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On the K-theory of truncated polynomial algebras over the integers

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Abstract

We show that $K_{2i}(\mathbb{Z}[x]/(x^m),(x))$ is finite of order $(mi)!(i!)^{m-2}$ and that $K_{2i+1}(\mathbb{Z}[x]/(x^m),(x))$ is free abelian of rank $m-1$. This is accomplished by showing that the equivariant homotopy groups $\mathrm{TR}^n_{q-\lambda}(\mathbb{Z};p)$ of the topological Hochschild $\mathbb{T}$-spectrum $T(\mathbb{Z})$ are free abelian for q even, and finite for q odd, and by determining their ranks and orders, respectively.

Introduction

It was proved by Soulé [**13**] and Staffeldt [**14**] that, for every non-negative integer q, the abelian group $K_q(\mathbb{Z}[x]/(x^m),(x))$ is finitely generated and that its rank is either 0 or $m-1$ according as q is even or odd. In this paper, we prove the following more precise result.

Theorem A. *Let m be a positive integer and let i be a non-negative integer. Then*
(i) *the abelian group $K_{2i+1}(\mathbb{Z}[x]/(x^m),(x))$ is free of rank $m-1$;*
(ii) *the abelian group $K_{2i}(\mathbb{Z}[x]/(x^m),(x))$ is finite of order $(mi)!(i!)^{m-2}$.*

In particular, the p-primary torsion subgroup of $K_{2i}(\mathbb{Z}[x]/(x^m),(x))$ is zero, for every prime number $p > mi$. At present, we do not know the group structure of the finite abelian group in degree $2i$ except for small values of i and m. We remark that the result agrees with the calculation by Geller and Roberts [**12**] of the group in degree two.

To prove Theorem A, we use the cyclotomic trace map of Bökstedt–Hsiang–Madsen [**4**] from the K-groups in the statement to the corresponding topological cyclic homology groups and a theorem of McCarthy [**11**], which shows that this map becomes an isomorphism after pro-finite completion. The third author and Madsen [**7**, Proposition 4.2.3], in turn, gave a formula for the topological cyclic homology groups in question in terms of the equivariant homotopy groups

$$\mathrm{TR}^r_{q-\lambda}(\mathbb{Z}) = [S^q \wedge (\mathbb{T}/C_r)_+, S^\lambda \wedge T(\mathbb{Z})]_{\mathbb{T}}$$

of the topological Hochschild $\mathbb{T}$-spectrum $T(\mathbb{Z})$. Here $\mathbb{T}$ is the multiplicative group of complex numbers of modulus 1, $C_r \subset \mathbb{T}$ is the finite subgroup of the indicated order, λ is a finite-dimensional complex $\mathbb{T}$-representation, and S^λ is the one-point compactification of λ. Since the groups $K_q(\mathbb{Z}[x]/(x^m),(x))$ and $\mathrm{TR}^r_{q-\lambda}(\mathbb{Z})$ are finitely generated by [**13**, **14**] and Lemma 1.3, respectively, these earlier results amount to a long exact sequence

$$\cdots \longrightarrow \lim_R \mathrm{TR}^{r/m}_{q-1-\lambda_d}(\mathbb{Z}) \xrightarrow{V_m} \lim_R \mathrm{TR}^r_{q-1-\lambda_d}(\mathbb{Z}) \longrightarrow K_q(\mathbb{Z}[x]/(x^m),(x)) \longrightarrow \cdots,$$

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where $d = d(m, r)$ is the integer part of $(r-1)/m$, and where λ_d is the sum

$$\lambda_d = \mathbb{C}(d) \oplus \mathbb{C}(d-1) \oplus \dots \oplus \mathbb{C}(1)$$

of the one-dimensional complex $\mathbb{T}$-representations defined by $\mathbb{C}(i) = \mathbb{C}$ with $\mathbb{T}$ acting from the left by $z \cdot w = z^i w$. The two limits range over the positive integers divisible by m and the positive integers, respectively, ordered under division. The structure maps R and the map V_m are explained in Section 1 below. In particular, we show that for every integer q there exists a positive integer $r = r(m, q)$ divisible by m such that the canonical projections

$$\lim_R \mathrm{TR}^r_{q-\lambda_d}(\mathbb{Z}) \longrightarrow \mathrm{TR}^r_{q-\lambda_d}(\mathbb{Z}), \qquad \lim_R \mathrm{TR}^{r/m}_{q-\lambda_d}(\mathbb{Z}) \longrightarrow \mathrm{TR}^{r/m}_{q-\lambda_d}(\mathbb{Z}),$$

are isomorphisms.

After localizing at a prime number p, the abelian groups $\mathrm{TR}^r_{q-\lambda}(\mathbb{Z})$ decompose as products of the p-typical equivariant homotopy groups

$$\mathrm{TR}^n_{q-\lambda}(\mathbb{Z}; p) = \mathrm{TR}^{p^{n-1}}_{q-\lambda}(\mathbb{Z}) = [S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, S^\lambda \wedge T(\mathbb{Z})]_{\mathbb{T}}.$$

In addition, the Verschiebung map V_m that appears in the long exact sequence above may be expressed in terms of the p-typical Verschiebung map $V = V_p$. The corresponding p-typical equivariant homotopy groups with $\mathbb{Z}/p\mathbb{Z}$-coefficients

$$\mathrm{TR}^n_{q-\lambda}(\mathbb{Z}; p, \mathbb{Z}/p\mathbb{Z}) = [S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, M_p \wedge S^\lambda \wedge T(\mathbb{Z})]_{\mathbb{T}},$$

were evaluated by the first and second author [**1**] and by Tsalidis [**16**]. More generally, the first and second author [**1**] evaluated the $RO(\mathbb{T})$-graded equivariant homotopy groups

$$\mathrm{TR}^n_{\alpha}(\mathbb{Z}; p, \mathbb{Z}/p\mathbb{Z}) = [S^\beta \wedge (\mathbb{T}/C_{p^{n-1}})_+, M_p \wedge S^\gamma \wedge T(\mathbb{Z})]_{\mathbb{T}},$$

where $\alpha \in RO(\mathbb{T})$ is any virtual finite-dimensional orthogonal $\mathbb{T}$-representation, and where β and γ are chosen actual representations with $\alpha = [\beta] - [\gamma]$. Based on these results, we prove the following.

THEOREM B. *Let p be a prime number, let n be a positive integer, and let λ be a finite-dimensional complex $\mathbb{T}$-representation. Then,*

(i) *for $q = 2i$ even, $\mathrm{TR}^n_{q-\lambda}(\mathbb{Z}; p)$ is a free abelian group whose rank is equal to the number of integers $0 \leqslant s < n$ such that $i = \dim_{\mathbb{C}}(\lambda^{C_{p^s}})$;*

(ii) *for $q = 2i - 1$ odd, $\mathrm{TR}^n_{q-\lambda}(\mathbb{Z}; p)$ is a finite abelian group whose order is determined, recursively, by*

$$\left|\mathrm{TR}^n_{q-\lambda}(\mathbb{Z}; p)\right| = \begin{cases} \left|\mathrm{TR}^{n-1}_{q-\lambda'}(\mathbb{Z}; p)\right| \cdot p^{n-1}(i - \dim_{\mathbb{C}}(\lambda)) & \text{if } i > \dim_{\mathbb{C}}(\lambda) \\ \left|\mathrm{TR}^{n-1}_{q-\lambda'}(\mathbb{Z}; p)\right| & \text{if } i \leqslant \dim_{\mathbb{C}}(\lambda), \end{cases}$$

where $\lambda' = \rho_p^* \lambda^{C_p}$ is the $\mathbb{T}/C_p$-representation λ^{C_p} viewed as a $\mathbb{T}$-representation via the isomorphism $\rho_p \colon \mathbb{T} \to \mathbb{T}/C_p$ given by the pth root;

(iii) *for every integer q, the Verschiebung map*

$$V \colon \mathrm{TR}^{n-1}_{q-\lambda}(\mathbb{Z}; p) \longrightarrow \mathrm{TR}^n_{q-\lambda}(\mathbb{Z}; p)$$

is injective, and for q even the cokernel is a free abelian group.

We remark that for $\lambda = 0$, the result is that

$$\left|\mathrm{TR}^n_{2i-1}(\mathbb{Z}; p)\right| = p^{n(n-1)/2}\, i^n,$$

while the even groups all are zero with the exception of $\mathrm{TR}^n_0(\mathbb{Z}; p)$, which is a free abelian group of rank n. In the case $n = 1$, which was proved by Bökstedt [**3**], the groups are all cyclic. For

$n > 1$, this is not the case. It remains a very interesting problem to determine the structure of these groups. We refer to [**5**, Theorem 18] for some partial results.

It follows from Theorem B that with $\mathbb{Z}/p\mathbb{Z}$-coefficients the Verschiebung map

$$V\colon \mathrm{TR}^{n-1}_{q-\lambda}(\mathbb{Z}; p, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \mathrm{TR}^{n}_{q-\lambda}(\mathbb{Z}; p, \mathbb{Z}/p\mathbb{Z})$$

is injective for q even. We do not know the value of this map for q odd. The calculation of this map, and hence, the groups $K_q(\mathbb{Z}[x]/(x^m), (x); \mathbb{Z}/p\mathbb{Z})$ claimed in [**16**, Proposition 7.7] is incorrect. Indeed, in loc. cit., it is only the map induced by V between the E^∞-terms of two spectral sequences that is evaluated.

The paper is organized as follows. In Section 1, we show that the groups $\mathrm{TR}^r_{q-\lambda}(\mathbb{Z})$ are finitely generated and determine their ranks. In Section 2, we recall the results of [**1**] and [**16**] and prove Theorem B. In the following Section 3, we evaluate the terms in the long exact sequence above and prove Theorem A. Finally, in Section 4, we specialize to the case of the dual numbers and determine the structure of the finite group $K_{2i}(\mathbb{Z}[x]/(x^2), (x))$ of order $(2i)!$ in low degrees.

1. *The groups* $\mathrm{TR}^r_{q-\lambda}(\mathbb{Z})$

In this section, we recall the groups $\mathrm{TR}^r_{q-\lambda}(\mathbb{Z})$ and the Frobenius, Verschiebung, and restriction operators that relate them. We refer to [**8**, Section 1] and [**5**, Section 2] for further details.

