}{\underline{k}} \oplus \textcolor{blue}{\underline{k}} \oplus \textcolor{red}{\underline{k^5}}$	$\textcolor{blue}{\underline{k}} \oplus \textcolor{blue}{\underline{k}} \oplus \textcolor{red}{\underline{k^6}}$
3	$k \oplus k \oplus 0$	$0 \oplus 0 \oplus k$	$0 \oplus 0 \oplus \textcolor{blue}{\underline{k^2}}$	$0 \oplus 0 \oplus \textcolor{blue}{\underline{\textcolor{red}{\underline{k^3}}}}$	$0 \oplus 0 \oplus \textcolor{blue}{\underline{\textcolor{red}{\underline{k^4}}}}$	$0 \oplus 0 \oplus k^5$	$0 \oplus 0 \oplus k^6$
2	$\textcolor{blue}{\underline{k}} \oplus \textcolor{blue}{\underline{k}} \oplus 0$	$\textcolor{blue}{\underline{k}} \oplus \textcolor{blue}{\underline{k}} \oplus \textcolor{red}{\underline{k}}$	$\textcolor{blue}{\underline{k}} \oplus \textcolor{blue}{\underline{k}} \oplus \textcolor{red}{\underline{k^2}}$	$k \oplus k \oplus k^3$	$k \oplus k \oplus k^4$	$k \oplus k \oplus k^5$	$k \oplus k \oplus k^6$
1	$k^\times$	0	$(*)$	$(*)$	$(*)$	$(*)$	$(*)$
0	$\mathbb{Z}^2$	0	$(\mathbb{Z}/2)^2$	0	$(\mathbb{Z}/2)^2$	0	$(\mathbb{Z}/2)^2$
	0	1	2	3	4	5	6

Figure 2.4: E_2-page of the limit spectral sequence for the Picard group of $\mathsf{StMod}(kD_8)$. For $t \geq 2$ we have indicated the contributions coming from each of the summands in Eq. (2.2.2). We have underlined in red which summands get at least partially killed by Hochschild–Serre differentials, and in blue the summands which get killed under Conjecture 2.2.3. $(*)$: These terms are 0 if the field k is perfect, and more complicated otherwise.

7	k^6	k	0	0	0	0	0
6	k^4	0	0	0	0	0	0
5	k^4	k	0	0	0	0	0
4	k^2	0	0	0	0	0	0
3	k^2	k	0	0	0	0	0
2	0	0	0	?	?	?	?
1	$k^\times$	0	?	?	?	?	?
0	$\mathbb{Z}^2$	0	?	0	?	0	?
	0	1	2	3	4	5	6

Figure 2.5: E_3-page of the limit spectral sequence for the Picard group of $\mathsf{StMod}(kD_8)$ under Conjecture 2.2.3. There are no remaining differentials that could influence the terms on the 0-line. All positive lines match up with the expected outcomes.

Through representation theory, the generators of the Picard group are explicitly known, as are their restrictions to the two subgroups $(C_2)^2$ of D_8. These are *not* two copies of $\Omega\, k$, but rather a copy of $\Omega\, k$ and a copy of $\Omega^2\, k$. We infer that E_∞^{00} must be an index-2 subgroup of E_2^{00}, and hence there is a single nonzero differential at some point.

2.3 Remarks on other finite groups

In this section, we briefly discuss how one might proceed from extraspecial p-groups to arbitrary finite p-groups, and further to any finite group.

When people initially tried to classify endotrivial modules for finite p-groups, they hoped to do this through a certain detection result, by which they meant that for any such group G, the natural restriction map $T(G) \to \prod_H T(H)$ is injective, where H ranges over those subgroups of G that are of a specific form. A famous result of Carlson–Thévenaz [CT04] shows that one must take those H that are elementary abelian of rank 2, extaspecial, or almost extaspecial.

Clearly, then, extraspecial groups play a special role in the theory of endotrivial modules. It is natural then to ask whether this can be seen from a homotopical point of view. One way to see extraspecial groups arise is as follows. Let P be a finite p-group. Then P has a nontrivial centre, within which we may find a central subgroup C which is isomorphic to the cyclic group C_p. Observe that we may take the family $\mathcal{A}$ of Theorem 1.2.7 to be those elementary abelian subgroups which contain C. These, then, are almost in one-to-one correspondence with the elementary abelian subgroups of the quotient G/C, though such a correspondence is muddled by potential extension problems. Still, it is tempting to believe that one can quantise in this way the relation between the endotrivial modules of G and those of G/C; continuing this process inductively, one eventually reaches an elementary abelian subgroup; the group preceding it is either abelian or extraspecial.

Now let us finally consider the case of a finite group G which is *not* a p-group. What can we still infer?

Let's focus on the torsionfree rank first. Consider the descent spectral sequence where we simply take the family $\mathcal{A}$ of Theorem 1.2.7 to be the family of all elementary abelian subgroups. I claim that the only source of torsionfree summands of $T(G)$ on the 0-line can occur on the $(0,0)$-th spot of the E_2-page

$$E_2^{st} = H^s(\mathcal{O}_\mathcal{A}, \pi_t\, \mathfrak{pic}\, \mathsf{StMod}(kH))$$

of the spectral sequence. Indeed for $t \geq 2$ the terms are all torsion as they are vector spaces over k. As for $t = 1$, a sufficiently large choice of k could perhaps yield nontrivial torsionfree rank of $H^1(\mathcal{O}_\mathcal{A}, k^\times)$ — however, we may assume without loss of generality that k is a subfield of $\overline{\mathbb{F}_p}$, as enlarging the field any further will not introduce new representation-theoretic phenomena, and $\overline{\mathbb{F}_p}^\times$ is entirely torsion.

As the targets of the differentials d_{00}^* all hit torsion groups, the rank of E_{00}^* is not altered as the spectral sequence progresses. As such, we may conclude that

$$\operatorname{rk} T(G) = \operatorname{rk} H^0(\mathcal{O}_\mathcal{A}, \pi_0\, \mathfrak{pic}\, \mathsf{StMod}(kH)).$$

We have encountered this limit in the case of extraspecial groups, cf. Remark 2.1.6, but it can be analysed for general G in much the same way. The functor $G/H \mapsto \operatorname{Pic} \mathsf{StMod}(kH)$ outputs $\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$ or 0 according to Lemma 1.2.14, and all morphisms send generators to generators. Consequently, the rank of this cohomology group is equal to the number of path components of the poset of elementary abelian subgroups of rank ≥ 2. This number can be analysed group-theoretically, thereby recovering a famous theorem of Carlson–Mazza–Nakano [CMN06]. Briefly, if n_G is the number of conjugacy classes of maximal rank-2 elementary abelian subgroups of G, then

$$\operatorname{rk} T(G) = \begin{cases} 0 & \text{if } G \text{ has } p\text{-rank } 1; \\ n_G & \text{if } G \text{ has } p\text{-rank } 2; \\ n_G + 1 & \text{if } G \text{ has } p\text{-rank} \geq 3. \end{cases}$$

Now let's instead take the family $\mathcal{A}$ to be the collection of p-subgroups of G. Denote the resulting orbit category by $\mathcal{O}_p$. Notice that we can do this even if G is itself a p-group, but the resulting decomposition will be trivial, as G would be in $\mathcal{A}$, thus giving $\mathcal{O}_p$ a final object and rendering the theorem vacuous. In the non-p-group case, however, the choice has substantial content: the functor $G/P \mapsto \widehat{H}^*(P;k)$ satisfies the hypotheses of [JM92, Prop. 5.14], and so its higher cohomology groups all vanish. Consequently, the only nonzero terms on the 0-line of the E_2-page are E_2^{00} and E_2^{11}, and the only relevant nontrivial differential is

$$d_2^{00} \colon \varprojlim_{\mathcal{O}_p^{\mathrm{op}}} \operatorname{Pic} \mathsf{StMod}(kP) \to H^2(\mathcal{O}_p(G); k^\times).$$

From the filtration associated to the convergence of the spectral sequence, we infer the exact sequence

$$0 \to H^1(\mathcal{O}_p; k^\times) \to \operatorname{Pic} \mathsf{StMod}(kG) \to \varprojlim_{\mathcal{O}_p^{\mathrm{op}}} \operatorname{Pic} \mathsf{StMod}(kP) \to H^2(\mathcal{O}_p; k^\times)$$

This recovers [Gro22, Thm. A]. The nontrivial differential is studied in more detail in upcoming work of Barthel, Grodal and Hunt.

2.A A Grothendieck spectral sequence for derived limits

Let $\mathcal{C}$ be a general category, and we write $\mathsf{PSh}(\mathcal{C})$ for the category of presheaves on $\mathcal{C}$ with values in some abelian category of (say) modules over a commutative rings. We are interested in computing cohomology groups of presheaves over $\mathcal{C}$. In our application, we take $\mathcal{C}$ to be a full subcategory of the orbit category $\mathcal{O}_G$ of some group G. Suppose $F \colon \mathcal{C} \to \mathcal{D}$ is a functor to some other category $\mathcal{D}$. Then there is an obvious pullback functor $F^* \colon \mathsf{PSh}(\mathcal{D}) \to \mathsf{PSh}(\mathcal{C})$ on presheaves.

Lemma 2.A.1. With the notation as above, the functor F^* has a right adjoint, which we shall denote by F_*.

To define the right adjoint in a succinct way, we recall the definition of (a special case of) the comma category $F \downarrow d$ for all objects $d \in D$. The category $F \downarrow d$ has as objects all pairs (c, φ) consisting of an object c in $\mathcal{C}$ and a map $\varphi \colon F(c) \to d$. A morphism $(c, \varphi) \to (c', \varphi')$ is an arrow $f \colon c \to c'$ in $\mathcal{C}$ such that

$$\begin{tikzcd} F(c) \arrow[rr, "F(f)"] \arrow[dr, "\varphi"'] & & F(c') \arrow[dl, "\varphi'"] \\ & d & \end{tikzcd}$$

commutes in $\mathcal{D}$. We shall take limits over $F \downarrow d$ in our definition of F_*.

Proof. The right adjoint has an explicit description. A presheaf $\mathcal{F}$ is sent to the presheaf

$$F_*\mathcal{F}(d) = \varprojlim_{F\downarrow d} \mathcal{F}(c).$$

To be more precise, the limit is taken over the functor $F \downarrow d \to \mathcal{A}$ sending a pair (c, φ) to $\mathcal{F}(c)$, and sending a map $f\colon c \to c'$ to the associated restriction map of $\mathcal{F}$. Let's now describe the restriction map of $F_*\mathcal{F}$ along an arrow $f\colon d \to d'$. Write an element of $F_*\mathcal{F}(d')$ as a tuple $(x_{c',\varphi'})$ parametrised over all (c', φ') in $F \downarrow d'$. We would like to associate to it a tuple $(y_{(c,\varphi)})$ parametrised by objects in $F \downarrow d$. Fix such an object (c, φ). Compose $\varphi\colon u(c) \to d$ with f to obtain an object $(c, f \circ \varphi)$ in $F \downarrow d'$. We define the element $y_{(c,\varphi)}$ to be $x_{(c,f\circ\varphi)}$.

Let's now verify that $F^* \dashv F_*$. We want to establish a natural bijection between $\mathrm{Hom}(F^*\mathcal{F}, \mathcal{G})$ and $\mathrm{Hom}(\mathcal{F}, F_*\mathcal{G})$. We start with a natural transformation $\alpha\colon F^*\mathcal{F} \to \mathcal{G}$. To associate to it a transformation $\beta\colon \mathcal{F} \to F_*\mathcal{G}$, we must describe maps $\beta(d)\colon \mathcal{F}(d) \to \varprojlim_{F\downarrow d} \mathcal{G}(c)$ for all objects d. In turn, to describe $\beta(d)$, we need to have maps $\mathcal{F}(d) \to \mathcal{G}(c)$ for all (c, φ) in $F \downarrow d$ in a natural way. These maps are given by the composition

$$\mathcal{F}(d) \xrightarrow{\mathcal{F}(\varphi)} \mathcal{F}(F(c)) \;=\!=\!=\; F^*\mathcal{F}(c) \xrightarrow{\alpha(c)} \mathcal{G}(c)$$

Naturality with respect to $F \downarrow d$ is immediate from naturality of the various $\alpha(c)$. Thus, we have the desired composed map $\beta(d)$. One must next verify that these $\beta(d)$ are natural in $\mathcal{D}$; in other words, if $f\colon d \to d'$ is any map, the diagram

$$\begin{tikzcd}
\mathcal{F}(d') \arrow[r] \arrow[d] & F_*\mathcal{G}(d') \arrow[r, equal] \arrow[d] & \varprojlim_{F\downarrow d'} \mathcal{G}(c') \arrow[d] \\
\mathcal{F}(d) \arrow[r] & F_*\mathcal{G}(d) \arrow[r, equal] & \varprojlim_{F\downarrow d} \mathcal{G}(c)
\end{tikzcd}$$

commutes. So start with an element $z \in \mathcal{F}(d')$. The image in $F_*\mathcal{G}(d')$ is a tuple $(x_{(c',\varphi')})$ where

$$x_{(c',\varphi')} = \big(\alpha(c') \circ \mathcal{F}(\varphi')\big)(z).$$

In turn, the image of $(x_{(c',\varphi')})$ in $F_*\mathcal{G}(d)$ is a tuple $(y_{(c,\varphi)})$ where

$$y_{(c,\varphi)} = x_{(c,f\circ\varphi)} = \big(\alpha(c) \circ \mathcal{F}(f \circ \varphi)\big)(z) = \big(\alpha(c) \circ \mathcal{F}(\varphi)\big)\big(f(z)\big),$$

which is clearly also what comes out had we gone through the diagram in the other way. This completes the construction of the map $\mathrm{Hom}(F^*\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}, F_*\mathcal{G})$.

The map in the other direction is somewhat easier, and we'll be more brisk. Start with a natural transformation $\beta\colon \mathcal{F} \to F_*\mathcal{G}$. For every c, the map $\beta\big(F(c)\big)\colon \mathcal{F}\big(F(c)\big) \to F_*\mathcal{G}(c)$ can be composed with the projection down towards the component corresponding to the element (c, Id) in $F \downarrow F(c)$. Assembling these maps for all c yields the desired natural transformation $F^*\mathcal{F} \to \mathcal{G}$. □

Lemma 2.A.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor, and let $\mathcal{F} \in \mathsf{PSh}(\mathcal{C})$. Then there exists a natural isomorphism

$$\varprojlim_{\mathcal{C}} \mathcal{F} \xrightarrow{\sim} \varprojlim_{\mathcal{D}} F_*\mathcal{F}.$$

Proof. We wish to show that $\varprojlim_{\mathcal{C}} \mathcal{F}(c) \simeq \varprojlim_{\mathcal{D}} \varprojlim_{F\downarrow d} \mathcal{F}(c)$. This is really an upshot of a more abstract situation. One has a collection of diagrams $J(i)$ parametrised by another diagram I. Each $J(i)$ admits a functor $F(i)$ to some category $\mathcal{C}$, and the functors are natural in I. One has a canonical isomorphism

$$\varprojlim_I \varprojlim_{J(i)} F(i) \simeq \varprojlim_{\varinjlim_I J} F,$$

where F denotes the unique functor $\varinjlim_I J \to \mathcal{C}$ arising from the $F(i)$. In our case, one has $I = \mathcal{D}$ and the $J(i)$ are the $F \downarrow d$. It's easy to check that $\varinjlim_{\mathcal{D}} F \downarrow d$ is isomorphic to $\mathcal{C}$, hence the result follows. ☐

The result above suggests that we may apply the Grothendieck spectral sequence to the composition $\varprojlim_{\mathcal{C}} \simeq \varprojlim_{\mathcal{D}} \circ F_*$. For this to work, F_* must take injective objects in $\mathsf{PSh}(\mathcal{C})$ to $\varprojlim_{\mathcal{D}}$-acyclic objects in $\mathsf{PSh}(\mathcal{D})$. This follows from the stronger fact that F_* preserves injectives, which in turn is a consequence of the following simple result.

Lemma 2.A.3. Let $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ be functors between abelian categories such that $F \dashv G$. Suppose that F preserves monomorphisms. Then G preserves injective objects.

