Copenhagen July 06 (1)

SURGERY (LAST LECTURE)

Lecture by E.K. Pedersen notes by N Wahl

$X^k \simeq \Omega B \quad H_*(X;\mathbb{Z})$ f.g AS ABELIAN GP

X^k IS A FINITE COMPLEX (FINITENESS OBSTRUCTION TRIVIAL FOR LOOP SPACES fd $\Rightarrow$ f.)

$\Downarrow$

X SATISFIES POINCARÉ DUALITY.

$X \subset \mathbb{R}^n \qquad X \subset N \subset \mathbb{R}^n$

REGULAR NBHD

DUAL OF X : $D(X_+) = N/\partial N$

BUT X H-SPACE $\Rightarrow X_+ \wedge X_+ = (X\times X)_+ \to X \times X \to X \to$

$\Rightarrow X_+ = D(X_+)$.

$\Rightarrow \quad \Sigma^d X_+ = N/\partial N$ (FOR n BIG ENOUGH) ($d = n-k$)

$$\begin{array}{ccccc} S^n & \longrightarrow & N/\partial N \simeq \Sigma^d(X_+) & & \\ \cup & & \cup & & \\ U & \longrightarrow & X \times \mathbb{R}^d & \longrightarrow & \mathbb{R}^d \\ \cup & & & & \Psi \\ M^k & \longrightarrow & X \times P & \longrightarrow & P \end{array}$$

CAN ASSUME THIS MAP IS SMOOTH

P REGULAR VALUE

MANIFOLD

$[M] \longmapsto [X]$ in H_k

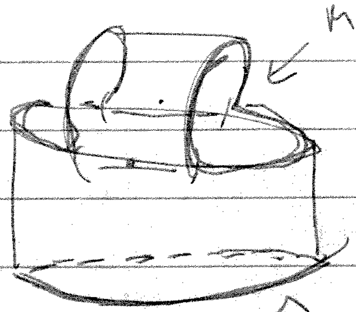
"DEGREE 1 NORMAL MAP"

$$S^i \times D^{\varepsilon - i} \subset M^{\varepsilon}$$

<u>SURGERY</u>: $M - (S^i \times \mathring{D}^{\varepsilon - i}) \cup (D^{i+1} \times S^{\varepsilon - i - 1})$

WANT TO USE THIS PROCESS TO CHANGE M TO GET A HOMOTOPY EQUIVALENCE TO X.

M AFTER SURGERY



M

ATTACHING A HANDLE:

$$M \times I \cup_{S^i \times D^{\varepsilon - i}} D^{i+1} \times D^{\varepsilon - i}$$

$\rightsquigarrow$ COBORDISM.

IN HOMOTOPY ~~VI~~ VIEW POINT. WE ARE JUST ATTACHING A CELL; $M \times I \simeq M$ $\quad S^i \times D^{\varepsilon - i} \simeq S^i$ ...

SO THAT WE CHANGE M BY A COBORDISM.
(EVERY COBORDISM IS A SEQUENCE OF SURGERIES
— VIA MORSE THEORY)

$$\begin{array}{ccc} M & \longrightarrow & X \\ \uparrow & & \uparrow \\ S^0 & \hookrightarrow & D^1 \\ & & \\ S^i & & D^{i+1} \end{array}$$

OBSTRUCTION TO HTPY EQUI is $\pi_*(X, M)$

IF $i < \frac{\dim M}{2}$, CAN ASSUME $S^i \hookrightarrow M$ EMBEDDING

COULD ALSO BE NON-TRIVIAL

$$\begin{array}{ccc} \nu_M & \longrightarrow & X \times \mathbb{R}^n \\ \downarrow & & \downarrow \\ M & \longrightarrow & X \end{array}$$

BDL MAP

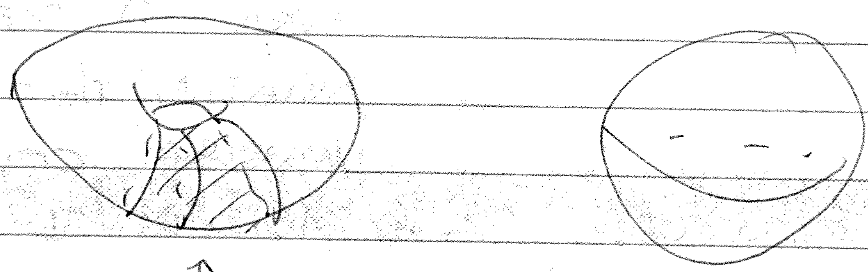
GIVES A TRIVIALIZATION OF NORMAL BDL OF $S^i \subset M$

