Topological cyclic homology and the Fargues–Fontaine curve

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Introduction

This paper is an elaboration of my lecture at the conference. The purpose is to explain how the Fargues–Fontaine curve and its decomposition into a punctured curve and the formal neighborhood of the puncture naturally appear from various forms of topological cyclic homology and maps between them. I make no claim of originality. My purpose here is to highlight some of the spectacular material contained in the papers of Nikolaus–Scholze [16], Bhatt–Morrow–Scholze [3], and Antieau–Mathew–Morrow–Nikolaus [1] on topological cyclic homology and in the book by Fargues–Fontaine [7] on their revolutionary curve.

1. The Fargues–Fontaine curve

We give a brief introduction to the Fargues–Fontaine curve and refer to their book [7] for details. The lecture notes by Lurie [12] are also helpful.

We define a completely valued field to be a pair $(C, \mathcal{O}_C)$ of a field C and a proper subring $\mathcal{O}_C \subset C$ that is a complete valuation ring of rank 1. The field C is the quotient field of $\mathcal{O}_C$ and the requirement that the inclusion $\mathcal{O}_C \subset C$ be proper is equivalent to the requirement that the value group of $\mathcal{O}_C$ be non-trivial. An isomorphism of completely valued fields $\psi\colon (C, \mathcal{O}_C) \to (C', \mathcal{O}_{C'})$ is an isomorphism of fields $\psi\colon C \to C'$ that restricts to an isomorphism of rings $\psi\colon \mathcal{O}_C \to \mathcal{O}_{C'}$.

We will be interested in the case, where C is algebraically closed and $\mathcal{O}_C$ is of residue characteristic $p > 0$. In this case, "tilting" is a correspondence that to an algebraically closed completely valued field $(C, \mathcal{O}_C)$ of residue characteristic $p > 0$ assigns the algebraically closed completely valued field $(C^\flat, \mathcal{O}_{C^\flat})$ of characteristic $p > 0$, where $\mathcal{O}_{C^\flat}$ is the limit of the diagram of $\mathbb{F}_p$-algebras

$$\cdots \xrightarrow{\varphi} \mathcal{O}_C/p \xrightarrow{\varphi} \mathcal{O}_C/p \xrightarrow{\varphi} \mathcal{O}_C/p$$

with φ the Frobenius. The non-trivial fact that $C^\flat$ is algebraically closed is proved in [7, Proposition 2.1.11]. The tilting correspondence is many-to-one. More precisely, given an algebraically closed completely valued field $(F, \mathcal{O}_F)$ of characteristic $p > 0$,

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an untilt of $(F, \mathcal{O}_F)$ is a pair $((C, \mathcal{O}_C), \iota)$ of an algebraically closed completely valued field $(C, \mathcal{O}_C)$ and an isomorphism of completely valued fields

$$(F, \mathcal{O}_F) \xrightarrow{\ \iota\ } (C^\flat, \mathcal{O}_{C^\flat}).$$

The isomorphism ι gives rise to an isomorphism of value groups

$$F^\times/\mathcal{O}_F^\times \longrightarrow C^\times/\mathcal{O}_C^\times$$

and the "perfectoid" nature of the respective valuation rings implies that for every pair of elements $\varpi_F \in \mathcal{O}_F$ and $\varpi_C \in \mathcal{O}_C$ of equal and sufficiently small but nonzero valuation, the isomorphism ι gives rise to an isomorphism of rings

$$\mathcal{O}_F/\varpi_F \longrightarrow \mathcal{O}_C/\varpi_C.$$

We note that the common ring is not a field, but rather is a highly non-noetherian nilpotent thickening of its (algebraically closed) residue field.

An isomorphism of untilts $\psi\colon ((C, \mathcal{O}_C), \iota) \to ((C', \mathcal{O}_{C'}, \iota')$ is an isomorphism of completely valued fields $\psi\colon (C, \mathcal{O}_C) \to (C', \mathcal{O}_{C'})$ with the property that the diagram

$$\begin{array}{ccc} & \overset{\iota}{\nearrow} & (C^\flat, \mathcal{O}_{C^\flat}) \\ (F, \mathcal{O}_F) & & \downarrow \psi^\flat \\ & \underset{\iota'}{\searrow} & (C'^\flat, \mathcal{O}_{C'^\flat}) \end{array}$$

commutes. Up to isomorphism, there is one untilt of $(F, \mathcal{O}_F)$ of characteristic p. But there are many non-isomorphic untilts of $(F, \mathcal{O}_F)$ of characteristic 0. We note that if $((C, \mathcal{O}_C), \iota)$ is an untilt of $(F, \mathcal{O}_F)$, then so is $((C, \mathcal{O}_C), \iota \circ \varphi)$, and that if C is of characteristic 0, then these two untilts are non-isomorphic. So the group $\varphi^{\mathbb{Z}}$ acts freely on the set of isomorphism classes of characteristic 0 untilts of F. Fargues and Fontaine show in [7, Théorème 6.5.2] that their curve

$$X = X_F \longrightarrow \operatorname{Spec}(\mathbb{Q}_p)$$

parametrizes the orbits of this action in the sense that there is a canonical bijection from the set $|X|$ of closed points onto the set of $\varphi^{\mathbb{Z}}$-orbits of isomorphism classes of characteristic 0 untilts of $(F, \mathcal{O}_F)$. Moreover, if $x \in |X|$ corresponds to the class of $((C, \mathcal{O}_C), \iota)$ via this bijection, then the residue field $k(x)$ is canonically isomorphic to C. The additional structure of the valuation ring $\mathcal{O}_C \subset C$ and the isomorphism $\iota\colon (F, \mathcal{O}_F) \to (C^\flat, \mathcal{O}_{C^\flat})$ arises from the "perfectoid" nature of the situation, a fact that is hidden away in the proof of [7, Théorème 6.5.2].

