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Topological Hochschild homology and the de Rham-Witt complex for $\mathbb{Z}_{(p)}$-algebras

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Abstract. This paper shows that for a $\mathbb{Z}_{(p)}$-algebra (p odd), the equivariant homotopy groups in degrees less than or equal to one of the topological Hochschild $\mathbb{T}$-spectrum are given, as functors, by the de Rham-Witt complex.

Introduction

Let A be a commutative and unital $\mathbb{Z}_{(p)}$-algebra with p odd. The topological Hochschild spectrum $\mathrm{TH}(A)$ has a natural action by the circle $\mathbb{T}$, and one defines

$$\mathrm{TR}^n(A;p) = \mathrm{TH}(A)^{C_{p^{n-1}}}$$

as the fixed point spectrum for the cyclic subgroup of the indicated order. Typically, the homotopy groups

$$\mathrm{TR}^{\cdot}_*(A;p) = \pi_* \mathrm{TR}^{\cdot}(A;p)$$

are very large, but they have a rich algebraic structure. There is a universal example of this structure, the de Rham-Witt complex, and the canonical map

$$\lambda\colon W.\,\Omega^*_A \to \mathrm{TR}^{\cdot}_*(A;p)$$

is in many ways analogous to the map from Milnor K-theory to Quillen K-theory. For example, it is an isomorphism of pro-abelian groups if A is a regular $\mathbb{F}_p$-algebra by [**4**, theorem B]. In this paper we prove the following result whose K-theory analog is well-known.

Theorem. *Let A be a $\mathbb{Z}_{(p)}$-algebra with p odd. Then the canonical map*

$$\lambda\colon W_n\,\Omega^q_A \to \mathrm{TR}^n_q(A;p)$$

is an isomorphism, for $q \leq 1$.

The statement was known previously in a number of special cases. The case of a polynomial algebra over $\mathbb{Z}_{(p)}$, proved in [**5**], is the starting point of the argument given here. We also mention that in his thesis [**9**], Kåre Nielsen has verified the case of a truncated polynomial algebra in a finite number of variables over a perfect field of characteristic $p > 0$. He shows further that if $A = \mathbb{F}_p[x]/(x^p)$ and $q = 2$,

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the map λ is not pro-isomorphism. However, it seems reasonable to expect that λ is a pro-isomorphism for $q = 2$, if A is regular.

In this paper, a pro-object in a category C is a functor from the set of positive integers, viewed as a category with one arrow from $n+1$ to n, to C, and a strict map between pro-objects is a natural transformation.

1. The de Rham-Witt complex

1.1. We briefly recall the definition of the de Rham-Witt complex and refer to [**5**] for details. This definition extends to $\mathbb{Z}_{(p)}$-algebras (with p odd) the original definition for $\mathbb{F}_p$-algebras of Bloch-Deligne-Illusie [**8**].

Let A be a $\mathbb{Z}_{(p)}$-algebra (with p odd). By a *Witt complex* over A, we mean the following data:

(i) a pro-differential graded ring $E^*_\cdot$ and a strict map of pro-rings

$$\lambda\colon W_\cdot(A) \to E^0_\cdot$$

from the pro-ring of Witt vectors in A;

(ii) a strict map of pro-graded rings

$$F\colon E^*_\cdot \to E^*_{\cdot-1},$$

such that $\lambda F = F\lambda$ and such that for all $a \in A$,

$$Fd\lambda[a]_n = \lambda[a]^{p-1}_{n-1} d\lambda[a]_{n-1},$$

where $[a]_n = (a, 0, \dots, 0) \in W_n(A)$ is the multiplicative representative;

(iii) a strict map of pro-graded $E^*_\cdot$-modules

$$V\colon F_* E^*_{\cdot-1} \to E^*_\cdot,$$

such that $\lambda V = V\lambda$ and such that $FV = p$ and $FdV = d$. (Here $F_* E^*_{n-1}$ denotes the E^*_{n-1}-module E^*_{n-1} considered as an E^*_n-module via $F\colon E^*_n \to E^*_{n-1}$.)