Let A be a unital associative ring. The topological Hochschild $\mathbb{T}$-spectrum $T(A)$ is, in particular, an orthogonal $\mathbb{T}$-spectrum in the sense of [**10**, Definition II.2.6]. Therefore, for every finite-dimensional orthogonal $\mathbb{T}$-representation λ and every finite subgroup $C_r \subset \mathbb{T}$, we have the equivariant homotopy group given by the following abelian group of maps in the homotopy category of orthogonal $\mathbb{T}$-spectra:

$$\mathrm{TR}^r_{q-\lambda}(A) = [S^q \wedge (\mathbb{T}/C_r)_+, S^\lambda \wedge T(A)]_{\mathbb{T}}.$$

For every divisor s of r with quotient $t = r/s$, there are maps,

$$F_s\colon \mathrm{TR}^r_{q-\lambda}(A) \longrightarrow \mathrm{TR}^t_{q-\lambda}(A) \quad \text{(Frobenius)},$$
$$V_s\colon \mathrm{TR}^t_{q-\lambda}(A) \longrightarrow \mathrm{TR}^r_{q-\lambda}(A) \quad \text{(Verschiebung)},$$

induced by maps $f_s\colon (\mathbb{T}/C_t)_+ \to (\mathbb{T}/C_r)_+$ and $v_s\colon (\mathbb{T}/C_r)_+ \to (\mathbb{T}/C_t)_+$ in the homotopy category of orthogonal $\mathbb{T}$-spectra. The map f_s is the map of suspension $\mathbb{T}$-spectra induced by the canonical projection $\mathrm{pr}\colon \mathbb{T}/C_t \to \mathbb{T}/C_r$ and the map v_s is the corresponding transfer map defined as follows. Let $\iota\colon \mathbb{T}/C_t \hookrightarrow \mu$ be an embedding into a finite-dimensional orthogonal $\mathbb{T}$-representation. Then the product embedding $(\iota, \mathrm{pr})\colon \mathbb{T}/C_t \to \mu \times (\mathbb{T}/C_r)$ has trivial normal bundle, and the linear structure of μ determines a canonical trivialization. Therefore, the Pontryagin–Thom construction gives a map of pointed $\mathbb{T}$-spaces

$$S^\mu \wedge (\mathbb{T}/C_r)_+ \longrightarrow S^\mu \wedge (\mathbb{T}/C_t)_+.$$

The induced map of suspension $\mathbb{T}$-spectra determines the homotopy class of maps of orthogonal $\mathbb{T}$-spectra $v_s\colon (\mathbb{T}/C_r)_+ \to (\mathbb{T}/C_t)_+$ and this homotopy class is independent of the choice of embedding ι as well as the choices made in forming the Pontryagin–Thom construction.

The orthogonal $\mathbb{T}$-spectrum $T(A)$ has the additional structure of a cyclotomic spectrum in the sense of [**8**, Definition 2.2]. This implies that, in the situation above, there is a map

$$R_s\colon \mathrm{TR}^r_{q-\lambda}(A) \longrightarrow \mathrm{TR}^t_{q-\lambda'}(A) \quad \text{(restriction)},$$

where $\lambda' = \rho_s^*(\lambda^{C_s})$ is the $\mathbb{T}/C_s$-representation λ^{C_s} considered as a $\mathbb{T}$-representation via the isomorphism $\rho_s\colon \mathbb{T} \to \mathbb{T}/C_s$ defined by $\rho_s(z) = z^{1/s}C_s$. Moreover, the map R_s admits a canonical factorization that we now explain. In general, let G be a compact Lie group and $\mathscr{F}$ a family of closed subgroups of G stable under conjugation and passage to subgroups. We recall that a universal $\mathscr{F}$-space is a G-CW-complex $E\mathscr{F}$ with the property that, for every closed subgroup $H \subset G$, the fixed point set $(E\mathscr{F})^H$ is contractible if $H \in \mathscr{F}$ and empty if $H \notin \mathscr{F}$. It was proved by tom Dieck [**15**, Satz 1] that a universal $\mathscr{F}$-space $E\mathscr{F}$ exists and that, if both $E\mathscr{F}$ and $E'\mathscr{F}$ are universal $\mathscr{F}$-spaces, then there exists a unique G-homotopy class of G-homotopy equivalences $f\colon E\mathscr{F} \to E'\mathscr{F}$. Given a universal $\mathscr{F}$-space $E\mathscr{F}$, the pointed G-space $\tilde{E}\mathscr{F}$ is defined to be the mapping cone of the map $\pi\colon E\mathscr{F}_+ \to S^0$ that collapses $E\mathscr{F}$ onto the non-base point such that we have a cofibration sequence of pointed G-spaces,

$$E\mathscr{F}_+ \xrightarrow{\pi} S^0 \xrightarrow{\iota} \tilde{E}\mathscr{F} \xrightarrow{\delta} \Sigma E\mathscr{F}_+.$$

If $N \subset G$ is a closed normal subgroup, we denote by $\mathscr{F}[N]$ the family of closed subgroups $H \subset G$ that do not contain N as a subgroup. Now, the map R_s admits a factorization as the composition of the map

$$\mathrm{TR}^r_{q-\lambda}(A) = [S^q \wedge (\mathbb{T}/C_r)_+, S^\lambda \wedge T(A)]_{\mathbb{T}} \longrightarrow [S^q \wedge (\mathbb{T}/C_r)_+, S^\lambda \wedge \tilde{E}\mathscr{F}[C_s] \wedge T(A)]_{\mathbb{T}},$$

induced by the map $\iota\colon S^0 \to \tilde{E}\mathscr{F}[C_s]$ and a canonical isomorphism

$$[S^q \wedge (\mathbb{T}/C_r)_+, S^\lambda \wedge \tilde{E}\mathscr{F}[C_s] \wedge T(A)]_{\mathbb{T}} \xrightarrow{\sim} [S^q \wedge (\mathbb{T}/C_t)_+, S^{\lambda'} \wedge T(A)]_{\mathbb{T}} = \mathrm{TR}^t_{q-\lambda'}(A),$$

induced from the cyclotomic structure of $T(A)$ and [**10**, Proposition V.4.17].

The group isomorphism $\rho_s\colon \mathbb{T} \to \mathbb{T}/C_s$ gives rise to an equivalence of categories ρ_s^* from the category of orthogonal $\mathbb{T}/C_s$-spectra to the category of orthogonal $\mathbb{T}$-spectra defined by

$$(\rho_s^* T)(\lambda) = \rho_s^*\big(T\big((\rho_s^{-1})^*\lambda\big)\big).$$

The following result is a generalization of [**8**, Theorem 2.2].

Proposition 1.1. *Let A be a unital associative ring, let r be a positive integer, and let λ be a finite-dimensional orthogonal $\mathbb{T}$-representation. Let p be a prime number that divides r, and let u and v be positive integers with $u + v = v_p(r) + 1$. Then there is a natural long exact sequence*

$$\cdots \longrightarrow \mathbb{H}_q\big(C_{p^u}, \mathrm{TR}^{r/p^u}_{\cdot-\lambda}(A)\big) \xrightarrow{N_{p^u}} \mathrm{TR}^r_{q-\lambda}(A) \xrightarrow{R_{p^v}} \mathrm{TR}^{r/p^v}_{q-\lambda'}(A) \xrightarrow{\partial} \mathbb{H}_{q-1}\big(C_{p^u}, \mathrm{TR}^{r/p^u}_{\cdot-\lambda}(A)\big) \longrightarrow \cdots,$$

where the left-hand term is the qth Borel homology group of the group C_{p^u} with coefficients in the orthogonal $\mathbb{T}$-spectrum defined by

$$\mathrm{TR}^{r/p^u}_{\cdot-\lambda}(A) = \rho^*_{r/p^u}(S^\lambda \wedge T(A))^{C_{r/p^u}}.$$

Proof. The cofibration sequence of pointed $\mathbb{T}$-spaces

$$E\mathscr{F}[C_{p^v}]_+ \xrightarrow{\pi} S^0 \xrightarrow{\iota} \tilde{E}\mathscr{F}[C_{p^v}] \xrightarrow{\delta} \Sigma E\mathscr{F}[C_{p^v}]_+$$

gives rise to a cofibration sequence of orthogonal $\mathbb{T}$-spectra

$$E\mathscr{F}[C_{p^v}]_+ \wedge T(A) \xrightarrow{\pi} T(A) \xrightarrow{\iota} \tilde{E}\mathscr{F}[C_{p^v}] \wedge T(A) \xrightarrow{\delta} \Sigma E\mathscr{F}[C_{p^v}]_+ \wedge T(A).$$

This, in turn, gives rise to a long exact sequence of equivariant homotopy groups that we now identify with the long exact sequence of the statement. By definition, we have

$$\mathrm{TR}^r_{q-\lambda}(A) = [S^q \wedge (\mathbb{T}/C_r)_+, S^\lambda \wedge T(A)]_{\mathbb{T}},$$

and, as recalled above, the restriction map R_{p^v} factors through the canonical isomorphism

$$[S^q \wedge (\mathbb{T}/C_r)_+, S^\lambda \wedge \tilde{E}\mathscr{F}[C_{p^v}] \wedge T(A)]_{\mathbb{T}} \xrightarrow{\sim} \mathrm{TR}^{r/p^v}_{q-\lambda'}(A).$$

This identifies the middle and right-hand terms of the long exact sequence. To identify the left-hand term, we recall from [**10**, Proposition V.2.3] the change-of-groups isomorphism,

$$[S^q \wedge (C_r/C_r)_+, S^\lambda \wedge E\mathscr{F}[C_{p^v}]_+ \wedge T(A)]_{C_r} \xrightarrow{\sim} [S^q \wedge (\mathbb{T}/C_r)_+, S^\lambda \wedge E\mathscr{F}[C_{p^v}]_+ \wedge T(A)]_{\mathbb{T}}.$$

On the left-hand side, the family $\mathscr{F}[C_{p^v}]$ is equal to the family of subgroups $C_s \subset C_r$ for which $v_p(s) < v$. Therefore, we may choose the universal space $E\mathscr{F}[C_{p^v}]$ to be a C_r-CW-complex that is non-equivariantly contractible and that only has cells of orbit-type $C_r/C_{r/p^u}$. Indeed, in this case we have

$$(E\mathscr{F}[C_{p^v}])^{C_s} = \begin{cases} E\mathscr{F}[C_{p^v}] & \text{if } v_p(s) < v \\ \varnothing & \text{if } v_p(s) \geqslant v, \end{cases}$$

as required. We then have canonical isomorphisms

$$\begin{aligned}[S^q, S^\lambda \wedge E\mathscr{F}[C_{p^v}]_+ \wedge T(A)]_{C_r} &\xleftarrow{\sim} [S^q, (S^\lambda \wedge E\mathscr{F}[C_{p^v}]_+ \wedge T(A))^{C_{r/p^u}}]_{C_r/C_{r/p^u}} \\ &\xleftarrow{\sim} [S^q, E\mathscr{F}[C_{p^v}]_+ \wedge (S^\lambda \wedge T(A))^{C_{r/p^u}}]_{C_r/C_{r/p^u}},\end{aligned}$$

where for the second isomorphism we use that $E\mathscr{F}[C_{p^v}]$ is chosen to be C_{r/p^u}-fixed. The group isomorphism $\rho_{r/p^u} : C_{p^u} \to C_r/C_{r/p^u}$ induces an isomorphism of categories ρ^*_{r/p^u} from the category of orthogonal C_{p^u}-spectra to the category of orthogonal $C_r/C_{r/p^u}$-spectra. In particular, this gives an isomorphism of the lower group above to the group

$$[S^q, \rho^*_{r/p^u} E\mathscr{F}[C_{p^v}]_+ \wedge \rho^*_{r/p^u}(S^\lambda \wedge T(A))^{C_{r/p^u}}]_{C_{p^u}} = \mathbb{H}_q\big(C_{p^u}, \mathrm{TR}^{r/p^u}_{\cdot-\lambda}(A)\big).$$

This is indeed the desired Borel homology group, since $\rho^*_{r/p^u} E\mathscr{F}[C_{p^v}]$ is a free C_{p^u}-CW-complex that is non-equivariantly contractible. □

We recall that the Borel homology groups that appear in the statement of Proposition 1.1 are the abutment of the first quadrant skeleton spectral sequence

$$E^2_{s,t} = H_s\big(C_{p^u}, \mathrm{TR}^{r/p^u}_{t-\lambda}(A)\big) \Rightarrow \mathbb{H}_{s+t}\big(C_{p^u}, \mathrm{TR}^{r/p^u}_{\cdot-\lambda}(A)\big), \tag{1.2}$$

from the group homology of C_{p^u} with coefficients in the trivial C_{p^u}-module $\mathrm{TR}^{r/p^u}_{t-\lambda}(A)$; see for instance [**5**, Section 4].

