STABLE TOPOLOGICAL CYCLIC HOMOLOGY IS TOPOLOGICAL HOCHSCHILD HOMOLOGY

By Lars Hesselholt

1.Introduction

1.1. Topological cyclic homology is the codomain of the cyclotomic trace from algebraic K-theory

$$\mathrm{trc}\colon K(L) \to \mathrm{TC}(L).$$

It was defined in [2] but for our purpose the exposition in [6] is more convenient. The cyclotomic trace is conjectured to induce a homotopy equivalence after p-completion for a certain class of rings including the rings of algebraic integers in local fields of possitive residue characteristic p. We refer to [11] for a detailed discussion of conjectures and results in this direction.

Recently B.Dundas and R.McCarthy have proven that the stabilization of algebraic K-theory is naturally equivalent to topological Hochschild homology,

$$K^S(R;M) \simeq T(R;M)$$

for any simplicial ring R and any simplicial R-module M, *cf.* [4]. We note that both functors are defined for pairs $(L;P)$ where L is a functor with smash product and P is an L-bimodule; *cf.* [12]. An outline of a proof in this setting and by quite different methods, has been given by R.Schwänzl, R.Staffelt and F.Waldhausen. Hence the following result is a necessary condition for the conjecture mentioned above to hold.

Theorem. *Let L be a functor with smash product and P an L-bimodule. Then there is a natural weak equivalence,* $\mathrm{TC}^S(L;P)^\wedge_p \simeq T(L;P)^\wedge_p$.

It is not surprising that we have to p-complete in the case of TC since the cyclotomic trace is really an invariant of the p-completion of algebraic K-theory, *cf.* 1.4 below. The rest of this paragraph recalls cyclotomic spectra, topological Hochschild homology, topological cyclic homology and stabilization. In paragraph 2 we decompose topological Hochschild homology of a split extension of *FSP*'s and approximate TC in a stable range. Finally in paragraph 3 we study free cyclic objects and use them to prove the theorem.

Throughout G denotes the circle group, equivalence means weak homotopy equivalence and a G-equivalence is a G-map which induces an equivalence of H-fixed sets for any closed subgroup $H \leq G$.

I want to thank my adviser Ib Madsen for much help and guidance in the preparation of this paper as well as in my graduate studies as a whole. Part of this work was done during a stay at the University of Bielefeld and it is a pleasure to thank the university and in particular Friedhelm Waldhausen for their hospitality. I also want to thank him and John Klein for many enlightening discussions.

1.2. Let L be an *FSP* and let P be an L-bimodule. Then $\mathrm{THH}(L;P)_\bullet$ is the simplicial space with k-simplices

$$\operatorname*{holim}_{\overrightarrow{I^{k+1}}} F(S^{i_0} \wedge \ldots \wedge S^{i_k}, P(S^{i_0}) \wedge L(S^{i_1}) \wedge \ldots \wedge L(S^{i_k}))$$

The author was supported by the Danish Natural Science Reseach Council

and Hochschild-type structure maps, *cf.* [12], and $\mathrm{THH}(L;P)$ is its realization. When $P=L$, considered as an L-bimodule in the obvious way, $\mathrm{THH}(L;L)$ is a cyclic space so $\mathrm{THH}(L;L)$ has a G-action. In both cases we use a thick realization to ensure that we get the right homotopy type, *cf.* the appendix. More generally if X is some space we let $\mathrm{THH}(L;P;X)_\bullet$ be the simplicial space

$$\operatorname*{holim}_{\overrightarrow{I^{k+1}}} F(S^{i_0}\wedge\ldots\wedge S^{i_k}, P(S^{i_0})\wedge L(S^{i_1})\wedge\ldots\wedge L(S^{i_k})\wedge X),$$

where X acts as a dummy for the simplicial structure maps. If X has a G-action then $\mathrm{THH}(L;P;X)$ becomes a G-space and $\mathrm{THH}(L;L;X)$ a $G\times G$-space. We shall view the latter as a G-space via the diagonal map $\Delta: G\to G\times G$ and then denote it $\mathrm{THH}(L;X)$.

We define a G-prespectrum $t(L;P)$ in the sense of [9] whose 0'th space is $\mathrm{THH}(L;P)$. Let V be any orthogonal G-representation, or more precisely, any f.d. sub inner product space of a fixed 'complete G-universe' U. Then

$$t(L;P)(V)=\mathrm{THH}(L;P;S^V),$$

with the obvious G-maps

$$\sigma: S^{W-V}\wedge t(L;P)(V)\to t(L;P)(W)$$

as prespectrum structure maps. Here S^V is the one-point compactification of V and $W-V$ is the orthogonal complement of V in W. We also define a G-spectrum $T(L;P)$ associated with $t(L;P)$, *i.e.* a G-prespectrum where the adjoints $\tilde\sigma$ of the structure maps are homeomorphisms. We first replace $t(L;P)$ by a thickened version $t^\tau(L;P)$ where the structure maps σ are closed inclusions. It has as V'th space the homotopy colimit over suspensions of the structure maps

$$t^\tau(L;P)(V)=\operatorname*{holim}_{\overrightarrow{Z\subset V}}\Sigma^{V-Z}t(L;P)(Z)$$

and as structure maps the compositions (t=t(L;P))

$$\Sigma^{W-V}\operatorname*{holim}_{\overrightarrow{Z\subset V}}\Sigma^{V-Z}t(Z)\cong\operatorname*{holim}_{\overrightarrow{Z\subset V}}\Sigma^{W-Z}t(Z)\to\operatorname*{holim}_{\overrightarrow{Z\subset W}}\Sigma^{W-Z}t(Z).$$

Here the last map is induced by the inclusion of a subcategory and as such is a closed cofibration, in particular it is a closed inclusion. Furthermore since V is terminal among $Z\subset V$ there is natural map $\pi: t^\tau(L;P)\to t(L;P)$ which is spacewise a G-homotopy equivalence. Next we define $T(L;P)$ by

$$T(L;P)(V)=\varinjlim_{W\subset U}\Omega^{W-V}t^\tau(L;P)(W)$$

with the obvious structure maps.

We can replace $\mathrm{THH}(L;P;S^V)$ by $\mathrm{THH}(L;S^V)$ above and get a G-prespectrum $t(L)$ and a G-spectrum $T(L)$. These possess some extra structure which allows the definition of $\mathrm{TC}(L)$ and we will now discuss this in some detail. For a complete account we refer to [6], see also [3].

1.3. Let C be a finite subgroup of G of order r and let J be the quotient. The r'th root $\rho_C: G\to J$ is an isomorphism of groups and allows us to view a J-space X as a G-space ρ_C^*X. Recall that the free loop space $\mathcal{L}X$ has the special property that $\rho_C\mathcal{L}X^C\cong_G\mathcal{L}X$ for any finite subgroup of G. Cyclotomic spectra, as defined in [3] and [6], is a class of G-spectra which have the analogous property in the world of spectra. This section recalls the defintion.