Proof. Consider a monomorphism $X \hookrightarrow Y$ in $\mathcal{A}$ along with a map $X \to G(I)$, with I injective. By the adjunction this corresponds to a map $F(X) \to I$. As F preserves injections, $F(X) \to F(Y)$ is injective, and so since I is injective, there's a natural map $F(Y) \to I$. The corresponding map $Y \to G(I)$ is precisely what's needed to show that $G(I)$ is injective. ☐

The Grothendieck spectral sequence for derived functors thus produces a spectral sequence

$$E_2^{pq} = H^p(\mathcal{D}, R^q F_* \mathcal{F}) \Rightarrow H^{p+q}(\mathcal{C}, \mathcal{F}).$$

The higher derived functors $R^q F_* \mathcal{F}$ can be described in terms of the cohomology of comma categories. Indeed, from the proof of Lemma 2.A.2 one finds that $R^q F_* \mathcal{F}(d)$ can also be described as $H^q\big(F \downarrow d, \mathcal{F}\big)$, and hence the spectral sequence can be rewritten as

$$E_2^{pq} = H^p\big(\mathcal{D}, H^q(F \downarrow (\,\cdot\,), \mathcal{F})\big) \Rightarrow H^{p+q}(\mathcal{C}, \mathcal{F}).$$

This spectral sequence can be of help whenever we can find a functor F to a simpler category $\mathcal{D}$ such that the comma categories $F \downarrow d$ are reasonably simple as well.

Example 2.A.4. Let G be a finite group with normal subgroup H. Derived limits over the categories BH and BG compute group cohomology over BH and BG. There is a natural functor $F\colon BG \to B(G/H)$. Since there's only one object in BG, there's only one comma category $F \downarrow *$. It has as objects all classes $[g]$ in G/H, and the morphisms are parametrised by H. This category is equivalent to BH. It follows that we obtain a spectral sequence

$$E_2^{pq} = H^p\big(G/H; H^q(H, \mathcal{F})\big) \Rightarrow H^{p+q}(G, \mathcal{F}).$$

This is, of course, the classical Hochschild–Serre spectral sequence.

Chapter 3

A genuine equivariant approach to the Dade group

Abstract. We investigate how the Dade group of endopermutation modules can be realised as the Picard group of a certain ∞-category of genuine equivariant spectra. On our way, we produce a general framework for studying modules whose endomorphisms are trivial up to a specified subcategory of the representation category. This produces invariants that interpolate between the group of endotrivial modules and the Dade group, as well as other more exotic invariants that are of independent interest.

3.1 Endopermutation modules

Let G be a finite group, and let M be a (finite-dimensional) kG-module. Then we call M an endopermutation module if $\mathrm{End}_k(M)$ is a permutation module. By this we mean a kG-module admitting a basis which is G-invariant. This notion, which was first introduced by Dade in [Dad78], evidently generalises the notion of endotrivial module as studied in the previous section.

Much as how endotrivial modules assemble into the Picard group of the stable module category, the endopermutation modules assemble into a group known as the Dade group of G. The goal of this section is to introduce the Dade group and present some of its properties. Most of the material can be found in [Dad78], but I have aimed to simplify the exposition and present some examples.

The results presented in this section work only when G is a p-group. Nonetheless, it will be useful to keep in mind the possibility that G is a general finite group of order divisible by p. The reason for this choice is that we will generalise the results in the next section, at which point we will have sufficient flexibility to produce a nontrivial theory for general finite groups.

Lemma 3.1.1. Endopermutation modules are closed under tensor products, duals, and internal Hom.

Proof. As all modules are presumed finite-dimensional, $\mathrm{End}_k(M)$ is canonically isomorphic to $M^* \otimes M$. Using this, we find that $\mathrm{End}_k(M \otimes_k N) \simeq \mathrm{End}_k(M) \otimes \mathrm{End}_k(N)$, so the first closure property is proved if we show that permutation modules are closed under tensor product. But this is easy enough: if X and Y are G-sets, then $k[X] \otimes_k k[Y]$ is isomorphic to $k[X \times Y]$. Closure under duals follows by noting that $\operatorname{End}_k(M)$ and $\operatorname{End}_k(M^*)$ are isomorphic. Finally, $\operatorname{Hom}_k(M,N)$ is isomorphic to $M^* \otimes N$, which is an endopermutation module by the first two results of this lemma. $\square$

Notice that endopermutation modules typically aren't closed under direct sums. Indeed if M and N are endopermutation modules, then the endomorphism algebra of $M \oplus N$ decomposes as

$$\operatorname{End}_k(M \oplus N) \simeq \operatorname{End}_k(M) \oplus \operatorname{Hom}_k(M,N) \oplus \operatorname{Hom}_k(N,M) \oplus \operatorname{End}_k(N),$$

and there's no reason to expect the two Hom groups to be permutation modules. On the other hand, we do have the following:

Lemma 3.1.2. If G is a p-group, then any summand of an endopermutation module is again an endopermutation module.

Proof. If M' is a summand of M then $\operatorname{End}_k(M')$ is a summand of $\operatorname{End}_k(M)$, and so the claim reduces to the analogous claim for permutation modules. We claim that if M is a permutation module of the form $k(G/H)$, then M is indecomposable. Suppose M were to split as $M' \oplus M''$. Recall that every modular representation of a p-group has a fixed point, as we mentioned in Remark 1.2.13, so that $\operatorname{Hom}_G(k, M' \oplus M'')$ is at least 2-dimensional. On the other hand,

$$\operatorname{Hom}_G\big(k, k(G/H)\big) \simeq \operatorname{Hom}_H(k,k) \simeq k,$$

which is 1-dimensional — a contradiction. $\square$

Remark 3.1.3. The statement breaks down for non-p-groups. Permutation modules over non-p-group typically decompose nontrivially, even when p divides the order of G. This can already be seen in a group as simple as C_6: its regular representation over $\mathbb{F}_2$ breaks up into two indecomposable summands, which are of dimension 2 and 4, respectively. Concretely, these two summands may be defined by

$$g \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad g \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

where g is a choice of generator of C_6. Already for dimension reasons, the latter cannot be a permutation module.

Lemma 3.1.4. Endopermutation modules are closed under restriction to subgroups and under inflation from quotients.

Proof. The endomorphism algebra of the restriction is simply the restriction of the original endomorphism algebras, and clearly restrictions of permutation modules are still permutation modules. The proof for inflation is essentially identical. $\square$

Remark 3.1.5. The reader will have noticed the absense of induction in the lemma above. Indeed if H is a subgroup of G, and M is a kH-module, then by the Mackey decomposition

$$\operatorname{End}_k\big(\operatorname{Ind}_H^G(M)\big) \simeq \bigoplus_{HgH} \operatorname{Ind}_{H \cap H^g}^G \operatorname{Hom}_k\big(\operatorname{Res}_{H \cap H^g}^H M, \operatorname{Res}_{H \cap H^g}^H M^g\big),$$

and there's no reason to expect these individual Hom summands to be permutation modules.

Example 3.1.6. As a concrete example, take H to be the quaternion group Q_8, sitting inside the semidihedral group $G = \mathrm{SD}_{16}$. Let M be the exotic endotrivial module of Q_8 over $k = \mathbb{F}_4$ given in Remark 1.4.6, and consider its induction to SD_{16}. A direct computation with GAP's MeatAxe module shows that $\mathrm{End}_k \mathrm{Ind}_H^G(M)$ decomposes into indecomposable summands of dimensions 18, 16 and 2, which rules out the possibility that $\mathrm{End}_k \mathrm{Ind}_H^G(M)$ is a permutation module.

In line with [Dad78], we shall call an endopermutation module M capped if the trivial module k is a direct summand of $\mathrm{End}_k(M)$. It is the *capped* endopermutation modules that will end up assembling into a group. This will become apparent from our categorical point of view as well: just as how endotrivial modules are effectively $\otimes$-invertible object upon trivialising the projective objects, so too are the capped endopermutation modules $\otimes$-invertible upon trivialising the permutation modules.

In his paper, Dade in fact gives a different definition of capped endopermutation modules. We shall verify that they are equivalent.

Lemma 3.1.7 ([Dad78, Prop. 3.9])**.** If G is a p-group, then an endopermutation module is capped if and only if it possesses an indecomposable summand with vertex G.

For a brief discussion on vertices, see Section 3.A. The proof below is paraphrased from [Dad78].

Proof. Consider an endopermutation module with an indecomposable summand M of vertex G. By Lemma 3.1.2, M is itself an endopermutation module. Define the two-sided ideal

$$I = \sum_{H<G} \mathrm{Tr}_H^G \mathrm{End}_{kH}(M) \subseteq \mathrm{End}_{kG}(M),$$

where Tr_H^G denotes the transfer along a subgroup. In modular representation theory, the quotient $\mathrm{End}_{kG}(M)/I$ is known as the Brauer quotient of M, and it's the representation-theoretic analogue of geometric fixed points.

Since $\mathrm{End}_k(M)$ is a permutation module, we can explicitly analyse I. For any nontrivial summand isomorphic to $k(G/H)$, the transfer map $\mathrm{Tr}_H^G \colon \big(k(G/H)\big)^H \to \big(k(G/H)\big)^G$ is surjective, but for trivial summands, all transfers $\mathrm{Tr}_H^G \colon k \to k$ are zero, because the index of H in G is always divisible by p. Thus, I is nontrivial if and only if $\mathrm{End}_k(M)$ has a trivial summand.

On the other hand, a subgroup H of G is a vertex of M if and only if H is minimal among the subgroups for which $\mathrm{Tr}_H^G \colon \mathrm{End}_{kH}(M) \to \mathrm{End}_{kG}(M)$ is surjective — cf. Lemma 3.A.2. Since M is indecomposable, $\mathrm{End}_{kG}(M)$ is a local ring; as such, M has vertex $H < G$ if and only if $I = \mathrm{End}_{kG}(M)$. Together with the previous paragraph, this proves the result. ☐

Now suppose M is a capped endopermutation module over a p-group G. Then we can take an indecomposable summand with maximal vertex. But what if we can take a different one? As it happens, the choice is irrelevant:

Lemma 3.1.8. Let M be an endopermutation G-module, where G is again a p-group. Suppose M_1 and M_2 are indecomposable summands of M with vertex G. Then M_1 and M_2 are isomorphic.

The proof as stated closely follows the one presented in [Dad78] — we'll revisit it in Lemma 3.2.3.

Proof. The only relevant ingredient is the fact that $\mathrm{Hom}_k(M_1, M_2)$ is a permutation module. From this we infer that

$$\begin{aligned} M_1 \otimes M_1^* \otimes M_2 &\simeq M_1 \otimes \mathrm{Hom}_k(M_1, M_2) \\ &\simeq \bigoplus (M_1 \otimes k(G/H)) \end{aligned}$$

The vertices of the indecomposable summands of $M_1 \otimes k(G/H)$ will be subgroups of H. Thus, if some indecomposable summand of $M_1 \otimes M_1^* \otimes M_2$ has vertex G, it must be isomorphic to M_1. On the other hand, M_2 is such a summand; indeed, by Lemma 3.1.7, $M_1 \otimes M_1^*$ has a trivial summand k, hence $(M_1 \otimes M_1^*) \otimes M_2$ has a summand $k \otimes M_2$. □

What this tells us is that a capped endopermutation module M has a *unique* indecomposable summand with maximal vertex. Fittingly, we will call this summand the cap of the module, denoted $\mathrm{cap}(M)$. We will say that two capped endopermutation modules are equivalent if their caps are isomorphic.

Theorem 3.1.9. The equivalence classes of capped endopermutation modules assemble into an abelian group under the tensor product. Its unit is k, and the inverse of any module is its dual.

The resulting abelian group is called the Dade group of G.

Proof. The tensor product is well defined on equivalence classes. Indeed if M_1 and M_2 are capped endopermutation modules, then both $\mathrm{cap}(\mathrm{cap}(M_1) \otimes \mathrm{cap}(M_2))$ and $\mathrm{cap}(M_1 \otimes M_2)$ are indecomposable summands of $M_1 \otimes M_2$ with maximal vertex, hence they are isomorphic by Lemma 3.1.8.

Under the tensor product, the trivial module k is clearly the unit. Moreover, if M is a capped endopermutation module, then $M^* \otimes M \simeq \mathrm{End}_k(M)$ has a k-summand by assumption, and thus summand acts as the cap. □

3.2 A generalised Dade group

The goal of this section is to give an abstraction of the construction of the Dade group. This produces invariants that interpolate between the group of endotrivial modules and the Dade group, as well as other more exotic invariants that are of independent interest.

Let $\mathcal{P}$ be a subcategory of the 1-category $\mathsf{Mod}^{\mathrm{fin}}(kG)$ of finite-dimensional kG-modules. A finite-dimensional module M is said to be $\mathcal{P}$-endotrivial if its endomorphism group $\mathrm{End}_k(M)$ splits as $k \oplus \mathcal{P}$, which for us will be shorthand for a splitting as $k \oplus P$ where P is an object in $\mathcal{P}$. We will always assume that $\mathcal{P}$ is closed under direct sums, tensor products, and taking duals, and that the $\otimes$-ideal $\langle \mathcal{P} \rangle$ generated by $\mathcal{P}$ is proper, i.e. it doesn't generate the whole category.

Remark 3.2.1. If G is a p-group and k is algebraically closed then a thick $\otimes$-ideal is proper if and only if all its objects have dimension divisible by p, as follows by [Ben20, Theorem 2.1].

Example 3.2.2. Let G be a p-group and let $\mathcal{P}$ be the subcategory of all permutation modules with proper isotropy. Then $\mathcal{P}$-endotrivial modules are precisely the capped endopermutation modules that we introduced in the previous section.

More generally, if G is any group, then we can take $\mathcal{P}$ to be the permutation modules with isotropy in a family $\mathcal{F}$ of subgroups of G. By abuse of notation, we will often write $\mathcal{F}$ instead of $\mathcal{P}$. By

Lemma 3.A.4 the ideal that it generates will be proper if and only if the subgroups in $\mathcal{F}$ are all strictly contained in the Sylow p-subgroup of G. Also notice that, if G is a p-group, then $\mathcal{F}$ is automatically thick because transitive permutation modules of a p-group are indecomposable, as we proved in Lemma 3.1.2.

The following result is a direct generalisation of Lemma 3.1.7 and Lemma 3.1.8.

Lemma 3.2.3. If M is $\mathcal{P}$-endotrivial, then M always has a unique indecomposable summand M_0 which lies outside of $\langle \mathcal{P} \rangle$.

Proof. Decompose M as a direct sum $M_0 \oplus \cdots \oplus M_r$ of indecomposable summands. $\mathrm{End}_k(M)$ then decomposes as $\bigoplus_{i,j} \mathrm{Hom}_k(M_i, M_j)$. The trivial summand of $\mathrm{End}_k(M)$ must be found in only one of the $\mathrm{Hom}_k(M_i, M_j)$, and since there's one and only one trivial summand, the i and j must be the same. Assume without loss of generality that $k \subseteq \mathrm{End}_k(M_0)$. Clearly, $M_0 \notin \langle \mathcal{P} \rangle$ by properness of $\langle \mathcal{P} \rangle$. Now take any other summand M_i for $i > 0$. We have

$$\begin{aligned} M_0 \otimes M_0^* \otimes M_i &\simeq M_0 \otimes \mathrm{Hom}_k(M_0, M_i) \\ &\simeq M_0 \otimes \mathcal{P} \end{aligned}$$

where '$M_0 \otimes \mathcal{P}$' is shorthand for '$M_0 \otimes$ (something in $\mathcal{P}$)' — an abuse of notation we'll frequency employ. We can thus conclude that $M_0 \otimes M_0^* \otimes M_i$ lives in the ideal $\langle \mathcal{P} \rangle$. On the other hand, we can also write

$$\begin{aligned} M_0 \otimes M_0^* \otimes M_i &\simeq (k \oplus \mathcal{P}) \otimes M_i \\ &\simeq M_i \oplus \langle \mathcal{P} \rangle \end{aligned}$$

Since $\langle \mathcal{P} \rangle$ is thick, this forces M_i to be in $\langle \mathcal{P} \rangle$. ☐

For the lack of better terminology, two $\mathcal{P}$-endotrivial modules M and M' will be called Dade equivalent if there's an equivalence $M_0 \simeq M_0'$ of underlying indecomposable summands as they appear in Lemma 3.2.3. Define the Dade group $D_{\mathcal{P}}(G)$ as the group of Dade equivalence classes of $\mathcal{P}$-endotrivial modules equipped with the tensor product.