One may wonder if all untilts of $(F, \mathcal{O}_F)$ have abstractly isomorphic underlying completely valued fields. More precisely, given an untilt $((C, \mathcal{O}_C), \iota)$, we let

$$\operatorname{Aut}(F, \mathcal{O}_F)/(\operatorname{Aut}(C, \mathcal{O}_C) \times \varphi^{\mathbb{Z}}) \longrightarrow |X|$$

be the map that to ψ assigns the class of $((C, \mathcal{O}_C), \iota \circ \psi)$ and ask if this map is a bijection. It follows from [7, Corollaire 2.2.23] and [17, Corollary 6] that the answer to this question is affirmative if and only if the valued field $(F, \mathcal{O}_F)$ is spherically complete (a.k.a. maximally complete) in the sense that every decreasing sequence of discs in F has non-empty intersection. This is a stronger condition than completeness, which is the condition that every decreasing sequence of discs, whose radii tend to 0, has non-empty intersection. For example, the completion $\mathbb{C}_p$ of an algebraic closure of $\mathbb{Q}_p$ with respect to the unique extention of the p-adic absolute value is not spherically complete, and neither is its tilt $\mathbb{C}_p^\flat$.

The Fargues–Fontaine curve behaves as a proper, regular, connected curve of genus 0 over $\mathbb{Q}_p$, except that it is not of finite type. Strikingly, the map

$$\mathbb{Q}_p \longrightarrow H^0(X, \mathcal{O}_X)$$

induced by the structure map is an isomorphism. Hence, there is a great discrepancy between the field $\mathbb{Q}_p$ of global sections and the residue fields $k(x)$ at closed points $x \in |X|$, all of which all are algebraically closed completely valued extensions of $\mathbb{Q}_p$ by [7, Théorème 6.5.2]. This phenomenon is one that the Fargues–Fontaine curve shares with the twistor projective line. The complex projective line $\mathbb{P}^1_{\mathbb{C}}$ has two real forms, namely, the real projective line $\mathbb{P}^1_{\mathbb{R}}$ and the twistor projective line

$$X = \widetilde{\mathbb{P}}^1_{\mathbb{R}} \longrightarrow \operatorname{Spec}(\mathbb{R}),$$

which is the Brauer–Severi variety of the quaternions $\mathbb{H}$. Its ring of global sections is $\mathbb{R}$, whereas the residue field at every closed point $x \in |X|$ is an algebraically closed field that contains $\mathbb{R}$. These residue fields are of course all isomorphic to $\mathbb{C}$. In particular, the twistor projective line has no real points.

2. Curves

In general, a connected curve over a field

$$X \longrightarrow \operatorname{Spec}(k)$$

can be understood as follows. If we choose a regular closed point $\infty \in |X|$, then the open complement $X \smallsetminus \{\infty\} \subset X$ is affine, and the quasi-coherent ideal

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \mathcal{O}_X \longrightarrow i_{\infty *}k(\infty) \longrightarrow 0$$

is invertible. Moreover, writing $\mathcal{O}_X(n)$ for its $(-n)$-fold tensor power, we have

$$X \simeq \operatorname{Proj}(P) \longrightarrow \operatorname{Spec}(k),$$

where P is the graded k-algebra

$$P = \textstyle\bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(n)).$$

Associated with the closed point $\infty \in |X|$, we have the maps

$$U \xrightarrow{\;i\;} X \xleftarrow{\;j\;} Y$$

of formal schemes over k, where i and j are the canonical maps from the open complement of $\{\infty\} \subset X$ and the formal completion along $\{\infty\} \subset X$, respectively. In Grothendieck's philosophy, the map j should be viewed as the open and affine immersion of a tubular neighborhood of $\{\infty\} \subset X$, whereas the map i should be viewed as the closed immersion of the closed complement of said neighborhood. This point of view is substantiated by the stable recollement

$$\operatorname{QCoh}(U) \underset{i^!}{\overset{i^*}{\leftrightarrows}} \xrightarrow{i_* \simeq i_!} \operatorname{QCoh}(X) \underset{j_*}{\overset{j_!}{\leftrightarrows}} \xrightarrow{j^! \simeq j^*} \operatorname{QCoh}(Y)$$

among the corresponding stable ∞-categories of quasi-coherent modules.[1] We give a proof of this is in [11, Theorem 1.7], but we also refer to [5, Lecture V] for an explanation of the open-closed reversal in this situation. So we have a cartesian diagram in the ∞-category $\mathrm{QCoh}(X)$ of quasi-coherent $\mathcal{O}_X$-modules

$$\begin{array}{ccc} \mathcal{O}_X(n) & \longrightarrow & i_*i^*\mathcal{O}_X(n) \\ \downarrow & & \downarrow \\ j_*j^*\mathcal{O}_X(n) & \longrightarrow & i_*i^*j_*j^*\mathcal{O}_X(n). \end{array}$$

We interpret $\mathcal{O}_X(n)$ as the sheaf of meromorphic functions on X that are regular away from ∞ and whose pole order at ∞ is at most n, and $i_*i^*\mathcal{O}_X(n)$ as the sheaf of meromorphic functions on X that are regular away from ∞. Similarly, we interpret $j_*j^*\mathcal{O}_X(n)$ as the sheaf of formal meromorphic functions near ∞ that are regular away from ∞ and whose pole order at ∞ is at most n, and $i_*i^*j_*j^*\mathcal{O}_X(n)$ as the sheaf of formal meromorphic functions near ∞ that are regular away from ∞. It follows that the k-vector spaces $H^0(X,\mathcal{O}_X(n))$ and $H^1(X,\mathcal{O}_X(n))$ are canonically identified with the limit and colimit, respectively, of the diagram

$$\begin{array}{ccc} & & H^0(X, i_*i^*\mathcal{O}_X(n)) \\ & & \downarrow \\ H^0(X, j_*j^*\mathcal{O}_X(n)) & \longrightarrow & H^0(X, i_*i^*j_*j^*\mathcal{O}_X(n)). \end{array} \tag{2.1}$$

As a simple example, let us first consider the complex projective line

$$X = \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathrm{Spec}(\mathbb{C}).$$