For a G-spectrum T there are two J-spectra T^C and $\Phi^C T$ each of which could be called the C-fixed spectrum of T. If $V\subset U^C$ is a C-trivial representation, then

$$T^C(V)=T(V)^C,\qquad \Phi^C T(V)=\varinjlim_{W\subset U}\Omega^{W^C-V}T(W)^C$$

and the structure maps are evident. There is a natural map $r_C\colon T^C \to \Phi^C T$ of J-spectra; $r_C(V)$ is the composition

$$T^C(V) \cong \varinjlim_{W\subset U} F(S^{W-V}, T(W))^C \xrightarrow{\iota^*} \varinjlim_{W\subset U} F(S^{W^C-V}, T(W)^C) = \Phi^C T(V)$$

where the map ι^* is induced by the inclusion of C-fixed points. The difference between T^C and $\Phi^C T$ is well illustated by the following example.

Example. Consider the case of a suspension G-spectrum $T = \Sigma^\infty_G X$,

$$T(V) = \varinjlim_{W\subset U} \Omega^{W-V}(S^W \wedge X).$$

We let $E_G H$ denote a universal H-free G-space, that is $E_G H^K \simeq *$ when $H\cap K = 1$ and $E_G H^K = \varnothing$ when $H \cap K \neq 1$. Then on the one hand we have the tom Dieck splitting

$$(\Sigma^\infty_G X)^C \simeq_J \bigvee_{H\leq C} \Sigma^\infty_J (E_{G/H}(C/H)_+ \wedge_{C/H} X^H),$$

and on the other hand the lemma shows that $\Phi^C(\Sigma^\infty_G X) \simeq_J \Sigma^\infty_J X^C$. Moreover the natural map $r_C\colon (\Sigma^\infty_G X)^C \to \Phi^C(\Sigma^\infty_G X)$ is the projection onto the summand $H = C$. $\square$

A J-spectrum D defines a G-spectrum $\rho_C^* D$. However this G-spectrum is indexed on the G-universe $\rho_C^* U^C$ rather than on U. To get a G-spectrum indexed on U we must choose an isometric isomorphism $f_C\colon U \to \rho_C^* U^C$, then $(\rho_C^* D)(f_C(V))$ is the V'th space of the required G-spectrum, which we denote it $\rho_C^\# D$.

We want the f_C's to be compatible for any pair of finite subgroups, that is the following diagram should commute

$$\begin{array}{ccc} U & \xrightarrow{f_{C_{rs}}} & \rho^*_{C_{rs}} U^{C_{rs}} \\ {\scriptstyle f_{C_r}}\downarrow & & \| \\ \rho^*_{C_r} U^{C_r} & \xrightarrow{\rho^*_{C_r}(f_{C_s})^{C_r}} & \rho^*_{C_r}(\rho^*_{C_s} U^{C_s})^{C_r}. \end{array}$$

Moreover the restriction of f_C to the G-trivial universe U^G induces an automorphism of U^G which we request be the identity. We fix our universe,

$$U = \bigoplus_{n\in\mathbb{Z}, \alpha\in\mathbb{N}} \mathbb{C}(n)_\alpha,$$

where $\mathbb{C}(n) = \mathbb{C}$ but with G acting through the n'th power map. The index α is a dummy. Since $\rho_C^*\mathbb{C}(n) = \mathbb{C}(nr)$, where r is the order of C, we obtain the required maps f_C by identifying $\mathbb{Z} = r\mathbb{Z}$.

Definition. ([6]) A *cyclotomic spectrum* is a G-spectrum indexed on U together with a G-equivalence

$$\varphi_C\colon \rho_C^\# \Phi^C T \to T$$

for every finite $C \subset G$, such that for any pair of finite subgroups the diagram

$$\begin{array}{ccc} \rho^\#_{C_r}\Phi^{C_r}\rho^\#_{C_s}\Phi^{C_s} T & = & \rho^\#_{C_{rs}}\Phi^{C_{rs}} T \\ {\scriptstyle \rho^\#_{C_r}\Phi^{C_r}\varphi_{C_s}}\downarrow & & \downarrow{\scriptstyle \varphi_{C_{rs}}} \\ \rho^\#_{C_r}\Phi^{C_r} T & \xrightarrow{\varphi_{C_r}} & T \end{array}$$

commutes.

We prove in [6] that the topological Hochschild spectrum $T(L)$ defined above is a cyclotomic spectrum. The rest of this section recalls the definition of the φ-maps for $T(L)$. The definition goes back to [2] and begins with the concept of edgewise subdivision.

The realization of a cyclic space becomes a G-space upon identifying G with $\mathbb{R}/\mathbb{Z}$, and hence C may be identified with $r^{-1}\mathbb{Z}/\mathbb{Z}$. Edgewise subdivision associates to a cyclic space $Z_\bullet$ a simplicial C-space $\operatorname{sd}_C Z_\bullet$. It has k-simplices $\operatorname{sd}_C Z_k = Z_{r(k+1)-1}$ and the generator $r^{-1}+\mathbb{Z}$ of C acts as τ^{k+1}. Moreover, there is a natural homeomorphism

$$D\colon |\operatorname{sd}_C Z_\bullet| \to |Z_\bullet|,$$

an $\mathbb{R}/r\mathbb{Z}$-action on $|\operatorname{sd}_C Z_\bullet|$ which extends the simplicial C-action, and D is G-equivariant when $\mathbb{R}/r\mathbb{Z}$ is identified with $\mathbb{R}/\mathbb{Z}$ through division by r.

We now consider the case of $\mathrm{THH}(L;X)_\bullet$. Let us write $G_k(i_0,\dots,i_k)$ for the pointed mapping space

$$F(S^{i_0}\wedge\ldots\wedge S^{i_k}, L(S^{i_0})\ldots\wedge L(S^{i_k})\wedge X).$$

Then the k-simplices of the edgewise subdivision is the homotopy colimit

$$\operatorname{sd}_C \mathrm{THH}(L;X)_k = \operatorname*{\underrightarrow{holim}}_{I^{r(k+1)}} G_{r(k+1)-1}.$$

The C-action on $\operatorname{sd}_C \mathrm{THH}(L;X)_k$ is not induced by one on $G_{r(k+1)-1}$. We consider instead the composite functor $G_{r(k+1)-1}\circ\Delta_r$ where $\Delta_r\colon I^{k+1}\to (I^{k+1})^r$ is the diagonal functor. It has C-action and the canonical map of homotopy colimits

$$b_k\colon \operatorname*{\underrightarrow{holim}}_{I^{k+1}} G_{r(k+1)-1}\circ\Delta_r \to \operatorname*{\underrightarrow{holim}}_{I^{r(k+1)}} G_{r(k+1)-1}$$

is a C-equivariant inclusion and induces a homeomorphism of C-fixed sets. Let Y and Z be two C-spaces and consider the mapping space $F(Y,Z)$. It is a C-space by conjugation and we have a natural map

$$\iota^*\colon F(Y,Z)^C \to F(Y^C,Z^C),$$

which takes a C-equivariant map $\psi\colon Y\to Z$ to the induced map of C-fixed sets. In the case at hand ι^* gives us a natural transformation

$$(G_{r(k+1)-1}\circ\Delta_r)^C \to G_k,$$

and the induced map on homotopy colimits defines a map of simplicial spaces

$$\tilde\phi_{C,\bullet}\colon \operatorname{sd}_C \mathrm{THH}(L;X)^C_\bullet \to \mathrm{THH}(L;X^C)_\bullet.$$