We now specialize to the case $A = \mathbb{Z}$ and recall from Bökstedt [**3**] that $\mathrm{TR}^1_q(\mathbb{Z})$ is zero if either q is negative or q is positive and even, a free abelian group of rank one if $q = 0$, and a finite cyclic group of order i if $q = 2i - 1$ is positive and odd; see also [**9**].

Lemma 1.3. *Let r be a positive integer, let q be an integer, and let λ be a finite-dimensional complex $\mathbb{T}$-representation. Then $\mathrm{TR}^r_{q-\lambda}(\mathbb{Z})$ is a finitely generated abelian group whose rank is equal to the number of positive divisors e of r for which $q = 2\dim_{\mathbb{C}}(\lambda^{C_e})$. The group is zero for $q < 2\dim_{\mathbb{C}}(\lambda^{C_r})$.*

Proof. Let $\ell(r, q, \lambda)$ denote the number of positive divisors e of r with $q = 2\dim_{\mathbb{C}}(\lambda^{C_e})$ and note that $\ell(r, q, \lambda)$ is zero, for q odd. We prove by induction on the number k of prime divisors in r that $\mathrm{TR}^r_{q-\lambda}(\mathbb{Z})$ is a finitely generated abelian group of rank $\ell(r, q, \lambda)$. If $k = 0$, or equivalently, if $r = 1$, the statement follows from the result of Bökstedt, which we recalled above. Indeed, it follows from [**10**, Proposition V.2.3] that, up to isomorphism,

$$\mathrm{TR}^1_{q-\lambda}(\mathbb{Z}) = \mathrm{TR}^1_{q-2\dim_{\mathbb{C}}(\lambda)}(\mathbb{Z}).$$

So we let $k \geqslant 1$ and assume that the lemma has been proved, for all q and λ as in the statement if r has $k - 1$ prime divisors. Let p be a prime divisor of r and write $r = p^n r'$ with r' not divisible by p. We consider the long exact sequence of Proposition 1.1 with $u = n$ and $v = 1$,

$$\cdots \longrightarrow \mathbb{H}_q\big(C_{p^n}, \mathrm{TR}^{r'}_{\cdot-\lambda}(\mathbb{Z})\big) \xrightarrow{N_{p^n}} \mathrm{TR}^r_{q-\lambda}(\mathbb{Z}) \xrightarrow{R_p} \mathrm{TR}^{r/p}_{q-\lambda'}(\mathbb{Z}) \xrightarrow{\partial} \mathbb{H}_{q-1}\big(C_{p^n}, \mathrm{TR}^{r'}_{\cdot-\lambda}(\mathbb{Z})\big) \longrightarrow \cdots.$$

Since r' has only $k-1$ prime divisors, the inductive hypothesis implies that in the skeleton spectral sequence (1.2), $E^2_{0,q}$ is a finitely generated abelian group of rank $\ell(q, r', \lambda)$ and that the groups $E^2_{s,t}$ with $s > 0$ are finite. Hence, the left-hand group in the long exact sequence is finitely generated of rank $\ell(q, r', \lambda)$. By further induction on $n \geqslant 0$, we may assume that the group $\mathrm{TR}^{r/p}_{q-\lambda'}(\mathbb{Z})$ is finitely generated of rank $\ell(q, r/p, \lambda')$. The first part of the lemma now follows from the formula

$$\ell(q, r', \lambda) + \ell(q, r/p, \lambda') = \ell(q, r, \lambda),$$

which holds since the two summands on the left-hand side count the number of positive divisors e of r with $q = 2\dim_{\mathbb{C}}(\lambda^{C_e})$ for which e is, respectively, prime to p and divisible by p. The second part of the lemma is proved in a similar manner. □

ADDENDUM 1.4. (i) *Let $m, r \geqslant 1$, $0 \leqslant \epsilon \leqslant 1$, and i be integers, and let $d = d(m, r)$ be the integer part of $(r-1)/m$. Then the canonical projection induces an isomorphism*

$$\lim_R \mathrm{TR}^r_{2i+\epsilon-\lambda_d}(\mathbb{Z}) \xrightarrow{\sim} \mathrm{TR}^r_{2i+\epsilon-\lambda_d}(\mathbb{Z}),$$

provided that $m(i+1) < p^{v_p(r)+1}$ for every prime number p.

(ii) *Let $m \geqslant 1$, $0 \leqslant \epsilon \leqslant 1$, and i be integers, let $r \geqslant 1$ be an integer divisible by m, and let $d = d(m, r)$. Then the canonical projection induces an isomorphism*

$$\lim_R \mathrm{TR}^{r/m}_{2i+\epsilon-\lambda_d}(\mathbb{Z}) \xrightarrow{\sim} \mathrm{TR}^{r/m}_{2i+\epsilon-\lambda_d}(\mathbb{Z})$$

provided that $i + 1 < p^{v_p(r/m)+1}$ for every prime number p.

Proof. We prove statement (i); the proof of statement (ii) is similar. It suffices to show that for every prime number p the restriction map

$$R_p\colon \mathrm{TR}^{pr}_{q-\lambda_{d(m,pr)}}(\mathbb{Z}) \longrightarrow \mathrm{TR}^r_{q-\lambda_{d(m,r)}}(\mathbb{Z})$$

is an isomorphism if $q = 2i + \epsilon$ with $m(i+1) < p^{v_p(r)+1}$. We write $r = p^{n-1}r'$ with r' not divisible by p and consider the long exact sequence of Proposition 1.1 with $u = n$ and $v = 1$,

$$\cdots \longrightarrow \mathbb{H}_q\big(C_{p^n}, \mathrm{TR}^{r'}_{\cdot-\lambda_{d(m,pr)}}(\mathbb{Z})\big) \xrightarrow{N_{p^n}} \mathrm{TR}^{pr}_{q-\lambda_{d(m,pr)}}(\mathbb{Z}) \xrightarrow{R_p} \mathrm{TR}^r_{q-\lambda_{d(m,r)}}(\mathbb{Z}) \longrightarrow \cdots$$

The skeleton spectral sequence (1.2) and Lemma 1.3 show that the left-hand group vanishes, provided that $q < 2\dim_{\mathbb{C}}(\lambda^{C_{r'}}_{d(m,pr)})$. Therefore, the map R_p is an isomorphism for

$$i < \dim_{\mathbb{C}}\big(\lambda^{C_{r'}}_{d(m,pr)}\big) = \lfloor d(m, pr)/r' \rfloor.$$

We claim that $d(m, p^n) \leqslant \lfloor d(m, pr)/r' \rfloor$. Indeed, this inequality is equivalent to the inequality $d(m, p^n) \leqslant d(m, pr)/r'$, which is equivalent to the inequality $r'd(m, p^n) \leqslant d(m, pr)$, which, in turn, is equivalent to the inequality $r'd(m, p^n) \leqslant (p^n r' - 1)/m$. We may rewrite this inequality as $mr'd(m, p^n) \leqslant p^n r' - 1$ or $mr'd(m, p^n) < p^n r'$ or $md(m, p^n) < p^n$. But this inequality is equivalent to the inequality $md(m, p^n) \leqslant p^n - 1$, which, in turn, is equivalent to the inequality $d(m, p^n) \leqslant (p^n - 1)/m$, which holds. The claim follows. Finally, a similar argument shows that the inequalities $i < d(m, p^n)$ and $m(i+1) < p^n$ are equivalent. □

2. *The p-typical groups* $\mathrm{TR}^n_{q-\lambda}(\mathbb{Z};p)$

In this section, we prove Theorem B of Section 1. We first show that after localization at the prime number p, the groups $\mathrm{TR}^r_{q-\lambda}(A)$ decompose as products of the p-typical groups:

$$\mathrm{TR}^n_{q-\lambda}(A;p) = \mathrm{TR}^{p^{n-1}}_{q-\lambda}(A) = [S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, S^\lambda \wedge T(A)]_{\mathbb{T}}.$$

Proposition 2.1. *Let A be a unital associative ring, let $r \geqslant 1$ and q be integers, and let λ be a finite-dimensional orthogonal $\mathbb{T}$-representation. Let p be a prime number and write $r = p^{n-1}r'$ with r' not divisible by p. Then the map*

$$\gamma\colon \mathrm{TR}^r_{q-\lambda}(A) \longrightarrow \textstyle\prod_{j\mid r'} \mathrm{TR}^n_{q-\lambda'}(A;p),$$

whose jth component is the composite map

$$\mathrm{TR}^r_{q-\lambda}(A) \xrightarrow{F_j} \mathrm{TR}^{r/j}_{q-\lambda}(A) \xrightarrow{R_{r'/j}} \mathrm{TR}^{p^{n-1}}_{q-\lambda'}(A) = \mathrm{TR}^n_{q-\lambda'}(A;p),$$

becomes an isomorphism after localization at p.