It turns out that if $\mathcal{P}$ is a (proper) $\otimes$-ideal, then the Dade group $D_{\mathcal{P}}(G)$ admits a different description:

Lemma 3.2.4. If $\mathcal{P}$ is a (proper) $\otimes$-ideal, then $D_{\mathcal{P}}(G)$ is isomorphic to the Picard group of the additive quotient $\mathsf{Mod}^{\mathrm{fin}}(kG)/\mathcal{P}$.

Proof. Two kG-modules M and N are equivalent in $\mathsf{Mod}^{\mathrm{fin}}(kG)/\mathcal{P}$ if and only if $M \oplus \mathcal{P} \simeq N \oplus \mathcal{P}$. This is a general fact for additive quotients of idempotent-complete additive categories. Clearly then, $\mathcal{P}$-endotrivial modules are invertible.

Conversely, suppose M is $\otimes$-invertible in $\mathsf{Mod}^{\mathrm{fin}}(kG)/\mathcal{P}$. We'll show that M is $\mathcal{P}$-endotrivial. The inverse of M must be M^* — this is a general fact for closed symmetric monoidal ∞-categories as witnessed by the adjunction

$$\begin{aligned} \mathrm{Map}(X, M^*) &\simeq \mathrm{Map}(X \otimes M, \mathbb{1}) \\ &\simeq \mathrm{Map}(X \otimes M \otimes M^{-1}, M^{-1}) \\ &\simeq \mathrm{Map}(X, M^{-1}) \end{aligned}$$

So if M is $\otimes$-invertible, then we know that $(M^* \otimes M) \oplus \mathcal{P} \simeq k \oplus \mathcal{P}$. By appealing to uniqueness of Krull–Schmidt decompositions, we may remove the $\mathcal{P}$ on the left-hand side to conclude the proof of the claim.

It remains to be verified that mod-$\mathcal{P}$ equivalence and Dade equivalence define the same equivalence relation. Let's denote by mod-$\mathcal{P}$ equivalence and Dade equivalence by $\sim_T$ and $\sim_D$, respectively. If $M \sim_D N$ then $M_0 \simeq N_0$, but clearly, $M \sim_T M_0$ and $N_0 \sim_T N$, so $M \sim_T N$. Conversely, if $M \sim_T N$, then $M \oplus \mathcal{P} \simeq N \oplus \mathcal{P}$. But it's easy to see that $M \sim_D M \oplus \mathcal{P}$ and $N \oplus \mathcal{P} \sim_D N$, so $M \sim_D N$. $\square$

Example 3.2.5. If $\mathcal{P}$ is the $\otimes$-ideal of projective kG-modules, then $D_{\mathcal{P}}(G)$ is the classical group of endotrivial modules, usually denoted $T(G)$. If G is a p-group then all projective modules are free so that we find back Example 3.2.2 in the case $\mathcal{F} = \{e\}$.

Remark 3.2.6. Somewhat paradoxically, if we enlarge the subcategory $\mathcal{P}$, say to a subcategory $\mathcal{Q}$, then the notion of being $\mathcal{P}$-endotrivial becomes less restrictive, yet the Dade equivalence relation doesn't, so we get to conclude the existence of an injection $D_{\mathcal{P}}(G) \hookrightarrow D_{\mathcal{Q}}(G)$.

In fact, the existence of an indecomposable summand outside $\langle \mathcal{P} \rangle$ as asserted in Lemma 3.2.3 becomes more restrictive as you enlarge $\mathcal{P}$. We might as well extrapolate this fact. If k is algebraically closed and G is a p-group then by Remark 3.2.1 there's a unique largest $\otimes$-ideal, which is the ideal (p) generated by *all* indecomposable modules of dimension divisible by p. This allows us to conclude, for instance, that if M is endotrivial, then M has a unique indecomposable summand of dimension not divisible by p.

The Dade group associated to this maximal $\otimes$-ideal (p) is rather mysterious. By Lemma 3.2.4 we can alternatively describe it as the Picard group of the additive quotient $\mathsf{Mod}^{\mathrm{fin}}(kG)/(p)$. In the same way that quotienting a ring by a maximal ideal produces a field, so does quotienting $\mathsf{Mod}^{\mathrm{fin}}(kG)$ by (p) produces a field-like category; more precisely, the resulting quotient $\mathsf{Mod}^{\mathrm{fin}}(kG)/(p)$ is precisely the semisimplification (in the sense of [EO22]) of $\mathsf{Mod}^{\mathrm{fin}}(kG)$.

The Dade group $D_{(p)}(G)$ has implicitly been studied by Benson in [Ben20], where the following conjecture is made: If G is a finite 2-group, and M is an odd-dimensional indecomposable kG-module, then $\mathrm{End}_k(M)$ is a direct sum of k and indecomposable modules of dimension divisible by 4. In particular, M is (2)-endotrivial. Thus, if $p = 2$, the group $D_{(p)}(G)$ is conjecturally generated by *all* odd-dimensional indecomposable modules. We will revisit this example in Remark 3.5.4.

Let's now take a look at some closure properties. If $f\colon G \to H$ is a group homomorphism, then it induces a symmetric monoidal pullback functor f^* on module categories. If $\mathcal{P}$ is a subcategory of $\mathsf{Mod}^{\mathrm{fin}}(kH)$, then $f^*(\mathcal{P})$ is a subcategory of $\mathsf{Mod}^{\mathrm{fin}}(kG)$. Thus, f^* sends $\mathcal{P}$-endotrivial modules to $f^*(\mathcal{P})$-endotrivial modules. Notice moreover that if $\mathcal{P}$ is closed under tensor products and duals then so is $f^*(\mathcal{P})$ — however, it could happen that $f^*(\mathcal{P})$ generates all of $\mathsf{Mod}^{\mathrm{fin}}(kG)$ as an ideal, which would invalidate most of the results above.

Example 3.2.7. If $f\colon G \hookrightarrow H$ is an inclusion then we take $\mathcal{P}$ to be the projective objects, which shows that endotrivial modules are closed under restriction. On the other hand, if f is a projection $G \to G/H$ and $\mathcal{P}$ is the category of permutation modules with isotropy in, say, the family of proper subgroups of G/H, then $f^*(\mathcal{P})$ consists of the permutation modules with isotropy in those proper subgroups of G containing H. In particular, endopermutation modules are closed under inflation.

The right adjoint pushforward functor f_* fails to be symmetric monoidal, and there's no reason to expect it to preserve $\mathcal{P}$-endotrivial modules for whatever reasonable $\mathcal{P}$ you might think of. Indeed if we recall Example 3.1.6 then we notice that the pushforward doesn't even have a trivial summand.

Although $\mathcal{P}$-endotrivial modules aren't closed under induction, they tend to be better preserved by tensor induction. Recall that if H is a subgroup of G, then the tensor induction of M is defined as the tensor product $\bigotimes_{G/H} M$ equipped with the diagonal G-action.

Example 3.2.8. Tensor induction preserves permutation modules, though the tensor induction of a free module need not remain free. So tensor induction preserves $\mathcal{F}$-endotrivial modules when $\mathcal{F}$ consists of all proper subgroups, but it doesn't when $\mathcal{F}$ consists of the trivial subgroup. This presents a striking difference in the structure of the classical Dade group (Example 3.2.2) versus the classical group of endotrivial modules (Example 3.2.5).

3.3 Relative stable module categories

Recall from Lemma 3.2.4 that the Dade group $D_{\mathcal{P}}(G)$ for a $\otimes$-ideal $\mathcal{P}$ could be defined as the Picard group of the additive quotient $\mathsf{Mod}^{\mathrm{fin}}(kG)/\mathcal{P}$. If $\mathcal{P}$ is the subcategory of projective kG-modules, then by [Ric89] this additive quotient is in fact the homotopy category of the stable module ∞-category $D^{b,\mathrm{fin}}(kG)/\mathsf{Perf}(kG)$, also denoted $\mathsf{StMod}(kG)$ (but see Remark 1.2.4). This observation gives us the opportunity to use homotopical methods in the study of endotrivial modules — an opportunity that we have made ample use of in the previous chapters.

We would like to emulate this process for more general Dade groups. One educated guess would perhaps be to start off with the (bounded) derived category of kG-modules, and taking the Verdier quotient by the $\otimes$-ideal generated not by the complexes of projectives (which would yield the ordinary stable module category) but rather by the complexes of objects in $\mathcal{P}$. This, however, does not work. If $\mathcal{P}$ is the class of permutation modules with isotropy in a sufficiently large family, then it is precisely the Quillen stratification that we used in Section 1.2 which prevents this from working: by [Mat16, Prop. 9.13], the Verdier quotient will be zero.

The approach we instead take is to go back to the ordinary category of kG-modules, and to alter what we mean by 'projective object'. We do this using the theory of exact categories, which we developed in Section 3.B. Specifically, we define an exact category on the category of kG-modules for which the projective objects are precisely the objects in the thick closure of $\mathcal{P}$. Of particular interest is the case where $\mathcal{P}$ consists of permutation modules, in which case the resulting category is morally similar to a construction conceived in [CPW98] for the purpose of studying support varieties. We review their construction in Section 3.D.

Let $\mathcal{A}$ be the abelian category $\mathsf{Mod}^{\mathrm{fin}}(kG)$ of finite kG-modules. Consider the exact structure $\mathcal{E}_{\mathcal{P}}$ as defined in Lemma 3.B.3, where $\mathcal{E}$ is the ordinary exact structure on $\mathcal{A}$, and $\mathcal{P}$ is the class of objects in our subcategory $\mathcal{P}$.

Lemma 3.3.1. With the notation as above, $\mathcal{E}_{\mathcal{P}}$ defines a Frobenius structure on $\mathsf{Mod}^{\mathrm{fin}}(kG)$ whose projectives / injectives are the objects in $\mathcal{P}$ along with their retracts.

Proof. To classify the projectives, we verify that the condition in Remark 3.B.5 is satisfied. This is indeed the case, though some care must be taken here, as $\mathsf{Mod}^{\mathrm{fin}}(kG)$ is not closed under arbitrary direct sums.

The fact that the injectives and projectives coincide follows by duality. If P is a projective object, and $M' \to M \to M''$ is an exact sequence, then we wish to show that the sequence

$$\mathrm{Hom}_{kG}(M'', P) \to \mathrm{Hom}_{kG}(M, P) \to \mathrm{Hom}_{kG}(M', P)$$

is again exact, thus proving that P is injective. Because finite-dimensional representations are reflexive (i.e. isomorphic to their double dual), this sequence is dual to

$$\mathrm{Hom}_{kG}(P, M') \to \mathrm{Hom}_{kG}(P, M) \to \mathrm{Hom}_{kG}(P, M'')$$

which is exact by projectivity of P. □

Remark 3.3.2. The existence of a categorical duality is in fact equivalent to the coincidence of projectives and injectives. More precisely, if R is an associative unital ring, then the following two properties are equivalent:

- Every projective R-module is injective and vice versa.
- R is Noetherian and every finitely generated R-module is reflexive.

Denote the (bounded) derived category of $\mathsf{Mod}^{\mathrm{fin}}(kG)$ with the exact structure $\mathcal{E}_{\mathcal{P}}$ by $D^{b,\mathrm{fin}}_{\mathcal{P}}(kG)$. Lemma 3.2.4 and Theorem 3.C.2 now let us conclude that, for any thick $\otimes$-ideal $\mathcal{P}$, the Dade group $D_{\mathcal{P}}(G)$ is isomorphic to the Picard group of the symmetric monoidal ∞-category $D^{b,\mathrm{fin}}_{\mathcal{P}}(kG)/D^b(\mathcal{P})$. We shall henceforth denote this quotient by $\mathsf{StMod}_{\mathcal{P}}(kG)$.

Remark 3.3.3. The classical Dade group is obtained by taking $\mathcal{P}$ to be the class of permutation modules with proper isotropy. This is not an ideal, and as stated in Remark 3.C.4, even though $D^{b,\mathrm{fin}}_{\mathcal{P}}(kG)$ is a well-defined category and even though it has a well-defined Verdier quotient, this Verdier quotient doesn't come with a tensor product.

A first approximation would be to simply replace the subcategory $\mathcal{P}$ of permutation modules with the thick $\otimes$-ideal generated by $\mathcal{P}$. This structure was conceived in [CPW98] and is reviewed in Section 3.D. The resulting Dade group has been studied in [Las11], where it is denoted by $T_V(G)$ for V the module $\bigoplus_{H<G} k[G/H]$, and at the very least it includes the classical Dade group as a subgroup.

In the next section, however, we will work with a slightly different setup. We still take $\mathcal{P}$ to be the collection of permutation modules, but we now simply close up the subcategory $D^b(\mathcal{P})$ and take its thick $\otimes$-ideal in $D^{b,\mathrm{fin}}_{\mathcal{P}}(kG)$. That is, we will study the (now symmetric monoidal) Verdier quotient $D^{b,\mathrm{fin}}_{\mathcal{P}}(kG)/\langle D^b(\mathcal{P})\rangle$.

3.4 Elmendorf's theorem in modular representation theory

When constructing the ∞-category of G-spaces using G-topological spaces, the 'correct' notion of equivalence is simply the equivariant analogue of homotopy equivalence. An equivariant version of Whitehead's theorem then tells you that this notion of equivalence can be described nonequivariantly by saying that we have a homotopy equivalence on all fixed-point spaces for all subgroups of G.

Since G-homotopy equivalence is detected on the fixed points, objects in $\mathcal{S}_G$ can be described entirely using abstract fixed-point data. Elmendorf's theorem makes this observation precise: $\mathcal{S}_G$ is equivalent to the category $\mathsf{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S})$. More generally, if $\mathcal{F}$ is a family of subgroups of G, and we define our weak equivalences on G-spaces as those maps which induce weak equivalences on H-fixed points for H in $\mathcal{F}$, then the resulting category is modelled by $\mathsf{Fun}(\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}, \mathcal{S})$.

Now take $\mathcal{P}$ to be the subcategory of $\mathsf{Mod}^{\mathrm{fin}}(kG)$ generated by the permutation modules with isotropy in a family $\mathcal{F}$. As a minor abuse of notation, denote the resulting derived category by $D^{b,\mathrm{fin}}_{\mathcal{F}}(kG)$. Observe that, for any subgroup H, we have an equivalence $\mathrm{Hom}_{kG}\big(k[G/H], M\big) \simeq M^H$. In view of Lemma 3.B.9, then, weak equivalences in $D^{b,\mathrm{fin}}_{\mathcal{F}}(kG)$ just amount to a 'classical' weak equivalence on the fixed points. It would therefore be tempting to suspect that there's a version of Elmendorf's theorem for the category $D^{b,\mathrm{fin}}_{\mathcal{F}}(kG)$.

A priori, one may be inclined to claim that $D^{b,\mathrm{fin}}_{\mathcal{F}}(kG)$ is modelled by $\mathsf{Fun}\big(\mathcal{O}_{\mathcal{F}}(G)^{\mathrm{op}}, \mathsf{Perf}(k)\big)$. This suspicion, however, isn't quite accurate. The orbit category $\mathcal{O}_G$ arises as the full subcategory of $\mathcal{S}_G$ on the 'generating' objects G/H — in contrast, the full subcategory of the 1-category $\mathsf{Mod}^{\mathrm{fin}}(kG)$ on the 'generating' objects $k[G/H]$ is an altogether different category. When we implement this correction, we get a statement which is actually correct.

Theorem 3.4.1. We have an equivalence

$$D^{b,\mathrm{fin}}_{\mathcal{F}}(kG) \simeq \mathsf{Fun}^{\pi}(\mathsf{Perm}_{\mathcal{F}}^{\mathrm{op}}, \mathsf{Sp})^{\mathrm{dual}}.$$

In words, $D^{b,\mathrm{fin}}_{\mathcal{F}}(kG)$ is equivalent to the dualisable objects in the functor category $\mathsf{Fun}^{\pi}(\mathsf{Perm}_{\mathcal{F}}^{\mathrm{op}}, \mathsf{Sp})$ of finite-product-preserving functors from the category $\mathsf{Perm}_{\mathcal{F}}$ of permutation kG-modules with isotropy in $\mathcal{F}$.

