

### 4.3 Fixed point spectra

Let  $T$  be a  $G$ -spectrum indexed on  $\mathcal{U}$  and let  $H \subset G$  be a closed subgroup. Then  $\mathcal{U}$  is also an  $H$ -universe, but  $T$  is not in  $H\mathcal{U}$  because we only have  $G$ -spaces  $T_V$  for  $V \in \mathcal{U}$  a  $G$ -representation. Define an indexing set  $\mathcal{A}$  in a  $G$ -universe  $\mathcal{U}$  to be a set of f.d.  $G$ -subrepr.  $V \in \mathcal{U}$  s.t. for every  $W \in \mathcal{U}$  f.d., there exists some  $V \in \mathcal{A}$  with  $W \subset V$ . We then have the obvious notions of  $G$ -prespectra and  $G$ -spectra indexed on  $\mathcal{A}$ , categories  $G\mathcal{P}\mathcal{A}$  and  $G\mathcal{S}\mathcal{A}$  and adjoint functors  $L \dashv R$ . By abuse of notation, we denote by  $\mathcal{U}$  the indexing set of all  $V \in \mathcal{U}$  f.d.

Lemma 4.3.1 Suppose that  $\mathcal{A}$  and  $\mathcal{B}$  are two indexing sets in  $\mathcal{U}$ . Then there is a canonical isomorphism of categories  $G\mathcal{S}\mathcal{A} \cong G\mathcal{S}\mathcal{B}$ .

PF It is enough to show that  $G\mathcal{S}\mathcal{A} \cong G\mathcal{S}\mathcal{U}$ . Let  $i: \mathcal{A} \hookrightarrow \mathcal{U}$  be the inclusion. Then we have an obvious forgetful functor  $i^*: G\mathcal{S}\mathcal{U} \rightarrow G\mathcal{S}\mathcal{A}$ . We define a left adjoint  $i_*: G\mathcal{S}\mathcal{A} \rightarrow G\mathcal{S}\mathcal{U}$  by

$$(i_* T)_V = \varinjlim_{V \subset W \in \mathcal{A}} \Omega^{W-V} T_W$$

with the evident structure maps. All maps in the limit system are homeomorphisms, so obviously the unit and counit of  $i_* \dashv i^*$  are isomorphisms  $\square$

Remark 4.3.2 In general  $G\mathcal{P}\mathcal{A} \not\cong G\mathcal{P}\mathcal{U}$ .

Let  $H \subset G$  be a closed subgroup, let  $NH \subset G$  be the normalizer and let  $W_H = NH/H$  be the Weyl group. If  $\mathcal{U}$  is a  $G$ -universe, then  $\mathcal{U}^H$  is both an  $NH$ -universe and a  $W_H$ -universe. Let  $i: \mathcal{U}^H \hookrightarrow \mathcal{U}$  be the inclusion, which is  $NH$ -equivariant. Using lemma 4.3.1 we have "forgetful" functors

$$G\text{-}\mathcal{U} \xrightarrow{U} NH\text{-}\mathcal{U} \xrightarrow{i^*} NH\text{-}\mathcal{U}^H.$$

Finally, taking  $H$ -fixed sets spacewise defines a functor  $NH\text{-}\mathcal{U}^H \rightarrow W_H\text{-}\mathcal{U}^H$ , which composed with the forgetful functor above yields the  $H$ -fixed point functor

$$(-)^H: G\text{-}\mathcal{U} \rightarrow W_H\text{-}\mathcal{U}^H. \quad (4.3.3)$$

We note that if  $V \subset \mathcal{U}$  is an  $H$ -fixed  $G$ -repr. and  $T$  is in  $G\text{-}\mathcal{U}$ , then  $T^H(V) = T(V)^H$ . More importantly, the functor  $i^*$  above has a left adjoint

$$i_*: NH\text{-}\mathcal{U}^H \rightarrow NH\text{-}\mathcal{U}. \quad (4.3.4)$$