Proof. The proof is by induction on the number k of positive divisors of r'. If $k=1$, or equivalently, if $r'=1$ then γ is the identity map and the statement holds trivially. So we let $k \geqslant 2$ and assume that the statement holds whenever r' has at most $k-1$ divisors. Let ℓ be a prime divisor of r', and let $v = v_\ell(r') = v_\ell(r)$. We show that the map

$$(R_\ell, F_{\ell^v})\colon \mathrm{TR}^r_{q-\lambda}(A) \longrightarrow \mathrm{TR}^{r/\ell}_{q-\lambda'}(A) \times \mathrm{TR}^{r/\ell^v}_{q-\lambda}(A)$$

becomes an isomorphism after localization at p. This will prove the induction step, since r/ℓ and r/ℓ^v have at most $k-1$ divisors and $\gamma = (\gamma \times \gamma) \circ (R_\ell, F_{\ell^v})$. Now, by Proposition 1.1, we have the long exact sequence

$$\cdots \longrightarrow \mathbb{H}_q\big(C_{\ell^v}, \mathrm{TR}^{r/\ell^v}_{\cdot-\lambda}(A)\big) \xrightarrow{N_{\ell^v}} \mathrm{TR}^r_{q-\lambda}(A) \xrightarrow{R_\ell} \mathrm{TR}^{r/\ell}_{q-\lambda'}(A) \xrightarrow{\partial} \mathbb{H}_{q-1}\big(C_{\ell^v}, \mathrm{TR}^{r/\ell^v}_{\cdot-\lambda}(A)\big) \longrightarrow \cdots.$$

Moreover, one readily shows that the composite map

$$\mathrm{TR}^{r/\ell^v}_{q-\lambda}(A) \xrightarrow{\epsilon} \mathbb{H}_q\big(C_{\ell^v}, \mathrm{TR}^{r/\ell^v}_{\cdot-\lambda}(A)\big) \xrightarrow{N_{\ell^v}} \mathrm{TR}^r_{q-\lambda}(A) \xrightarrow{F_{\ell^v}} \mathrm{TR}^{r/\ell^v}_{q-\lambda}(A),$$

where ϵ is the edge homomorphism of the skeleton spectral sequence (1.2), is equal to the composition $F_{\ell^v}V_{\ell^v}$ of the Frobenius and Verschiebung maps, which, in turn, is equal to the map given by multiplication by ℓ^v. Hence, after localization at p, F_{ℓ^v} is the projection onto a direct summand of the group $\mathrm{TR}^r_{q-\lambda}(Z)$. The long exact sequence shows that the map (R_ℓ, F_{ℓ^v}) becomes an isomorphism after localization at p as desired. □

The maps F_s, V_s, and R_s may also be expressed as products of their p-typical analogs:

$$\begin{aligned}
F = F_p\colon\ & \mathrm{TR}^n_{q-\lambda}(A;p) \longrightarrow \mathrm{TR}^{n-1}_{q-\lambda}(A;p) && \text{(Frobenius)}\\
V = V_p\colon\ & \mathrm{TR}^{n-1}_{q-\lambda}(A;p) \longrightarrow \mathrm{TR}^n_{q-\lambda}(A;p) && \text{(Verschiebung)}\\
R = R_p\colon\ & \mathrm{TR}^n_{q-\lambda}(A;p) \longrightarrow \mathrm{TR}^{n-1}_{q-\lambda'}(A;p) && \text{(restriction)}
\end{aligned}$$

Suppose that $r = st$ and write $s = p^v s'$ and $t = p^{n-v-1}t'$ with s' and t' not divisible by p. Then there are three commutative square diagrams,

$$\begin{array}{ccc} \mathrm{TR}^r_{q-\lambda}(A) & \xrightarrow{\gamma} & \prod_{j\mid r'} \mathrm{TR}^n_{q-\lambda'}(A;p) \\ {\scriptstyle F_s}\downarrow\uparrow{\scriptstyle V_s} & & {\scriptstyle F_s^\gamma}\downarrow\uparrow{\scriptstyle V_s^\gamma} \\ \mathrm{TR}^t_{q-\lambda}(A) & \xrightarrow{\gamma} & \prod_{j\mid t'} \mathrm{TR}^{n-v}_{q-\lambda'}(A;p) \end{array} \qquad \begin{array}{ccc} \mathrm{TR}^r_{q-\lambda}(A) & \xrightarrow{\gamma} & \prod_{j\mid r'} \mathrm{TR}^n_{q-\lambda'}(A;p) \\ \downarrow{\scriptstyle R_s} & & \downarrow{\scriptstyle R_s^\gamma} \\ \mathrm{TR}^t_{q-\lambda'}(A) & \xrightarrow{\gamma} & \prod_{j\mid t'} \mathrm{TR}^{n-v}_{q-\lambda''}(A;p), \end{array} \tag{2.2}$$

where the maps F_s^γ, V_s^γ, and R_s^γ are defined as follows. The map F_s^γ takes the factor indexed by a divisor j of r' that is divisible by s' to the factor indexed by the divisor j/s' of t' by the map F^v and annihilates the remaining factors. The map V_s^γ takes the factor indexed by the divisor j of t' to the factor indexed by the divisor $s'j$ of r' by the map $s'V^v$. Finally, the map R_s^γ takes the factor indexed by a divisor j of t' to the factor indexed by the same divisor j of t' by the map R^v and annihilates the factors indexed by divisors j of r' that do not divide t'.

Let M_p be the equivariant Moore spectrum defined by the mapping cone of the multiplication by p map on the sphere $\mathbb{T}$-spectrum. The equivariant homotopy groups with $\mathbb{Z}/p\mathbb{Z}$-coefficients

$$\mathrm{TR}^n_{q-\lambda}(\mathbb{Z};p,\mathbb{Z}/p\mathbb{Z}) = [S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, M_p \wedge S^\lambda \wedge T(\mathbb{Z})]_{\mathbb{T}}$$

were evaluated for p odd by Tsalidis [**16**], and for all p by the first and second authors [**1**]. We recall the result.

THEOREM 2.3 (Angeltveit-Gerhardt [**1**], Tsalidis [**16**]). *Let p be a prime number, let n be a positive integer, let λ be a finite-dimensional complex $\mathbb{T}$-representation, and define*

$$\delta_p(\lambda) = (1-p)\sum_{s\geqslant 0} \dim_{\mathbb{C}}(\lambda^{C_{p^s}})p^s.$$

Then the finite $\mathbb{Z}_{(p)}$-modules $\mathrm{TR}^n_{q-\lambda}(\mathbb{Z};p,\mathbb{Z}/p\mathbb{Z})$ have the following structure.

(i) *For $q \geqslant 2\dim_{\mathbb{C}}(\lambda)$, $\mathrm{TR}^n_{q-\lambda}(\mathbb{Z};p,\mathbb{Z}/p\mathbb{Z})$ has length n, if q is congruent to $2\delta_p(\lambda)$ or $2\delta_p(\lambda)-1$ modulo $2p^n$, and $n-1$, otherwise.*

(ii) *For $2\dim_{\mathbb{C}}(\lambda^{C_{p^s}}) \leqslant q < 2\dim_{\mathbb{C}}(\lambda^{C_{p^{s-1}}})$ with $1 \leqslant s < n$, $\mathrm{TR}^n_{q-\lambda}(\mathbb{Z};p,\mathbb{Z}/p\mathbb{Z})$ has length $n-s$, if q is congruent to $2\delta_p(\lambda^{C_{p^s}})$ or $2\delta_p(\lambda^{C_{p^s}})-1$ modulo $2p^{n-s}$, and $n-s-1$, otherwise.*

(iii) *For $q < 2\dim_{\mathbb{C}}(\lambda^{C_{p^{n-1}}})$, $\mathrm{TR}^n_{q-\lambda}(\mathbb{Z};p,\mathbb{Z}/p\mathbb{Z})$ is zero.*

We show in Corollary 2.7 below that the groups $\mathrm{TR}^n_{q-\lambda}(\mathbb{Z};p,\mathbb{Z}/p\mathbb{Z})$ have exponent p for all prime numbers p.

Proof of Theorem B (i). By Lemma 1.3, $\mathrm{TR}^n_{2i-\lambda}(\mathbb{Z};p)$ is a finitely generated abelian group, and hence, it suffices to show that it is torsion free. We first show that the p-torsion subgroup is trivial. Comparing Theorem 2.3 and Lemma 1.3, we find that for all integers i,

$$\mathrm{length}_{\mathbb{Z}_{(p)}}\mathrm{TR}^n_{2i-\lambda}(\mathbb{Z};p,\mathbb{Z}/p\mathbb{Z}) - \mathrm{length}_{\mathbb{Z}_{(p)}}\mathrm{TR}^n_{2i-1-\lambda}(\mathbb{Z};p,\mathbb{Z}/p\mathbb{Z}) = \mathrm{rk}_{\mathbb{Z}}\,\mathrm{TR}^n_{2i-\lambda}(\mathbb{Z};p).$$

Moreover, Lemma 1.3 shows that for every integer i, $\mathrm{TR}^n_{2i-1-\lambda}(\mathbb{Z};p)_{(p)}$ is a finite p-primary torsion group. By a Bockstein spectral sequence argument, we conclude that $\mathrm{TR}^n_{2i-\lambda}(\mathbb{Z};p)_{(p)}$ is torsion free; compare [**5**, Proposition 13]. This shows that the group $\mathrm{TR}^n_{2i-\lambda}(\mathbb{Z};p)$ has no p-torsion. To see that it has no prime to p torsion, we use that, by Proposition 2.1, the map

$$(R^{n-1-s}F^s)\colon \mathrm{TR}^n_{2i-\lambda}(\mathbb{Z};p) \longrightarrow \prod_{0\leqslant s<n} \mathrm{TR}^1_{2i-\lambda^{(n-1-s)}}(\mathbb{Z};p)$$

becomes an isomorphism after inverting p. Therefore, Bökstedt's result recalled earlier shows that also $\mathrm{TR}^n_{2i-\lambda}(\mathbb{Z};p)[1/p]$ is torsion free. This completes the proof. □

Lemma 2.4. *Let p be a prime number, let n be a positive integer, and let λ be a finite-dimensional complex $\mathbb{T}$-representation. Then the restriction map*

$$R\colon \mathrm{TR}^n_{q-\lambda}(\mathbb{Z};p) \longrightarrow \mathrm{TR}^{n-1}_{q-\lambda'}(\mathbb{Z};p)$$

is surjective for every even integer q.

Proof. We see as in the proof of Addendum 1.4 that the map of the statement is an epimorphism, for $q \leqslant 2\dim_{\mathbb{C}}(\lambda)$. Moreover, Lemma 1.3 and Theorem B(i) show that $\mathrm{TR}^{n-1}_{q-\lambda'}(\mathbb{Z};p)$ is zero, for $q > 2\dim_{\mathbb{C}}(\lambda')$ and even. The lemma follows, since $\dim_{\mathbb{C}}(\lambda') \leqslant \dim_{\mathbb{C}}(\lambda)$. □

Proposition 2.5. *Let p be a prime number, let n be a positive integer, and let λ be a finite-dimensional complex $\mathbb{T}$-representation. Then in the skeleton spectral sequence*

$$E^2_{s,t} = H_s\big(C_{p^{n-1}}, \mathrm{TR}^1_{t-\lambda}(\mathbb{Z};p)\big) \Rightarrow \mathbb{H}_{s+t}\big(C_{p^{n-1}}, \mathrm{TR}^1_{\cdot-\lambda}(\mathbb{Z};p)\big),$$

every non-zero differential $d^r\colon E^r_{s,t} \to E^r_{s-r,t+r-1}$ is supported in odd total degree.