We define a G-equivariant map

$$\phi_C(V)\colon \rho_C^* t(L)(V)^C \to t(L)(f_C^{-1}(\rho_C^* V^C))$$

as the composite

$$\begin{aligned}\rho_C^*|\mathrm{THH}(L;S^V)|^C &\xrightarrow{D^{-1}} |\operatorname{sd}_C \mathrm{THH}(L;S^V)|^C \xrightarrow{\tilde\phi_C} |\mathrm{THH}(L;S^{\rho_C^* V^C})| \\ &\xrightarrow{(f_C^{-1})_*} |\mathrm{THH}(L;S^{f_C^{-1}\rho_C^* V^C})|.\end{aligned}$$

Indeed it is G-equivariant by [2] lemma 1.11. Next we define a G-map

$$\varphi_C(V)\colon \rho_C^* T(L)(V)^C \to T(f_C^{-1}(\rho_C^* V^C))$$

as the map on colimits over $W\subset U$ induced by the composition

$$\begin{aligned}\rho_C^*(\Omega^{W-V} t^\tau(L)(W))^C &\xrightarrow{i^*} \rho_C^*(\Omega^{W^C-V^C} t^\tau(L)(W)^C) \\ &\xrightarrow{\phi_C(W)_*} \Omega^{\rho_C^*(W^C-V^C)} t^\tau(L)(f_C^{-1}(\rho_C^* W^C)) \\ &\xrightarrow{f_C^*} \Omega^{f_C^{-1}(\rho_C^*(W-V)^C)} t^\tau(L)(f_C^{-1}(\rho_C^* W^C)).\end{aligned}$$

Then the required maps $\varphi_C\colon \rho_C^{\#}\Phi^C T\to T$ of G-spectra are evident in view of the definitions. Furthermore [2] 1.12 shows that the diagram which relates the φ-maps for a pair of finite subgroups of G commutes. We refer to [6] for the proof that the φ-maps are G-equivalences.

1.4. Let $j\colon U^G \to U^C$ be the inclusion of the trivial G-universe and let D be a J-spectrum. The underlying non-equivariant spectrum of D is the spectrum j^*D with its J-action forgotten. By abuse of notation we usually denote this D again.

Let T be a cyclotomic spectrum, then r_{C_r} and φ_{C_r} induce a map of G-spectra

$$\rho^{\#}_{C_{rs}} T^{C_{rs}} = \rho^{\#}_{C_s}(\rho^{\#}_{C_r} T^{C_r})^{C_s} \to \rho^{\#}_{C_s}(\rho^{\#}_{C_r} \Phi^{C_r} T)^{C_s} \to \rho^{\#}_{C_s} T^{C_s}.$$

It gives a map $\Phi_r\colon T^{C_{rs}} \to T^{C_s}$ of underlying non-equivariant spectra and the compatibility condition in definition 1.3 implies that $\Phi_r\Phi_s = \Phi_{rs}$. The inclusion of the fixed set of a bigger group in that of a smaller also defines a map of non-equivariant spectra $D_r\colon T^{C_{rs}} \to T^{C_s}$, and these satisfies that $D_rD_s = D_{rs}$. Moreover $D_r\Phi_s = \Phi_sD_r$.

Topological cyclic homology of an *FSP* was defined in [2]; the presentation here is due to T. Goodwillie [5]. Let $\mathbb{I}$ be the category with $\operatorname{ob}\mathbb{I} = \{1, 2, 3, \dots\}$ and two morphisms $\Phi_r, D_r\colon n \to m$, whenever $n = rm$, subject to the relations

$$\begin{aligned} &\Phi_1 = D_1 = \mathrm{id}_n, \\ &\Phi_r\Phi_s = \Phi_{rs},\ D_rD_s = D_{rs}, \\ &\Phi_rD_s = D_s\Phi_r. \end{aligned}$$

For a prime p we let $\mathbb{I}_p$ denote the full subcategory with $\operatorname{ob}\mathbb{I}_p = \{1, p, p^2, \dots\}$. The discussion above shows that a cyclotomic spectrum T defines a functor from $\mathbb{I}$ to the category of non-equivariant spectra, which takes n to T^{C_n}.

Definition. ([2]) $\mathrm{TC}(T) = \underset{\overleftarrow{\mathbb{I}}}{\operatorname{holim}}\, T^{C_n}, \quad \mathrm{TC}(T;p) = \underset{\overleftarrow{\mathbb{I}_p}}{\operatorname{holim}}\, T^{C_{p^s}}.$

If L is a functor with smash product then $\mathrm{TC}(L)$ and $\mathrm{TC}(L;p)$ are the connective covers of $\mathrm{TC}(T(L))$ and $\mathrm{TC}(T(L);p)$ respectively. It is often useful to have the definition of $\mathrm{TC}(T;p)$ in the form it is given in [2],

$$\mathrm{TC}(T;p) \cong [\underset{\overleftarrow{D_p}}{\operatorname{holim}}\, T^{C_{p^s}}]^{h\langle\Phi_p\rangle} \cong [\underset{\overleftarrow{\Phi_p}}{\operatorname{holim}}\, T^{C_{p^s}}]^{h\langle D_p\rangle}.$$

Here $\langle D_p\rangle$ is the free monoid on D_p and $X^{h\langle D_p\rangle}$ stands for the $\langle D_p\rangle$-homotopy fixed points of X. It is naturally equivalent to the homotopy fiber of $1 - D_p$.

The functor $\mathrm{TC}(-)$ is really not a stronger invariant than the $\mathrm{TC}(-;p)$'s. Indeed we have the following result, which will be proved elsewhere.

Proposition. *The projections* $\mathrm{TC}(T) \to \mathrm{TC}(T;p)$ *induce an equivalence of* $\mathrm{TC}(T)$ *with the fiber product of the* $\mathrm{TC}(T;p)$*'s over* T. *Moreover the* p*-complete theories agree,* $\mathrm{TC}(T)^\wedge_p \simeq \mathrm{TC}(T;p)^\wedge_p$. □

Remark. In [2] the authors define a space $\mathrm{TC}(L;p)$ and a Γ-space structure on it. Furthermore they show that the cyclotomic trace $\mathrm{trc}\colon K(L) \to \mathrm{TC}(L;p)$ is a map of Γ-spaces. We show in [6] that the spectrum $\mathrm{TC}(L;p)$ defined above is equivalent to the one determined by the Γ-space structure. □

1.5. Stable K-theory of simplicial rings was defined by Waldhausen in [15], see also [8]. We conclude this paragraph with the definition of stable TC of a *FSP* and leave it to reader to see that stable K-theory also may be defined in this generality.

Definition. Let P be an L-bimodule and K a space. The *shift* $P[K]$ of P by K is the functor given by $P[K](X) = K \wedge P(X)$ with structure maps

$$l^{P[K]}_{X,Y} = \mathrm{id}_K \wedge l^P_{X,Y} \circ \mathrm{tw} \wedge \mathrm{id}_{P(Y)}, \qquad r^{P[K]}_X, Y = \mathrm{id}_K \wedge r^P_{X,Y}.$$

We shall write $P[n]$ for $P[S^n]$.

We define a new *FSP* denoted $L \oplus P$ which is to be thought of as an extension of L by a square zero ideal P.