The classical proof of Elmendorf's theorem, based on the two-sided bar construction, carries over to this setting. For this to work, one needs to realise $D^{b,\mathrm{fin}}_{\mathcal{F}}(kG)$ as a simplicial model category, which can indeed be done thanks to the work of [CH02]. However, we choose to take a different approach to this result.

Let us begin by observing that $D^{b,\mathrm{fin}}_{\mathcal{F}}(kG)$ describes the dualisable objects in the larger derived category $D_{\mathcal{F}}(kG)$ of complexes of kG-modules with weak equivalences defined, as before, on the level of fixed points. As such, it suffices to describe $D_{\mathcal{F}}(kG)$ as $\mathsf{Fun}^{\pi}(\mathsf{Perm}_{\mathcal{F}}^{\mathrm{op}}, \mathsf{Sp})$.

Let $\mathcal{C}$ be an ∞-category which admits all small colimits. Recall that an object X in $\mathcal{C}$ is called compact if $\mathrm{Hom}(X, \cdot\,)$ commutes with filtered colimits; X is called projective if $\mathrm{Hom}(X, \cdot\,)$ commutes with geometric realisations. An object is compact projective if and only if $\mathrm{Hom}(X, \cdot\,)$ commutes with sifted colimits.

Lemma 3.4.2 ([Lur09, Proposition 5.5.8.25])**.** Let $\mathcal{P}$ be a set of compact projective generators of $\mathcal{C}$, by which we mean that $\mathcal{P}$ generates $\mathcal{C}$ under small colimits. Assume that $\mathcal{P}$ is closed under finite coproducts, or otherwise add them to your set. Then $\mathcal{C}$ is equivalent to the category $\mathsf{Fun}^{\pi}(\mathcal{P}^{\mathrm{op}}, \mathcal{S})$ of finite-product-preserving functors from $\mathcal{P}^{\mathrm{op}}$ to spaces. Moreover, every compact projective is either an object in $\mathcal{P}$, or a retract thereof.

The intuition here is that the functor category $\mathsf{Fun}^{\pi}(\mathcal{P}^{\mathrm{op}}, \mathcal{S})$ is the closure of $\mathcal{P}$ under sifted colimits.

Example 3.4.3. The compact projective objects in $\mathcal{S}$ are given by the finite sets. In this case, Lemma 3.4.2 reduces to a tautology.

Lemma 3.4.2 is manifestly about unstable categories. To apply it to our context, we need to consider first the connective part $D^{\geq 0}_{\mathcal{F}}(kG)$.

Lemma 3.4.4. The compact projective objects of $D^{\geq 0}_{\mathcal{F}}(kG)$ are the permutation modules with isotropy in $\mathcal{F}$ (or possible nontrivial summands thereof, in case G is not a p-group).

To prove this result, we use the following technical lemma:

Lemma 3.4.5 ([Lur12, Corollary 4.7.3.18])**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories with right adjoint G. Assume that G is conservative and preserves sifted colimits. If $\mathcal{C}$ admits a set of compact projective generators, then so does $\mathcal{D}$; moreover, the compact projective objects in $\mathcal{D}$ are precisely the retracts of $F(P)$ where P is compact projective in $\mathcal{C}$.

Example 3.4.6. The compact projective objects in the ∞-category $\mathcal{S}_G$ are given by the finite G-sets, as follows by applying Lemma 3.4.5 to the functor $G\colon \mathcal{S} \to \mathcal{S}_G$ sending X to $\bigsqcup_H G/H \times X$. Thus, Lemma 3.4.2 recovers Elmendorf's theorem. More generally, in the ∞-category $\mathcal{S}_{\mathcal{F}}$ of G-spaces with weak equivalences defined in terms of H-fixed points for all $H \in \mathcal{F}$, the compact projectives are the finite G-sets with isotropy in $\mathcal{F}$.

Example 3.4.7. The description of the category Sp_G of genuine G-spectra as spectral Mackey functors can be viewed as an instance of Lemma 3.4.2. The connective part $\mathsf{Sp}^{\geq 0}_G$ has compact projective generators $\Sigma^\infty_+ G/H$ as can be seen by applying Lemma 3.4.5 to the loop space functor $\Omega^\infty\colon \mathsf{Sp}_G \to \mathcal{S}_G$. The full subcategory spanned by the suspension spectra $\Sigma^\infty_+ G/H$ is precisely the Burnside category. To obtain the description of Sp_G as a functor category to spectra, we simply pass to the stabilisation of our prestable ∞-category.

Example 3.4.8. If R is an associative ring, then the compact projective objects of $D^{\geq 0}(R)$ are precisely the finitely generated free modules, i.e. the finite coproducts of R, as follows by applying Lemma 3.4.5 to the forgetful functor $F\colon D^{\geq 0}(R) \to \mathsf{Sp}^{\geq 0} \to \mathcal{S}$, and recalling that the compact projectives in $\mathcal{S}$ are just the finite discrete sets. This implies that the derived ∞-category of R can also be described as the category $\mathsf{Fun}^\pi(\mathsf{Lat}_R^{\mathrm{op}}, \mathsf{Sp})$ of product-preserving functors from the category of finitely generated free R-modules. In fact, this example readily generalises to arbitrary connective $\mathbb{E}_1$-rings.

We adapt the example above to prove our assertion about compact projectives in $D^{b,\geq 0}_{\mathcal{F}}(kG)$:

Proof of Lemma 3.4.4 and Theorem 3.4.1. Analogous to Example 3.4.8, we observe that there's a forgetful functor $D^{\geq 0}_{\mathcal{F}}(kG) \to \mathcal{S}_{\mathcal{F}}$, to which we apply Lemma 3.4.5. This yields the desired classification of compact projectives. The result follows by passing to the stabilisation. □

Remark 3.4.9. Notice that $\mathsf{Perm}_{\mathcal{F}}$ is really just the classical 1-category of permutation modules, despite arising as a full subcategory of the ∞-category $D_{\mathcal{F}}(kG)$. The intuitive reason for this is that the permutation modules are projective with respect to our chosen exact structure, and hence the mapping spaces are in fact discrete.

Remark 3.4.10. More generally, let $\mathcal{P}$ be a suitable thick subcategory of $\mathsf{Mod}^{\mathrm{fin}}(kG)$. It would be tempting to believe that the derived category $D_{\mathcal{P}}(kG)$ can be modelled likewise as a functor category $\mathsf{Fun}^\pi(\mathcal{P}^{\mathrm{op}}, \mathsf{Sp})$. I don't know whether this is the case.

We proceed to take a closer look at the category $\mathsf{Perm}_{\mathcal{F}}$. Whenever we have a G-map $f\colon G/H \to G/K$, there's a k-linearised map $f\colon k[G/H] \to k[G/K]$. But f moreover gives rise to a transfer map $\mathrm{Tr}_f\colon k[G/K] \to k[G/H]$, in which we take sums over the pre-images of f. This rules out the possibility of describing $\mathsf{Perm}_{\mathcal{F}}$ as a k-linearisation of the orbit category, and rather brings us closer to the Burnside category of spans of G-sets.

Remark 3.4.11. This observation about transfers gives us an a posteriori reason why we couldn't have expected $D^{b,\mathrm{fin}}_{\mathcal{F}}(kG)$ to be described in terms of functors from the orbit category: $D^{b,\mathrm{fin}}_{\mathcal{F}}(kG)$ has transfers. In hindsight, this could already have been inferred from the original description in terms of chain complexes.

The relationship to the Burnside category can be made more precise. Write $\mathsf{Span}_{\mathcal{F}}$ for the full subcategory of the Burnside category on those G-sets that have isotropy in $\mathcal{F}$. Then there's a functor $\pi\colon \mathsf{Span}_{\mathcal{F}} \to \mathsf{Perm}_{\mathcal{F}}$, defined by

$$\begin{tikzcd} & G/K \arrow[dl, "f"'] \arrow[dr, "g"] & & & \\ G/H & & G/L \end{tikzcd} \quad \mapsto \quad k[G/H] \xrightarrow{\mathrm{Tr}_f} k[G/K] \xrightarrow{g} k[G/L]$$

Upon k-linearising $\mathsf{Span}_{\mathcal{F}}$, this functor becomes full and essentially surjective, but not faithful. To see this, consider a map $f\colon G/H \to G/K$ corresponding to a subconjugation $H^g \subseteq K$, and notice that the composition $f \circ \mathrm{Tr}_f$ is equal to multiplication by the index $[H : K]$. If G is a p-group, and k has characteristic p, then this always produces 0. In fact the 'kernel' of our functor $\mathsf{Span}_{\mathcal{F}} \to \mathsf{Perm}_{\mathcal{F}}$ is generated by these relations.

With this in mind, it seems reasonable to conclude that $\mathsf{Fun}^{\pi}(\mathsf{Perm}_{\mathcal{F}}^{\mathrm{op}}, \mathsf{Sp})$ may be thought of as G-spectra but with an enforced relation between the restriction and the transfer. We now proceed to formalise this observation:

Theorem 3.4.12. $\mathsf{Fun}^{\pi}(\mathsf{Perm}_{\mathcal{F}}^{\mathrm{op}}, \mathsf{Sp})$ is equivalent to the category $\mathsf{Mod}_{\mathsf{Sp}_{\mathcal{F}}}(k)$ of modules in the ∞-category $\mathsf{Sp}_{\mathcal{F}}$ of $\mathcal{F}$-complete G-spectra over the Eilenberg–MacLane Mackey functor k.

Before we give the proof, observe that we have an equivalence

$$\mathsf{Sp}_{\mathcal{F}} \simeq \mathsf{Fun}^{\pi}(\mathsf{Span}_{\mathcal{F}}^{\mathrm{op}}, \mathsf{Sp}),$$

which explains why $\mathsf{Sp}_{\mathcal{F}}$ appears in the theorem statement. This equivalence is again a direct consequence of Lemma 3.4.2, as the compact projective generators in $\mathsf{Sp}_{\mathcal{F}}$ are precisely the suspension spectra $\Sigma^{\infty}_{+} G/H$ for $H \in \mathcal{F}$.

Proof. The functor $\pi\colon \mathsf{Span}_{\mathcal{F}} \to \mathsf{Perm}_{\mathcal{F}}$ that we introduced before gives rise to an adjunction $\pi_! \dashv \pi^* \dashv \pi_*$ at the level of functor categories. At this point we invoke the following result.

Lemma 3.4.13 ([MNN17, Proposition 5.29])**.** Suppose we have a symmetric monoidal functor of stable categories $L\colon \mathcal{C} \to \mathcal{D}$ with right adjoint R. Assume that $L \dashv R$ satisfies the projection formula, that R is conservative, and that R commutes with colimits. Then the natural adjunction $\mathsf{Mod}_{\mathcal{C}}\big(R(\mathbb{1}_{\mathcal{D}})\big) \rightleftarrows \mathcal{D}$ is an inverse equivalence of symmetric monoidal ∞-categories.

The adjunction $\pi_! \dashv \pi^*$ satisfies the hypotheses of this result. The fact that $\pi_!$ is symmetric monoidal is a general fact about Day convolutions. Conservativity of π^* follows from the fact that equivalences in $D_{\mathcal{F}}(kG)$ and in $\mathsf{Sp}_{\mathcal{F}}$ are both governed by Whitehead's theorem. π^* commutes with colimits because it has a further right adjoint. As for the projection formula, consider objects X and Y, and assume that X is dualisable. We have the following sequence of adjunctions:

$$\begin{aligned}
\operatorname{map}\big(Z, \pi^*(\pi_! X \otimes Y)\big) &\simeq \operatorname{map}\big(\pi_! Z, \pi_! X \otimes Y\big) && \pi_! \dashv \pi^* \\
&\simeq \operatorname{map}\big(\pi_! Z, \underline{\operatorname{map}}(\mathbb{D}\pi_! X, Y)\big) && \pi_! \text{ preserves dualisables} \\
&\simeq \operatorname{map}\big(\pi_! Z, \underline{\operatorname{map}}(\pi_! \mathbb{D} X, Y)\big) && \otimes\text{-functors commute with } \mathbb{D} \\
&\simeq \operatorname{map}\big(\pi_! Z \otimes \pi_! \mathbb{D} X, Y\big) && \text{tensor} \dashv \text{Hom} \\
&\simeq \operatorname{map}\big(\pi_!(Z \otimes \mathbb{D} X), Y)\big) && \pi_! \text{ symmetric monoidal} \\
&\simeq \operatorname{map}\big(Z \otimes \mathbb{D} X, \pi^* Y\big) && \pi_! \dashv \pi^* \\
&\simeq \operatorname{map}\big(Z, \underline{\operatorname{map}}(\mathbb{D} X, \pi^* Y)\big) && \text{tensor} \dashv \text{Hom} \\
&\simeq \operatorname{map}\big(Z, X \otimes \pi^* Y\big) && \text{by definition}
\end{aligned}$$

This establishes the projection formula under the dualisability assumption. As every object in $\mathsf{Sp}_{\mathcal{F}}$ is built up as a colimit of dualisables (specifically, the $\Sigma^\infty_+ G/H$), the formula holds for all X.

We are now done if we show that the unit in $\mathsf{Fun}^\pi(\mathsf{Perm}_{\mathcal{F}}^{\mathrm{op}}, \mathsf{Sp})$ is an Eilenberg–MacLane Mackey functor. It suffices to check this for $\mathsf{Fun}^\pi(\mathsf{Perm}_{\mathcal{F}}^{\mathrm{op}}, \mathcal{S})$. This follows from the general fact that the Yoneda embedding $\mathcal{P} \to \mathsf{Fun}^\pi(\mathcal{P}^{\mathrm{op}}, \mathcal{S})$ is symmetric monoidal when the target is equipped with the Day convolution. In particular, the unit is represented by $\mathbb{1}_{\mathcal{P}}$, and in the case where $\mathcal{P} = \mathsf{Perm}_{\mathcal{F}}$, this just returns the constant functor k. □

Remark 3.4.14. Alternatively, Theorem 3.4.12 could be proved by investigating the compact projective generators of $\mathsf{Mod}_{\mathsf{Sp}_{\mathcal{F}}}(k)$. Example 3.4.8 generalises to the category $\mathsf{LMod}_{\mathcal{C}}(A)$ of left modules over an $\mathbb{E}_1$-algebra object A in a symmetric monoidal ∞-category $\mathcal{C}$. This allows us to see that the objects $k[G/H]$ for $H \in \mathcal{F}$ form a collection of compact projective generators in $\mathsf{Mod}_{\mathsf{Sp}_{\mathcal{F}}}(k)$, and so we're done if we show that the full subcategory on these objects is $\mathsf{Perm}_{\mathcal{F}}$.

Remark 3.4.15. In the discrete case, Mackey functors with this relationship between restriction and transfer are known as cohomological Mackey functors. By a theorem of Yoshida, these may also be described as modules over the Hecke algebra $\mathrm{End}_{kG} \bigoplus_{H \in \mathcal{F}} k[G/H]$. There's an analogous description for $D_{\mathcal{F}}(kG)$, e.g. as a consequence of the Schwede–Shipley theorem; in fact the ring remains discrete, cf. Remark 3.4.9. Notice that the Hecke algebra isn't commutative; to retain the symmetric monoidal structure, one needs to remember the Hopf algebra structure of the Hecke algebra, and view $D_{\mathcal{F}}(kG)$ as a comodule category instead.

Remark 3.4.16. From Theorem 3.4.12, we may infer that our category $D_{\mathcal{F}}(kG)$ has both genuine and geometric fixed points — but this could also have been seen directly from the description as $\mathsf{Fun}^\pi(\mathsf{Perm}_{\mathcal{F}}^{\mathrm{op}}, \mathsf{Sp})$. Genuine H-fixed points is the expected thing, namely the functor corepresented by $k[G/H]$. Geometric fixed points ought to be a colimit-preserving functor compatible with suspension spectra, and as such, we'd expect Φ^H to be the left Kan extension of an association $k[G/K] \mapsto k[(G/K)^H]$. If $\mathcal{F}$ is contained in the Sylow p-subgroup of G then this association can be made functorial and Φ^H is well-defined.

Now let X be a G-set with isotropy in $\mathcal{F}$ and consider the permutation representation $k[X]$, in other words '$\Sigma^\infty_+ X$'. Then the genuine fixed points $k[X]^H$ is just the classical fixed points of the representation. Notice that $k[X]^H$ is spanned by the H-orbits of X. We can tautologically split these up into size-1 orbits and larger orbits, yielding

$$k[X]^H \simeq \Phi^H k[X] \oplus (\text{stuff spanned by nontrivial orbits}).$$

This is analogous to the tom Dieck splitting.