Proof. We must show that if $s+t$ is even then $d^r\colon E^r_{s,t} \to E^r_{s-r,t+r-1}$ is zero. Suppose first that s and t are both even. Then $E^2_{s,t}$ is zero unless $s=0$ and $t = 2\dim_{\mathbb{C}}(\lambda)$. Therefore, in this case, $d^r\colon E^r_{s,t} \to E^r_{s-r,t+r-1}$ is zero. Suppose next that s and t are both odd and that r is even. Then $E^2_{s,t}$ is zero unless $t > 2\dim_{\mathbb{C}}(\lambda)$, and $E^2_{s-r,t+r-1}$ is zero unless $t+r-1 = 2\dim_{\mathbb{C}}(\lambda)$. It follows that, also in this case, $d^r\colon E^r_{s,t} \to E^r_{s-r,t+r-1}$ is zero. It remains to prove that if r, s, and t are all odd then $d^r\colon E^r_{s,t} \to E^r_{s-r,t+r-1}$ is zero.

To this end, we use that, for all integers $n' \geqslant n \geqslant 1$, the iterated Frobenius map

$$F^{n'-n}\colon \mathbb{H}_q\big(C_{p^{n'-1}}, \mathrm{TR}^1_{\cdot-\lambda}(\mathbb{Z};p)\big) \longrightarrow \mathbb{H}_q\big(C_{p^{n-1}}, \mathrm{TR}^1_{\cdot-\lambda}(\mathbb{Z};p)\big)$$

induces a map of skeleton spectral sequences that we write

$$F^{n'-n}\colon E^r_{s,t}(n',\lambda) \longrightarrow E^r_{s,t}(n,\lambda).$$

The map of E^2-terms is given by the transfer map in group homology and is readily evaluated; see for instance [**5**, Lemma 6]. It is surjective if s is odd, and zero if s is even, t is odd, and $n'-n$ is sufficiently large. We prove by induction on $r \geqslant 2$ that, for all odd integers s and t and for all n and λ as in the statement, the differential $d^r\colon E^r_{s,t}(n,\lambda) \to E^r_{s-r,t+r-1}(n,\lambda)$ is zero and the Frobenius map $F\colon E^r_{s,t}(n+1,\lambda) \to E^r_{s,t}(n,\lambda)$ is surjective. The case $r=2$ has been proved above. So we let $r \geqslant 3$ and assume, inductively, that the statement has been proved for $r-1$. Since the differential $d^{r-1}\colon E^{r-1}_{s,t}(n,\lambda) \to E^{r-1}_{s-(r-1),t+r-2}(n,\lambda)$ is zero by the inductive hypothesis, the canonical isomorphism

$$H\big(E^{r-1}_{s+r-1,t-(r-2)}(n,\lambda) \xrightarrow{d^{r-1}} E^{r-1}_{s,t}(n,\lambda) \xrightarrow{d^{r-1}} E^{r-1}_{s-(r-1),t+r-2}(n,\lambda)\big) \xrightarrow{\sim} E^r_{s,t}(n,\lambda)$$

gives rise to a canonical surjection $\pi\colon E^{r-1}_{s,t}(n,\lambda) \twoheadrightarrow E^r_{s,t}(n,\lambda)$. Moreover, by naturality of the skeleton spectral sequence, the diagram

$$\begin{array}{ccc} E^{r-1}_{s,t}(n+1,\lambda) & \xrightarrow{\pi} & E^r_{s,t}(n+1,\lambda) \\ \big\downarrow F & & \big\downarrow F \\ E^{r-1}_{s,t}(n,\lambda) & \xrightarrow{\pi} & E^r_{s,t}(n,\lambda) \end{array}$$

commutes. Since the left-hand vertical map F is surjective by the inductive hypothesis, we conclude that the right-hand vertical map F is surjective as desired. It remains to be proved that the differential $d^r\colon E^r_{s,t}(n,\lambda) \to E^r_{s-r,t+r-1}(n,\lambda)$ is zero. The case where r is even has been proved above, and in the case where r is odd, we consider the following commutative diagram:

$$\begin{array}{ccc} E^r_{s,t}(n',\lambda) & \xrightarrow{d^r} & E^r_{s-r,t+r-1}(n',\lambda) \\ \big\downarrow{\scriptstyle F^{n'-n}} & & \big\downarrow{\scriptstyle F^{n'-n}} \\ E^r_{s,t}(n,\lambda) & \xrightarrow{d^r} & E^r_{s-r,t+r-1}(n,\lambda). \end{array}$$

We have just proved that the left-hand vertical map is surjective. Moreover, as recalled above, the right-hand vertical map is zero if $n'-n$ is sufficiently large. Indeed, $s-r$ is even and $t-r+1$ is odd. Hence, the lower horizontal map d^r is zero as desired. This proves the induction step and the proposition. □

REMARK 2.6. The proof of Proposition 2.5 also shows that in the Tate spectral sequence

$$\hat{E}^2_{s,t} = \hat{H}^{-s}\big(C_{p^{n-1}}, \mathrm{TR}^1_{t-\lambda}(\mathbb{Z};p)\big) \Rightarrow \hat{\mathbb{H}}^{-s-t}\big(C_{p^{n-1}}, \mathrm{TR}^1_{\cdot-\lambda}(\mathbb{Z};p)\big),$$

every non-zero differential is supported in even total degree. It remains an important problem to determine the differential structure of the two spectral sequences.

Proof of Theorem B (ii). First, for $i \leqslant \dim_{\mathbb{C}}(\lambda)$, we see as in the proof of Lemma 2.4 that the restriction map

$$R\colon \mathrm{TR}^n_{q-\lambda}(\mathbb{Z};p) \longrightarrow \mathrm{TR}^{n-1}_{q-\lambda'}(\mathbb{Z};p)$$

is an isomorphism, so the statement holds in this case. Next, for $i = \dim_{\mathbb{C}}(\lambda)+1$, Proposition 1.1 and Lemmas 1.3 and 2.4 give a short exact sequence

$$0 \longrightarrow \mathbb{H}_q(C_{p^{n-1}}, \mathrm{TR}^1_{\cdot-\lambda}(\mathbb{Z};p)) \longrightarrow \mathrm{TR}^n_{q-\lambda}(\mathbb{Z};p) \longrightarrow \mathrm{TR}^{n-1}_{q-\lambda'}(\mathbb{Z};p) \longrightarrow 0,$$

and the skeleton spectral sequence (1.2) shows that the left-hand group has order p^{n-1}. So the statement also holds in this case. Finally, for $i > \dim_{\mathbb{C}}(\lambda)+1$, Proposition 1.1 and Lemma 1.3 give a four-term exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathbb{H}_q\big(C_{p^{n-1}}, \mathrm{TR}^1_{\cdot-\lambda}(\mathbb{Z};p)\big) \longrightarrow \mathrm{TR}^n_{q-\lambda}(\mathbb{Z};p) \longrightarrow \mathrm{TR}^{n-1}_{q-\lambda'}(\mathbb{Z};p) \\ &\longrightarrow \mathbb{H}_{q-1}\big(C_{p^{n-1}}, \mathrm{TR}^1_{\cdot-\lambda}(\mathbb{Z};p)\big) \longrightarrow 0, \end{aligned}$$

which shows that the orders of the four groups satisfy the equality

$$\big|\mathrm{TR}^n_{q-\lambda}(\mathbb{Z};p)\big| / \big|\mathrm{TR}^{n-1}_{q-1-\lambda'}(\mathbb{Z};p)\big| = \big|\mathbb{H}_q\big(C_{p^{n-1}}, \mathrm{TR}^1_{\cdot-\lambda}(\mathbb{Z};p)\big)\big| / \big|\mathbb{H}_{q-1}\big(C_{p^{n-1}}, \mathrm{TR}^1_{\cdot-\lambda}(\mathbb{Z};p)\big)\big|.$$

To evaluate the ratio on the right-hand side, we consider the skeleton spectral sequence (1.2). We may write the ratio in question as

$$\big|\mathbb{H}_q\big(C_{p^{n-1}}, \mathrm{TR}^1_{\cdot-\lambda}(\mathbb{Z};p)\big)\big| / \big|\mathbb{H}_{q-1}\big(C_{p^{n-1}}, \mathrm{TR}^1_{\cdot-\lambda}(\mathbb{Z};p)\big)\big| = \Big(\prod_{s+t=q} |E^\infty_{s,t}|\Big) \Big/ \Big(\prod_{s+t=q-1} |E^\infty_{s,t}|\Big).$$

By Proposition 2.5, for all $r \geqslant 2$ and all s and t with $s+t$ being odd, we have an exact sequence,

$$0 \longrightarrow E^{r+1}_{s,t} \longrightarrow E^r_{s,t} \xrightarrow{d^r} E^r_{s-r,t+r-1} \longrightarrow E^{r+1}_{s-r,t+r-1} \longrightarrow 0.$$

Hence, by induction on r, we find that

$$\Big(\prod_{s+t=q} |E^\infty_{s,t}|\Big) \Big/ \Big(\prod_{s+t=q-1} |E^\infty_{s,t}|\Big) = \Big(\prod_{s+t=q} |E^2_{s,t}|\Big) \Big/ \Big(\prod_{s+t=q-1} |E^2_{s,t}|\Big),$$

and the ratio on the right-hand side is readily seen to be equal to

$$|E^2_{0,q}| \cdot |E^2_{q-2\dim_{\mathbb{C}}(\lambda),2\dim_{\mathbb{C}}(\lambda)}| = (i - \dim_{\mathbb{C}}(\lambda)) \cdot p^{n-1}.$$

This completes the proof. □

Proof of Theorem B (iii). First, for q odd, we use that the Verschiebung map in question is equal to the composition of the edge homomorphism

$$\epsilon\colon \mathrm{TR}^{n-1}_{q-\lambda}(\mathbb{Z};p) \longrightarrow \mathbb{H}_q\big(C_p, \mathrm{TR}^{n-1}_{\cdot-\lambda}(\mathbb{Z};p)\big)$$

of the skeleton spectral sequence (1.2) and the norm map N_{n-1} in the long exact sequence

$$\cdots \longrightarrow \mathbb{H}_q\big(C_p, \mathrm{TR}^{n-1}_{\cdot-\lambda}(\mathbb{Z};p)\big) \xrightarrow{N_{n-1}} \mathrm{TR}^n_{q-\lambda}(\mathbb{Z};p) \xrightarrow{R^{n-1}} \mathrm{TR}^1_{q-\lambda^{(n-1)}}(\mathbb{Z};p) \longrightarrow \cdots$$

from Proposition 1.1. Since q is odd, Lemma 2.4 shows that the latter map is injective. Hence, it will suffice to show that also the edge homomorphism is injective, or equivalently, that in the skeleton spectral sequence, all differentials of the form

$$d^r : E^r_{r,q-r+1} \longrightarrow E^r_{0,q}$$

are zero. If r is even, then $q-r+1$ is even and Theorem B (i) shows that the group $E^r_{r,q-r+1}$ is zero. Hence, d^r is zero in this case. If r is odd, we consider the iterated Frobenius map

$$F^{v'-v} : \mathbb{H}_q\big(C_{p^{v'}}, \mathrm{TR}^{n-1}_{\cdot-\lambda}(\mathbb{Z};p)\big) \longrightarrow \mathbb{H}_q\big(C_{p^{v}}, \mathrm{TR}^{n-1}_{\cdot-\lambda}(\mathbb{Z};p)\big).$$