If we choose a coordinate t^{-1} at $\infty \in |X|$, then the diagram (2.1) takes the form

$$\begin{array}{ccc} & & \mathbb{C}[t] \\ & & \downarrow \\ t^n\mathbb{C}[[t^{-1}]] & \longrightarrow & \mathbb{C}((t^{-1})) \end{array} \tag{2.2}$$

with the two maps given by the canonical inclusions. So for $n \geq 0$, we have

$$H^0(X,\mathcal{O}_X(n)) = \mathbb{C}\cdot\{1,t,\dots,t^n\}$$
$$H^1(X,\mathcal{O}_X(n)) = 0,$$

where $\mathbb{C}\cdot S$ is the $\mathbb{C}$-vector space generated by S, whereas for $n < 0$, we have

$$H^0(X,\mathcal{O}_X(n)) = 0$$
$$H^1(X,\mathcal{O}_X(n)) = \mathbb{C}\cdot\{t^{-1},\dots,t^{n+1}\}.$$

In particular, we find that the graded $\mathbb{C}$-algebra P is given by

$$P \simeq \bigoplus\nolimits_{n\geq 0} \mathbb{C}\cdot\{1,t,\dots,t^n\} \longrightarrow \mathbb{C}[x,y],$$

where the nth component of the right-hand isomorphism takes t^i to x^iy^{n-i}.

[1] Let $Y^{(n)} \subset X$ be the nth infinitesimal neighborhood of $\{\infty\} \subset X$. We define $\mathrm{QCoh}(Y)$ to be the colimit $\mathrm{colim}_n\,\mathrm{QCoh}(Y^{(n)})$ in the ∞-category of presentable ∞-categories and right adjoint functors, or equivalently, as the limit $\lim_n \mathrm{QCoh}(Y^{(n)})$ in the ∞-category of presentable ∞-categories and left adjoint functors. The functor $j^! \simeq j^*$ is Grothendieck's local cohomology.

In the case of the twistor projective line

$$X = \widetilde{\mathbb{P}}^1_{\mathbb{R}} \longrightarrow \operatorname{Spec}(\mathbb{R}),$$

the diagram (2.1) takes the form

$$\begin{array}{ccc} & & \mathbb{R}[u,v]/(u^2+v^2+1) \\ & & \big\downarrow \\ t^n\mathbb{C}[[t^{-1}]] & \longrightarrow & \mathbb{C}((t^{-1})) \end{array} \tag{2.3}$$

with the horizontal map given by the canonical inclusion and with the vertical map obtained by solving the complex linear system of equations

$$\begin{aligned} u + iv &= t \\ u - iv &= -t^{-1}. \end{aligned}$$

We remark that now the identification of the terms in the bottom row uses the fact that a complete discrete valuation ring of residue characteristic 0 admits a unique coefficient field, which, in general, relies on the axiom of choice. Thus, the $\mathbb{C}$-algebra structure on the two rings is somewhat of a mirage, as opposed to the $\mathbb{C}$-algebra structure on the associated graded rings for the t-adic filtrations. This filtration, in turn, induces a $\mathbb{Z}$-graded ascending filtration

$$\cdots \subset \operatorname{Fil}_{m-1} H^i(X, \mathcal{O}_X(n)) \subset \operatorname{Fil}_m H^i(X, \mathcal{O}_X(n)) \subset \cdots$$

of the cohomology groups in question, and we find that for $n \geq 0$,

$$\operatorname{gr}_m H^0(X, \mathcal{O}_X(n)) \simeq \begin{cases} \mathbb{R} \cdot 1 & \text{if } m = 0 \\ \mathbb{C} \cdot t^m & \text{if } 0 < m \leq n \end{cases}$$

are the only nonzero graded pieces, whereas for $n < 0$,

$$\operatorname{gr}_m H^1(X, \mathcal{O}_X(n)) \simeq \begin{cases} \mathbb{C} \cdot t^m & \text{if } n+1 \leq m \leq -1 \\ \mathbb{C}/\mathbb{R} \cdot 1 & \text{if } m = 0 \end{cases}$$

are the only nonzero graded pieces. In particular, the quotient $\mathbb{C}/\mathbb{R}$ of the residue field at ∞ by the subfield of global sections appears as $H^1(X, \mathcal{O}_X(-1))$. We find that there is an isomorphism of graded $\mathbb{R}$-algebras

$$P = \bigoplus\nolimits_{n \geq 0} H^0(X, \mathcal{O}_X(n)) \longrightarrow \mathbb{R}[x,y,z]/(x^2+y^2+z^2)$$

whose nth component takes the class of u^iv^j to $x^iy^jz^{n-(i+j)}$.

In the case of the Fargues–Fontaine curve

$$X = X_F \longrightarrow \operatorname{Spec}(\mathbb{Q}_p),$$

the diagram (2.1) is expressed in terms of Fontaine's period rings[2] as

$$\begin{array}{ccc} & & B_e \\ & & \big\downarrow \\ \operatorname{Fil}_n B_{\mathrm{dR}} & \longrightarrow & B_{\mathrm{dR}} \end{array} \tag{2.4}$$

[2] The notation B_e stems from [4, (3.7.2)]. The subscript e indicates the exponential case.

The ring B_{dR} is a complete discrete valuation field with residue field $C = k(\infty)$,[3] and the horizontal map is the canonical inclusion of the $(-n)$th power of the maximal ideal of its valuation ring $B_{\mathrm{dR}}^+ = \mathrm{Fil}_0 B_{\mathrm{dR}}$. The ring B_e is a principal ideal domain, the nonzero ideals of which are in canonical one-to-one correspondence with the closed points $x \in |X \smallsetminus \{\infty\}|$. This non-trivial fact, instigated by Berger's discovery that every finitely generated ideal in B_e is principal [2, Proposition 1.1.9], was the discovery which led Fargues and Fontaine to realize that they had a curve in their hands. The proof is given in [7, Théorème 6.5.2], and Colmez' recounting of this discovery process in [7, Préface] is quite illuminating. The discrete valuation on B_{dR} again gives rise to an ascending $\mathbb{Z}$-graded filtration

$$\cdots \subset \mathrm{Fil}_{m-1} H^i(X, \mathcal{O}_X(n)) \subset \mathrm{Fil}_m H^i(X, \mathcal{O}_X(n)) \subset \cdots$$

of the cohomology groups, and for $n \geq 0$,

$$\mathrm{gr}_m H^0(X, \mathcal{O}_X(n)) = \begin{cases} \mathbb{Q}_p \cdot 1 & \text{if } m = 0 \\ C \cdot t^m & \text{if } 0 < m \leq n \end{cases}$$