To see this, first define  $i_*: NH\text{-}\mathcal{U}^H \rightarrow NH\text{-}\mathcal{U}$  by

$$(i_* t)_V = \sum^{V-V^H} t_{V^H}$$

with structure maps

$$\sum^{W-V} \sum^{V-V^H} t_{V^H} \cong \sum^{W-W^H} \sum^{W^H-V^H} t_{V^H} \rightarrow \sum^{W-W^H} t_{W^H}$$

and note that this is left adjoint to the forgetful functor  $i^*: NH\text{-}\mathcal{U} \rightarrow NH\text{-}\mathcal{U}^H$ . Now

define  $i_*: \text{NH}\mathcal{S}\mathcal{U}^H \rightarrow \text{NH}\mathcal{S}\mathcal{U}$  by  $i_*T = L(i_*\ell T)$ .

Then  $i_* \dashv i^*$  and since  $i^*$  and  $F(X, -)$  commute, we by category theory that

$$i_*(X \wedge T) = X \wedge i_*T,$$

compare 4.2.14.

Example 4.3.5 If  $S_{\mathcal{U}^H}^0$  is the sphere  $\text{NH}$ -spectrum for the  $\text{NH}$ -universe  $\mathcal{U}^H$ , then  $(i_* S_{\mathcal{U}^H}^0) \cong S_{\mathcal{U}}^0$  is the  $\text{NH}$ -sphere for  $\mathcal{U}$ . In contrast,  $i^* S_{\mathcal{U}}^0$  is not  $S_{\mathcal{U}^H}^0$ .

If  $a$  is an indexing sequence in the  $G$ -universe  $\mathcal{U}$ , then  $a^H := \{V^H \mid V \in a\}$  is an indexing sequence for the  $\text{WH}$ -universe  $\mathcal{U}^H$ . We define a functor  $\Phi^H: G\mathcal{O}a \rightarrow \text{WH}\mathcal{O}a^H$  by

$$(\Phi^H t)_{V^H} = (t_V)^H$$

with structure maps

$$\sum^{W^H \dashv V^H} (t_V)^H = (\sum^{W \dashv V} t_V)^H \xrightarrow{\sigma^H} (t_W)^H.$$

The geometric fixed point functor

$$\Phi^H: G\mathcal{S}\mathcal{U} \rightarrow \text{WH}\mathcal{S}\mathcal{U}^H \quad (4.3.6)$$

is then defined by

$$\Phi^H T := L(\Phi^H \ell T).$$

The restriction to  $H$ -fixed sets defines a map

$$(\Omega^{V-V^H} T_V)^H \xrightarrow{\tau_0} (T_V)^H$$

which in turn induces a natural transformation

$$S_H : T^H \rightarrow \Phi^H T \quad (4.3.7)$$

Lemma 4.3.8 Suppose  $t$  is a good  $G$ -prespectrum and let  $T = Lt$ . Then there is a canonical homeomorphism

$$(\Phi^H T)_V \cong \varinjlim_{W \subset V} \Omega^{W^H-V} (t_W)^H$$

and the maps in the limit system are closed inclusion.

Pf We evaluate  $L'(\Phi^H Lt)$ . Since  $t$  is good,

$$T_W = \varinjlim_{Z \supset W} \Omega^{Z-W} t_Z$$

so

$$\begin{aligned} \varinjlim_{W \supset V} \Omega^{W^H-V} (T_W)^H &= \varinjlim_{W \supset V} \Omega^{W^H-V} \left( \varinjlim_{Z \supset W} \Omega^{Z-W} t_Z \right)^H \\ &\cong \varinjlim_{\substack{Z \supset W \supset V \\ W \supset V}} \Omega^{W^H-V} (\Omega^{Z-W} t_Z)^H \end{aligned}$$