It induces a map of spectral sequences that we write

$$F^{v'-v} : E^r_{s,t}(v',\lambda) \longrightarrow E^r_{s,t}(v,\lambda).$$

As in the proof of Theorem B (ii), an induction on $r \geqslant 2$ shows that, for all $v' \geqslant v \geqslant 1$, the left-hand vertical map and the horizontal maps in the diagram

$$\begin{array}{ccc} E^r_{r,q-r+1}(v',\lambda) & \xrightarrow{d^r} & E^r_{0,q}(v',\lambda) \\ \big\downarrow{\scriptstyle F^{v'-v}} & & \big\downarrow{\scriptstyle F^{v'-v}} \\ E^r_{r,q-r+1}(v,\lambda) & \xrightarrow{d^r} & E^r_{0,q}(v,\lambda) \end{array}$$

are surjective and zero, respectively. The proof of the induction step uses that, for $v'-v$ sufficiently large, the right-hand vertical map is zero. This proves the statement for q odd.

Finally, suppose that q is even. We let μ be the direct sum of λ and the one-dimensional complex representation $\mathbb{C}(p^{n-2})$. Then [**6**, Proposition 4.2] gives a long exact sequence

$$\cdots \longrightarrow \mathrm{TR}^n_{q+1-\mu}(\mathbb{Z};p) \xrightarrow{(F,-Fd)} \begin{array}{c}\mathrm{TR}^{n-1}_{q-1-\lambda}(\mathbb{Z};p)\\ \oplus \\ \mathrm{TR}^{n-1}_{q-\lambda}(\mathbb{Z};p)\end{array} \xrightarrow{dV+V} \mathrm{TR}^n_{q-\lambda}(\mathbb{Z};p) \xrightarrow{\iota_*} \mathrm{TR}^n_{q-\mu}(\mathbb{Z};p) \longrightarrow \cdots.$$

Now, we proved in Theorem B (i) that $\mathrm{TR}^n_{q+1-\mu}(\mathbb{Z};p)$ and $\mathrm{TR}^{n-1}_{q-1-\lambda}(\mathbb{Z};p)$ are finite abelian groups while $\mathrm{TR}^{n-1}_{q-\lambda}(\mathbb{Z};p)$, $\mathrm{TR}^n_{q-\lambda}(\mathbb{Z};p)$, and $\mathrm{TR}^n_{q-\mu}(\mathbb{Z};p)$ are free abelian groups. Hence, we obtain the exact sequence of free abelian groups

$$0 \longrightarrow \mathrm{TR}^{n-1}_{q-\lambda}(\mathbb{Z};p) \xrightarrow{V} \mathrm{TR}^n_{q-\lambda}(\mathbb{Z};p) \xrightarrow{\iota_*} \mathrm{TR}^n_{q-\mu}(\mathbb{Z};p),$$

which shows that for q even, the Verschiebung map is the inclusion of a direct summand as stated. This concludes the proof of Theorem B. □

COROLLARY 2.7. *Let n be a positive integer, let p be a prime number, and let λ be a finite-dimensional complex $\mathbb{T}$-representation. Then $\mathrm{TR}^n_{q-\lambda}(\mathbb{Z};p,\mathbb{Z}/p\mathbb{Z})$ has exponent p for every integer q.*

Proof. We first let $p=2$ and consider the coefficient long exact sequence

$$\cdots \longrightarrow \mathrm{TR}^n_{q-\lambda}(\mathbb{Z};2) \xrightarrow{2} \mathrm{TR}^n_{q-\lambda}(\mathbb{Z};2) \xrightarrow{\iota} \mathrm{TR}^n_{q-\lambda}(\mathbb{Z};2,\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\beta} \mathrm{TR}^n_{q-1-\lambda}(\mathbb{Z};2) \longrightarrow \cdots.$$

It is proved in [**2**, Theorem 1.1] that the composition

$$\mathrm{TR}^n_{q-\lambda}(\mathbb{Z};2,\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\beta} \mathrm{TR}^n_{q-1-\lambda}(\mathbb{Z};2) \xrightarrow{\eta} \mathrm{TR}^n_{q-\lambda}(\mathbb{Z};2) \xrightarrow{\iota} \mathrm{TR}^n_{q-\lambda}(\mathbb{Z};2,\mathbb{Z}/2\mathbb{Z})$$

is equal to multiplication by 2. Now, Theorem B shows that for q odd the map β is zero, and that for q even the map η is zero. Hence, the group $\mathrm{TR}^n_{q-\lambda}(\mathbb{Z};2,\mathbb{Z}/2\mathbb{Z})$ is annihilated by multiplication by 2 as stated. Finally, for p odd, [**2**, Theorem 1.1] shows that the multiplication by p map on $\mathrm{TR}^n_{q-\lambda}(\mathbb{Z};p,\mathbb{Z}/p\mathbb{Z})$ is equal to zero. □

3. *The groups* $K_q(\mathbb{Z}[x]/(x^m),(x))$

In this section, we prove Theorem A of Section 1.

PROPOSITION 3.1. *Let m and r be positive integers, let i be a non-negative integer, and let $d=d(m,r)$ be the integer part of $(r-1)/m$. Then*
(i) *the abelian group $\lim_R \mathrm{TR}^r_{2i-\lambda_d}(\mathbb{Z})$ is free of rank m;*
(ii) *the abelian group $\lim_R \mathrm{TR}^r_{2i-1-\lambda_d}(\mathbb{Z})$ is finite of order $(mi)!(i!)^m$.*

Proof. It follows from Lemma 1.3 and Addendum 1.4 that the groups $\lim_R \mathrm{TR}^r_{q-\lambda_d}(\mathbb{Z})$ are finitely generated. Hence, it suffices to show that for every prime number p, the $\mathbb{Z}_{(p)}$-module $\lim_R \mathrm{TR}^r_{2i-\lambda_d}(\mathbb{Z})_{(p)}$ has finite rank m and the $\mathbb{Z}_{(p)}$-module $\lim_R \mathrm{TR}^r_{2i-1-\lambda_d}(\mathbb{Z})_{(p)}$ has finite length $v_p((mi)!(i!)^m)$. We fix a prime number p and let I_p be the set of positive integers not divisible by p. It follows from Proposition 2.1 that there is a canonical isomorphism

$$\lim_R \mathrm{TR}^r_{q-\lambda_d}(\mathbb{Z})_{(p)} \xrightarrow{\sim} \prod_{j\in I_p} \lim_R \mathrm{TR}^n_{q-\lambda_d}(\mathbb{Z};p)_{(p)},$$

where, on the left-hand side, the limit ranges over the set of positive integers ordered under division and $d=d(m,r)$, and where, on the right-hand side, the limits range over the set of non-negative integers ordered additively and $d=d(m,p^{n-1}j)$. Moreover, on the jth factor of the product, the canonical projection

$$\lim_R \mathrm{TR}^n_{q-\lambda_d}(\mathbb{Z};p)_{(p)} \longrightarrow \mathrm{TR}^s_{q-\lambda_d}(\mathbb{Z};p)_{(p)}$$

is an isomorphism for $q<2d(m,p^sj)$; see [**6**, Lemma 2.6]. The requirement that $2i-2$ and $2i-1$ be strictly smaller than $2d(m,p^sj)$ is equivalent to the requirement that $mi<p^sj$. Hence, for $q=2i-2$ or $q=2i-1$, we have a canonical isomorphism

$$\lim_R \mathrm{TR}^r_{q-\lambda_d}(\mathbb{Z})_{(p)} \xrightarrow{\sim} \prod_{\substack{1\leqslant j\leqslant mi \\ j\in I_p}} \mathrm{TR}^s_{q-\lambda_d}(\mathbb{Z};p)_{(p)},$$

where $s=s_p(m,i,j)$ is the unique integer such that

$$p^{s-1}j \leqslant mi < p^sj.$$

Now, from Theorem B (i) we find that

$$\begin{aligned}\mathrm{rk}_{\mathbb{Z}_{(p)}} \lim_R \mathrm{TR}^r_{2i-\lambda_d}(\mathbb{Z})_{(p)} &= |\{j \in I_p \mid i = d(m, p^{n-1}j), \text{ for some } n \geqslant 1\}| \\ &= |\{r \in \mathbb{N} \mid i = d(m,r)\}| = |\{mi+1, mi+2, \dots, mi+m\}| = m,\end{aligned}$$

which proves statement (i). Similarly, we have

$$\mathrm{length}_{\mathbb{Z}_{(p)}} \lim_R \mathrm{TR}^r_{2i-1-\lambda_d}(\mathbb{Z})_{(p)} = \sum_{\substack{1\leqslant j\leqslant mi\\ j\in I_p}} \mathrm{length}_{\mathbb{Z}_{(p)}} \mathrm{TR}^s_{2i-1-\lambda_d}(\mathbb{Z};p)_{(p)},$$

where $s = s_p(m,i,j)$, and Theorem B (ii) shows that the right-hand side is equal to

$$\begin{aligned}&\sum_{\substack{1\leqslant j\leqslant mi\\ j\in I_p}} \sum_{1\leqslant t\leqslant s} (v_p(i - d(m, p^{t-1}j)) + t - 1) = \sum_{1\leqslant k\leqslant mi} (v_p(i-d(m,k)) + v_p(k)) \\ &\quad = m\sum_{0\leqslant l<i} v_p(i-l) + \sum_{1\leqslant k\leqslant mi} v_p(k) = m\sum_{1\leqslant k\leqslant i} v_p(k) + \sum_{1\leqslant k\leqslant mi} v_p(k) = v_p((i!)^m(mi)!),\end{aligned}$$

which proves statement (ii). □

Proposition 3.2. *Let m and r be positive integers, let i be a non-negative integer, and let $d = d(m,r)$ be the integer part of $(r-1)/m$. Then*

(i) *the abelian group $\lim_R \mathrm{TR}^{r/m}_{2i-\lambda_d}(\mathbb{Z})$ is free of rank 1;*