$$\rightsquigarrow S^i \times D^{2-i} \subset M$$

~~(WE ARE USING IMMERSION THEORY... IMMERSIONS ARE DETERMINED BY BUNDLE INFORMATION)~~

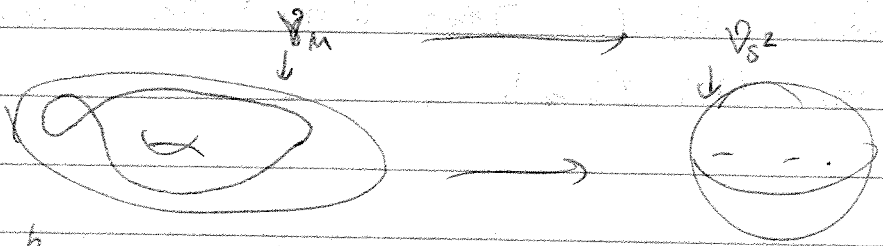
~~IF $S^i \times D^{2-i} \to M^2$ IS COVERED BY A GOOD BDL MAP $\rightsquigarrow$ CAN CHANGE THE MAP TO AN IMMERSION.~~

Eg.:



OBVIOUS SURGERY

BUT $\exists$ OBSTRUCTION IN DIM 2:



BDL DATA ~~GIVE~~ GIVES THE IMMERSIONS OF S^1

$\to$ MIGHT NOT BE ABLE TO DO SURGERY

WOULD NEED TO "REFRAME THE PB" – CHANGE THE BDL – TO BE ABLE TO DO SURGERY

$$\mathrm{IMMERSIONS}(\overset{\text{OPEN}}{\overset{\cup}{M^k}}, N^k)$$

$$\downarrow \simeq$$

$$\left\{ \begin{array}{ccc} \nu_M & \longrightarrow & \nu_N \\ \downarrow & & \downarrow \\ M^k & \longrightarrow & N^k \end{array} \right\}$$

EVERY MAP COVERED BY A BDL MAP IS HOMOTOPIC TO AN IMMERSION.

$$\begin{array}{ccc} M^k & \longrightarrow & X^k \\ \text{SURGERY} \Big\} & \Big\} \nearrow & \\ M' & & \end{array}$$

$\longrightarrow \left[\frac{k}{2}\right]$ - CONNECTED.

BY P.D., IF k IS EVEN $\exists$ ONE NON-TRIVIAL RELATIVE H_*-GP (ON UNIVERSAL COVER)

IF k IS ODD $\exists$ 2 NON-TRIVIAL LEFT.

$\longrightarrow$ SURGERY OBSTRUCTION

(ATTACHING CELLS RATHER THAN HANDLES MAKES US STAY IN MANIFOLDS)

WALL'S π-π THEOREM: $\underset{\text{MFD}}{(M, \partial M)} \longrightarrow \underset{\text{PD-PAIR}}{\overset{\text{CONNECTED}}{(X, \partial X)}}$

IF $\pi_1 \partial X \cong \pi_1 X$, THEN CAN ALWAYS DO SURGERY: GET $\underset{\text{COBORDANT TO } (M, \partial M)}{(M', \partial M')} \xrightarrow{\simeq} (X, \partial X)$

(NOTE: $\partial X \neq \emptyset$)

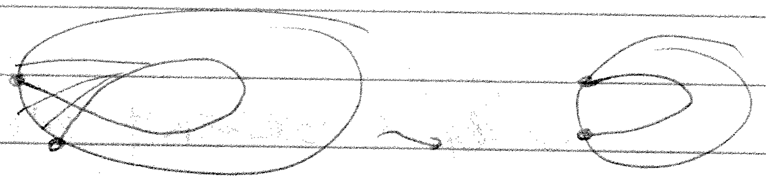
FOR LOOP SPACES, TILLMAN'S THESIS ~~DESCRIBES~~ THE DUAL OF SPACES ... CAN USE THE π-π THEOREM

$\downarrow$

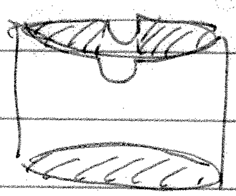
LOTS OF SELF-DUAL SPACES

(π-π)-THEOREM: CAN GET RID OF SELF-INTERSECT

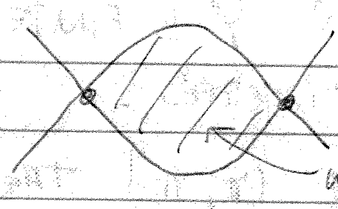
IN MIDDLE DIMENSION, MIGHT HAVE SOME DOUBLE POINTS WE WANT TO GET RID OFF.