Definition. Let L be an *FSP* and P an L-bimodule. We define the *extension* of L by P as $L \oplus P(X) = L(X) \vee P(X)$ with multiplication

$$\begin{aligned} L \oplus P(X) \wedge L \oplus P(Y) &\to L(X) \wedge L(Y) \vee L(X) \wedge P(Y) \vee P(X) \wedge L(Y) \vee P(X) \wedge P(Y) \\ &\to L(X \wedge Y) \vee P(X \wedge Y) \vee P(X \wedge Y) \to L \oplus P(X \wedge Y). \end{aligned}$$

The first map is the canonical homeomorphism, the second is $\mu_{X,Y} \vee l_{X,Y} \vee r_{X,Y} \vee *$ and the last is convolution. Finally the unit in $L \oplus P$ is the composite

$$X \to L(X) \to L \oplus P(X).$$

One verifies immediately that $L \oplus P$ is in fact an *FSP* and that it contains L as a retract. We shall write $\widetilde{\mathrm{TC}}(L \oplus P)$ for the homotopy fiber of the induced retraction $\mathrm{TC}(L \oplus P) \to \mathrm{TC}(L)$.

Lemma. *If K is contractible then so is $\widetilde{\mathrm{TC}}(L \oplus P[K])$. Furthermore a contraction of K induces one of $\widetilde{\mathrm{TC}}(L \oplus P[K])$.*

Proof. Let us write F instead of $L \oplus P[K]$. If $h\colon I_+ \wedge K \to K$ is a contraction we can define $h(X)\colon I_+ \wedge F(X) \to F(X)$ by the composition

$$I_+ \wedge (L(X) \vee K \wedge P(X)) \cong I_+ \wedge L(X) \vee I_+ \wedge K \wedge P(X) \xrightarrow{\mathrm{pr}_2 \vee h \wedge \mathrm{id}} L(X) \vee K \wedge P(X).$$

It is compatible with the multiplication and unit in F, that is the following diagrams commute

$$\begin{array}{ccc} I_+ \wedge (F(X) \wedge F(Y)) & \xrightarrow{\mathrm{id} \wedge \mu_{X,Y}} & I_+ \wedge F(X \wedge Y) \\ {\scriptstyle \Delta \wedge \mathrm{id}} \downarrow & & {\scriptstyle h_{X \wedge Y}} \downarrow \\ (I \times I)_+ \wedge F(X) \wedge F(Y) & & F(X \wedge Y) \\ {\scriptstyle \mathrm{id} \wedge \mathrm{tw}\, \mathrm{id}} \downarrow & & {\scriptstyle \mu_{X,Y}} \uparrow \\ I_+ \wedge F(X) \wedge I_+ \wedge F(Y) & \xrightarrow{h_X \wedge h_Y} & F(X) \wedge F(Y). \end{array}$$

and

$$\begin{array}{ccc} I_+ \wedge X & \xrightarrow{\mathrm{id} \wedge \mathbf{1}_X} & I_+ \wedge F(X) \\ {\scriptstyle \mathrm{pr}_2} \downarrow & & {\scriptstyle h(X)} \downarrow \\ X & \xrightarrow{\mathbf{1}_X} & F(X). \end{array}$$

Therefore the composition

$$\begin{aligned} I_+ \wedge (F(S^{i_0}) \wedge \ldots \wedge F(S^{i_k})) &\xrightarrow{\mathrm{tw} \circ (\Delta \wedge \mathrm{id})} I_+ \wedge F(S^{i_0}) \wedge \ldots \wedge I_+ \wedge F(S^{i_k}) \\ &\xrightarrow{h(S^{i_0}) \wedge \ldots \wedge h(S^{i_k})} F(S^{i_0}) \wedge \ldots \wedge F(S^{i_k}) \end{aligned}$$

gives rise to a cyclic map $h_{V\bullet}\colon I_+ \wedge \mathrm{THH}(F; F; S^V)_\bullet \to \mathrm{THH}(F; F; S^V)_\bullet$ whose realization is a G-equivariant homotopy

$$h_V\colon I_+ \wedge t(F)(V) \to t(F)(V).$$

Furthermore these are compatible with the structure maps in the prespectrum such that we get a G-equivariant homotopy

$$H\colon I_+ \wedge T(F) \to T(F).$$

This gives in turn a homotopy $I_+ \wedge \mathrm{TC}(F) \to \mathrm{TC}(F)$ from the identity to the retraction onto the image of $\mathrm{TC}(L)$. $\square$

If we apply $\widetilde{\mathrm{TC}}(L \oplus P[-])$ to the cocartesian square of spaces

$$\begin{array}{ccc} S^n & \longrightarrow & D_+^{n+1} \\ \downarrow & & \downarrow \\ D_-^{n+1} & \longrightarrow & S^{n+1}. \end{array}$$

we get a map from $\widetilde{\mathrm{TC}}(L \oplus P[n])$ to the homotopy limit of

$$\widetilde{\mathrm{TC}}(L \oplus P[D_+^{n+1}]) \to \widetilde{\mathrm{TC}}(L \oplus P[S^{n+1}], p) \leftarrow \widetilde{\mathrm{TC}}(L \oplus P[D_-^{n+1}]).$$

By the lemma the radial contrations of the discs D^{n+1} give a preferred contraction of $\widetilde{\mathrm{TC}}(L \oplus P[D^{n+1}])$. Hence we obtain a natural map from the homotopy limit above to $\Omega\widetilde{\mathrm{TC}}(L \oplus P[n+1])$. All in all we get a stabilization map

$$\sigma \colon \widetilde{\mathrm{TC}}(L \oplus P[n]) \to \Omega\widetilde{\mathrm{TC}}(L \oplus P[n+1])$$

which is natural in L and P.

Definition. Let L be an *FSP* and P an L-bimodule. Then

$$\mathrm{TC}^S(L; P) = \underset{\overrightarrow{n}}{\mathrm{holim}}\, \Omega^{n+1}\widetilde{\mathrm{TC}}(L \oplus P[n]),$$

with the colimit taken over the stabilization maps.

2. Stable Approximation of $\mathrm{TC}(L \oplus P)$

2.1. In the rest of this paper the prime p is fixed and we shall always consider the functor $\mathrm{TC}(-; p)$ rather than $\mathrm{TC}(-)$.

Recall that by definition $L \oplus P(S^i) = L(S^i) \vee P(S^i)$. Thus we can decompose the smash product

$$L \oplus P(S^{i_0}) \wedge \ldots \wedge L \oplus P(S^{i_k})$$

into a wedge of summands of the form

$$F_0(S^{i_0}) \wedge \ldots \wedge F_k(S^{i_k}),$$

where $F_i = L, P$. A summand where $\#\{i | F_i = P\} = a$ will be called an a-configuration and the one-point space $*$ will be considered an a-configuration for any $a \geq 0$.