By a map of Witt complexes we mean a strict map of pro-differential graded rings $f\colon E^*_\cdot \to E'^*_\cdot$ such that $f\lambda = \lambda' f$, $fF = F'f$, and $fV = V'f$. We write R for the structure map in the pro-graded ring $E^*_\cdot$ and call it the restriction map.

By definition, the de Rham-Witt complex $W_\cdot\,\Omega^*_A$ is the universal Witt complex over A. The existence is proved in [**5**, theorem A], which also shows that the canonical map

$$\Omega^*_{W_\cdot(A)} \twoheadrightarrow W_\cdot\,\Omega^*_A$$

is surjective. Hence, every element of $W_n\,\Omega^q_A$ can be written, non-uniquely, as a sum of forms $\omega = a_0 da_1 \dots da_q$ with $a_i \in W_n(A)$. In particular, the restriction map is surjective. For the de Rham-Witt complex, the structure map

$$\lambda\colon W_n(A) \xrightarrow{\sim} W_n\,\Omega^0_A$$

is an isomorphism, and therefore, we frequently omit it from the notation.

1.2. The definition of the ring $W_n(I)$ of Witt vectors does not require that the ring I be unital.

Lemma 1.2.1. *Let A be a ring and $I \subset A$ an ideal. Then $W_n(I) \subset W_n(A)$ is an ideal and the natural projection induces an isomorphism*

$$W_n(A)/W_n(I) \xrightarrow{\sim} W_n(A/I).$$

Proof. Only the last statement needs proof. We argue by induction on n starting from the case $n = 1$ which is trivial. In the induction step we consider the 3x3-diagram with exact columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle V^{n-1}} & & \downarrow{\scriptstyle V_{n-1}} & & \downarrow{\scriptstyle V^{n-1}} & & \\
0 & \longrightarrow & W_n(I) & \longrightarrow & W_n(A) & \longrightarrow & W_n(A/I) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle R} & & \downarrow{\scriptstyle R} & & \downarrow{\scriptstyle R} & & \\
0 & \longrightarrow & W_{n-1}(I) & \longrightarrow & W_{n-1}(A) & \longrightarrow & W_{n-1}(A/I) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The top and bottom row are exact by induction. Hence, so is the middle row. □

The following result is [**2**, lemma 2.2.1]; for the convenience of the reader we include it here.

Lemma 1.2.2. *Let $I \subset A$ be an ideal, and let $W_n\,\Omega^*_{(A,I)}$ be the differential graded ideal of $W_n\,\Omega^*_A$ generated by $W_n(I) \subset W_n(A)$. Then the canonical projection*

$$W_n\,\Omega^q_A / W_n\,\Omega^q_{(A,I)} \xrightarrow{\sim} W_n\,\Omega^q_{A/I}$$

is an isomorphism.

Proof. We first show that $W.\,\Omega^*_A / W.\,\Omega^*_{(A,I)}$ is a Witt complex over A/I. The definition and lemma 1.2.1 show that it is a pro-differential graded ring with underlying pro-ring $W.(A/I)$. Hence, we need only show that the operators F, R and V on $W.\,\Omega^*_A$ descend to operators on $W.\,\Omega^*_A / W.\,\Omega^*_{(A,I)}$, or equivalently, that

$$\begin{aligned}
RW_n\,\Omega^q_{(A,I)} &\subset W_{n-1}\,\Omega^q_{(A,I)}, \\
FW_n\,\Omega^q_{(A,I)} &\subset W_{n-1}\,\Omega^q_{(A,I)}, \\
VW_n\,\Omega^q_{(A,I)} &\subset W_{n+1}\,\Omega^q_{(A,I)}.
\end{aligned}$$