EX OF SURGERY WITH BOUNDARY:



NEED dim ≥ 5 BECAUSE NEED TO EMBED A 2-DISC.



WANT EMBEDDED DISC TO SLIDE ONE THROUGH THE OTHER.

IDEAL WORLD:

X LOOP SPACE

Y FINITE CPX

$\pi_1 X \cong \pi_1 Y$

$$\begin{array}{c} S^1 \longrightarrow X \\ \downarrow \\ Y \end{array}$$

$\Rightarrow Y$ SATISFIES P.D. (USING SERRE SS)

$G_2 \simeq O(2) \longrightarrow S^1$-FIBRATION IS S^1-BDL.

→ ∃ DISC BDL

$$\begin{tikzcd} S^1 \arrow[r, hook] \arrow[d] & D^2 \arrow[d] \\ X \arrow[r, hook] \arrow[d] & E \arrow[dl] \\ Y & \end{tikzcd}$$

$\pi_1(X) \cong \pi_1(E)$

→ APPLY $(\pi\text{-}\pi)$ - THM TO (E, X)

THOM'S THESIS : Y is SELF-DUAL

→ SET-UP A SURGERY PROBLEM ON Y : $M \to Y$, PULL IT BACK TO (E, X).

⇝ ON BDRY

$$\begin{tikzcd} \nu_{M'} \arrow[r] \arrow[d] & X \times \mathbb{R}^n \arrow[d] \\ M' \arrow[r, "\simeq"] & X \end{tikzcd}$$

+

→ NORMAL BDL STABLY PARALLELIZABLE

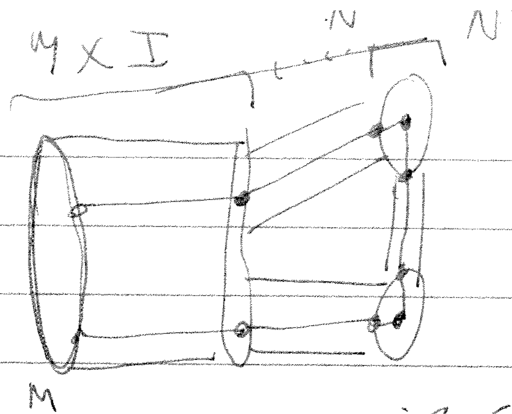
<u>REALITY</u>: DON'T KNOW Y IS FINITE, BUT KNOW IS FINITELY DOMINATED.

USE A GENERALIZATION OF (π, π) - THEOREM -

<u>SURGERY BELOW THE MIDDLE DIMENSION</u> ($\pm$ AS IN WALL'S BOOK - SIMPLER -)

$$\begin{tikzcd} \nu_M \arrow[r] \arrow[d] & \xi \arrow[d] \\ M \arrow[r, hook] & X \end{tikzcd}$$

MAKE THIS INTO AN INCLUSION BY REPLACING X BY THE MAPPING CYLINDER.



(Hold onto it!)

$\boxed{\text{POINTS OF } X^{(0)} \text{ NOT in } M}$

S^0 CAN MISS $M \cup X^{(0)}$ BY ~~GENERAL~~ POSITION + PUSH TO $\partial(N^{(0)} - (M \cup X^{(0)}))$

$$\begin{array}{ccccccc} M & \hookrightarrow & M \cup X^{(0)} & \hookrightarrow & M \cup X^{(1)} & \hookrightarrow & M \cup X^{(2)} \cdots \\ & & \text{IS} & & \text{IS} & & \\ & & N^{(0)} & & N^{(1)} & & \end{array}$$

$\overline{M \quad \underline{N} \quad M'}$

$N \simeq M' \cup$ CELLS ABOVE MID DIMENSION

IS

$M \cup X^{(\frac{n}{2})} \longrightarrow X$

$\Rightarrow X = M' \cup$ CELLS ABOVE MID-DIMENSION

$\Rightarrow M' \longrightarrow X$ HIGHLY CONNECTED.