Recall from 1.3 the functor $G_k = G_k(L \oplus P; X)$ whose homotopy colimit is $\mathrm{THH}(L \oplus P; X)_k$. The a-configurations define subspaces

$$G_{a,k}(i_0, \ldots, i_k) \subset G_k(i_0, \ldots, i_k)$$

preserved under $G_k(f_0, \ldots, f_k)$, *i.e.* we get a functor $G_{a,k} = G_{a,k}(L \oplus P; X)$. The spaces

$$\mathrm{THH}_a(L \oplus P; X)_k = \underset{\overrightarrow{I_{k+1}}}{\mathrm{holim}}\, G_{a,k}(L \oplus P; X)$$

form a cyclic subspace $\mathrm{THH}_a(L \oplus P; X)_{\bullet} \subset \mathrm{THH}(L \oplus P; X)_{\bullet}$ with realization $\mathrm{THH}_a(L \oplus P; X)$. Like in 1.2 we can define a G-prespectrum $t_a(L \oplus P)$ and a G-spectrum $T_a(L \oplus P)$. Then $T_a(L \oplus P)$ is a retract of $T(L \oplus P)$. We show below that as a G-spectrum $T(L \oplus P)$ is the wedge sum of the $T_a(L \oplus P)$'s.

Lemma. *Let j be a G-prespectrum and let J be the G-spectrum associated with j^τ. If $J^\Gamma \simeq *$ for any finite subgroup $\Gamma \subset G$ and $j(V)^G \simeq *$ for any $V \subset U$ then $J \simeq_G *$.*

Proof. Let $\mathcal{F}$ be the family of finite subgroups of the circle, then J is $\mathcal{F}$-contractible. Since $J \wedge E\mathcal{F}_+ \to J$ is an $\mathcal{F}$-equivalence, $J \wedge E\mathcal{F}_+$ is also $\mathcal{F}$-contractible. However $J \wedge E\mathcal{F}_+$ is G-equivalent to an $\mathcal{F}$-CW-spectrum and therefore it is in fact G-contractible by the $\mathcal{F}$-Whitehead theorem, [9] p.63. Now

$$(J \wedge E\mathcal{F}_+)(V) \cong \varinjlim_W \Omega^W(j^\tau(V+W) \wedge E\mathcal{F}_+),$$

and $j^\tau(V) \wedge E\mathcal{F}_+ \to j^\tau(V)$ is an G-equivalence since $j(V)^G \simeq *$. Therefore $J \simeq_G J \wedge E\mathcal{F}_+$ and we have already seen that the latter is G-contractible. □

Lemma. *Let H be a compact Lie group, let X a finite H-CW-complex and let Y_a a family of H-spaces. For $K \leq H$ a closed subgroup we let $n(K) = \min_a\{\mathrm{conn}(Y_a^K)\}$. Then the inclusion*

$$\bigvee_a F(X, Y_a)^H \to F(X, \bigvee_a Y_a)^H$$

is $2\min\{n(K) - \dim(X^K) | K \leq H\} + 1$-connected.

Proof. The inclusion above fits into a commutative square

$$\begin{array}{ccc} \bigvee_a F(X,Y_a)^C & \longrightarrow & F(X, \bigvee_a Y_a)^C \\ \downarrow & & \downarrow \\ \prod'_a F(X,Y_a)^C & \xrightarrow{\cong} & F(X, \prod'_a Y_a)^C, \end{array}$$

where $\prod'$ is the weak product, *i.e.* the subspace of the product where all but a finite number of coordinates are at the basepoint. The lower horizontal map is a homeomorphism because X is finite, and the connectivity of the vertical maps may be estimated by elementary equivariant obstruction theory. For example the connectivity of an equivariant mapping space satisfies

$$\mathrm{conn}(F(X,Y)^H) \geq \min\{\mathrm{conn}(Y^K) - \dim(X^K) | K \leq H\}.$$

Therefore the left vertical map is $2\min\{n(K) - \dim(X^K) | K \leq H\} + 1$-connected. □

Proposition. $T(L \oplus P) \simeq_G \bigvee_a T_a(L \oplus P)$.

Proof. We apply the first lemma with j the G-prespectrum whose V'th space is the homotopy fiber of the inclusion

$$\bigvee_{a=0}^{\infty} t_a(L \oplus P)(V) \to t(L \oplus P)(V).$$

We first consider a finite subgroup $\Gamma \subset G$ and show that $J^\Gamma \simeq *$. It suffices to show that $j(V)^C$ is $\dim(V^C) + k(V,C)$-connected, where $k(V,C) \to \infty$ as V runs through the f.d. sub inner product spaces of U, for any subgroup $C \subset \Gamma$. We use edgewise subdivision to get a simplicial C-action, that is we can identify $j(V)^C$ with the homotopy fiber of

$$|\bigvee_a \mathrm{sd}_C\, \mathrm{THH}_a(L \oplus P; S^V)^C_\bullet| \to |\mathrm{sd}_C\, \mathrm{THH}(L \oplus P; S^V)^C_\bullet|.$$

As in the 1.3 we consider the diagonal functor $\Delta_r\colon I^{k+1} \to (I^{k+1})^r$. Then the second lemma shows that the inclusion

$$\bigvee_a (G_{a,r(k+1)-1} \circ \Delta_r(i_0, \ldots, i_k))^C \to (G_{r(k+1)-1} \circ \Delta_r(i_0, \ldots, i_k))^C$$

is $2\dim(V^C)-1$-connected. By [1] theorem 1.5 the same is true for the homotopy colimits over I^{k+1}. Hence the inclusion map

$$\bigvee_a \operatorname{sd}_C \operatorname{THH}_a(L\oplus P; S^V)^C_k \to \operatorname{sd}_C \operatorname{THH}(L\oplus P; S^V)^C_k$$

is $2\dim(V^C)-1$-connected. Finally the spectral sequence of [13] shows that the induced map on realizations is $2\dim(V^C)-1$-connected. It follows that $J^\Gamma \simeq *$.

We have only left to show that $j(V)^G \simeq *$. If $X_\bullet$ is a cyclic space, then $|X_\bullet|^G$ is homeomorphic to the subspace $\{x\in X_0 | s_0x = \tau_1 s_1 x\}$ of the 0-simplices. For the domain and the codomain of $j(V)$ this is S^{V^G} and $j(V)$ is the identity. □

2.2. Let us write $a = p^s k$ with $(k,p)=1$ and denote $T_a(L\oplus P)$ by $T^k_s(L\oplus P)$. Then the cyclotomic structure map $\varphi = \varphi_{C_p}$ induces a G-equivalence

$$\varphi_s\colon \rho^\#_{C_p}\Phi^{C_p}T^k_s(L\oplus P) \to T^k_{s-1}(L\oplus P), \quad s\geq 0,$$

where for convenience $T^k_{-1}(L\oplus P)$ denotes the trivial G-spectrum $*$.

Lemma. i) *The cyclotomic structure map induces a map of underlying non-equivariant spectra*

$$T^k_s(L\oplus P[n])^{C_{p^r}} \to T^k_0(L\oplus P[n])^{C_{p^{r-s}}}$$

which is kpn-connected.
ii) $T^k_0(L\oplus P[n])^{C_{p^r}}$ *is kn-connected.*

Proof. Let $\tilde{E}G$ be the mapping cone of the map $\pi\colon EG_+ \to S^0$ which collapses EG to the non-basepoint of S^0. It comes with a G-map $\iota\colon S^0 \to \tilde{E}G$ and a G-null homotopy of the composition

$$EG_+ \xrightarrow{\pi} S^0 \xrightarrow{\iota} \tilde{E}G.$$

We can also describe $\tilde{E}G$ as the unreduced suspension of EG and ι as the inclusion of S^0 as the two cone vertices. Finally we note that $\tilde{E}G$ is non-equivariantly contractible while $\tilde{E}G^C = S^0$ for any non-trivial subgroup $C\leq G$.