(ii) *the abelian group $\lim_R \mathrm{TR}^{r/m}_{2i-1-\lambda_d}(\mathbb{Z})$ is finite of order $(i!)^2$.*

Proof. It follows from Lemma 1.3 and Addendum 1.4 that the groups $\lim_R \mathrm{TR}^{r/m}_{q-\lambda_d}(\mathbb{Z})$ are finitely generated. We fix a prime number p and write $m = p^v m'$ with m' not divisible by p. Then for $q = 2i-2$ and $q = 2i-1$, there is a canonical isomorphism

$$\lim_R \mathrm{TR}^{r/m}_{q-\lambda_d}(\mathbb{Z})_{(p)} \xrightarrow{\sim} \prod_{\substack{1\leqslant j\leqslant mi\\ j\in m'I_p}} \mathrm{TR}^{s-v}_{q-\lambda_d}(\mathbb{Z};p)_{(p)},$$

where $s = s_p(m,i,j)$. From Theorem B (i) we find

$$\begin{aligned}\mathrm{rk}_{\mathbb{Z}_{(p)}} \lim_R \mathrm{TR}^{r/m}_{2i-\lambda_d}(\mathbb{Z})_{(p)} &= |\{j \in m'I_p \mid i = d(m, p^{n+v-1}j), \text{ for some } n \geqslant 1\}| \\ &= |\{r \in m\mathbb{N} \mid i = d(m,r)\}| = |\{mi+m\}| = 1,\end{aligned}$$

which proves statement (i). Similarly, from Theorem B (ii), we have

$$\begin{aligned}\mathrm{length}_{\mathbb{Z}_{(p)}} \lim_R \mathrm{TR}^r_{2i-1-\lambda_d}(\mathbb{Z})_{(p)} &= \sum_{\substack{1\leqslant j\leqslant mi\\ j\in m'I_p}} \mathrm{length}_{\mathbb{Z}_{(p)}} \mathrm{TR}^{s-v}_{2i-1-\lambda_d}(\mathbb{Z};p)_{(p)} \\ &= \sum_{\substack{1\leqslant j\leqslant mi\\ j\in m'I_p}} \sum_{1\leqslant t\leqslant s-v} (v_p(i - d(m, p^{t+v-1}j)) + t - 1) = \sum_{1\leqslant k\leqslant i} (v_p(i-d(m,km)) + v_p(k)) \\ &= \sum_{0\leqslant l<i} v_p(i-l) + \sum_{1\leqslant k\leqslant i} v_p(k) = 2\sum_{1\leqslant k\leqslant i} v_p(k) = v_p((i!)^2),\end{aligned}$$

which proves statement (ii). □

PROPOSITION 3.3. *Let m and r be positive integers, and let $d = d(m,r)$ be the integer part of $(r-1)/m$. Then the Verschiebung map*

$$V_m \colon \lim_R \mathrm{TR}^{r/m}_{q-\lambda_d}(\mathbb{Z}) \longrightarrow \lim_R \mathrm{TR}^{r}_{q-\lambda_d}(\mathbb{Z})$$

is injective for all integers q, and has free abelian cokernel for all even integers q.

Proof. We fix a prime number p and show that the Verschiebung map

$$V_m \colon \lim_R \mathrm{TR}^{r/m}_{q-\lambda_d}(\mathbb{Z})_{(p)} \longrightarrow \lim_R \mathrm{TR}^{r}_{q-\lambda_d}(\mathbb{Z})_{(p)}$$

is injective for all integers q, and has cokernel a free $\mathbb{Z}_{(p)}$-module for all even integers q. We write $m = p^v m'$ with m' not divisible by p. Then for $q = 2i-2$ and $q = 2i-1$ the map V_m is canonically isomorphic to the map

$$m'V^v \colon \prod_{\substack{1\leqslant j\leqslant mi \\ j\in m'I_p}} \mathrm{TR}^{s-v}_{q-\lambda_d}(\mathbb{Z};p)_{(p)} \longrightarrow \prod_{\substack{1\leqslant j\leqslant mi \\ j\in I_p}} \mathrm{TR}^{s}_{q-\lambda_d}(\mathbb{Z};p)_{(p)},$$

where $s = s_p(m,i,j)$. The statement now follows from Theorem B (iii). □

Proof of Theorem A. The statement follows immediately from the long exact sequence recalled in Section 1 together with Propositions 3.1, 3.2, and 3.3. □

4. *The dual numbers*

It follows from Theorem B that $K_{2i}(\mathbb{Z}[x]/(x^2),(x))$ is a finite abelian group of order $(2i)!$. In this section, we investigate the structure of these groups in low degrees.

THEOREM 4.1. *There are isomorphisms*

$$\begin{aligned} K_2(\mathbb{Z}[x]/(x^2),(x)) &\approx \mathbb{Z}/2\mathbb{Z}, \\ K_4(\mathbb{Z}[x]/(x^2),(x)) &\approx \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}, \\ K_6(\mathbb{Z}[x]/(x^2),(x)) &\approx \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}. \end{aligned}$$

Proof. We know from Theorem B that the orders of these three groups are as stated. Hence, it suffices to show that the two-primary torsion subgroup of the groups in degree four and six and the three-primary torsion subgroup of the group in degree six are as stated.

We first consider the group in degree four, which is given by the short exact sequence

$$0 \longrightarrow \lim_R \mathrm{TR}^{r/2}_{3-\lambda_d}(\mathbb{Z}) \xrightarrow{V_2} \lim_R \mathrm{TR}^{r}_{3-\lambda_d}(\mathbb{Z}) \longrightarrow K_4(\mathbb{Z}[x]/(x^2),(x)) \longrightarrow 0.$$

The middle term in the short exact sequence decomposes two-locally as the direct sum

$$\lim_R \mathrm{TR}^{r}_{3-\lambda_d}(\mathbb{Z})_{(2)} \xrightarrow{\sim} \mathrm{TR}^{3}_{3-\lambda_1}(\mathbb{Z};2)_{(2)} \oplus \mathrm{TR}^{1}_{3-\lambda_1}(\mathbb{Z};2)_{(2)},$$

where the first and second summands on the right-hand side correspond to $j=1$ and $j=3$, respectively. Similarly, the left-hand term in the short exact sequence decomposes two-locally as

$$\lim_R \mathrm{TR}^{r/2}_{3-\lambda_d}(\mathbb{Z})_{(2)} \xrightarrow{\sim} \mathrm{TR}^{2}_{3-\lambda_1}(\mathbb{Z};2)_{(2)} \oplus \mathrm{TR}^{0}_{3-\lambda_1}(\mathbb{Z};2)_{(2)}.$$

The summands corresponding to $j = 3$ are both zero. Hence, the two-primary torsion subgroup of $K_4(\mathbb{Z}[x]/(x^2),(x))$ is canonically isomorphic to the cokernel of the Verschiebung map

$$V\colon \mathrm{TR}^2_{3-\lambda_1}(\mathbb{Z};2)_{(2)} \longrightarrow \mathrm{TR}^3_{3-\lambda_1}(\mathbb{Z};2)_{(2)}.$$

To evaluate this cokernel, we consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{TR}^1_3(\mathbb{Z};2)_{(2)} & \xrightarrow{V} & \mathrm{TR}^2_3(\mathbb{Z};2)_{(2)} & \xrightarrow{\iota_*} & \mathrm{TR}^2_{3-\lambda_1}(\mathbb{Z};2)_{(2)} & \longrightarrow & 0 \\
 & & \| & & \downarrow V & & \downarrow V & & \\
0 & \longrightarrow & \mathrm{TR}^1_3(\mathbb{Z};2)_{(2)} & \xrightarrow{V^2} & \mathrm{TR}^3_3(\mathbb{Z};2)_{(2)} & \xrightarrow{\iota_*} & \mathrm{TR}^3_{3-\lambda_1}(\mathbb{Z};2)_{(2)} & \longrightarrow & 0,
\end{array}$$

where the rows are exact by [**6**, Proposition 4.2]. It follows from [**5**, Theorem 18] that this diagram is isomorphic to the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{4} & \mathbb{Z}/8\mathbb{Z} & \xrightarrow{1} & \mathbb{Z}/4\mathbb{Z} & \longrightarrow & 0 \\
 & & \| & & \downarrow (a,b) & & \downarrow (a,b) & & \\
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{(0,4)} & \mathbb{Z}/8\mathbb{Z}\oplus\mathbb{Z}/8\mathbb{Z} & \xrightarrow{1\oplus 1} & \mathbb{Z}/8\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z} & \longrightarrow & 0,
\end{array}$$

where $a \in 2\mathbb{Z}/8\mathbb{Z}$ and $b \in (\mathbb{Z}/8\mathbb{Z})^*$. Now, the cokernels of the middle and right-hand vertical maps are isomorphic to $\mathbb{Z}/8\mathbb{Z}$. This shows that $K_4(\mathbb{Z}[x]/(x^2),(x))_{(2)}$ is as stated.

We next consider the group in degree six, which is given by the short exact sequence

$$0 \longrightarrow \lim_R \mathrm{TR}^{r/2}_{5-\lambda_d}(\mathbb{Z}) \xrightarrow{V_2} \lim_R \mathrm{TR}^r_{5-\lambda_d}(\mathbb{Z}) \longrightarrow K_6(\mathbb{Z}[x]/(x^2),(x)) \longrightarrow 0,$$

and begin by evaluating the two-primary torsion subgroup. The middle term in the short exact sequence decomposes two-locally as the direct sum

$$\lim_R \mathrm{TR}^r_{5-\lambda_d}(\mathbb{Z})_{(2)} \xrightarrow{\sim} \mathrm{TR}^3_{5-\lambda_1}(\mathbb{Z};2)_{(2)} \oplus \mathrm{TR}^2_{5-\lambda_2}(\mathbb{Z};2)_{(2)} \oplus \mathrm{TR}^1_{5-\lambda_2}(\mathbb{Z};2)_{(2)},$$

where the three summands on the right-hand side correspond to $j = 1$, $j = 3$, and $j = 5$, respectively. Similarly, the left-hand term in the short exact sequence decomposes two-locally as

$$\lim_R \mathrm{TR}^{r/2}_{5-\lambda_d}(\mathbb{Z})_{(2)} \xrightarrow{\sim} \mathrm{TR}^2_{5-\lambda_1}(\mathbb{Z};2)_{(2)} \oplus \mathrm{TR}^1_{5-\lambda_2}(\mathbb{Z};2)_{(2)} \oplus \mathrm{TR}^0_{5-\lambda_2}(\mathbb{Z};2)_{(2)}.$$