are the only nonzero graded pieces, whereas for $n < 0$,

$$\mathrm{gr}_m H^1(X, \mathcal{O}_X(n)) = \begin{cases} C \cdot t^m & \text{if } n+1 \leq m \leq -1 \\ C/\mathbb{Q}_p \cdot 1 & \text{if } m = 0 \end{cases}$$

are the only nonzero graded pieces. Here $t^{-1} \in \mathrm{Fil}_{-1} B_{\mathrm{dR}}$ is a local parameter at ∞, that is, a generator of this B_{dR}^+-module, which is free of rank 1. Again, the quotient $C/\mathbb{Q}_p$ of the residue field at ∞ and the subfield of global sections appears as $H^1(X, \mathcal{O}_X(-1))$, but this now an infinite dimensional $\mathbb{Q}_p$-vector space, reflecting the fact that the Fargues–Fontaine curve is not of finite type over $\mathbb{Q}_p$. Nevertheless, Fargues and Fontaine show in [7, Théorème 6.2.1] that the graded $\mathbb{Q}_p$-algebra

$$P = \bigoplus\nolimits_{n \geq 0} H^0(X, \mathcal{O}_X(n))$$

is a graded unique factorization domain, all of whose irreducible elements are of degree 1. Thus, closed points $x \in |X|$ are in canonical one-to-one correspondence with $\mathbb{Q}_p$-lines in $H^0(X, \mathcal{O}_X(1))$, or equivalently, extensions of the form

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow H^0(X, \mathcal{O}_X(1)) \longrightarrow k(x) \longrightarrow 0;$$

see [7, Théorème 6.5.2].

3. The formal neighborhood of $\infty \in |X|$

We proceed to explain how to obtain the diagram (2.4) for the Fargues–Fontaine curve from a diagram of spectra. We begin with the lower line in the diagram, and recall some general theory following Nikolaus–Scholze [16, Theorem I.4.1].

We will use the language of ∞-categories following Joyal and Lurie [14], but we will use the term "anima" or "animated set" for what Lurie calls a space to emphasize that these should be viewed as sets with internal symmetries rather than anything resembling a topological space.

If $f \colon T \to S$ is any map of anima, then we have the restriction along f, $f^! \simeq f^*$, and the left and right Kan extensions along f, $f_!$ and f_*, between the corresponding ∞-categories of spectrum-valued presheaves:

[3] While there does exist an isomorphism $B_{\mathrm{dR}} \simeq C((t^{-1}))$ of complete discrete valuation fields, such an isomorphism is highly non-canonical and not useful in practice.

$$\mathrm{Sp}^T \;\underset{f_*}{\overset{f_!}{\rightrightarrows}}\; \mathrm{Sp}^S, \qquad \mathrm{Sp}^T \xleftarrow{\;f^! \simeq f^*\;} \mathrm{Sp}^S$$

In fact, these functors are part of a six-functor formalism on the ∞-category of anima in the sense of Mann [**15**, Definition A.5.7]. Indeed, this claim is a trivial instance of [**15**, Proposition A.5.10], where every map of anima $f\colon T \to S$ is declared to be a local isomorphism, and where a map of anima $f\colon T \to S$ is declared to be proper if its fibers are compact projective anima, or equivalently, anima that are equivalent to finite sets, including the empty set. In this six-functor formalism, we have $f^! \simeq f^*$ for every map $f\colon T \to S$, which reflects the fact that every map of anima is a local isomorphism. Now, the ∞-categories Sp^S and Sp^T are compactly generated presentable stable ∞-categories, and the "homology" functor $f_!$ preserves compact objects, because its right adjoint $f^!$ preserves all colimits. However, the "cohomology" functor f_* does not preserve compact objects, and by [**16**, Theorem I.4.1], there is a unique a map $f_* \to f_*^T$ to a "Tate cohomology" functor that takes all compact objects to zero and that is initial with this property. The fiber of this map preserves colimits, which implies that it necessarily is of the form $X \mapsto f_!(X \otimes D_f)$ for a unique $D_f \in \mathrm{Sp}^T$.

We recall from [**13**, Theorem 5.6.2.10] that every group in anima G is the loop group ΩBG of a pointed connected anima

$$1 \xrightarrow{\;s\;} BG,$$

and we first apply the general theory above to the unique map

$$BG \xrightarrow{\;f\;} 1.$$

If G is the group in anima underlying a compact Lie group G, then $D_f \simeq S^{\mathfrak{g}}$ is the suspension spectrum of the one-point compactification of its Lie algebra with the adjoint G-action. So in this situation, the defining fiber sequences takes the form

$$f_!(S^{\mathfrak{g}} \otimes X) \longrightarrow f_*(X) \longrightarrow f_*^T(X),$$

where $X \in \mathrm{Sp}^{BG}$. This sequence is commonly written as

$$(S^{\mathfrak{g}} \otimes X)_{hG} \longrightarrow X^{hG} \longrightarrow X^{tG}.$$

The Postnikov filtration of X gives rise to the spectral sequences

$$E^2_{i,j} = H^{-i}(BG, \pi_j(s^*(X))) \Rightarrow \pi_{i+j}(X^{hG})$$
$$E^2_{i,j} = \hat{H}^{-i}(BG, \pi_j(s^*(X))) \Rightarrow \pi_{i+j}(X^{tG}),$$

and the edge homomorphism of the former,

$$\pi_j(X^{hG}) \xrightarrow{\;\theta\;} \pi_j(s^*(X)),$$

is induced by the unit map

$$f_*(X) \xrightarrow{\;\theta\;} f_*s_*s^*(X) \simeq s^*(X).$$

The groups in the two E^2-terms can be interpreted as the cohomology and Tate cohomology of BG with coefficients in the local system $\pi_j(s^*(X))$. If G is finite, then these can be identified with the group cohomology and Tate cohomology of the group G with coefficients in the G-module $\pi_j(s^*(X))$.