Let us write T_s for $T^k_s(L\oplus P[n])$. We can smash the sequence above with T_s and take C_{p^r}-fixed points. Then we get maps of underlying non-equivariant spectra

$$[EG_+\wedge T_s]^{C_{p^r}} \xrightarrow{\pi_*} T_s^{C_{p^r}} \xrightarrow{\iota_*} [\tilde{E}G\wedge T_s]^{C_{p^r}}$$

and a preferred null homotopy of their composition. These data specifies a map from $[EG_+\wedge T_s]^{C_{p^r}}$ to the homotopy fiber of ι_* and this an equivalence.

We identify the right hand term. Recall the natural map $r_{C_p}\colon T_s^{C_p} \to \Phi^{C_p}T_s$ from 1.3. It factors as a composition

$$T_s^{C_p} \xrightarrow{\pi_*} [\tilde{E}G\wedge T_s]^{C_{p^r}} \xrightarrow{\bar{r}_C} \Phi^{C_p}T_s,$$

where $\bar{r}_C(V)$ is induced from the restriction map

$$F(S^{W-V}, \tilde{E}G\wedge T_s(W))^{C_p} \to F(S^{W^{C_p}-V}, T(W)^{C_p}).$$

Moreover $\bar{r}_{C_p}(V)$ is a fibration with fiber the equivariant (pointed) mapping space

$$F(S^{W-V}/S^{W^{C_p}-V}, \tilde{E}G\wedge T(W))^{C_p}.$$

If we regard W as a C_{p^r}-space, then W^{C_p} is the singular set, so $S^{W-V}/S^{W^{C_p}-V}$ is a free C_{p^r}-CW-complex in the pointed sense. Since $\tilde{E}G$ is non-equivariantly contractible it follows that $\bar{r}_{C_p}$ is a C_{p^r}/C_p-equivalence. The map Φ_p of underlying non-equivariant spectra defined in 1.4 restricts to a map

$$T_s^{C_{p^r}} \xrightarrow{r_{C_p}^{C_{p^r}/C_p}} (\Phi^{C_p} T_s)^{C_{p^r}/C_p} = (\rho_{C_p}^{\#} \Phi^{C_p} T_s)^{C_{p^{r-1}}} \xrightarrow{\varphi_{C_p}^{C_{p^{r-1}}}} T_{s-1}^{C_{p^{r-1}}}.$$

Our calculation above shows that its homotopy fiber is equivalent to the underlying non-equivariant spectrum of $[EG_+ \wedge T_s]^{C_{p^r}}$. We contend that this is as highly connected as is T_s. Indeed the skeleton filtration of EG gives rise to a first quadrant spectral sequence

$$E^2_{s,t} = H_s(C_{p^r}; \pi_t(T_s)) \Rightarrow \pi_{s+t}([EG_+ \wedge T_s]^{C_{p^r}}),$$

where $\pi_t(T_s)$ is a trivial C_{p^r}-module. The identification of the E^2-term uses the transfer equivalence of [9] p. 89. □

Proposition. *In the stable range $\le 2n$ we have*

$$\widetilde{\mathrm{TC}}(L \oplus P[n]) \simeq_{2n} \underset{\overleftarrow{r}}{\mathrm{holim}}\, T_1(L \oplus P[n]; p)^{C_{p^r}},$$

with the limit taken over the inclusion maps D.

Proof. We get from the connectivity statements in the lemma that

$$\begin{aligned}\tilde{T}(L \oplus P[n])^{C_{p^r}} &\simeq_{2n} T^1(L \oplus P[n])^{C_{p^r}} = \bigvee_{s=0}^{\infty} T_s^1(L \oplus P[n])^{C_{p^r}} \\ &\simeq_{2n} \bigvee_{s=0}^{r} T_0^1(L \oplus P[n])^{C_{p^{r-s}}} = \bigvee_{t=0}^{r} T_0^1(L \oplus P[n])^{C_{p^t}}.\end{aligned}$$

Under these equivalences $\Phi\colon \tilde{T}(L \oplus P[n])^{C_{p^r}} \to \tilde{T}(L \oplus P[n])^{C_{p^{r-1}}}$ becomes projection onto the first r summands. Therefore

$$\widetilde{\mathrm{TC}}(L \oplus P[n]; p) = [\underset{\overleftarrow{\Phi}}{\mathrm{holim}}\, \tilde{T}(L \oplus P[n])^{C_{p^r}}]^{h\langle D\rangle} \simeq_{2n} [\prod_{t=0}^{\infty} T_0^1(L \oplus P[n])^{C_{p^t}}]^{h\langle D\rangle}.$$

The latter spectrum is naturally equivalent to the homotopy limit stated above. □

Remark. When $P = L$ there is an unstable formula for $\widetilde{\mathrm{TC}}(L \oplus L[n])$. It was found in [6] and used to evaluate TC of rings of dual numbers over finite fields.

3. Free cyclic objects

3.1. In this paragraph we examine the cyclic spaces $t_1(L \oplus P)(V)_\bullet$ we introduced in 2.2. They turn out to be the free cyclic spaces generated by the simplicial spaces $t(L; P)(V)_\bullet$ from 1.2. First we study free cyclic objects.

Suppose $K\colon I \to J$ is a functor between small categories and $\mathbb{C}$ a category which have all colimits. Then the functor $K^*\colon \mathbb{C}^J \to \mathbb{C}^I$ has a left adjoint F. If $X\colon I \to \mathbb{C}$ is a functor then

$$FX(j) = \underrightarrow{\lim}((K \downarrow j) \xrightarrow{\mathrm{pr}_1} I \xrightarrow{X} \mathbb{C}),$$

where $(K \downarrow j)$ is the category of objects K-over j. It is called the left Kan extension of X along K, *cf.* [10]. As an instance of this construction suppose I and J are monoids, *i.e.* categories with one object, and $\mathbb{C}$ the category of (unbased) spaces. Then a functor $X\colon I \to \mathbb{C}$ is just an I-space and FX is the J-space $J \times_I X$.

Definition. Let $X_\bullet$ be a simplicial object in $\mathbb{C}$. The *free cyclic object* generated by $X_\bullet$ is the left Kan extension of $X_\bullet$ along the forgetfull functor $K\colon \mathbf{\Delta}^{\rm op} \to \mathbf{\Lambda}^{\rm op}$. It is denoted $FX_\bullet$.

If X is an object in $\mathbb{C}$ and S is a set, then we let $S \ltimes X$ denote the coproduct of copies of X indexed by S. We give a concrete description of $FX_\bullet$.

Lemma. *Let $C_{n+1} = \{1, \tau_n, \tau_n^2, \dots, \tau_n^n\}$. Then $FX_\bullet$ has n-simplices*

$$FX_n \cong C_{n+1} \ltimes X_n,$$

and the cyclic structure maps are

$$\begin{aligned} d_i(\tau_n^s \ltimes x) &= \tau_{n-1}^s \ltimes d_{i+s}x \quad , \text{ if } i+s \le n \\ &= \tau_{n-1}^{s-1} \ltimes d_{i+s}x \quad , \text{ if } i+s > n \\ s_i(\tau_n^s \ltimes x) &= \tau_{n+1}^s \ltimes s_{i+s}x \quad , \text{ if } i+s \le n \\ &= \tau_{n+1}^{s+1} \ltimes s_{i+s}x \quad , \text{ if } i+s > n \\ t_n(\tau_n^s \ltimes x) &= \tau_n^{s-1} \ltimes x. \end{aligned}$$

All indicices are to be understood as the principal representatives modulo $n+1$.