Theorem 3.4.12 allows us to express the derived category $D^{b,\mathrm{fin}}_{\mathcal{F}}(kG)$ as a module category internal to $\mathcal{F}$-complete spectra. It would be beneficial if we could lift this to a statement internal to all G-spectra. This can be done, at least on the level of dualisable objects.

Theorem 3.4.17. We have an equivalence $\mathsf{Mod}_{\mathsf{Sp}_{\mathcal{F}}}(k)^{\mathrm{dual}} \simeq \mathsf{Mod}_{\mathsf{Sp}_G}(k^\wedge_{\mathcal{F}})^{\mathrm{dual}}$, where $k^\wedge_{\mathcal{F}}$ denotes the $\mathcal{F}$-completion of k.

Proof. $\mathsf{Mod}_{\mathsf{Sp}_G}(k^\wedge_{\mathcal{F}})$ is generated under colimits by the $k^\wedge_{\mathcal{F}}[G/H]$. Its dualisable objects are compact, hence the identity functor on a dualisable object factors through a finite colimit, which shows that the dualisable objects in $\mathsf{Mod}_{\mathsf{Sp}_G}(k^\wedge_{\mathcal{F}})$ are finite colimits of the permutation modules $k^\wedge_{\mathcal{F}}[G/H]$, or potential retracts thereof. From the sequence of identifications

$$\begin{aligned}
\underline{\mathrm{map}}(E\mathcal{F}_+, k^\wedge_{\mathcal{F}}[G/H]) &\simeq \underline{\mathrm{map}}(E\mathcal{F}_+, \mathrm{Ind}^G_H \mathrm{Res}^G_H k^\wedge_{\mathcal{F}}) \\
&\simeq \mathrm{Ind}^G_H \mathrm{Res}^G_H \underline{\mathrm{map}}(E\mathcal{F}_+, k^\wedge_{\mathcal{F}}) \\
&\simeq \underline{\mathrm{map}}(E\mathcal{F}_+, k^\wedge_{\mathcal{F}})[G/H] \\
&\simeq k^\wedge_{\mathcal{F}}[G/H]
\end{aligned}$$

we infer that the $k^\wedge_{\mathcal{F}}[G/H]$ (and hence their finite colimits) are $\mathcal{F}$-complete. Identifying with the objects $k[G/H]$ in $\mathsf{Mod}_{\mathsf{Sp}_{\mathcal{F}}}(k)^{\mathrm{dual}}$, we are by observing that $\mathsf{Mod}_{\mathsf{Sp}_{\mathcal{F}}}(k)^{\mathrm{dual}}$ is generated (under finite colimits) by these same objects $k[G/H]$ (but see Remark 3.5.1). ☐

We now turn our attention to the $\mathcal{F}$-stable module category $\mathsf{StMod}_{\mathcal{F}}(kG)$, obtained by taking the Verdier quotient of $D_{\mathcal{F}}(kG)$ by the thick $\otimes$-ideal generated by the $k[G/H]$ for $H \in \mathcal{F}$. This is entirely analogous to the Verdier quotient of Sp_G by $\langle \Sigma^\infty_+ G/H \rangle_{H \in \mathcal{F}}$. The latter has been studied in [MNN17] where it is called the $\mathcal{F}^{-1}$-localisation of Sp_G.

Theorem 3.4.18. We have an equivalence $\mathsf{StMod}_{\mathcal{F}}(kG)^{\mathrm{dual}} \simeq \mathsf{Mod}_{\mathsf{Sp}_G}(k^\wedge_{\mathcal{F}} \otimes \widetilde{E\mathcal{F}})^{\mathrm{dual}}$.

Proof. $\mathcal{F}^{-1}$-localisation is a smashing localisation, hence $\mathcal{F}^{-1}$-local modules over a ring are equivalently modules over the $\mathcal{F}^{-1}$-localised ring. The result thus follows from Theorem 3.4.17. ☐

This description of $\mathsf{StMod}_{\mathcal{F}}(kG)$ allows us to give an alternative description of the mapping spectra in $\mathsf{StMod}_{\mathcal{F}}(kG)$. The following result is a direct analogue of [Kra20, Lemma 4.2].

Corollary 3.4.19. On the level of dualisable objects, $\mathsf{StMod}_{\mathcal{F}}(kG)$ can be described as follows. If M and N are dualisable kG-modules, then

$$\mathrm{map}_{\mathsf{StMod}_{\mathcal{F}}(kG)}(M, N) \simeq \left(\underline{\mathrm{map}}_k(M, N)\right)^{t_{\mathcal{F}}G}.$$

Proof. A symmetric monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$ between closed symmetric monoidal ∞-categories commutes with internal Homs of dualisable objects, as evidenced by the series of identities

$$\begin{aligned}\underline{\mathrm{map}}\big(F(X), F(Y)\big) &\simeq \mathbb{D}F(X) \otimes F(Y) \\ &\simeq F(\mathbb{D}X) \otimes F(Y) \\ &\simeq F(\mathbb{D}X \otimes Y) \\ &\simeq F\big(\underline{\mathrm{map}}(X, Y)\big)\end{aligned}$$

In particular, if R is a commutative algebra object in $\mathcal{C}$, then we may take $F\colon \mathcal{C} \to \mathsf{Mod}_{\mathcal{C}}(R)$ to be the functor sending X to $R \otimes X$. Apply this to the case $\mathcal{C} = \mathsf{Mod}_{\mathsf{Sp}_G}(k)$ and $R = k^{\wedge}_{\mathcal{F}} \otimes \widetilde{E\mathcal{F}}$. For dualisable k-modules M and N with G-action, we then know that

$$\underline{\mathrm{map}}_{\mathsf{StMod}_{\mathcal{F}}(kG)}(M, N) \simeq \underline{\mathrm{map}}_k(M, N) \otimes k^{\wedge}_{\mathcal{F}} \otimes \widetilde{E\mathcal{F}}.$$

By an argument analogous to that in the proof of Theorem 3.4.17 we find that

$$\underline{\mathrm{map}}_k(M, N) \otimes k^{\wedge}_{\mathcal{F}} \otimes \widetilde{E\mathcal{F}} \simeq \underline{\mathrm{map}}_k(M, N)^{\wedge}_{\mathcal{F}} \otimes \widetilde{E\mathcal{F}}$$

To get the mapping spectrum, we simply pass to genuine G-fixed points. By definition, this is the $\mathcal{F}$-Tate construction of $\mathrm{map}_k(M, N)$. ☐

3.5 Final remarks

We end with some open-ended remarks. As they are largely speculative, I am compelled to be more imprecise in my use of language. I hope that the remarks are nonetheless of some value to the reader.

Remark 3.5.1. Throughout this section we have worked with a discrete base field. It is tempting to ask whether the results carry over to more general base rings or ring spectra. At least two complications arise.

In Remark 1.2.4 we assured that it's irrelevant whether we take the small or large stable module category, but this fails when we consider other base rings. As pointed out in [Kra20, Remark 4.3], $\mathsf{StMod}(RG)$ tends to exhibit nonsplit idempotents when R is a ring of integers. In upcoming work, Grodal and Krause establish the failure of $\mathsf{StMod}(RG)$ to be idempotent-complete in terms of character theory.

Second, we cannot pass from $\mathsf{Sp}_{\mathcal{F}}$ to Sp_G because Theorem 3.4.17 fails to hold. The complication arises at the very end, when we say that $\mathsf{Mod}_{\mathsf{Sp}_{\mathcal{F}}}(k)^{\mathrm{dual}}$ is generated under finite colimits by the objects $k[G/H]$. By [Tre15, Theorem A.4], this fact is true if you take the base ring to be a discrete regular Noetherian ring of finite Krull dimension, but there are counterexamples otherwise:

In Example A.3 of the same paper, Mathew considers the case $\mathcal{F} = \{e\}$, $G = C_2$, with base ring $R = \mathbb{F}_2[\varepsilon]/(\varepsilon^2)$. Then the free R-module of rank 1 with C_2-action specified by multiplication by $1 + \varepsilon$ defines a $\otimes$-invertible object in the functor category which does not belong to the thick subcategory generated by the permutation modules.

Passing to non-discrete $\mathbb{E}_\infty$-rings, the result fails even for the most well-behaved regular rings, such as the sphere spectrum. The functor category $\mathsf{Fun}\big(BC_p, \mathsf{Perf}(\mathbb{S})\big)$ has many exotic $\otimes$-invertible objects: by the Atiyah–Segal theorem, maps $BC_p \to BO^{\wedge}_p$ are classified by the p-adically completed real representation ring $\mathrm{RO}(C_p)$; composing these maps with the p-completed J-homomorphism $BO_p^\wedge \to \mathsf{Sp}$ produces uncountably many Picard elements.

Remark 3.5.2. In Example 3.2.8 we observed that $\mathcal{F}$-endotrivial modules are closed under tensor induction if $\mathcal{P}$ has all proper subgroups, but not if $\mathcal{P}$ has only the trivial subgroup. On the level of stable module categories, this manifests itself as the existence of symmetric monoidal functors between the $\mathsf{StMod}_{\mathcal{F}}(kG)$ for varying G and large enough $\mathcal{F}$. These functors seem to be analogous to the Hill–Hopkins–Ravenel norm maps N_H^G found in equivariant homotopy theory.

Remark 3.5.3. In [Kra20], A. Krause develops a mechanism of 'isotropy separation for compact objects', manifesting into a pullback square

$$\begin{tikzcd}
\mathsf{Mod}_{\mathsf{Sp}_G}(R_G)^\omega/\mathcal{G} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & \mathsf{Mod}_{\mathsf{Sp}_G}(R_G)^\omega/\big(\mathcal{G} \cup \{G/K\}\big) \arrow[d] \\
\mathsf{Fun}\big(BW_G(K), \mathsf{Mod}_{\mathsf{Sp}}(R)^\omega\big) \arrow[r] & \mathsf{StMod}\big(RW_G(K)\big)
\end{tikzcd}$$

for any connective $\mathbb{E}_\infty$-ring R, family $\mathcal{G}$, and subgroup K such that $K' \in \mathcal{G}$ for all $K' \subsetneq K$. Unfortunately, $\mathsf{Mod}_{\mathsf{Sp}_{\mathcal{F}}}(k)$ doesn't quite fit into the mechanism, because, despite what the notation may suggest, the Eilenberg–MacLane Mackey functor k isn't the inflation of a non-equivariant spectrum. Nonetheless, it's reasonable to expect a similar-looking square to exist. Think of $D_{\mathcal{F}}(kG)$ as $\mathsf{Fun}^\pi(\mathsf{Perm}_{\mathcal{F}}^{\mathrm{op}}, \mathsf{Sp})$. Killing a family $\mathcal{G}$ is akin to restricting the 'domain of definition' from $\mathsf{Perm}_{\mathcal{F}}$ to the open complement $\mathsf{Perm}_{\mathcal{F}} \setminus \mathsf{Perm}_{\mathcal{G}}$, and killing an additional element G/K is akin to taking an open complement of the 'closed point' $B\,\mathrm{Aut}\big(k[G/K]\big)$. Perhaps, then, there's some kind of square

$$\begin{tikzcd}
D_{\mathcal{F}}(G)^{\mathrm{dual}}/\mathcal{G} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & D_{\mathcal{F}}(G)^{\mathrm{dual}}/\big(\mathcal{G} \cup \{G/K\}\big) \arrow[d] \\
\mathsf{Fun}^\pi\big(\big\{\bigoplus k[G/K]\big\}^{\mathrm{op}}, \mathsf{Sp}\big)^{\mathrm{dual}} \arrow[r] & \mathsf{Fun}^\pi\big(\big\{\bigoplus k[G/K]\big\}^{\mathrm{op}}, \mathsf{Sp}\big)^{\mathrm{dual}}/\langle k[G/K]\rangle
\end{tikzcd}$$

Given an element of the Dade group $\mathrm{Pic}\, D_{\mathcal{F}}(G)^{\mathrm{dual}}/\mathcal{F}$, we can ask how far it can climb up the isotropy separation ladder. This yields a kind of filtration, indexed by the conjugacy classes of subgroups of G. Classical literature often relates the Dade group to Weyl groups of subgroups of G. For instance, [Bou06, Theorem 8.2] establishes a direct sum decomposition $D(G)_{\mathrm{tors}} \simeq \bigoplus_K T\big(W_G(K)\big)_{\mathrm{tors}}$ where K ranges over certain subgroups of G. It seems reasonable that the filtration and the direct sum decomposition are related.

Remark 3.5.4. Throughout this section we have worked with $\mathsf{StMod}_{\mathcal{P}}(kG)$ where $\mathcal{P}$ is generated by permutation modules, but it would be interesting to examine the nature of this stable module category for other $\mathcal{P}$. Of particular interest, at least to me, is the case $\mathcal{P} = (p)$, as introduced in Remark 3.2.6. Benson's conjecture would then say that $\mathsf{StMod}_{(2)}(kG)$ has the curious field-like property that all objects are direct sums of $\otimes$-invertibles.

The special nature of 2-groups can already be observed by considering the classical stable module category of the cyclic group C_p. This stable module category is equivalent to $\mathsf{Mod}(k^{tC_p})$, and

$$\pi_*\, k^{tC_p} \simeq \begin{cases} k[x^{\pm 1}] \text{ with } |x| = 1 & \text{if } p = 2; \\ k[x^{\pm 1}] \otimes \Lambda(y) \text{ with } |x| = 2,\ |y| = 1 & \text{if } p \text{ is odd.} \end{cases}$$

The ring k^{tC_2} is the most field-like because $\pi_* k^{tC_2}$ is a graded field. That said, k^{tC_3} is still somewhat field-like in the sense that all its compact modules are free. (In the literature, $\mathbb{A}_\infty$-rings with this property are known as semisimple ring spectra.) For larger primes, $\mathsf{Mod}(k^{tC_p})$ can still be analysed directly — the modular representations theory is tame for all p thanks to Jordan normal form theory — but the complexity of $\mathsf{Mod}(k^{tC_p})$ increases exponentially in p.

3.A The vertex of a representation

The goal of this appendix is to define the vertex of a modular representation, and verify some of its basic properties. To define vertices, we first need to recall the notion of H-projectivity — a topic which we will get back to in Section 3.D. All of the material in this appendix is well known.

Let P be a finite-dimensional kG-module, and let H be a subgroup of G. Then we say that M is H-projective if P is isomorphic to a direct summand of a module induced up from H.

Example 3.A.1. The usual notion of projectivity is clearly equivalent to H-projectivity where H is the trivial subgroup. At the opposite end of the scale, any kG-module is S-projective for S a Sylow p-subgroup of G, which can be seen more easily after the lemma below.

Lemma 3.A.2. Let P be a kG-module. Then P is H-projective if and only if any, and hence all, of the following properties are satisfied.

(a) P satisfies the following lifting property: We are given a homomorphism $g\colon N' \to N$ and a surjective homomorphism $f\colon P \to N$. If f admits a lift along g as an H-module, then in fact it admits a lift along g as a G-module.

(b) P satisfies the following splitting property: If a surjective G-homomorphism $f\colon Q \to P$ admits a splitting as an H-homomorphism, then in fact it splits as a G-homomomorphism.

(c) P is a direct summand of $\operatorname{Ind}_H^G \operatorname{Res}_H^G P$.

(d) There exists a kG-module M such that P is a direct summand of $k(G/H) \otimes_k M$.

Proof. We prove that H-projectivity implies (a). First assume that $P = \operatorname{Ind}_H^G M$ for some H-module M. We are given a commutative diagram of H-modules

$$\begin{tikzcd}
& \operatorname{Res}_H^G \operatorname{Ind}_H^G M \arrow[dl, "\overline{f}"'] \arrow[d, "f"] \\
\operatorname{Res}_H^G N' \arrow[r, two heads, "g"'] & \operatorname{Res}_H^G N
\end{tikzcd}$$

To find a lift of f as a G-module, precompose $\overline{f}$ with the unit $M \to \operatorname{Res}_H^G \operatorname{Ind}_H^G M$; the adjoint maps, by naturality, then form the desired lifting diagram of G-modules.

More generally, P is merely a summand of a kG-module $\operatorname{Ind}_H^G M$. Let's write ι and π for the implied inclusion and projection. Apply the same method as before to obtain a lifting of the composition $f \circ \pi$ and precompose the resulting lift with ι to obtain the desired result.