Here the colimit is over pairs  $Z, W \subset \mathcal{U}$  such that  $V \subset W \subset Z$ . In this index category the pairs with  $Z=W$  is cofinal, so

$$\varinjlim_{W \supset V} \Omega^{W^H-V} (T_W)^H \cong \varinjlim_{Z \supset V} \Omega^{Z^H-V} (t_Z)^H$$

The spaces on the right form a spectrum, so  $L'(\Phi^H Lt)$  is a spectrum, which is therefore  $\Phi^H T$  by 4.2.3.  $\square$



Let  $H \subset G$  be a closed subgroup and let  $\mathcal{U}$  be a  $G$ -universe. Then as in the space case, the forgetful functor

$$U: G\mathcal{S}\mathcal{U} \rightarrow H\mathcal{S}\mathcal{U}$$

has both a left adjoint  $G_+ \hat{\wedge}_H -$  and a right adjoint  $F_H(G_+, -)$  defined as follows: for  $H$ -prespectra

$$(G_+ \hat{\wedge}_H t)_V = G_+ \hat{\wedge}_H t_V \quad (4.3.9)$$

with  $G$  acting by left multiplication in the 1st variable and with structure maps

$$(G_+ \hat{\wedge}_H t)_V \wedge S^{W-V} \xrightarrow[\cong]{\xi} G_+ \hat{\wedge}_H (t_V \wedge S^{W-V}) \xrightarrow{G_+ \hat{\wedge}_H \sigma} G_+ \hat{\wedge}_H t_W.$$

where  $\xi(g, t, v) = (g, t, gv)$ . For  $H$ -spectra indexed on  $\mathcal{U}$ ,  $G_+ \hat{\wedge}_H T := L(G_+ \hat{\wedge}_H tT)$ . The right adjoint is defined for  $H$ -spectra or prespectra by

$$F_H(G_+, T)_V = F_H(G_+, T_V) \quad (4.3.10)$$

with the (left)  $G$ -action induced from  $G$ 's action by right multiplication on itself, and with structure maps

$$F_H(G_+, T_V) \xrightarrow{\omega} F_H(G_+, \Omega^{W-V} T_W) \xleftarrow[\cong]{\phi} \Omega^{W-V} F_H(G_+, T_W),$$

where  $\phi(\omega)(g)(v) = \omega(g^{-1}v)(g)$ .

Exercise 4.3.11 Show that if  $T$  is a  $G$ -spectrum, then

$$G_+ \hat{\wedge}_H T \cong G/H_+ \wedge T, \quad F_H(G_+, T) \cong F(G/H_+, T),$$

the smash product and function spectra from §4.2.

We now assume that the  $G$ -universe  $\mathcal{U}$  is complete and choose a  $G$ -embedding  $j: G/H \hookrightarrow V$  with  $V \subset \mathcal{U}$ . The tangent space  $L = T_{1,H}(G/H)$  is an  $H$ -repr. and  $j$  induces an embedding of  $L$  as a sub  $H$ -repr. of  $V$ . The complement  $V-L$  is then also a sub  $H$ -repr. and left translation specifies an isomorphism of the normal bdl. of  $j$  with bundle

$$V-L \rightarrow G \times_H (V-L) \rightarrow G/H.$$

Hence the Thom-Pontryagin construction gives a  $G$ -map

$$TP: S^V \rightarrow G_+ \wedge_H S^{V-L}$$

and one may check that the composite  $G$ -map

$$\begin{aligned} F_H(G_+, E_W) \wedge S^V &\xrightarrow{1 \wedge TP} F_H(G_+, E_W) \wedge G_+ \wedge_H S^{V-L} \\ &\xleftarrow[\cong]{\cong} G_+ \wedge_H (F_H(G_+, E_W) \wedge S^{V-L}) \xrightarrow{ev_1} G_+ \wedge_H (E_W \wedge S^{V-L}) \end{aligned}$$

defines, for every  $H$ -spectrum  $E$  indexed on  $\mathcal{U}$ , a map in  $G\mathcal{P}\mathcal{U}$

$$\tau: \Sigma^V F_H(G_+, E) \rightarrow G_+ \wedge_H \Sigma^{V-L} E. \quad (4.3.13)$$