It follows from [16, Theorem I.4.1] that if $E \in \mathrm{CAlg}(\mathrm{Sp}^{BG})$ is a commutative algebra in spectra with G-action, then the map $f_*(E) \to f_*^T(E)$ promotes to a map of commutative algebras in spectra. This implies that the spectral sequences become spectral sequences of bigraded anticommutative rings and that the edge homomorphism θ becomes an map of graded anticommutative rings; see [9]. If the G-action on E is trivial in the sense that $E \simeq f^*s^*(E)$, then a choice of trivialization and the unit map combine to give a section

$$s^*(E) \xrightarrow{\sigma} f_*f^*s^*(E) \simeq f_*(E)$$

of the edge homomorphism θ. So in this situation, the map $f_*(E) \to f_*^T(E)$ becomes a map of commutative algebras in $s^*(E)$-modules in spectra. So the induced map on homotopy groups becomes a map of anticommutative graded $\pi_*(s^*(E))$-algebras, and the spectral sequences above become spectral sequences of anticommutative bigraded $\pi_*(s^*(E))$-algebras.

We are interested in the group $G \simeq U(1)$, and since it is abelian, the G-action on its Lie algebra is trivial. So in this case, the fiber sequence above becomes

$$\Sigma E_{hG} \longrightarrow E^{hG} \longrightarrow E^{tG}.$$

For instance, in the case of $E \simeq \mathrm{HH}(A/R)$, this gives Connes' sequence

$$\Sigma\, \mathrm{HC}(A/R) \longrightarrow \mathrm{HC}^-(A/R) \longrightarrow \mathrm{HP}(A/R)$$

relating cyclic homology, negative cyclic homology, and periodic cyclic homology. Choosing a generator $\bar{v} \in H^2(BU(1),\mathbb{Z})$, the spectral sequences take the form[4]

$$\begin{aligned} E^2 &= \pi_*(s^*(E))[\bar{v}] \Rightarrow \pi_*(E^{hU(1)}) \\ E^2 &= \pi_*(s^*(E))[\bar{v}^{\pm 1}] \Rightarrow \pi_*(E^{tU(1)}). \end{aligned}$$

We refer to the filtrations of the respective abutments induced by the two spectral sequences as the "Nygaard" filtrations.

We will consider $E \in \mathrm{CAlg}(\mathrm{Sp}^{BU(1)})$ equipped with a "Bott" element

$$\beta \in \pi_2(E^{hU(1)}) \longrightarrow \pi_2(E^{tU(1)})$$

and obtain the desired filtered ring from the filtered graded ring $\pi_*(E^{tU(1)})$ by the following procedure:

(1) Invert the Bott element.
(2) Complete with respect to the Nygaard filtration.
(3) Extract the filtered subring consisting of homogeneous elements of degree 0.

The image of the induced ring homomorphism

$$(\pi_*(E^{hU(1)})[\beta^{-1}]^\wedge)_0 \longrightarrow (\pi_*(E^{tU(1)})[\beta^{-1}]^\wedge)_0$$

agrees with the 0th stage of the Nygaard filtration. If $\pi_*(s^*(E))$ is concentrated in even degrees, then so are $\pi_*(E^{hU(1)})$ and $\pi_*(E^{tU(1)})$ and all differentials in the respective spectral sequences are zero. In this case, the map above is injective.

[4] Here $\pi_*(-)$ indicates homotopy groups and not pushforward along some map π.

Suppose first that the $U(1)$-action on E is trivial. If the homotopy groups $\pi_*(s^*(E))$ are concentrated in even degrees, then there are isomorphisms

$$\pi_*(E^{hU(1)}) \simeq \pi_*(s^*(E))[[v]] \simeq \lim_n \pi_*(s^*(E))[v]/(v^{n+1})$$
$$\pi_*(E^{tU(1)}) \simeq \pi_*(s^*(E))((v)) \simeq \pi_*(s^*(E))[[v]][v^{-1}]$$

of anticommutative graded $\pi_*(s^*(E))$-algebras. Here the limit is calculated in the category of graded $\pi_*(s^*(E))$-algebras and $v \in \pi_{-2}(E^{hU(1)})$ is a choice of lift of $\bar{v}$, that is, a "complex orientation" of E. So with $t^{-1} = \beta v$, we have

$$(\pi_*(E^{hU(1)})[\beta^{-1}])_0 \simeq (\pi_*(s^*(E))[\beta^{-1}])_0[[t^{-1}]]$$
$$(\pi_*(E^{tU(1)})[\beta^{-1}])_0 \simeq (\pi_*(s^*(E))[\beta^{-1}])_0((t^{-1}))$$

Hence, writing $R = (\pi_*(s^*(E))[\beta^{-1}])_0$, we see that the graded $\pi_*(s^*(E))$-algebra $\pi_*(E^{tU(1)})$ with the Nygaard filtration gives rise to the R-algebra

$$R((t^{-1}))$$

with the t-adic filtration. This is the filtered R-algebra that we would obtain from the formal neighborhood of an R-valued point $\infty \in |X|$ of a curve $X \to \operatorname{Spec}(R)$.

Let us now consider the Fargues–Fontaine curve

$$X = X_F \longrightarrow \operatorname{Spec}(\mathbb{Q}_p)$$

associated to an algebraically closed completely valued field $(F, \mathcal{O}_F)$ of characteristic $p > 0$, and let $\infty \in |X|$ be a closed point corresponding to an untilt $((C, \mathcal{O}_C), \iota)$ of $(F, \mathcal{O}_F)$. We consider the commutative algebra in spectra with $U(1)$-action

$$E = \operatorname{THH}(\mathcal{O}_C, \mathbb{Z}_p) \in \operatorname{CAlg}(\operatorname{Sp}^{BU(1)})$$

given by the p-adic completion of the topological Hochschild homology of $\mathcal{O}_C$. In this case, Bhatt–Morrow–Scholze show in [3, Proposition 6.2] that

$$\pi_*(E^{hU(1)}) \simeq \operatorname{TC}^-_*(\mathcal{O}_C, \mathbb{Z}_p) \simeq A_{\inf}(\mathcal{O}_F)[u,v]/(uv - \xi)$$
$$\pi_*(E^{tU(1)}) \simeq \operatorname{TP}_*(\mathcal{O}_C, \mathbb{Z}_p) \simeq A_{\inf}(\mathcal{O}_F)[v^{\pm 1}],$$

where $u \in \operatorname{TC}^-_2(\mathcal{O}_C, \mathbb{Z}_p)$ and $v \in \operatorname{TC}^-_{-2}(\mathcal{O}_C, \mathbb{Z}_p)$, and where ξ is a generator of the kernel of the edge homomorphism

$$\operatorname{TC}^-_0(\mathcal{O}_C, \mathbb{Z}_p) \simeq A_{\inf}(\mathcal{O}_F) \xrightarrow{\ \theta\ } \mathcal{O}_C.$$