To prove that (a) implies (b), start with a section σ_H of the surjective H-module map $f\colon \operatorname{Res}^G_H Q \to \operatorname{Res}^G_H P$, and apply the lifting property of (a) to the diagram

$$\begin{tikzcd}
 & \operatorname{Res}^G_H P \arrow[ld, "\sigma_H"'] \arrow[d, equal] \\
\operatorname{Res}^G_H Q \arrow[r, "f"'] & \operatorname{Res}^G_H P
\end{tikzcd}$$

To prove that (b) implies (c), observe that the map $\operatorname{Ind}^G_H \operatorname{Res}^G_H M \to M$ must split as a kG-homomorphism, hence M is a direct summand of $\operatorname{Ind}^G_H \operatorname{Res}^G_H M$. This in turns implies (d) thanks to the series of adjunctions

$$\begin{aligned}
\operatorname{Hom}_{kG}\big(M \otimes_k k(G/H), N\big) &\simeq \operatorname{Hom}_{kG}\big(M, \operatorname{Hom}_k(k(G/H), N)\big) && \text{tensor} \dashv \text{Hom} \\
&\simeq \operatorname{Hom}_{kG}\big(M, \operatorname{CoInd}^G_H(\operatorname{Res}^G_H N)\big) && \text{by definition} \\
&\simeq \operatorname{Hom}_{kH}\big(\operatorname{Res}^G_H M, \operatorname{Res}^G_H(M)\big) && \text{restriction} \dashv \text{coinduction} \\
&\simeq \operatorname{Hom}_{kG}\big(\operatorname{Ind}^G_H(M) \operatorname{Res}^G_H(M), N\big) && \text{induction} \dashv \text{restriction}
\end{aligned}$$

In turn, any module satisfying (d) is clearly H-projective, so we're done. ☐

A subgroup H of an indecomposable module M is called a vertex of M if it is a minimal with respect to the property that M is H-projective.

Lemma 3.A.3. The vertex of a module M is well-defined up to conjugacy. Moreover, it is always a p-subgroup.

Proof sketch. By our discussion in Example 3.A.1, H will be contained within the Sylow p-subgroup of G, hence it's a p-subgroup. As for uniqueness, if M is both H-projective and K-projective, then it will also be $(H \cap K^g)$-projective for a choice of g, as can be seen by applying Lemma 3.A.2 (c) to both H and K and invoking the Mackey formula. ☐

We leave the following lemma for future reference.

Lemma 3.A.4. Assume that k is algebraically closed. Let M be an indecomposable kG-module with vertex H. Then $\dim_k(M)$ is divisible by the index of H in the Sylow p-subgroup of G.

We sketch a proof of this lemma by reducing it to the following well-known theorem of Green.

Theorem 3.A.5 (Green Indecomposability Theorem)**.** Assume that k is algebraically closed. Let L be an indecomposable kH-module, where H is a normal subgroup of G such that $[G : H] = p$. Then the induced module $\operatorname{Ind}^G_H L$ is indecomposable.

Proof sketch of Lemma 3.A.4. If S is a Sylow p-subgroup of G, then the indecomposable summands of $\operatorname{Res}^G_S M$ have their vertex contained in H; thus, it suffices to consider the case where G is a p-group, which we henceforth assume.

M is a summand of $\operatorname{Ind}^G_H L$ for some H-module L which by the Krull–Schmidt theorem may be assumed to be indecomposable. I claim that $\operatorname{Ind}^G_H L$ must itself be indecomposable. Since G is a p-group, any subgroup is properly contained in its normaliser; as such, there's a subnormal series from H to G whose factors are cyclic of order p. The result now follows by iteratively applying Theorem 3.A.5. ☐

3.B Generalities on exact categories

This section serves as a catch-all repository for basic results within the theory of exact categories. Nothing that is written in this section is new.

Let $\mathcal{A}$ be an additive category. A kernel–cokernel pair in $\mathcal{A}$ is a pair (i, p) of morphisms $i\colon A' \xrightarrow{i} A$ and $p\colon A \to A''$, such that i is the cokernel of p and p is the kernel of i. Fix a class $\mathcal{E}$ of kernel–cokernel pairs in $\mathcal{A}$. With respect to $\mathcal{E}$, a morphism i is called admissible monic if there exists a p such that $(i, p) \in \mathcal{E}$; dually, p is called admissible epic.

A class $\mathcal{E}$ of kernel–cokernel pairs in $\mathcal{A}$ is called an exact structure on $\mathcal{A}$ if the following axioms are satisfied.

- For all $A \in \mathcal{A}$, the identity morphism Id_A is admissible monic and admissible epic;
- the admissible monics and admissible epics are closed under composition;
- admissible monics are closed under pushouts, and admissible epics are closed under pullbacks.

Together, $(\mathcal{A}, \mathcal{E})$ is often called an exact category, and kernel–cokernel pairs in $\mathcal{E}$ are referred to as short exact sequences. If $(\mathcal{A}, \mathcal{E})$ and $(\mathcal{A}', \mathcal{E}')$ are two exact categories, then an exact functor is an additive functor $\mathcal{A} \to \mathcal{A}'$ sending $\mathcal{E}$ into $\mathcal{E}'$.

As is well known, the axioms of an exact category encapsulate the properties of short exact sequences sufficiently well that most standard results of homological algebra generalise to exact categories. We refer to [Büh10] for more information.

Lemma 3.B.1 ([Büh10, Prop. 2.16])**.** Let $i\colon A \to B$ be a morphism in $\mathcal{A}$ admitting a cokernel. If there exists a morphism $j\colon B \to C$ such that $j \circ i$ is an admissible monic, then i is also an admissible monic. Dually, starting with $j\colon B \to C$, if there's an $i\colon A \to B$ such that $j \circ i$ is an admissible epic, then so is j.

Proof sketch. We prove the first statement only, as the second is formally dual. Write $k\colon B \to \operatorname{Coker} i$ for the cokernel of $i\colon A \to B$. Now consider the following diagram:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, equal] & B \arrow[r, "k"] \arrow[d, "\mathrm{Id} \oplus 0"'] \arrow[dr, phantom, very near start, "\ulcorner"] & \operatorname{Coker} i \arrow[d, "\mathrm{Id} \oplus 0"] \\
A \arrow[r, "i \oplus 0"'] & B \oplus C \arrow[r, "k \oplus \mathrm{Id}"'] & \operatorname{Coker} i \oplus C
\end{tikzcd}$$

In this diagram, $k \oplus \mathrm{Id}_C$ may be exhibited as the cokernel of $i \oplus 0$, from which one infers that it is an admissible epic. But the right square is a pullback square, forcing k to be admissible epic as well. □

Let $\mathcal{A}$ be an exact category. An object P is called $\mathcal{E}$-projective, or just projective for short, if $\mathrm{Hom}_{\mathcal{A}}(P, \cdot\,)\colon \mathcal{A} \to \mathsf{Ab}$ is exact, where Ab is endowed with the standard exact structure; dually, an object is called $\mathcal{E}$-injective if $\mathrm{Hom}_{\mathcal{A}}(\,\cdot\,, I)\colon \mathcal{A}^{\mathrm{op}} \to \mathsf{Ab}$ is exact.

Lemma 3.B.2. An object P in an exact category $\mathcal{A}$ is projective if and only if it satisfies the usual lifting property along admissible epimorphisms. Dually, an object is injective if and only if it satisfies the lifting property along admissible monomorphisms.

Proof. The standard proof (for abelian categories with the standard exact structure) carries over virtually without change. □

Let $\mathcal{P}$ be a set of objects in an exact category $(\mathcal{A}, \mathcal{E})$. Then we can define a new exact structure on $\mathcal{A}$ by saying that a sequence $A' \to A \to A''$ is exact if and only if it is $\mathcal{E}$-exact *and* the sequences $\mathrm{Hom}(P, A') \to \mathrm{Hom}(P, A) \to \mathrm{Hom}(P, A'')$ are short exact in Ab for every object P in $\mathcal{P}$. Let us denote by $\mathcal{E}_{\mathcal{P}}$ the resulting exact category.

Lemma 3.B.3. The class $\mathcal{E}_{\mathcal{P}}$ defined above indeed defines an exact structure.

Proof sketch. The proof involves some diagram chasing techniques. Let's first clear out the easy parts. Clearly the identity is admissible mono and epi with respect to $\mathcal{E}_{\mathcal{P}}$. Admissible epimorphisms are closed under composition and pullbacks because $\mathrm{Hom}(P, \cdot\,)$ preserves kernels and pullbacks.

Suppose $f\colon A \to B$ and $g\colon B \to C$ are admissible monomorphisms in $\mathcal{E}_{\mathcal{P}}$. To show that their composition is again admissible, consider the commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B/A \arrow[d] \\
A \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C/A \arrow[d] \\
0 \arrow[r] & C/B \arrow[r] & C/B
\end{tikzcd}$$

where the quotient notation is just a shorthand for the cokernel. I claim that, upon applying $\mathrm{Hom}(P, \cdot\,)$, all columns, as well as the top and bottom row, become short exact sequences. For the most part, this is obvious, perhaps with the exception of the rightmost column, where we surjectivity of $\mathrm{Hom}(P, C/A) \to \mathrm{Hom}(P, C/B)$ follows by precomposing with $\mathrm{Hom}(P, C) \to \mathrm{Hom}(P, C/A)$ and noting that the map $C \to C/B$ was $\mathcal{E}_{\mathcal{P}}$-admissible epic. At this point, apply the Nine Lemma to conclude that the middle row is exact as well.

Finally, consider an admissible monomorphism $i\colon A \to B$ and take its pushout along a map $f\colon A \to A'$. We have a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near end] & B \arrow[r] \arrow[d] & B/A \arrow[d] \\
A' \arrow[r] & B' \arrow[r] & B'/A'
\end{tikzcd}$$

Now apply $\mathrm{Hom}(P, \cdot\,)$. Surely the top row remains short exact, while the bottom row is a priori merely left-exact. However, surjectivity of the map $\mathrm{Hom}(P, B') \to \mathrm{Hom}(P, B'/A')$ follows by precomposing with $\mathrm{Hom}(P, B) \to \mathrm{Hom}(P, B')$ and observing that the natural map $B/A \to B'/A'$ was an isomorphism by virtue of the pushout construction. □

Remark 3.B.4. More generally, if $F\colon (\mathcal{A}, \mathcal{E}) \to (\mathcal{A}', \mathcal{E}')$ is a functor of exact categories which isn't exact but preserves admissible kernels, then the same proof technique shows that the collection of kernel–cokernel pairs $A' \to A \to A''$ in $\mathcal{E}$ which F sends to a kernel–cokernel pair in $\mathcal{E}'$ defines a new exact structure on $\mathcal{A}$. At two points one needs to invoke the dual of Lemma 3.B.1 to make the proof go through; additionally, one must check that the Nine Lemma still makes sense in general exact categories; cf. [Büh10, Lem. 3.6].

Remark 3.B.5. In full generality, it is not clear what the projectives of $\mathcal{E}_{\mathcal{P}}$ ought to be, but they are 'what you'd expect' under a mild assumption that there are 'enough' objects in $\mathcal{P}$. To make this precise, assume that $\mathcal{A}$ admits direct sums. Starting with an object X, consider the object $\bigoplus_{P \to X} P$, where the direct sum is taken over all maps $P \to X$ with $P \in \mathcal{P}$. We impose the assumption that the natural map $\bigoplus_{P \to X} P \to X$ is an admissible epimorphism with respect to $\mathcal{E}$. Notice that it is then automatically also an admissible epimorphism with respect to $\mathcal{E}_{\mathcal{P}}$. Now if X were $\mathcal{E}_{\mathcal{P}}$-projective, then this map would split, so X would be a summand of $\bigoplus_{P \to X} P$. We conclude that the projective objects with respect to $\mathcal{E}_{\mathcal{P}}$ are direct sums of objects in $\mathcal{P}$, along with any additional summands. I learned this argument from [CH02, Lem. 1.5].

We say $\mathcal{A}$ has enough projectives if, for every $A \in \mathcal{A}$, there exists an admissible epic $P \to A$ from an $\mathcal{E}$-projective object P; dually, $\mathcal{A}$ has enough injectives if any object A admits an admissible mono $A \to I$ into an $\mathcal{E}$-injective object I.

Lemma 3.B.6. Let $\mathcal{A}$ be an exact category with enough projectives. Then a sequence $A' \to A \to A''$ is $\mathcal{E}$-exact if and only if $\mathrm{Hom}(P, A') \to \mathrm{Hom}(P, A) \to \mathrm{Hom}(P, A'')$ is a short exact sequence of abelian groups for every projective object P in $\mathcal{A}$.

Proof. We first observe that $A \to A''$ is an admissible epimorphism. As $\mathcal{A}$ has enough projectives, there exists a projective cover $P \to A''$. By surjectivity of $\mathrm{Hom}(P, A) \to \mathrm{Hom}(P, A'')$, there exists a lift of this cover to a map $P \to A$. Lemma 3.B.1 now implies the desired result.

Next, we prove that $A' \to A$ is a monomorphism. Suppose that we're given two maps $B \rightrightarrows A'$ which become the same upon composing with A. Take an admissible epimorphism $P \to B$. Then the two maps $P \to B \rightrightarrows A' \to A$ are the same, and so by injectivity of $\mathrm{Hom}(P, A') \to \mathrm{Hom}(P, A)$, we conclude that the two maps $P \to B \rightrightarrows A'$ coincide as well. As $P \to B$ was an epimorphism, the maps $B \rightrightarrows A'$ coincide, as desired.

Now pick a projective cover $P \to A'$, and notice that the composition $P \to A' \to A \to A''$ is the zero map, from which we infer that $A' \to A \to A''$ is the zero map as well; consequently, $A' \to A$ factors through the kernel K of $A \to A''$. The map $A' \to K$ is an admissible epimorphism, as one can see by taking a projective cover $P \to K$, lifting it to A', and applying Lemma 3.B.1; on the other hand, it is also a monomorphism, since so is $A' \to A$.

Because the map $A' \to K$ is both a monomorphism and an admissible epi, it fits in the kernel–cokernel pair $0 \to A' \to K$. Applying the Five Lemma [Büh10, Cor. 3.2] to the morphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & K \arrow[d] \\
0 \arrow[r] & K \arrow[r, "\mathrm{Id}"'] & K
\end{tikzcd}$$

shows that $A' \to K$ is an isomorphism, which proves the result. □

Let $\mathcal{A}$ be an additive category. Write $\mathsf{Ch}(\mathcal{A})$ for the category of chain complexes in $\mathcal{A}$. If $\mathcal{A}$ admits an exact structure, then so does $\mathsf{Ch}(\mathcal{A})$: the short exact sequences are declared to be the chain maps which are exact in each degree. In $\mathsf{Ch}(\mathcal{A})$, the notion of chain homotopy is defined as usual, leading to the homotopy category $K(\mathcal{A})$, in which we take $\mathsf{Ch}(\mathcal{A})$ and mod out by the nullhomotopic maps.

This is well known to be a triangulated category, the triangulation being given by the shift functor $\Sigma A_* = A_{*-1}$.

A chain complex A_* in an exact category $\mathcal{A}$ is called $\mathcal{E}$-acyclic if the following properties are satisfied.

- Every differential $A_n \to A_{n-1}$ factors as a decomposition $A_n \to Z_n \to A_{n-1}$ such that $A_n \to Z_n$ is an admissible epi and $Z_n \to A_{n-1}$ an admissible mono;
- every resulting composition $Z_n \to A_n \to Z_{n-1}$ is exact.

Lemma 3.B.7 ([Büh10, Cor. 10.5])**.** The homotopy category $\mathsf{Ac}(\mathcal{A})$ of acyclic complexes forms a triangulated subcategory of $K(\mathcal{A})$.

If $\mathcal{A}$ is assumed to be idempotent-complete, then a bit more can be said. In all cases of relevance for us, $\mathcal{A}$ will be idempotent-complete anyway.

Lemma 3.B.8 ([Büh10, Cor. 10.11])**.** If $\mathcal{A}$ is idempotent-complete, then the acyclic complexes are closed under isomorphisms in $K(\mathcal{A})$ and $\mathsf{Ac}(\mathcal{A})$ forms a thick subcategory of $K(\mathcal{A})$.