Remark 4.3.14 Recall that a  $G$ -prospectrum  $t$  is good if  $\Sigma^{W-V} t_V \rightarrow t_W$  is a closed inclusion, for all  $V \subset W \subset \mathcal{U}$ . We define a  $G$ -spectrum  $T$  to be good if  $T = Lt$  for some good  $G$ -prospectrum  $t$ . Then if  $T$  is good, so is  $\mathbb{F}^H T$  and  $X \wedge T$ , see (4.3.2) and (4.2.14). However, in general  $T^H$  and  $F(X, T)$  are not.

False - see  
appendix

The map  $LT^c \rightarrow T$  is a weak  $G$ -equivalence, for any  $T$  in  $G\mathcal{U}$ , and its domain is good. Finally, if  $T$  is good then  $T \rightarrow \Omega^V \Sigma^V T$  is a weak  $G$ -equivalence. If  $T$  is bounded below this follows from the equivariant suspension theorem. See [LMS] in general.  $\square$

The following result is known as the Adams - Wirthmüller isomorphism.

Theorem 4.3.15 Let  $E$  be a <sup>well-pointed</sup>  $H$ -spectrum indexed on a complete  $G$ -universe  $\mathcal{U}$ . Then there is a natural chain of weak  $G$ -equivalences

$$F_H(G_+, \Sigma^L E) \sim G_+ \hat{\wedge}_H E,$$

where  $L = T_{1,H}(G/H)$ .

Prf The chain of maps we have in mind is

$$\begin{aligned} F_H(G_+, \Sigma^L E) &\xleftarrow{\sim} L F_H(G_+, \Sigma^L E)^c \xrightarrow{\sim} \Omega^V L \Sigma^V F_H(G_+, \Sigma^L E)^c \\ &\rightarrow \Omega^V L (G_+ \hat{\wedge}_H \Sigma^V E)^c \xleftarrow[\cong]{\sim} \Omega^V L \Sigma^V (G_+ \hat{\wedge}_H E)^c \\ &\xleftarrow{\sim} L (G_+ \hat{\wedge}_H E)^c \xrightarrow{\sim} G_+ \hat{\wedge}_H E. \end{aligned}$$

We must show that the 3<sup>rd</sup> map is a weak  $G$ -equivalence. We shall now assume that  $G$  is finite, and refer the reader to [LMS] for the general case. Let  $[-, -]_G$  denote set of  $G$ -homotopy classes of maps of  $G$ -spectra. We shall prove that for a good  $H$ -spectrum  $E$  and  $G$ -spectrum

There is an isomorphism

$$[T, E]_H \cong [T, G_+ \hat{\wedge}_H E]_G;$$

Since also  $[T, E]_H \cong [T, F_H(G_+, E)]_G$ , the theorem follows. Note that since  $E$  is good,  $[T, E]_H = [\Sigma^V T, \Sigma^V E]_H$  and similarly  $[D, T]_G \cong [\Sigma^V D, \Sigma^V T]_G$ . Hence (4.3.13) gives a well-defined homotopy class in  $G\text{-}\mathcal{S}\mathcal{U}$

$$\gamma: T \rightarrow G_+ \hat{\wedge}_H T.$$

We define a map in  $H\text{-}\mathcal{S}\mathcal{U}$

$$\alpha: G_+ \hat{\wedge}_H E \rightarrow E$$

which on the prespectrum level is given by

$$\alpha(g, e) = \begin{cases} ge & , \text{ if } g \in H \\ * & , \text{ otherwise.} \end{cases}$$