The elements of $W_n\,\Omega^q_{(A,I)}$ are sums of forms $\omega = a_0 da_1 \dots da_q$, where $a_i \in W_n(A)$, for all i, and where $a_i \in W_n(I)$, for some i. The statement for the Verschiebung map then follows from the formula

$$V(\omega) = V(a_0\, FdV(a_1) \dots FdV(a_q)) = V(a_0) dV(a_1) \dots dV(a_q).$$

In the case of the Frobenius map, we first note that

$$F(\omega) = Fa_0 \cdot Fda_1 \cdot \ldots \cdot Fda_q.$$

If $a_0 \in W_n(I)$ then $F(a_0) \in W_{n-1}(I)$. If $a_i \in W_n(I)$, for some $1 \le i \le q$, we write out a_i in Witt coordinates,

$$a_i = [a_{i,0}]_n + V[a_{i,1}]_{n-1} + \cdots + V^{n-1}[a_{i,n-1}]_1.$$

We then have

$$Fda_i = [a_{i,0}]_{n-1}^{p-1} d[a_{i,0}]_{n-1} + d[a_{i,1}]_{n-1} + dV[a_{i,2}]_{n-2} + \cdots + dV^{n-2}[a_{i,n-1}]_1,$$

which shows that $Fda_i \in W_{n-1}\,\Omega^1_{(A,I)}$. Hence, in either case $F(\omega) \in W_{n-1}\,\Omega^q_{(A,I)}$. The statement for R is clear. This shows that $W.\,\Omega^*_A / W.\,\Omega^*_{(A,I)}$ is a Witt complex over A/I. One immediately verifies that it is the universal one. □

COROLLARY 1.2.3. *The abelian group $W_n\,\Omega^1_{(A,I)}$ is generated by elements of the form $dV^s[x]_{n-s}$, $V^s([x]_{n-s}d[a]_{n-s})$ and $V^s([a]_{n-s}d[x]_{n-s})$ with $0 \le s < n$, $a \in A$, and $x \in I$.*

PROOF. The group $W_n\,\Omega^1_{(A,I)}$ is generated by elements of the form adx and xda with $a \in W_n(A)$ and $x \in W_n(I)$. Writing a and x out in Witt coordinates,

$$a = [a_0]_n + V[a_1]_{n-1} + \cdots + V^{n-1}[a_{n-1}]_1,$$
$$x = [x_0]_n + V[x_1]_{n-1} + \cdots + V^{n-1}[x_{n-1}]_1,$$

we see that only the generators $V^s[a]_{n-s} \cdot dV^t[x]_{n-t}$ and $V^s[x]_{n-s} \cdot dV^t[a]_{n-t}$ with $a \in A$, $x \in I$, and $0 \le s,t < n$, are needed. The generators $V^s[a]_{n-s} \cdot dV^t[x]_{n-t}$ with $s \ge t$ may be rewritten

$$V^s[a]_{n-s} \cdot dV^t[x]_{n-t} = V^s([a]_{n-s}F^s dV^t[x]_{n-t}) = V^s([a]_{n-s}[x]_{n-s}^{p^{s-t}-1} d[x]_{n-s}),$$

which is of the desired form. And if $s \le t$, we have

$$\begin{aligned} V^s[a]_{n-s} \cdot dV^t([x]_{n-t} &= d(V^s([a]_{n-s}V^t([x]_{n-t}))) - dV^s([a]_{n-s})V^t([x]_{n-t}) \\ &= dV^t([a]_{n-t}^{p^{t-s}}[x]_{n-t}) - V^t([a]_{n-t}^{p^{t-s}-1}[x]_{n-t}d[a]_{n-t}). \end{aligned}$$

Similarly, for the generators $V^s[x]_{n-s} \cdot dV^t[a]_{n-t}$. □

2. The relative theory $\mathrm{TR}^{\cdot}_*(A, I; p)$

2.1. Let $I \subset A$ be an ideal and let

$$\mathrm{TH}(A, I) = \mathrm{TH}(A \to A/I)$$

be the (cyclotomic) spectrum defined in [**1**, appendix]. Then for all $n \ge 1$, there is a natural cofibration sequence of $\mathrm{TR}^n(A;p)$-module spectra

$$\mathrm{TR}^n(A, I; p) \to \mathrm{TR}^n(A; p) \to \mathrm{TR}^n(A/I; p) \to \Sigma\,\mathrm{TR}^n(A, I; p).$$

This is proved in *loc.cit.* under a certain connectivity requirement. But, as a consequence of [**10**, theorem 4.2.8], this requirement may be dropped. For the module structure we refer to [**6**, section 2.7]; see also [**3**, appendix].