At this point it makes sense to define the derived category of an exact category $\mathcal{A}$, denoted $D(\mathcal{A})$, as the triangulated Verdier quotient of $K(\mathcal{A})$ by the $\mathcal{E}$-acyclic complexes. More generally, if $\mathcal{A}$ fails to be idempotent-complete, we should take the thick closure of the $\mathcal{E}$-acyclics. The various cousins of $D(\mathcal{A})$ that ask for certain boundedness conditions on the chain complexes are defined analogously. This Verdier quotient is obtained by formally inverting the morphisms whose cofibre is $\mathcal{E}$-acyclic — such a chain map is also called a quasi-isomorphism.

If $\mathcal{A}$ has enough projectives, then acyclicity can be measured by considering mappings out of projectives. We will make use of this in Section 3.4 to understand the quasi-isomorphisms of the relative derived category.

Lemma 3.B.9. Let $\mathcal{A}$ be an idempotent-complete exact category, and assume that it has enough projective objects. Then a bounded chain complex A_* is $\mathcal{E}$-acyclic if and only if $\mathrm{Hom}(P, A_*)$ is an exact sequence for every projective object P. Consequently, a chain map $A_* \to B_*$ is a quasi-isomorphism if and only if the induced map $\mathrm{Hom}(P, A_*) \to \mathrm{Hom}(P, B_*)$ is a quasi-isomorphism in the classical sense.

Proof. The Z_n are unique in that they must be the kernel of $A_{n-1} \to A_{n-2}$, as well as the image of $A_n \to A_{n-1}$. Therefore, the claim about acyclicity reduces to checking exactness of the compositions $Z_n \to A_n \to Z_{n-1}$ by mapping out of projectives, which is the content of Lemma 3.B.6. ☐

Remark 3.B.10. Although we will not explicitly need it, it is worth pointing out that one may frequently endow the category of chain complexes $\mathsf{Ch}(\mathcal{E})$ of our exact category $\mathcal{E}$ with a model structure with respect to which the quasi-isomorphisms become weak equivalences. In the setup of Lemma 3.B.9, fibrations can be defined by mapping out of projectives. That is, the fibrations are those chain maps $f\colon A_* \to B_*$ for which the induced map $\mathrm{Hom}(P, A_*) \to \mathrm{Hom}(P, B_*)$ is a fibration of chain complexes of abelian groups, i.e. a degreewise surjection; as usual, the cofibrations are simply taken to be those maps with a suitable lifting property along trivial fibrations.

Remark 3.B.11. The exact categories that are of relevance to us are moreover equipped with a symmetric monoidal structure, and it is reasonable to ask how these fit together. Notably, it is not to be expected in general that the acyclic complexes form a $\otimes$-ideal; consequently, the Verdier quotient need not be symmetric monoidal anymore. In the context of model categories, the existence of a monoidal structure on the derived category has been considered in [CH02, Section 2.1], where criteria are given under which their model structure yields a monoidal model category. For what it's worth these criteria are satisfied in the examples that are of interest to us.

Remark 3.B.12. In the language of exact ∞-categories, there's a forgetful functor $\mathsf{Cat}_{\mathsf{stable}} \to \mathsf{Cat}_{\mathsf{exact}}$, and the derived category of an exact category is simply the left adjoint.

3.C Frobenius categories

In this section we study Frobenius categories, which are a special class of exact categories. We refer the reader to Section 3.B for the necessary generalities on the theory of exact categories.

Following [Hap88], we define a Frobenius category to be an exact category with enough projectives and injectives, which satisfies the property that the projectives and injectives coincide. If $\mathcal{A}$ is a Frobenius category, then we define the stable category $\mathsf{St}(\mathcal{A})$ to be the additive quotient of $\mathcal{A}$ by the maps which factor through a projective. That is, the objects of $\mathsf{St}(\mathcal{A})$ are those of $\mathcal{A}$, and

$$\mathrm{Hom}_{\mathsf{St}(\mathcal{A})}(A, B) = \mathrm{Hom}_{\mathcal{A}}(A, B) / \, \mathrm{PHom}_{\mathcal{A}}(A, B),$$

where $\mathrm{PHom}_{\mathcal{A}}(A, B)$ is the collection of those maps $A \to B$ which can be made to factor through a projective object.

Lemma 3.C.1 ([Hap88, Section I.2])**.** $\mathsf{St}(\mathcal{A})$ is a triangulated category.

Proof sketch. The auto-equivalence Σ of $\mathsf{St}(\mathcal{A})$ is defined as follows. If X is an object in $\mathcal{A}$, then there exists an admissible mono $X \hookrightarrow I(X)$ for some injective object $I(X)$. Pick a corresponding admissible epi $I(X) \twoheadrightarrow Y$. Then we declare ΣX to be the object Y. The inverse Σ^{-1} can likewise be obtained by taking the admissible kernel of a projective cover $P(X) \twoheadrightarrow X$. Notice that the assignment can be made functorial.

We now consider the triangulation on $\mathsf{St}(\mathcal{A})$. Starting with a map $f \colon X \to Y$, let's construct a diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "f"'] & I(X) \arrow[r] \arrow[d] & \Sigma X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & Z \arrow[r] & \Sigma X \arrow[r] & 0
\end{tikzcd}$$

in $\mathcal{A}$, where the map $Z \to \Sigma X$ is defined through the universal property of Z. We simply declare the distinguished triangles to be those isomorphic to the triangles $X \to Y \to Z \to \Sigma X$ arising from the procedure we outlined.

(TR1) and (TR3) are satisfied for obvious reasons; (TR2) and (TR4) are verified in a straightforward but tedious manner. Let's sketch the verification of (TR2), referring the reader to the reference for (TR4). If $X \to Y \to Z \to \Sigma X$ is a distinguished triangle, then exactness of the first two columns in the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & Y \arrow[r] \arrow[d] & I(Y) \arrow[r] \arrow[d] & \Sigma Y \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z \arrow[r] \arrow[d] & I(Y) \oplus \Sigma X \arrow[r] \arrow[d] & \Sigma Y \arrow[r] & 0 \\
 & \Sigma Y \arrow[r, equal] \arrow[d] & \Sigma Y \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

implies that $I(Y) \oplus \Sigma X$ is a pushout in the top left square, so that $Y \to Z \to I(Y) \oplus \Sigma X \to \Sigma(Y)$ is a distinguished triangle, and in $\mathsf{St}(\mathcal{A})$, this is isomorphic to $Y \to Z \to \Sigma X \to \Sigma Y$. $\square$

The triangulation on $\mathsf{St}(\mathcal{A})$ strongly suggests that it arises as the homotopy category of a suitable stable ∞-category, and indeed this turns out to be the case. The following result is well known in the case that $\mathcal{A}$ is the abelian category of modules over a Frobenius ring, where it is a theorem of [Ric89].

Following Section 3.B, we define $D^b(\mathcal{A})$ to be the bounded derived category of the exact category $\mathcal{A}$. Let us define the perfect complexes to be the complexes which are quasi-isomorphic to a bounded complex of *projective* objects. They form a full subcategory of $D^b(\mathcal{A})$ that we will denote by $D^b(P_{\mathcal{A}})$.

Theorem 3.C.2. The stable category $\mathsf{St}(\mathcal{A})$ is equivalent to the Verdier quotient of $D^b(\mathcal{A})$ by the subcategory of perfect complexes.

Proof. Observe first that $D^b(P_{\mathcal{A}})$ forms a triangulated subcategory which is closed under direct summands, so that the Verdier quotient inherits a natural triangulated structure from $D^b(\mathcal{A})$.

Now take a look at the additive functor $\mathcal{A} \to D^b(\mathcal{A})/D^b(P_{\mathcal{A}})$ sending an object to its complex concentrated in degree 0. This map clearly kills projectives, so that it factors through $\mathsf{St}(\mathcal{A})$ to produce an additive functor $F\colon \mathsf{St}(\mathcal{A}) \to D^b(\mathcal{A})/D^b(P_{\mathcal{A}})$.

We claim that F is exact. Before we prove this, first observe the following. If X is in $\mathcal{A}$, then we may form the short exact sequence $0 \to X \to I(X) \to \Sigma X \to 0$. Now apply F. Effectively by definition, in $D^b(\mathcal{A})$, a short exact sequence gets sent to a distinguished triangle. On the other hand, as projectives get killed upon quotienting, $I(X)$ gets killed, and we find that $X \to 0 \to \Sigma X \to \Sigma X$ is a distinguished triangle in $D^b(\mathcal{A})/D^b(P_{\mathcal{A}})$. From this we may infer that $\Sigma X \simeq X[1]$.

Now let's form a distinguished triangle in $\mathsf{St}(\mathcal{A})$ from the diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & I(X) \arrow[r] \arrow[d] & \Sigma X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] & Z \arrow[r] & \Sigma X \arrow[r] & 0
\end{tikzcd}$$

Let's apply F to this diagram. The bottom row gets sent to the distinguished triangle $Y \to Z \to \Sigma X \to Y[1]$, which by the previous remark is isomorphic to $Y \to Z \to X[1] \to Y[1]$. Now, by shifting, we find that $X \to Y \to Z \to X[1]$ is a distinguished triangle as well. This proves exactness.

Fullness of F is easy to see, but faithfulness is harder to show. As a first step, let's check that F is 'injective on objects', in that if $F(X) \simeq 0$ then also $X \simeq 0$. What this effectively says is that no nonprojective object in $\mathcal{A}$ is quasi-isomorphic to a finite complex of projectives. This may seen counterintuitive — indeed if $P_* \to A$ is a finite projective resolution of a nonprojective object A, then P_* and A are quasi-isomorphic. But this can never happen. Indeed, suppose that $0 \to P_n \to \cdots \to P_0 \to A$ is such a resolution. Then $P_n \hookrightarrow P_{n-1}$ is an admissible monomorphism, and as P_n and P_{n-1} are also injective, this forces P_n to become a summand P_{n-1}, allowing us to delete P_n from the resolution it and shorten it. Now iterate this procedure to conclude that A is in fact projective.

We're now in a position to check that F is faithful. Suppose $\alpha\colon X \to Y$ is a map in $\mathsf{St}(\mathcal{A})$ such that $F(\alpha) = 0$. Then we prove that $\alpha = 0$. First, we put α in a distinguished triangle $X \hookrightarrow \alpha Y \hookrightarrow \beta Z \to \Sigma X$. Now apply F to this triangle. As α gets mapped to 0, by general triangulated nonsense there must be a map $g\colon FZ \to FY$ such that $g \circ F\beta = \mathrm{Id}$. Since F is full, $g = F\varepsilon$ for some $\varepsilon\colon Z \to Y$. We see that $F(\varepsilon \circ \beta) = \mathrm{Id}$, that F commutes with cones, and that the cone of Id is 0, and so the cone of $\varepsilon \circ \beta$ must get sent to 0. But by 'injectivity on objects', the cone of $\varepsilon \circ \beta$ must already be 0, hence $\varepsilon \circ \beta$ is an isomorphism. By yet more general triangulated nonsense, this tells us that $Y \simeq X \oplus \Sigma Z$, and so our triangle becomes isomorphic to a direct sum of $0 \to Y \to Y \to 0$ and $X \to 0 \to \Sigma X \to \Sigma X$. We find that α must be 0.

Finally, we show that F is essentially surjective. Take an object X in $D^b(\mathcal{A})/D^b(P_{\mathcal{A}})$. Lift it to a representative in $D^b(A)$, and rewrite it, by taking projective resolutions, as a complex of projectives

$$P_* = \cdots \to P_r \to P_{r-1} \to \cdots \to P_s \to 0 \to \cdots,$$

where P_* has trivial homology above degree r for some sufficiently large r. Now consider the map from P_* to

$$\widetilde{P}_* = \cdots \to P_{r+2} \to P_{r+1} \to P_r \to 0 \to \cdots.$$

This map is an isomorphism in $D^b(\mathcal{A})/D^b(P_{\mathcal{A}})$ because the mapping cone is a bounded complex of projectives. But $\widetilde{P}_*$ has nonzero homology only at position r, hence $\widetilde{P}_*$ is quasi-isomorphic to the stalk complex $\Sigma^{-r}M$ for some M in $\mathcal{A}$, and this object clearly gets hit by F. □

Remark 3.C.3. Classically, the perfect complexes coincide with the compact objects in the derived ∞-category. To what extent does this generalise? On the one hand, we'll see in Section 3.4 that this generalises to certain 'relative' derived categories of kG-modules. On the one hand, let $\mathcal{A}$ be the category of R-modules endowed with the 'trivial' exact structure in which only the split exact sequences are declared to be exact. Then every object is both projective and injective. The derived category $D(\mathcal{A})$ is in fact the homotopy category $K(R)$ of chain complexes of R-modules, which is not compactly generated let alone every object being compact.

Remark 3.C.4. We have conveniently ignored the symmetric monoidal structure. To ensure that the Verdier quotient $D^b(\mathcal{A})/D^b(P_{\mathcal{A}})$ inherits a symmetric monoidal structure from $D^b(\mathcal{A})$, one wants the subcategory of perfect complexes to form a $\otimes$-ideal, and as such we want the projectives to be 'absorbing' in that tensoring with projectives always yields something projective again. Although this is true for the construction in Section 3.D, it will *fail* in the construction presented in Section 3.3. We work around this issue by simply closing up the subcategory under the tensor product.

Remark 3.C.5. The stable category of a Frobenius category is an additive construction; the exact structure merely serves as a crutch for the proofs. It would therefore be reasonable to anticipate a statement that doesn't involve the language of exact categories.

Here is one such line of thought. Given an inclusion $\mathcal{A} \hookrightarrow \mathcal{B}$ of additive categories, the additive quotient may be defined ∞-categorically as a Dwyer–Kan localisation — or perhaps as a cofibre in a suitable ∞-category of (semi-)additive ∞-categories. We now state a (by necessity rather stringent) condition on $\mathcal{A}$ under which the additive quotient $\mathcal{B}/\mathcal{A}$ will be stable.

Let's call a map $X \to Y$ $\mathcal{A}$-injective if, for any object A in $\mathcal{A}$ and any map $X \to A$, there exists a lifted map $Y \to A$ making the relevant diagram commute. Dually, call it $\mathcal{A}$-surjective if any map $A \to Y$ admits a lifting to a map $A \to X$. It's straightforward to verify that the quotient functor $\mathcal{B} \to \mathcal{B}/\mathcal{A}$ preserves pushouts along $\mathcal{A}$-injectives, and pullbacks along $\mathcal{A}$-surjectives. Now take an object X in $\mathcal{B}$, and suppose that it admits an $\mathcal{A}$-injective map $X \to A$ into an object of $\mathcal{A}$. Let $I(X)$ be the cofibre of this map. Then in $\mathcal{B}/\mathcal{A}$, this objects in the suspension of X. Dually, the fibre $P(X)$ of an $\mathcal{A}$-surjective map $A \to X$ yields the loop space of X in $\mathcal{B}/\mathcal{A}$. The category $\mathcal{B}/\mathcal{A}$ is stable if these two constructions are mutual inverses, which can be stated intrinsically in $\mathcal{B}$ by requiring that the natural map $X \to P\big(I(X)\big)$ splits.

Notice that this condition is satisfied in the case of a Frobenius exact category, where $P(X)$ and $I(X)$ are the projective cover and injective hull of an object X, respectively.

3.D The Carlson–Peng–Wheeler exact category

The goal of this section is to take a look at the construction of a 'relative' stable module category, which generalises the usual stable module category of a finite group. This category was first conceived by Carlson–Peng–Wheeler in their study of the theory of support varieties [CPW98]. We generalise the construction in Section 3.3 but we proceed in Section 3.4 to study a category closely related, but likely inequivalent, to that of Carlson–Peng–Wheeler.

Let G be a finite group, and let k be a field of modular characteristic p. We take $\mathcal{A}$ to be the abelian category $\mathsf{Mod}^{\mathrm{fin}}(kG)$ of finite-dimensional kG-modules, and we define on $\mathcal{A}$ an exact structure that we call the $\mathcal{F}$-exact structure, where $\mathcal{F}$ refers to a family of subgroups of G. This, as we will show, defines a Frobenius exact structure, and the resulting stable category is equivalent to the category studied in [CPW98]. The usual stable module category is obtained by setting $\mathcal{F}$ equal to $\{e\}$.