Proof. Both $\mathbf{\Delta}$ and $\mathbf{\Lambda}$ has objects the finite ordered sets $\mathbf{n} = \{0, \dots, n\}$ but $\mathbf{\Lambda}$ has more morphism than $\mathbf{\Delta}$. Specifically $\mathbf{\Lambda}(\mathbf{n}, \mathbf{m}) = \mathbf{\Delta}(\mathbf{n}, \mathbf{m}) \times \mathrm{Aut}_{\mathbf{\Lambda}}(\mathbf{n})$ and $\mathrm{Aut}_{\mathbf{\Lambda}}(\mathbf{n})$ is a cyclic group of order $n+1$. As a generator for $\mathrm{Aut}_{\mathbf{\Lambda}}(\mathbf{n})$ we choose the cyclic permutation $\tau_n\colon \mathbf{n} \to \mathbf{n}$; $\tau_n(i) = i-1 \pmod{n+1}$.

Consider the full subcategory $C(\mathbf{n}) \subset (K \downarrow \mathbf{n})$ whose objects are the automorphisms $K\mathbf{n} \to \mathbf{n}$, *i.e.* $\mathrm{ob}\, C(\mathbf{n}) = C_{n+1}$. The restriction of colimits comes with a map

$$\varinjlim(C(\mathbf{n}) \xrightarrow{\mathrm{pr}_1} \mathbf{\Delta}^{\rm op} \xrightarrow{X_\bullet} \mathbb{C}) \to \varinjlim((K \downarrow \mathbf{n}) \xrightarrow{\mathrm{pr}} \mathbf{\Delta}^{\rm op} \xrightarrow{X_\bullet} \mathbb{C}) = FX_n,$$

and from the definitions one may readily show that this is an isomorphism. Since in $\mathbf{\Delta}^{\rm op}$ there are no automorphisms of $\mathbf{n}$ apart from the identity, the category $C(\mathbf{n})$ is a discrete category, *i.e.* any morphism is an identity. We conclude that

$$FX_n \cong \coprod_{\mathrm{ob}\, C(\mathbf{n})} X_n = C_{n+1} \ltimes X_n.$$

It is straightforward to check that the cyclic structure maps are as claimed. □

Example. Suppose $\mathbb{C}$ is the category of commutative rings, where the coproduct is tensor product of rings, and $R_\bullet = R$ is a constant simplicial ring. Then the complex associated with FR is the standard Hochschild complex $Z(R)$ whose homology is $\mathrm{HH}_*(R)$.

3.2. We now take $\mathbb{C}$ to be the category of pointed topological spaces and study the relation between F and realization.

Lemma. *There is a natural G-homeomorphism $|FX_\bullet| \cong G_+ \wedge |X_\bullet|$.*

Proof. Consider the standard cyclic sets $\Lambda[n] = \mathbf{\Lambda}(-, n)$ and their realizations Λ^n. From [7], 3.4 we know that as cocyclic spaces $\Lambda^\bullet \cong G \times \Delta^\bullet$, so we may view $\Lambda^\bullet$ as a cocyclic G-space. Now suppose Y is a (based) G-space. We can define a cyclic space $C_\bullet(Y)$ as the equivariant mapping space

$$C_\bullet(Y) = F_G(\Lambda^\bullet, Y),$$

with the compact open topology. Then one immediately verifies that $C_\bullet$ is right adjoint to the realization functor $|-|$. The realization functor for simplicial spaces also has a right adjoint. It is given as $S_\bullet(X) = F(\Delta^\bullet, X)$ with the compact open topology. Finally the forgetfull functor U from G-spaces to spaces is right adjoint to the functor $G_+ \wedge -$.

By a very general principle in category theory called conjunction, to prove the lemma we may as well show that $S_\bullet(UY) = K^* C_\bullet(Y)$ for any G-space Y. But this is evident since $F_G(G_+ \wedge X, Y) \cong F(X, UY)$ □

Proposition. *There is a natural equivalence of G-spectra*

$$G_+ \wedge T(L;P) \simeq_G T_1(L \oplus P).$$

The V'th space in the smash product G-spectrum on the left is naturally homeomorphic to $\varinjlim_W \Omega^{W-V}(G_+ \wedge t^\tau(L;P)(W))$, where G acts diagonally on $G_+ \wedge t^\tau(L;P)(W)$.

Proof. The smash product $P(S^{i_0}) \wedge L(S^{i_1}) \wedge \ldots \wedge L(S^{i_k})$ is a 1-configuration, *cf.* 2.1. Thus we have an inclusion map $\mathrm{THH}(L;P;X)_k \hookrightarrow \mathrm{THH}_1(L \oplus P;X)_k$ and these commutes with the simplicial structure maps. By definition we get a map of cyclic spaces

$$j(X)_\bullet \colon F\,\mathrm{THH}(L;P;X)_\bullet \to \mathrm{THH}_1(L \oplus P;X)_\bullet$$

and lemma 3.2 shows that on realizations this gives rise to a G-equivariant map

$$j(X)\colon G_+ \wedge \mathrm{THH}(L;P;X) \to \mathrm{THH}_1(L \oplus P;X).$$

When X runs through the spheres S^V these maps form a map j of G-prespectra. Let us write $G_+ \wedge t^\tau(L;P)$ for the G-spectrum whose V'th space is the colimit

$$\varinjlim_{W \subset U} \Omega^{W-V}(G_+ \wedge t^\tau(L;P)(W)).$$

Then j induces a map $J\colon G_+ \wedge t^\tau(L;P) \to T_1(L \oplus P)$ and an argument completely analogous to the proof of proposition 2.1 shows that this is a G-equivalence. Finally the canonical inclusion

$$G_+ \wedge t^\tau(L;P)(V) \to G_+ \wedge T(L;P)(V)$$

gives a map $G_+ \wedge t^\tau(L;P) \to G_+ \wedge T(L;P)$ and this is a homeomorphism, *cf.* the appendix. □

3.3. Before we prove our main theorem we need the following key lemma, also used extensively in [6].

Lemma. *Let T be a G-spectrum. Then there is a natural equivalence of non-equivariant spectra*

$$[T \wedge G_+]^{C_{p^r}} \simeq T \vee \Sigma T,$$

and the inclusion $D\colon [T \wedge G_+]^{C_{p^r}} \hookrightarrow [T \wedge G_+]^{C_{p^{r-1}}}$ becomes $p \vee \mathrm{id}$. Here p denotes multiplication by p.