LEMMA 2.1.1. *There is a canonical isomorphism $I \xrightarrow{\sim} \mathrm{TH}_0(A, I)$, and as an abelian group, $\mathrm{TH}_1(A, I)$ is generated by elements of the form xda and adx with $a \in A$ and $x \in I$.*

PROOF. The spectrum $\mathrm{TH}(A, I)$ is defined as the geometric realization of a simplicial symmetric spectrum $[s] \mapsto \mathrm{TH}(A, I)_s$. The spectrum $\mathrm{TH}(A, I)_s$ in simplicial degree s has the homotopy type of the homotopy colimit of the punctured s-cube which to $T \subsetneq \{1, 2, \dots, s\}$ associates the smash product

$$A_1 \wedge A_2 \wedge \cdots \wedge A_s,$$

where $A_i = A$ (resp. $A_i = I$) if $i \in T$ (resp. if $i \notin T$). (Here we denote a ring and its Eilenberg-MacLane spectrum by the same symbol.) It follows that in the skeleton spectral sequence

$$E^1_{s,t} = \pi_t(\mathrm{TH}(A,I)_s) \Rightarrow \mathrm{TH}_{s+t}(A,I),$$

we have

$$E^1_{0,t} = \begin{cases} I, & \text{if } t = 0, \\ 0, & \text{if } t > 0, \end{cases}$$

$$E^1_{1,0} = I \otimes A \oplus_{I\otimes I} A \otimes I.$$

Since A is commutative, the differential

$$d^1 \colon E^1_{0,1} \to E^1_{0,0}$$

is zero, and hence, the edge-homomorphism $I \xrightarrow{\sim} \mathrm{TH}_0(A,I)$ is an isomorphism. Finally, the elements xda (resp. adx) with $a \in A$ and $x \in I$ are represented in the spectral sequence by $x \otimes a$ (resp. $a \otimes x$) in $E^1_{1,0}$. □

Remark 2.1.2. The generators xda and adx of $\mathrm{TH}_1(A,I)$ are subject to the additivity relations

$$(a_1 + a_2)d(x_1 + x_2) = a_1dx_1 + a_1dx_2 + a_2dx_1 + a_2dx_2,$$
$$(x_1 + x_2)d(a_1 + a_2) = x_1da_1 + x_1da_2 + x_2da_1 + x_2da_2,$$

where $a_1, a_2 \in A$ and $x_1, x_2 \in I$, and to the Leibniz rule

$$a_0d(a_1a_2) = a_0a_1da_2 + a_2a_0da_1,$$

where $a_i \in A$, for all $i = 0, 1, 2$, and $a_i \in I$, for some $i = 0, 1, 2$. Indeed, the additivity relations follows from the definition of the tensor product, which gives $E^1_{1,0}$, and the Leibniz rule follows from the differential $d^1 \colon E^1_{0,2} \to E^1_{0,1}$.

Since $\mathrm{TH}(A,I)$ is *cyclotomic* in the sense of [**6**, definition 2.2], we have a natural cofibration sequence

$${}_h\mathrm{TR}^n(A,I;p) \xrightarrow{N} \mathrm{TR}^n(A,I;p) \xrightarrow{R} \mathrm{TR}^{n-1}(A,I;p) \xrightarrow{\partial} \Sigma_h\mathrm{TR}^n(A,I;p);$$

see [**6**, theorem 2.2]. The left hand term

$${}_h\mathrm{TR}^n(A,I;p) = \mathbb{H}.(C_{p^{n-1}}, \mathrm{TH}(A,I))$$

is the group homology spectrum (or Borel construction) whose homotopy groups are the abutment of a first quadrant spectral sequence

$$E^2_{s,t} = H_s(C_{p^{n-1}}, \mathrm{TH}_t(A,I)) \Rightarrow {}_h\mathrm{TR}^n_{s+t}(A,I;p).$$

We refer the reader to [**7**, paragraph 4] for a thorough treatment of the construction of this spectral sequence. The $C_{p^{n-1}}$-module $\mathrm{TH}_t(A,I)$ is trivial, since the $C_{p^{n-1}}$-action on $\mathrm{TH}(A,I)$ comes from a circle action.