We shall prove that the two composites

$$T \xrightarrow{\gamma} G_+ \hat{\wedge}_H T \xrightarrow{\alpha} T$$

$$G_+ \hat{\wedge}_H E \xrightarrow{\gamma} G_+ \hat{\wedge}_H (G_+ \hat{\wedge}_H E) \xrightarrow{G_+ \hat{\wedge}_H \alpha} G_+ \hat{\wedge}_H E$$

are  $H$ -homotopic and  $G$ -homotopic to the identity, respectively. This will give the isomorphism we wanted. Since  $G$  is finite, we may take  $V$  to be  $\mathbb{R}[G/H]$  such that the composite

$$t: S^V \xrightarrow{TP} G/H \wedge S^V \xrightarrow[\cong]{\cong} G/H_+ \wedge S^V$$

is the Thom-Pontryagin construction corresponding to the trivialization of the normal bdl. of  $G/H \subset V$  given by parallel transport of the frame in  $O$  to each  $gH \in V$ . Now let  $\tilde{\gamma}: G/H_+ \rightarrow S^0$  be the map which sends  $1 \cdot H$  to the non-basepoint and all the cosets to the basepoint in  $S^0$ . Then the composite

$$t: S^V \longrightarrow G/H_+ \wedge S^V \xrightarrow{\tilde{\gamma} \wedge 1} S^V$$

takes a small ball around  $1 \cdot H$  and blows it up to all of  $V$ . This is  $H$ -homotopic to the identity, since the line segment  $t(1 \cdot H)$ ,  $t \in [0, 1]$ , is  $H$ -fixed. Now the strictly commutative diagram

$$\begin{array}{ccccc} T_W \wedge S^V & \xrightarrow{1 \wedge t} & T_W \wedge G/H_+ \wedge S^V & \xrightarrow{1 \wedge \tilde{\gamma} \wedge 1} & T_W \wedge S^V \\ & \searrow \gamma \wedge 1 & \cong \uparrow \tilde{\gamma} & \nearrow \alpha \wedge 1 & \\ & & T_{W/H} \wedge G/H_+ \wedge S^V & & \end{array}$$

shows that  $\alpha \circ \gamma \underset{H}{\cong} id$ .

Next, we define

$$\delta: G/H_+ \wedge G_+ \longrightarrow G_+$$

by  $\delta(g_1 H, g_2) = g_2$ , if  $g_2 \in g_1 H$ , and  $*$  else, and consider the diagram

$$\begin{array}{ccccc}
 S^V \wedge_{G_+ \wedge H} E_W & \xrightarrow{t \wedge 1} & S^V \wedge G/H_+ \wedge_{G_+ \wedge H} E_W & \xrightarrow{1 \wedge \delta \wedge 1} & S^V \wedge_{G_+ \wedge H} E_W \\
 & & \cong \uparrow \xi & & \\
 1 \wedge \gamma & \searrow & & \nearrow & 1 \wedge \alpha \\
 & & S^V \wedge_{G_+ \wedge H} (G_+ \wedge_H E_W) & & 
 \end{array}$$

This is easily seen to commute, so we are left with showing that the composite

$$S^V \wedge G_+ \xrightarrow{t \wedge 1} S^V \wedge G/H_+ \wedge_{G_+} \xrightarrow{1 \wedge \delta} S^V \wedge G_+$$

is  $G \times H$ -homotopic to the identity. Here  $G$  acts diagonally from the left and  $H$  acts by right multiplication in the factor  $G_+$ . Now  $(1 \wedge \delta) \circ (t \wedge 1)$  is the map which collapses to a point a tubular neighborhood of  $G$  embedded in  $V \times G$  by  $g \mapsto (gH, g)$ . But this is  $G \times H$ -homotopic to the map which collapses a nbhd of  $G$  embedded by  $g \mapsto (o, g)$  via the linear homotopy given by  $g \mapsto (thg, g)$ . The latter is obviously  $G \times H$ -homotopic to the identity.  $\square$