The element u is called a Bökstedt element. It is not a Bott element, but rather a divided Bott element. More precisely, the element

$$\beta = \varphi^{-1}(\mu)u$$

is a Bott element, but $\varphi^{-1}(\mu) \in A_{\inf}(\mathcal{O}_F)$ is not a unit; see also [10, Section 3.2]. So by inverting β, we also invert $\varphi^{-1}(\mu)$, which, in particular, inverts p, since

$$\theta(\varphi^{-1}(\mu)) = \zeta_p - 1 \in \mathcal{O}_C$$

is a pseudo-uniformizer. It follows that

$$(\pi_*(E^{tU(1)})[\beta^{-1}]^\wedge)_0 \simeq (\operatorname{TP}_*(\mathcal{O}_C, \mathbb{Z}_p)[\beta^{-1}]^\wedge)_0 \simeq B_{\mathrm{dR}}$$

as a filtered $\mathbb{Q}_p$-algebra. It is a discrete valuation field with residue field C. We stress that, after inverting β, we do not subsequently p-complete, which would leave us with zero. So this procedure is distinct from Morava $K(1)$-localization.

4. The affine complement of $\infty \in |X|$

We will next explain how to obtain the right-hand vertical map in (2.4) from the "crystalline Chern character" of Antieau–Mathew–Morrow–Nikolaus [1]. The definition of this map relies on the modern approach to topological cyclic homology and cyclotomic spectra due to Nikolaus–Scholze [16].

To briefly recall the definition, let $p\colon BU(1) \to BU(1)$ be the map of pointed anima induced by the p-power map of the abelian group $U(1)$, and let

$$\mathrm{Sp}_p^{BU(1)} \xrightarrow{p_*^T} \mathrm{Sp}_p^{BU(1)}$$

be the associated Tate cohomology functor. A p-complete cyclotomic spectrum is a pair (X, φ) of a p-complete spectrum[5] with $U(1)$-action and a "Frobenius" map

$$X \xrightarrow{\varphi} p_*^T(X)$$

of spectra with $U(1)$-action. Nikolaus and Scholze show that p-complete cyclotomic spectra can be organized into a stable symmetric monoidal ∞-category equipped with a conservative symmetric monoidal forgetful functor

$$\mathrm{CycSp}_p \longrightarrow \mathrm{Sp}_p^{BU(1)}$$

to the symmetric monoidal stable ∞-category of p-complete spectra with $U(1)$-action. The tensor unit is given by the pair $\mathbf{1} \simeq (f^*(\mathbb{S}_p), \varphi)$ of the p-complete sphere spectrum with trivial $U(1)$-action and the composition

$$f^*(\mathbb{S}_p) \longrightarrow p_* f^*(\mathbb{S}_p) \longrightarrow p_*^T f^*(\mathbb{S}_p)$$

of the adjunct of the equivalence $p^*f^* \simeq (fp)^* \simeq f^*$ and the canonical map. Now, p-complete topological cyclic homology is the symmetric monoidal functor

$$\mathrm{CycSp}_p \xrightarrow{\mathrm{TC}} \mathrm{Sp}_p$$

corepresented by $\mathbf{1} \in \mathrm{CycSp}_p$. If (X, φ) is a p-complete cyclotomic spectrum with X bounded below, then $\mathrm{TC}(X, \varphi)$ is given by the equalizer

$$\mathrm{TC}(X,\varphi) \longrightarrow \mathrm{TC}^-(X) \simeq f_*(X) \underset{\mathrm{can}}{\overset{\varphi}{\rightrightarrows}} \mathrm{TP}(X) \simeq f_*^T(X)$$

of the canonical map "can" and the cyclotomic Frobenius "φ" defined as follows. Since $fp \simeq f$, we have a diagram of p-complete spectra

$$\begin{array}{ccccc}
\Sigma f_! p_!(X) \simeq \Sigma f_!(X) & \longrightarrow & f_*(X) & \longrightarrow & f_*^T(X) \\
\downarrow & & \| & & \downarrow \\
f_* p_!(X) & \longrightarrow & f_* p_*(X) & \longrightarrow & f_* p_*^T(X)
\end{array}$$

[5] The inclusion $j_*\colon \mathrm{Sp}_p \to \mathrm{Sp}$ of the full subcategory of p-complete spectra admits a left adjoint $j^*\colon \mathrm{Sp} \to \mathrm{Sp}_p$, and the unit map $\eta\colon X \to j_*j^*(X)$ is p-completion. The functor j^* admits an essentially unique promotion to a symmetric monoidal functor, which, in turn, promotes j_* to a lax symmetric monoidal functor. Hence, we obtain an induced adjunction of the associated ∞-categories of commutative algebras.

in which the two rows are fiber sequences. Since X is bounded below, the Tate orbit lemma [16, Lemma I.2.1] shows that the left-hand vertical map is an equivalence, and hence, so is the right-hand vertical map. So in this case, we have the map

$$X^{hU(1)} \simeq f_*(X) \xrightarrow{f_*(\varphi)} f_*p^T_*(X) \simeq f^T_*(X) \simeq X^{tU(1)},$$

which, by abuse of notation, we also denote by $\varphi\colon \mathrm{TC}^-(X) \to \mathrm{TP}(X)$.

We recall the most economical way of associating to a commutative algebra in p-complete spectra R a commutative algebra in p-complete cyclotomic spectra

$$(\mathrm{THH}(R, \mathbb{Z}_p), \varphi)$$

following [16, Section IV.2].