The summands corresponding to $j = 5$ are both zero. Hence, the two-primary torsion subgroup of $K_6(\mathbb{Z}[x]/(x^2),(x))$ is canonically isomorphic to the direct sum of the cokernels of

$$V\colon \mathrm{TR}^2_{5-\lambda_1}(\mathbb{Z};2)_{(2)} \longrightarrow \mathrm{TR}^3_{5-\lambda_1}(\mathbb{Z};2)_{(2)},$$
$$V\colon \mathrm{TR}^1_{5-\lambda_2}(\mathbb{Z};2)_{(2)} \longrightarrow \mathrm{TR}^2_{5-\lambda_2}(\mathbb{Z};2)_{(2)}.$$

We show that these are isomorphic to $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$, respectively. The statement for the latter cokernel follows directly from Theorems B and 2.3. The two theorems also show that the group $\mathrm{TR}^2_{5-\lambda_1}(\mathbb{Z};2)_{(2)}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ and that the group $\mathrm{TR}^3_{5-\lambda_1}(\mathbb{Z};2)_{(2)}$ is isomorphic to either $\mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/8\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. We will prove that the latter group is isomorphic to $\mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$ by showing that it contains $\mathbb{Z}/4\mathbb{Z}$ as a direct summand. To this end, we consider the diagram

$$\begin{array}{ccccccc}
0 \longrightarrow \mathbb{H}_5\big(C_2, \mathrm{TR}^1_{\cdot-\lambda_1}(\mathbb{Z};2)\big)_{(2)} & \longrightarrow & \mathrm{TR}^2_{5-\lambda_1}(\mathbb{Z};2)_{(2)} & \longrightarrow & \mathrm{TR}^1_5(\mathbb{Z};2)_{(2)} \longrightarrow 0 \\
\downarrow V & & \downarrow V & & \downarrow V \\
0 \longrightarrow \mathbb{H}_5(C_4, \mathrm{TR}^1_{\cdot-\lambda_1}(\mathbb{Z};2))_{(2)} & \longrightarrow & \mathrm{TR}^3_{5-\lambda_1}(\mathbb{Z};2)_{(2)} & \longrightarrow & \mathrm{TR}^2_5(\mathbb{Z};2)_{(2)} \longrightarrow 0 \\
\downarrow F^2 & & \downarrow F^2 & & \\
\mathrm{TR}^1_3(\mathbb{Z};2)_{(2)} & = & \mathrm{TR}^1_3(\mathbb{Z};2)_{(2)}, & &
\end{array}$$

where the rows, but not the columns, are exact. It follows from Theorem B that the top middle and right-hand vertical maps V are injective. Hence, also the top left-hand vertical map V is injective. Moreover, [**6**, Proposition 4.2] and [**5**, Proposition 15] show that the bottom left-hand vertical map F^2 is surjective. Hence, also the bottom right-hand vertical map F^2 is surjective. The skeleton spectral sequence

$$E^2_{s,t} = H_s\big(C_4, \mathrm{TR}^1_{t-\lambda_1}(\mathbb{Z};2)\big)_{(2)} \Rightarrow \mathbb{H}_{s+t}\big(C_4, \mathrm{TR}^1_{\cdot-\lambda_1}(\mathbb{Z};2)\big)_{(2)}$$

shows that the middle left-hand group is an extension of $E^\infty_{2,3} = \mathbb{Z}/4\mathbb{Z}$ by $E^\infty_{0,5} = \mathbb{Z}/2\mathbb{Z}$ and the diagram above shows that the extension is split. It follows from [**5**, Lemma 6] that

$$F\colon \mathbb{H}_5\big(C_4, \mathrm{TR}^1_{\cdot-\lambda_1}(\mathbb{Z};2)\big)_{(2)} \longrightarrow \mathbb{H}_5(C_2, \mathrm{TR}^1_{\cdot-\lambda_1}(\mathbb{Z};2))_{(2)}$$

maps the generator of the summand $E^\infty_{0,5} = \mathbb{Z}/2\mathbb{Z}$ to zero. Hence, the lower left-hand vertical map F^2 in the diagram above maps the generator of the summand $E^\infty_{0,5} = \mathbb{Z}/2\mathbb{Z}$ to zero. But the map F^2 is surjective, and therefore, maps a generator of the summand $E^\infty_{3,2} = \mathbb{Z}/4\mathbb{Z}$ non-trivially. It follows that $\mathrm{TR}^3_{5-\lambda_1}(\mathbb{Z};2)_{(2)}$ contains direct summand isomorphic to $\mathbb{Z}/4\mathbb{Z}$, and hence, is isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. This shows that the cokernel of the upper middle vertical map V is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and hence, that $K_6(\mathbb{Z}[x]/(x^2),(x))_{(2)}$ is as stated.

It remains to evaluate $K_6(\mathbb{Z}[x]/(x^2),(x))_{(3)}$. This group is canonically isomorphic to the direct sum of $\mathrm{TR}^2_{5-\lambda_1}(\mathbb{Z};3)_{(3)}$ and $\mathrm{TR}^1_{5-\lambda_2}(\mathbb{Z};3)_{(3)}$. It follows from Theorem B that the former group has order nine and that the latter group is zero and from Theorem 2.3 that the former group is cyclic. This completes the proof. □

Theorem 4.2. *Let p be an odd prime number. Then for $2i < p^2$ the p-primary torsion subgroup of $K_{2i}(\mathbb{Z}[x]/(x^2),(x))$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{r_1} \oplus (\mathbb{Z}/p^2\mathbb{Z})^{r_2}$, where*

$$(r_1, r_2) = \begin{cases} (0, \lfloor i/p \rfloor) & \text{if } 2i+1 \equiv 0 \text{ modulo } p, \\ (\lfloor 2i/p \rfloor - 2, 1) & \text{if } 2i+1 \equiv j \text{ modulo } p \text{ with } 1 \leqslant j \leqslant 2i/p \text{ odd}, \\ (\lfloor 2i/p \rfloor, 0) & \text{otherwise.} \end{cases}$$

Here $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

Proof. After localizing at the odd prime number p, the short exact sequence

$$0 \longrightarrow \lim_R \mathrm{TR}^{r/2}_{2i-1-\lambda_d}(\mathbb{Z}) \xrightarrow{V_2} \lim_R \mathrm{TR}^r_{2i-1-\lambda_d}(\mathbb{Z}) \longrightarrow K_{2i}(\mathbb{Z}[x]/(x^2),(x)) \longrightarrow 0$$

induces a canonical isomorphism

$$\bigoplus_j \mathrm{TR}^s_{2i-1-\lambda_d}(\mathbb{Z};p)_{(p)} \xrightarrow{\sim} K_{2i}(\mathbb{Z}[x]/(x^2),(x))_{(p)},$$

where the sum runs over integers $1 \leqslant j \leqslant 2i$ coprime to both 2 and p and $s = s_p(2, i, j)$ is the unique integer that satisfies $p^{s-1}j \leqslant 2i < p^s j$. Since $2i < p^2$ we find

$$s_p(2, i, j) = \begin{cases} 2 & \text{if } 1 \leqslant j \leqslant 2i/p, \\ 1 & \text{if } 2i/p < j \leqslant 2i. \end{cases}$$

If $2i/p < j \leqslant 2i$ we have $d = (j-1)/2$, and hence

$$\text{length}_{\mathbb{Z}_{(p)}} \text{TR}^1_{2i-1-\lambda_d}(\mathbb{Z}; p)_{(p)} = v_p(i - d) = v_p(2i + 1 - j).$$

The length is at most 1 since $2i < p^2$. If $1 \leqslant j \leqslant 2i/p$ we have $d = (pj - 1)/2$, and in this case Theorem B shows that

$$\begin{aligned} \text{length}_{\mathbb{Z}_{(p)}} \text{TR}^2_{2i-1-\lambda_d}(\mathbb{Z}; p)_{(p)} &= \text{length}_{\mathbb{Z}_{(p)}} \text{TR}^1_{2i-1-\lambda'_d}(\mathbb{Z}; p)_{(p)} + v_p(i - d) + 1 \\ &= v_p(2i + 1 - j) + v_p(2i + 1) + 1, \end{aligned}$$

where we have used that $\lambda'_d = \lambda_{d'}$ with $d' = (j - 1)/2$. The length is at most 2 since $2i < p^2$ and since j is coprime to p. We claim that the group $\text{TR}^2_{2i-1-\lambda_d}(\mathbb{Z}; p)_{(p)}$ is always cyclic. By Theorem 2.3, the claim is equivalent to the congruence

$$2i - 1 \not\equiv 2\delta_p(\lambda_d) - 1 \mod 2p^2.$$

We compute that modulo p^2

$$\delta_p(\lambda_d) \equiv (1 - p)((pj - 1)/2 + (j - 1)/2 \cdot p).$$

Hence, we have $2i - 1 \equiv 2\delta_p(\lambda_d) - 1$ modulo $2p^2$ if and only if $2i + 1 \equiv 2pj + p^2$ modulo $2p^2$. This is possible only if $2i + 1$ is congruent to 0 modulo p. If we write $2i + 1 = ap$ then $1 \leqslant a \leqslant p$ and $1 \leqslant j < a$. Hence, $p \leqslant ap \leqslant p^2$ and $2p + p^2 \leqslant 2pj + p^2 < 2ap + p^2 \leqslant 3p^2$, which implies that the congruence $ap \equiv 2pj + p^2$ modulo $2p^2$ is equivalent to the equality $ap + p^2 = 2pj$. But then $a + p = 2j$, which contradicts that $j < a$. The claim follows.

We now show that the integers (r_1, r_2) are as stated. It follows from Theorem A that

$$r_1 + 2r_2 = v_p((2i)!) = \lfloor 2i/p \rfloor.$$

In the case where $2i + 1$ is congruent to 0 modulo p, we proved above that $r_1 = 0$. If we write $2i + 1 = ap$ then a is odd and

$$r_2 = \lfloor 2i/p \rfloor/2 = (a - 1)/2 = \lfloor (a - 1)/2 + (p - 1)/2p \rfloor = \lfloor i/p \rfloor,$$

as stated. In the case where $2i + 1 \equiv j$ modulo p with $1 \leqslant j \leqslant 2i/p$ odd, we proved above that $r_2 = 1$. Hence, $r_1 = \lfloor 2i/p \rfloor - 2$. Finally, in the remaining case, $r_2 = 0$, and hence, $r_1 = \lfloor 2i/p \rfloor$. This completes the proof. □

EXAMPLE 4.3. Let p be an odd prime number. We spell out the statement of Theorem 4.2, for $i \leqslant p + 1$. The p-primary torsion subgroup of $K_{2i}(\mathbb{Z}[x]/(x^2), (x))$ is zero for $i \leqslant (p - 1)/2$, is cyclic of order p for $(p + 1)/2 \leqslant i < p$, and is cyclic of order p^2 for $i = p$. The structure of the p-primary torsion subgroup of $K_{2p+2}(\mathbb{Z}[x]/(x^2), (x))$ depends on the odd prime p. It is cyclic of order 9 if $p = 3$, and the direct sum of two cyclic groups of order p if $p \geqslant 5$.

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