Lemma 2.1.3. *The map $I \oplus \mathrm{TH}_1(A,I) \to {}_h\mathrm{TR}^n_1(A,I;p)$, which to (x,ω) associates $dV^{n-1}x + V^{n-1}\omega$, is a surjection.*

Proof. The spectral sequence above amounts to an exact sequence

$${}_{p^{n-1}}I \xrightarrow{d^2} \mathrm{TH}_1(A,I) \xrightarrow{V^{n-1}} {}_h\mathrm{TR}^n_1(A,I;p) \xrightarrow{\pi} I/p^{n-1}I \to 0,$$

where π is the edge-homomorphism to the baseline. Moreover, the composite

$$I \xrightarrow{V^{n-1}} {}_h\mathrm{TR}^n_0(A,I;p) \xrightarrow{d} {}_h\mathrm{TR}^n_1(A,I;p) \xrightarrow{\pi} I/p^{n-1}I$$

may be identified with the map $H_0(C_{p^{n-1}}, I) \to H_1(C_{p^{n-1}}, I)$ given by multiplication by the fundamental class $[\mathbb{T}/C_{p^{n-1}}]$. It is well-known that this map is an epimorphism. □

PROPOSITION 2.1.4. *As a non-unital ring,* $\mathrm{TR}_0^n(A, I; p)$ *is canonically isomorphic to* $W_n(I)$*, and as an abelian group,* $\mathrm{TR}_1^n(A, I; p)$ *is generated by elements of the form* $dV^s([x]_{n-s})$*,* $V^s([x]_{n-s}d[a]_{n-s})$ *and* $V^s([a]_{n-s}d[x]_{n-s})$*, where* $0 \leq s < n$*,* $a \in A$*, and* $x \in I$*.*

PROOF. The first statement follows from the proof of [**6**, theorem F]. Indeed, it is not necessary for this proof that the ring I be unital. The second statement follows from lemmas 2.1.1 and 2.1.3 by an induction argument based on the exact sequence

$$_h\,\mathrm{TR}_1^n(A, I; p) \to \mathrm{TR}_1^n(A, I; p) \to \mathrm{TR}_1^{n-1}(A, I; p) \to 0.$$

The maps in the sequence commute with d and V (and F). □

PROOF OF THE THEOREM. The statement for $q = 0$ is [**6**, theorem F], so consider $q = 1$. If A is a polynomial algebra over $\mathbb{Z}_{(p)}$, the statement was proved in [**5**]. In the general case, we write $A = R/I$ with R a polynomial algebra over $\mathbb{Z}_{(p)}$ and consider the following diagram with exact rows

$$\begin{array}{ccccccc}
0 \to & W_n\,\Omega^1_{(R,I)} & \to & W_n\,\Omega^1_R & \to & W_n\,\Omega^1_{R/I} & \to 0 \\
 & & & \downarrow \wr & & \downarrow & \\
 & \mathrm{TR}_1^n(R, I; p) & \to & \mathrm{TR}_1^n(R; p) & \to & \mathrm{TR}_1^n(R/I; p) & \to 0
\end{array}$$

Then the middle vertical map is an isomorphism, and hence, it suffices to show that the image of the composite

$$W_n\,\Omega^1_{(R,I)} \to W_n\,\Omega^1_R \xrightarrow{\sim} \mathrm{TR}_1^n(R; p)$$

coincides with the image of the canonical map

$$\mathrm{TR}_1^n(R, I; p) \to \mathrm{TR}_1^n(R; p).$$

But corollaries 1.2.3 and 2.1.4 identifies both images with the subgroup generated by elements of the form $dV^s[x]_{n-s}$, $V^s([x]_{n-s}d[a]_{n-s})$ and $V^s([a]_{n-s}d[x]_{n-s})$, where $0 \leq s < n$, $a \in R$, and $x \in I$. This concludes the proof. □

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