In general, let G be a group in anima, and let $s\colon 1 \to BG$ be the corresponding connected pointed anima. If $\mathcal{C}$ is any presentable ∞-category, then we have

$$\mathcal{C} \underset{s^*}{\overset{s_!}{\rightleftarrows}} \mathcal{C}^{BG},$$

where the right adjoint s^* takes an object of $\mathcal{C}$ with G-action to the underlying object of $\mathcal{C}$, and where the left adjoint $s_!$ takes an object of $\mathcal{C}$ to the free object of $\mathcal{C}$ with G-action. We have a cartesian square of pointed anima

$$\begin{array}{ccc} G & \xrightarrow{s'} & 1 \\ \downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle s} \\ 1 & \xrightarrow{s} & BG \end{array}$$

and the base-change formula applied to this square shows that

$$s^*s_! \simeq s'_!s'^*$$

as endofunctors of $\mathcal{C}$. Informally, if X is an object of $\mathcal{C}$, then $s^*s_!(X)$ is the object of $\mathcal{C}$ underlying the free object of $\mathcal{C}$ with G-action associated with X, whereas $s'_!s'^*(X)$ is the colimit in $\mathcal{C}$ of the constant diagram with value X indexed by G.

We apply this with $G \simeq U(1)$ and $\mathcal{C} \simeq \mathrm{CAlg}(\mathrm{Sp}_p)$. Given $R \in \mathrm{CAlg}(\mathrm{Sp}_p)$, we define its p-complete topological Hochschild homology to be the commutative algebra in p-complete spectra with $U(1)$-action given by

$$\mathrm{THH}(R, \mathbb{Z}_p) \simeq s_!(R) \in \mathrm{CAlg}(\mathrm{Sp}_p)^{BU(1)}$$

and the base-change formula shows that its underlying commutative algebra in p-complete spectra is given by

$$s^*\,\mathrm{THH}(R, \mathbb{Z}_p) \simeq s^*s_!(R) \simeq s'_!s'^*(R) \simeq R^{\otimes U(1)},$$

where the right-hand term is suggestive notation for the colimit in $\mathrm{CAlg}(\mathrm{Sp}_p)$ of the constant diagram with value R indexed by the anima underlying $U(1)$.

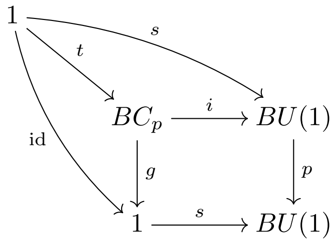
We next recall the definition of the Frobenius map

$$\mathrm{THH}(R, \mathbb{Z}_p) \xrightarrow{\varphi} p^T_*(\mathrm{THH}(R, \mathbb{Z}_p)),$$

where we abuse notation and also write p^T_* for the endofunctor of

$$\mathrm{CAlg}(\mathrm{Sp}_p)^{BU(1)} \simeq \mathrm{CAlg}(\mathrm{Sp}_p^{BU(1)})$$

induced by the lax symmetric monoidal endofunctor p^T_* of $\mathrm{Sp}_p^{BU(1)}$. We consider the following diagram of pointed anima



with the square cartesian. As part of the structure of any symmetric monoidal ∞-category $\mathcal{C}$, there is a symmetric monoidal functor

$$\mathcal{C} \xrightarrow{t_!^{\otimes}} \mathcal{C}^{BC_p}$$

which, informally, takes X to $X^{\otimes p}$ together with an equivalence between the induced functor of commutative algebras and

$$\mathrm{CAlg}(\mathcal{C}) \xrightarrow{t_!} \mathrm{CAlg}(\mathcal{C})^{BC_p} \simeq \mathrm{CAlg}(\mathcal{C}^{BC_p}).$$

We now recall from [**16**, Proposition III.3.1] that there is a unique lax symmetric monoidal transformation called the Tate diagonal[6]

$$X \xrightarrow{\delta} g^T_* t_!^{\otimes}(X).$$

So for $R \in \mathrm{CAlg}(\mathrm{Sp}_p)$, we get the composite map

$$R \xrightarrow{\delta} g^T_* t_!(R) \xrightarrow{\eta} g^T_* i^* i_! t_!(R) \simeq g^T_* i^* s_!(R) \simeq s^* p^T_* s_!(R)$$

of commutative algebras in p-complete spectra, whose adjunct

$$s_!(R) \xrightarrow{\varphi} p^T_* s_!(R)$$

is the desired map of commutative algebras in p-complete spectra with $U(1)$-action.

The crystalline Chern character of [**1**] is obtained as follows. Given a p-complete cyclotomic spectrum $X = (X, \varphi)$, we may form the tensor product

$$X \otimes \mathrm{THH}(\mathbb{F}_p, \mathbb{Z}_p)$$

in the symmetric monoidal ∞-category CycSp_p of p-complete cyclotomic spectra and consider the canonical map

$$\mathrm{TC}(X \otimes \mathrm{THH}(\mathbb{F}_p, \mathbb{Z}_p)) \longrightarrow \mathrm{TC}^-(X \otimes \mathrm{THH}(\mathbb{F}_p, \mathbb{Z}_p))$$

from its topological cyclic homology to its negative topological cyclic homology. We consider this map for $X \simeq \mathrm{THH}(\mathcal{O}_C, \mathbb{Z}_p)$, take homotopy groups, invert the Bott element, Nygaard complete the target, and extract the subrings of homogeneous elements of degree 0. By [**1**, Theorem 2.12], the resulting map takes the form

$$(\mathrm{TC}_*(\mathcal{O}_C/p, \mathbb{Z}_p)[\beta^{-1}])_0 \longrightarrow (\mathrm{TP}_*(\mathcal{O}_C, \mathbb{Z}_p)[\beta^{-1}]^\wedge)_0,$$

---

[6] The generalized Segal conjecture for C_p proved in [**16**, Theorem III.1.7] states that if X is bounded below, then the Tate diagonal is an equivalence.

and this map is the desired crystalline Chern character. Finally, the identification of this map with the map $B_e \to B_{\mathrm{dR}}$ in (2.4) is a consequence of [3, Corollary 8.23] and [3, Proposition 6.2]. We conclude that the diagram

$$\begin{array}{ccc} & & (\mathrm{TC}_*(\mathcal{O}_C/p,\mathbb{Z}_p)[\beta^{-1}])_0 \\ & & \downarrow \\ \mathrm{Fil}_n(\mathrm{TP}_*(\mathcal{O}_C,\mathbb{Z}_p)[\beta^{-1}]^\wedge)_0 & \longrightarrow & (\mathrm{TP}_*(\mathcal{O}_C,\mathbb{Z}_p)[\beta^{-1}]^\wedge)_0 \end{array} \tag{4.1}$$

obtained from the various flavors of topological cyclic homology and maps between them is canonically identified with the diagram of period rings (2.4).