Corollary 4.3.16 <sup>well-pointed</sup> let  $T$  be a  $G$ -spectrum indexed on a complete universe. Then there is a natural chain of weak  $G$ -equivalences

$$F(G/H_+, \Sigma^L T) \sim G/H_+ \wedge T$$

$\square$  This follows from 4.3.11 and 4.3.15.  $\square$

Let  $j: \mathcal{U}^G \hookrightarrow \mathcal{U}$  be the inclusion of the trivial universe and note that a  $G$ -spectrum  $D$  in  $G\mathcal{U}^G$  is the same as a spectrum with a  $G$ -action. We also say that  $D$  is a naive  $G$ -spectrum. Let  $T$  be in  $G\mathcal{U}$ . We write  $\pi_* T$  for  $\pi_*(i^*T)$  and  $|T|$  for  $i^*T$  with its  $G$ -action forgotten.

Proposition 4.3.17 Let  $G$  be a finite group and let  $T$  be a well-pointed  $G$ -spectrum indexed on a complete universe  $\mathcal{U}$ . Then there is a strongly convergent right half plane spectral sequence

$$E_{k,l}^2 = H_k(G; \pi_l T) \Rightarrow \pi_{k+l}(EG_+ \wedge T)^G$$

from the group homology of the  $G$ -module  $\pi_* T$ .

Pf Since  $T$  is well-pointed the  $G$ -skeleton filtration of  $EG$  gives rise to an exact couple

$$D_{k,l} = \pi_{k+l}(EG_+^{(k)} \wedge T)^G$$

$$E_{k,l} = \pi_{k+l}(EG^{(k)} / EG^{(k-1)} \wedge T)^G$$

and thus a spectral sequence for  $\pi_*(EG_+ \wedge T)^G$ ; we determine the  $E^2$ -term. As our model for  $EG$  we may take the thick realization of the bar-construction  $B.(G, G, *)$  with  $G$  acting in the

first variable. Then as a  $G$ -space

$$EG^{(k)} / EG^{(k-1)} \cong \sum_k (G \times \underbrace{G \times \dots \times G}_k)_+$$

so by 4.3.16

$$(EG^{(k)} / EG^{(k-1)} \wedge T)^G \sim F(G_+, \underbrace{G \times \dots \times G}_k \wedge \Sigma^k T)^G$$

$$\xrightarrow[\cong]{ev_1} \underbrace{G \times \dots \times G}_k \wedge \Sigma^k |T|.$$

Taking  $\pi_{k+l}$  we get

$$E_{k,l} \cong \underbrace{\mathbb{Z}G \otimes \dots \otimes \mathbb{Z}G}_k \otimes \pi_l T.$$

Now  $[k] \mapsto E_{k,l}$  is a simplicial abelian group and we claim that in fact

$$E_{\bullet,l} \cong B(\mathbb{Z}G, \mathbb{Z}G, \mathbb{Z}) \otimes \pi_l T, \quad (4.3.18)$$

where  $B(\mathbb{Z}G, \mathbb{Z}G, \mathbb{Z})$  is viewed as a right  $\mathbb{Z}G$ -module via the antipode  $\mathbb{Z}G \rightarrow \mathbb{Z}G^{\text{op}}$ . This will prove that the  $E^2$ -term is as stated. It follows by naturality that  $(*)$  commutes with all simplicial operators other than  $d_0$ . Let  $S = \underbrace{G \times \dots \times G}_{k-1} \wedge \Sigma^k T$ . We must then evaluate

$$(G_+ \wedge \underbrace{G \times \dots \times G}_{k-1} \wedge S)^G \rightarrow (G_+ \wedge S)^G$$

under the identification in 4.3.16. We can do



this one  $g \in G$  at a time. Let  $r_g: G \rightarrow G$  denote right multiplication by  $g$ . One may show that the diagram

$$\begin{array}{ccc} F(G_+, S) & \xrightarrow{\tau} & G_+ \wedge S \\ \downarrow (r_{g^{-1}})^* & & \downarrow g \wedge 1 \\ F(G_+, S) & \xrightarrow{\tau} & G_+ \wedge S \end{array}$$

is  $G$ -homotopy commutative, cf. [HM], 8.1. Taking fixed points and evaluating in  $1 \in G$ , we get