Proof. The Thom collaps $t\colon S^{\mathbb{C}} \to S^{i\mathbb{R}} \wedge G_+$ of $S(\mathbb{C}) \subset \mathbb{C}$ gives rise to a G-equivariant transfer map

$$\tau\colon F(G_+, \Sigma T) \to G_+ \wedge T$$

which is a G-homotopy equivalence, *cf.* [9], p.89. There is a cofibration sequence of C_{p_r}-spaces

$$C_{p^r+} \hookrightarrow G_+ \to C_{p^r+} \wedge S^1$$

where S^1 is C_{p^r}-trivial. We may apply $F_{C_{p^r}}(-, \Sigma T)$ and get a cofibration sequence of spectra

$$F(S^1, \Sigma T) \longrightarrow F_{C_{p^r}}(G_+, \Sigma T) \xrightarrow{\mathrm{ev}_\zeta} \Sigma T.$$

Finally ev_ζ is naturally split by the adjoint of the G-action $G_+ \wedge \Sigma T \to \Sigma T$. □

Proof of theorem. If we compare proposition 3.2 and lemma 3.3 we find that

$$T_1(L \oplus P)^{C_{p^r}} \simeq T(L;P) \vee \Sigma T(L;P).$$

Now holim of a string of maps

$$\cdots \xrightarrow{f_i} X_n \xrightarrow{f_{i-1}} \cdots \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0$$

where every $f_i = pg_i$ for some g_i vanishes after p-completion, so by proposition 2.2 and lemma 3.3 we get

$$\widetilde{\mathrm{TC}}(L \oplus P[n]) \simeq_{2n} \Sigma T(L;P[n]).$$

The functor $T(L;P)$ is linear in the second variable, *cf.* [12] 2.13, so therefore

$$\Omega^{n+1}\widetilde{\mathrm{TC}}(L \oplus P[n]) \simeq_n \Omega^{n+1}\Sigma T(L;P[n]) \simeq T(L;P).$$

It remains only to check that the stabilization maps defined in 1.5 induce an equivalence of $T(L;P)$. They do. □

Appendix

A.1. Let $\mathbb{C}$ be either of the categories $\mathbf{\Delta}$ or $\mathbf{\Lambda}$ and let $X: \mathbb{C} \to \text{Top}_*$ be a functor to pointed spaces. We define a new functor $\bar{X}: \mathbb{C} \to \text{Top}_*$ by the homotopy colimit

$$\underrightarrow{\text{holim}}((- \downarrow \mathbb{C})^{\text{op}} \xrightarrow{\text{pr}_2^{\text{op}}} \mathbb{C}^{\text{op}} \xrightarrow{X} \text{Top}_*),$$

where $(\mathbf{n} \downarrow \mathbb{C})$ is the category under $\mathbf{n}$, *cf.* [10]. If $\theta: \mathbf{n} \to \mathbf{m}$ is a morphism in $\mathbf{\Delta}$ (not $\mathbb{C}$), which is surjective, then $\theta^*: (\mathbf{m} \downarrow \mathbb{C}) \to (\mathbf{n} \downarrow \mathbb{C})$ is an inclusion functor. In general inclusions of index categories induces closed cofibrations on homotopy colimits. In particular $\theta^*: \bar{X}_m \to \bar{X}_n$ is a closed cofibration, so $\bar{X}$ is good in the sense of [14]. Moreover we have a homotopy equivalence $\bar{X}_n \to X_n$ because $\text{id}: \mathbf{n} \to \mathbf{n}$ is initial in $(\mathbf{n} \downarrow \mathbb{C})$.

A.2. This section explains a technical point in the passage from G-prespectra to G-spectra. Let $G\mathcal{P}U$ denote the category of G-prespectra indexed on the universe U and let $G\mathcal{S}U$ be the full subcategory of G-spectra. In [9] the authors prove that the forgetful functor $l: G\mathcal{S}U \to G\mathcal{P}U$ has a left adjoint $L: G\mathcal{P}U \to G\mathcal{S}U$. We call this functor spectrification and if $t \in G\mathcal{P}U$ then we call Lt the associated G-spectrum. Such a functor is needed since many constructions such as $X \wedge -$ and any (homotopy) colimits do not preserve G-spectra. However L has the serious drawback that in general it looses (weak) homotopy type, *i.e.* the homotopy type of $(Lt)(V)$ cannot be described in terms of that of the spaces $t(W)$. To control the homotopy type the G-prespectrum t has to be an inclusion G-prespectrum, that is the structure maps $\tilde{\sigma}: t(V) \to \Omega^{W-V} t(W)$ must be inclusions, then

$$(Lt)(V) = \varinjlim_{W \subset U} \Omega^{W-V} t(W).$$

This is the case for example if the adjoints $\sigma: \Sigma^{W-V} t(V) \to t(W)$ are closed inclusions. The thickening functor $(-)^\tau$ defined in 1.2 produces G-prespectra of this kind. Therefore $L(t^\tau)$ has the right homotopy type.

If $a: G\mathcal{P}U \to G\mathcal{P}U$ is a functor we define $A: G\mathcal{S}U \to G\mathcal{S}U$ as the composite functor Lal and if a has a right adjoint b, then B is the right adjoint of A. Suppose b preserves G-spectra, then $b(lT) \cong lB(T)$ for any $T \in G\mathcal{S}U$. By conjugation we get

$$A(Lt) \cong La(t)$$

for any $t \in G\mathcal{P}U$. The functors a we consider take a G-prespectrum, whose structure maps σ are closed inclusions, to a G-prespectrum of the same kind. Hence the homotopy type of $La(t^\tau)$ and therefore $A(L(t^\tau))$ may be calculated. This shows that all G-spectra considered in this paper have the right homotopy type.

References

[1] M.Bökstedt, *Topological Hochschild Homology*, to appear in Topology.

[2] M.Bökstedt,W.C.Hsiang,I.Madsen, *The Cyclotomic Trace and Algebraic K-theory of Spaces*, Invent. Math. (1993).

[3] M.Bökstedt,I.Madsen, *Topological Cyclic Homology of the Integers*, these proceedings.

[4] B.Dundas,R.McCarthy, *Stable K-theory and Topological Hochschild Homology*, preprint Brown University.

[5] T.Goodwillie, *Notes on the Cyclotomic Trace*, MSRI (unpublished).

[6] L.Hesselholt, I.Madsen, *Topological Cyclic Homology of Dual Numbers over Finite Fields*, preprint series, Aarhus University (1993).

[7] J.D.S.Jones, *Cyclic Homology and Equivariant Homology*, Invent.math. **87** (1987), 403-423.

[8] C.Kassel, *La K-théorie Stable*, Bull.Soc.math., France **110** (1982), 381-416.

[9] L.G.Lewis,J.P.May,M.Steinberger, *Stable Equivariant Homotopy Theory*, LNM 1213.

[10] S.MacLane, *Categories for the Working Mathematician*, GTM 5, Springer-Verlag.

[11] I.Madsen, *The Cyclotomic Trace in Algebraic K-theory*, Proc. ECM, Paris, France (1992).

[12] T.Pirashvili, F.Waldhausen, *MacLane Homology and Topological Hochschild Homology*, J.Pure Appl.Alg **82** (1992), 81-99.

[13] G.Segal, *Classifying Spaces and Spectral Sequences*, Publ.Math.I.H.E.S. **34** (1968), 105-112.

[14] G.Segal, *Categories and Cohomology Theories*, Topology **13** (1974), 293-312.

[15] F.Waldhausen, *Algebraic K-theory of Spaces II*, Algebraic Topology Aarhus 1978, LNM **763**, 356-394.