5. Dreams

We would like to similarly obtain the diagram (2.3) for the twistor projective line from a diagram analogous to (4.1). Given a closed point $\infty \in |X|$ with residue field $k(\infty) \simeq \mathbb{C}$, one might hope to obtain the bottom line in (2.3) from the periodic cyclic homology $\mathrm{HP}_*(\mathbb{C}/\mathbb{R})$. This will not work, however, due to the lack of a Bott element $\beta \in \mathrm{HC}_2^-(\mathbb{C}/\mathbb{R})$. The liquid theory of Clausen–Scholze [6] points to an alternative. Let $0 < r < 1$ be a real number. The Harbater ring is the subring

$$\mathbb{Z}((T))_{>r} \subset \mathbb{Z}((T))$$

consisting of the Laurent series that for some $r' > r$ converge absolutely on a disc of radius r' centered at the origin. Harbater shows in [8] that this ring is a principal ideal domain; see also [6, Theorem 7.1]. In particular, if $x \in \mathbb{C}$ and $|x| \leq r$, then the kernel of the continuous ring homomorphism

$$\mathbb{Z}((T))_{>r} \longrightarrow \mathbb{C}$$

that to T assigns x is a principal ideal, and hence, the Hochschild homology

$$\mathrm{HH}_*(\mathbb{C}/\mathbb{Z}((T))_{>r}) = \mathbb{C}\langle \bar{u} \rangle = \mathbb{C}[\bar{u}]$$

is a (divided) polynomial algebra on a generator $\bar{u}$ of degree 2. So with

$$E = \mathrm{HH}(\mathbb{C}/\mathbb{Z}((T))_{>r}) \in \mathrm{CAlg}(\mathrm{Sp}^{BU(1)}),$$

we find that

$$\begin{aligned} \pi_*(E^{hU(1)}) &\simeq \mathrm{HC}_*^-(\mathbb{C}/\mathbb{Z}((T))_{>r}) \simeq \mathbb{C}[[\xi]][u,v]/(uv-\xi) \\ \pi_*(E^{tU(1)}) &\simeq \mathrm{HC}_*^-(\mathbb{C}/\mathbb{Z}((T))_{>r}) \simeq \mathbb{C}[[\xi]][v^{\pm 1}] \end{aligned}$$

with $u \in \mathrm{HC}_2^-(\mathbb{C}/\mathbb{Z}((T))_{>r})$ and $v \in \mathrm{HC}_{-2}^-(\mathbb{C}/\mathbb{Z}((T))_{>r})$ and with ξ is a generator of the kernel of the edge homomorphism

$$\mathrm{HC}_0^-(\mathbb{C}/\mathbb{Z}((T))_{>r}) \simeq \mathbb{C}[[\xi]] \xrightarrow{\ \theta\ } \mathbb{C},$$

analogously to $E \simeq \mathrm{THH}(\mathcal{O}_C,\mathbb{Z}_p)$. Hence, for a "Bott element" of the form $\beta = fu$ with $f \in \mathbb{C}[[\xi]]^\times$ a unit, we obtain the bottom row in (2.3) with $t^{-1} = \beta v$.

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References

[1] B. Antieau, A. Mathew, M. Morrow, and T. Nikolaus, *On the Beilinson fiber square*, Duke Math. J. (to appear).

[2] L. Berger, *Construction de (ϕ, Γ)-modules: représentations p-adiques et B-paires*, Algebra Number Theory **2** (2008), 91–120.

[3] B. Bhatt, M. Morrow, and P. Scholze, *Topological Hochschild homology and integral p-adic Hodge theory*, Publ. Math. Inst. Hautes Études Sci. **129** (2019), 199–310.

[4] S. Bloch and K. Kato, *L-functions and Tamagawa numbers of motives*, The Grothendieck Festschrift, Vol. I,, Progr. Math., vol. 86, Birkhäuser Boston, Boston, MA, 1990, pp. 333–400.

[5] D. Clausen and P. Scholze, *Condensed mathematics and complex geometry*, Lecture notes, Bonn–Copenhagen, 2022.

[6] ———, *Lectures on analytic geometry*, Lecture notes, Bonn, 2020.

[7] L. Fargues and J.-M. Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge p-adique*, Astérisque **406** (2018), With a preface by Pierre Colmez.

[8] D. Harbater, *Convergent arithmetic power series*, Amer. J. Math. **106** (1984), 801–846.

[9] A. Hedenlund and J. Rognes, *A multiplicative Tate spectral sequence for compact Lie group actions*, arxiv:2008.09095.

[10] L. Hesselholt and T. Nikolaus, *Topological cyclic homology*, Handbook of Homotopy Theory, CRC Press/Chapman Hall Handb. Math. Ser., CRC Press, Boca Raton, FL, 2020, pp. 619–656.

[11] L. Hesselholt and P. Pstragowski, *Dirac geometry II: Coherent cohomology*, in preparation.

[12] J. Lurie, *The Fargues–Fontaine curve*, available at math.ias.edu/∼lurie.

[13] ———, *Higher algebra*, available at math.ias.edu/∼lurie.

[14] ———, *Higher topos theory*, Ann. of Math. Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

[15] L. Mann, *A p-adic 6-functor formalism in rigid-analytic geometry*, arxiv:2206.02022.

[16] T. Nikolaus and P. Scholze, *On topological cyclic homology*, Acta Math. **221** (2018), 203–409.

[17] B. Poonen, *Maximally complete fields*, Enseign. Math. **39** (1993), 87–106.

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