We first begin by investigating a special case, which is well known in the literature. Fix a subgroup H of G. A sequence $M' \to M \to M''$ is called H-exact if it is a short exact sequence in the usual sense, and it splits after tensoring with $k(G/H)$. We will soon prove that this defines an exact structure on $\mathsf{Mod}^{\mathrm{fin}}(kG)$. We begin with some simple remarks.

Lemma 3.D.1. The $\{e\}$-exact sequences are precisely the usual short exact sequences.

Proof. The $\{e\}$-short exact sequences are just the short exact sequences which split after tensoring with kG. But any short exact sequence admits such a splitting because tensoring with kG yields a projective kG-module. $\square$

One can alternatively formulate H-exact sequences in terms of restrictions — a description which appears to be used more frequently in the literature. Our choice of definition will however be more convenient to generalise when we introduce $\mathcal{F}$-exact sequences in a moment.

Lemma 3.D.2. Let $M' \to M \to M''$ be a short exact sequence of kG-modules. Then the sequence is H-exact if and only if $M|_H \to M|_H \to M''|_H$ admits a splitting of H-modules.

Proof. In fact this is a consequence of a more general fact. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between abelian categories, with right adjoint G. Suppose that we're given a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{D}$ such that $0 \to FG(A) \to FG(B) \to FG(C) \to 0$ is split exact. Then $0 \to G(A) \to G(B) \to G(C) \to 0$ is split exact as well. In view of the sequence of adjunctions in the proof of Lemma 3.A.2, our lemma will be proved if we apply our claim to the case where F is Ind_H^G, and G is Res_H^G (see the sequence of adjunctions in the proof of Lemma 3.A.2).

To prove our claim, start with a section $s\colon FG(C) \to FG(B)$ of $FG(g)$. Consider the map

$$G(C) \xrightarrow{\widetilde{s}} GFG(B) \xrightarrow{G(\varepsilon)} G(B)$$

where $\widetilde{s}$ is the adjoint of s and ε is the counit of the adjunction. This map, we claim, is a section of $G(g)$. To verify this, consider the diagram

$$\begin{tikzcd}
 & G(C) \arrow[dd, "\widetilde{s}"] & \\
G(B) \arrow[ur, "G(g)"] & & G(B) \arrow[ul, "G(g)"'] \arrow[dl, "\eta(GB)"] \\
 & GFG(B) \arrow[ul, "G(\varepsilon B)"] &
\end{tikzcd}$$

Note that $G(\varepsilon B) \circ \eta(GB)$ is the identity map by the triangle identity. Moreover, the right triangle commutes by naturality of the adjunction. This implies that $G(\varepsilon) \circ \widetilde{s} \circ G(g) = \mathrm{Id}$. Compose both sides with $G(g)$. Since $G(g)$ is surjective, it is right-cancellative, and we may deduce that $G(g) \circ G(\varepsilon) \circ \widetilde{s} = \mathrm{Id}$, as desired. ☐

Corollary 3.D.3. If $H' \leq H$ are subgroups of G, then any H-exact sequence is also H'-exact. If H and H' are conjugate, then a sequence is H-exact if and only if it is H'-exact.

Proof. The first part is obvious in view of Lemma 3.D.2. As for the second part, simply note that G/H and G/H' are isomorphic as G-sets. ☐

We now consider the general definition. Let $\mathcal{F}$ be a collection of subgroups of G, and consider a sequence $M' \to M \to M''$ of kG-modules. Then we call this sequence $\mathcal{F}$-exact if it is short exact in the classical sense and it splits upon tensoring with $\bigoplus_{H \in \mathcal{F}} k(G/H)$. Let us henceforth denote this direct sum as $k(G/\mathcal{F})$. Note that in view of Corollary 3.D.3, there is no harm in assuming that $\mathcal{F}$ is in fact a *family* of subgroups, i.e. that $\mathcal{F}$ is closed under conjugations and taking subgroups.

The following lemma shows that the general notion is by no means more exotic than the special case we considered first.

Lemma 3.D.4. A sequence $M' \to M \to M''$ is $\mathcal{F}$-exact if and only if it is H-exact for every $H \in \mathcal{F}$.

Proof. This is a direct consequence of the following observation, whose proof is elementary: Suppose that $0 \to A' \to A \to A'' \to 0$ and $0 \to B' \to B \to B'' \to 0$ are two short exact sequences, labelled E_A and E_B. Then $E_A \oplus E_B$ is again a short exact sequence, and it is split if and only if both E_A and E_B are split. ☐

Theorem 3.D.5. The $\mathcal{F}$-exact sequences define an exact structure on the category $\mathsf{Mod}^{\mathrm{fin}}(kG)$.

By Lemma 3.D.4 it would suffice to prove this for the H-exact structure, where it is known from the literature. Nonetheless we opt for a direct proof.

Proof. We use the axioms presented in Section 3.B. We confine ourselves to the statements about admissible monomorphisms, as those about admissible epimorphisms are dual. The fact that identity maps are admissible monic is easy to see. To prove that admissible monics are closed under composition, take two admissible monics $M_1 \to M_2$ and $M_2 \to M_3$. Then they are monomorphisms in the classical sense, hence so is the composition $M_1 \to M_3$. In addition, the two maps admit a splitting upon tensoring with $k(G/\mathcal{F})$, and this splitting can be composed. Finally, consider a pushout diagram

$$\begin{tikzcd}
M' \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[d, "\widetilde{g}"] \arrow[dl, phantom, "\lrcorner", very near end] \\
N' \arrow[r, "\widetilde{f}"'] & N
\end{tikzcd}$$

where f is admissible mono. Surely $\widetilde{f}$ is a monomorphism. Upon tensoring with $k(G/\mathcal{F})$, f admits a splitting, say ε. And indeed so does $\widetilde{f}$: to specify a map $N \otimes k(G/\mathcal{F}) \to N' \otimes k(G/\mathcal{F})$, by the pushout property it suffices to find maps $M \otimes k(G/\mathcal{F}) \to N' \otimes k(G/\mathcal{F})$ and $N' \otimes k(G/\mathcal{F}) \to N' \otimes k(G/\mathcal{F})$, and $(g \circ \varepsilon) \otimes \mathrm{Id}$ and $\mathrm{Id} \otimes \mathrm{Id}$ do the job. ☐

Let's now investigate the projectives and injectives relative to the $\mathcal{F}$-exact structure. We shall accordingly call them the $\mathcal{F}$-projectives and $\mathcal{F}$-injectives, respectively. The proof of the following result can also be found in [CPW98, Section 2].

Theorem 3.D.6. The $\mathcal{F}$-exact sequences define a Frobenius category; that is, the $\mathcal{F}$-projectives and $\mathcal{F}$-injectives coincide. In addition, they are characterised as being the summands of all kG-modules of the form $k(G/\mathcal{F}) \otimes M$ for some kG-module M.

Before we present a proof, we will first need to treat the following key observation.

Lemma 3.D.7. If V is a finite-dimensional kG-module, then the evaluation map $V \otimes V^* \to k$ splits upon tensoring with V. In particular, the evaluation map $k(G/\mathcal{F}) \otimes k(G/\mathcal{F})^* \to k$ is an admissible epimorphism with respect to the $\mathcal{F}$-exact structure.

Proof. The evaluation map is obviously an epimorphism. Now take a basis $\{v_1, \ldots, v_n\}$ of V, with dual basis $\{v_1^*, \ldots, v_n^*\}$. Then the map $\mathrm{ev} \otimes \mathrm{Id} \colon V \otimes V^* \otimes V \to V$ has a right inverse which we define by sending v to $\sum_i v_i \otimes v_i^* \otimes v$. ☐

Proof of Theorem 3.D.6. Suppose P is $\mathcal{F}$-projective. Then we claim that P is a summand of $k(G/\mathcal{F}) \otimes M$ for some kG-module M; in fact, we may take M to be $k(G/\mathcal{F})^* \otimes P$. To prove this, consider the evaluation map $\mathrm{ev} \colon k(G/\mathcal{F}) \otimes k(G/\mathcal{F})^* \to k$. By Lemma 3.D.7 this map is admissible epi, and it clearly remains so after tensoring with P. Now apply the lifting property (Lemma 3.B.2) to the diagram

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d, "\mathrm{Id}"] \\ k(G/\mathcal{F}) \otimes k(G/\mathcal{F})^* \otimes P \arrow[r, "\mathrm{ev} \otimes \mathrm{Id}"'] & P \end{tikzcd}$$

to conclude the desired result.

Conversely, suppose that P is a summand of $k(G/\mathcal{F}) \otimes M$ for a kG-module M. We verify that it satisfies the usual lifting property along admissible epimorphisms. When constructing a lift, we may as well assume without loss of generality that P is $k(G/\mathcal{F}) \otimes M$. So suppose that we're given an admissible epimorphism $f \colon N' \to N$. Then upon tensoring with $k(G/\mathcal{F})$, this map admits a splitting. But let us instead fix a basis $\{v_1, \ldots, v_n\}$ of $k(G/\mathcal{F})$, and identify $k(G/\mathcal{F})$ with $k(G/\mathcal{F})^*$ according to this basis. Then the map $N' \to N$ admits a splitting after tensoring with $k(G/\mathcal{F})^*$. Now consider the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(k(G/\mathcal{F}) \otimes M, N') \arrow[r, "\sim"] \arrow[d, "f_*"'] & \mathrm{Hom}(M, k(G/\mathcal{F})^* \otimes N') \arrow[d, "(\mathrm{Id} \otimes f)_*"] \\ \mathrm{Hom}(k(G/\mathcal{F}) \otimes M, N) \arrow[r, "\sim"'] & \mathrm{Hom}(M, k(G/\mathcal{F})^* \otimes N) \end{tikzcd}$$

Thanks to the splitting, $(\mathrm{Id} \otimes f)_*$ admits a section, hence so does f_*, which is what we were after.

Finally, observe that $\mathsf{Mod}^{\mathrm{fin}}(kG)$ has enough $\mathcal{F}$-projectives; indeed if M is a kG-module, then the map $\mathrm{ev} \otimes M \to k(G/\mathcal{F}) \otimes k(G/\mathcal{F})^* \otimes M \to M$ defines a projective cover of M.

Dually to Lemma 3.D.7, the coevaluation map $k \to k(G/\mathcal{F}) \otimes k(G/\mathcal{F})^*$ is an admissible monomorphism. We may use this to dualise all our proofs to conclude the corresponding statements for injectives. However, some care must be taken when doing so, as there is a key step in which the symmetry would normally break. Namely, while tensoring with a module always preserves epimorphisms, it typically doesn't preserve monomorphisms. Fortunately, since kG is a Frobenius ring, all injective kG-modules (in the classical sense) are also projective kG-modules, so no troubles arise, and all proofs go through. $\square$

Example 3.D.8. In the more familiar case where $\mathcal{F}$ is the family of subgroups of H, $\mathcal{F}$-projectivity reduces to the H-projectivity as discussed in Section 3.A.

Denote by $D^{b,\mathrm{fin}}_{\mathcal{F}}(kG)$ the bounded derived category of the exact category of finite-dimensional kG-modules with its $\mathcal{F}$-exact sequences; and denote by $\mathsf{StMod}_{\mathcal{F}}(kG)$ the stable category of this exact category. By Theorem 3.C.2, the latter may also be described as the Verdier quotient of $D^{b,\mathrm{fin}}_{\mathcal{F}}(kG)$ by the perfect complexes. By the discussion in Remark 3.C.4, the derived category and its stable Verdier quotient are symmetric monoidal. It is the Picard group of $\mathsf{StMod}_{\mathcal{F}}(kG)$ which, for our purposes, is the invariant of principal interest, because it relates directly to the Dade group of G.

What can we concretely say about $D^{b,\mathrm{fin}}_{\mathcal{F}}(kG)$? To begin with, some of the Hom groups can be understood quite explicitly. To this end, we begin by recalling a well-known construction. For a family $\mathcal{F}$ of subgroups of G, we write $E\mathcal{F}$ for the homotopy type of a G-space characterised by the property that

$$E\mathcal{F}^H \simeq \begin{cases} * & \text{if } H \in \mathcal{F}; \\ \varnothing & \text{if } H \notin \mathcal{F}. \end{cases}$$

This G-space has an explicit simplicial model. Let X be the G-set $\bigsqcup_{H\in\mathcal{F}} G/H$. Now consider the simplicial set whose n-simplices are $(n+1)$-tuples $(x_0, \ldots, x_n)$, with face maps

$$d_i(x_0, \ldots, x_n) = (x_0, \ldots, \widehat{x_i}, \ldots, x_n)$$

and degeneracy maps

$$s_i(x_0, \ldots, x_n) = (x_0, \ldots, x_i, x_i, \ldots, x_n).$$

There is an obvious G-action on X; to wit, g sends the n-simplex $(x_0, \ldots, x_n)$ to $(gx_0, \ldots, gx_n)$. Clearly if $H \notin \mathcal{F}$ then there are no H-fixed points; on the other hand, suppose that $H \in \mathcal{F}$. Pick an element $v \in X$ which is fixed by H. Consider the simplicial homotopy sending an H-fixed simplex $(x_0, \ldots, x_n)$ to $(x_0, \ldots, x_n, v)$. This is a nullhomotopy contracting the space onto v.

The simplicial chain complex on our model for $E\mathcal{F}$ is the chain complex $C_*(X)$, where $C_n(X)$ is the k-vector space with basis X_{n+1}, and with

$$\partial_n(x_0, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \ldots, \widehat{x_i}, \ldots, x_n).$$

Lemma 3.D.9. $C_*(X)$ defines an $\mathcal{F}$-projective resolution of the trivial module k.

Proof. $\mathcal{F}$-projectivity of the $C_i(X)$ is immediate from the characterisation of $\mathcal{F}$-projective modules presented in Theorem 3.D.6. So we are done if we verify that the chain complex is $\mathcal{F}$-exact. By Lemma 3.D.4, it suffices to check that the chain complex is H-exact for every $H \in \mathcal{F}$. To check this, we use a method analogous to how we observed that $E\mathcal{F}$ is contractible: we observe that there is an H-nullhomotopy on $C_*(X)$. (Any such H-nullhomotopy will in particular imply that $C_*(X)$ is exact in the classical sense, but this is also immediate from the contractibility of $E\mathcal{F}$.)

More precisely, fix a subgroup $H \in \mathcal{F}$, and pick an element $v \in X$ which is fixed by H. Consider the simplicial homotopy h_* sending a simplex $(x_0, \ldots, x_n)$ to $(x_0, \ldots, x_n, v)$. This is a kH-module map and it is a nullhomotopy contracting the space onto v. Notice now that h_* defines a section of $\partial_n \colon C_{n+1} \to \operatorname{Ker}(\partial_n)$ — at least upon viewing it as a kH-module map. In particular, $\operatorname{Ker}(\partial_n)$ forms an H-module summand of C_{n+1}. We now see that the equivalent description presented in Lemma 3.D.2 is satisfied, and we are done. □

Write $B\mathcal{F}$ for the orbit space of the G-space $E\mathcal{F}$.

Corollary 3.D.10. With respect to the $\mathcal{F}$-exact structure, $\operatorname{Tor}_*^{\mathcal{F}}(k,k) \simeq H_*(B\mathcal{F};k)$ and $\operatorname{Ext}_{\mathcal{F}}^*(k,k) \simeq H^*(B\mathcal{F};k)$.

Proof. The simplicial chain complex of $B\mathcal{F}$ may be taken to be the G-module of orbits of $C_*(X)$. However, this is equivalent to taking $C_*(X) \otimes_{kG} k$. This proves the first isomorphism, and the second isomorphism is proved in an analogous fashion. □

We can further rewrite this in terms of orbit categories.

Lemma 3.D.11. $B\mathcal{F}$ is homotopy equivalent to the nerve of the orbit category $\mathcal{O}_{\mathcal{F}}(G)$.

Proof. To see this it is helpful to view $E\mathcal{F}$ through a different model. Namely, we can realise $E\mathcal{F}$ as the nerve of the category whose objects are pairs (H, x) for $H \in \mathcal{F}$ and $x \in G/H$, and whose morphisms $(H, x) \to (H', x')$ are those G-maps $G/H \to G/H'$ such that x gets sent to x'. The G-action on $E\mathcal{F}$ is the obvious one: g sends (H, x) to (H, gx). With respect to this model, the result is easily seen by comparing simplices. ☐

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