$$\begin{array}{ccc} F(G_+, S)^G & \xrightarrow{\text{ev}_1} & |S| \\ \downarrow (r_{g^{-1}})^* & & \downarrow g^{-1} \\ F(G_+, S)^G & \xrightarrow{\text{ev}_1} & |S| \end{array}$$

which shows that (4.3.18) commutes with  $d_0$ .  $\square$

Definition 4.3.19 For  $T$  in  $G\mathcal{U}$ , we write

$$H^*(G; T) = (EG_+ \wedge T)^G$$

and call it the Borel homology spectrum.

Let  $G$  be a finite group and let  $\mathcal{U}$  be a complete  $G$ -universe. Suppose that  $S \rightarrow T$  is a map in  $G\mathcal{U}$  between well-pointed  $G$ -spectra and that the induced map  $|S| \xrightarrow{\sim} |T|$  is a w.e. Then by 4.3.17, so is

$$H_*(G; S) \xrightarrow{\sim} H_*(G; T) \quad (4.3.20)$$

The counit  $\varepsilon: i_* i^* T \rightarrow T$  is a non-equivariant

weak equivalence whenever  $T$  is well-pointed, and therefore induces a w.e. on  $H_1(G; -)$ . In other words  $H_1(G; T)$  only depends on  $i^*T$  in  $G\text{-}\mathcal{S}\mathcal{U}^G$ .

Remark 4.3.21 Let  $T$  be a well-pointed  $G$ -spectrum. Then there is a weak equivalence ( $G$  finite)

$$H_1(G; T) \sim EG_+ \wedge_G i^*T$$

given by an equivariant transfer map. Here the right hand side is the spectrum associated with the prespectrum  $V \mapsto EG_+ \wedge_G T_V$ . This is proved in [LMS], but the reader should first look at the proof in [Adams] of the case where  $T$  is a suspension spectrum:  $H_1(G; \Sigma^\infty X) \sim \Sigma^\infty EG_+ \wedge_G X$ .  $\square$

Let  $\tilde{E}G$  be the reduced mapping cone of the map  $EG_+ \rightarrow S^0$  which collapses  $EG$  to the non-basepoint in  $S^0$  such that we have a cofibration sequence of pointed  $G$ -spaces

$$EG_+ \rightarrow S^0 \rightarrow \tilde{E}G. \quad (4.3.22)$$

Alternatively,  $\tilde{E}G$  is the unreduced suspension of  $EG$ . It is a (non-equivariantly) contractible space and

$$\tilde{E}G^H = S^0, \quad \text{if } \phi \neq H \subset G.$$

Now let  $T$  be a well-pointed  $G$ -spectrum indexed

on a complete universe. Smashing (4.3.22) with  $T$  we obtain a cofibration sequence in  $GSU$ ,

$$EG_+ \wedge T \rightarrow T \rightarrow \tilde{E}G \wedge T,$$

which is also a fibration sequence, i.e. the map from  $EG_+ \wedge T$  to the homotopy fiber of the right hand map, induced by the preferred null-homotopy of the composition, is a weak  $G$ -equivalence. Hence so is

$$(EG_+ \wedge T)^G \rightarrow T^G \rightarrow (\tilde{E}G \wedge T)^G,$$

where the spectrum on the left is  $H_*(G; T)$ . We turn to the right hand spectrum when  $G = C_{p^n}$  is a cyclic  $p$ -group.

Proposition 4.3.22 Let  $t$  be a well-pointed good  $C_{p^n}$ -prespectrum indexed on a complete universe and let  $T = Lt$ . Then there is a fibration sequence

$$H_*(C_{p^n}; T) \rightarrow T^{C_{p^n}} \xrightarrow{s} (\Phi^* T)^{C_{p^n}/C_p}$$

of non-equivariant spectra.

Pf Note that by 4.3.8 and 4.2.14

$$\Phi^*(\tilde{E}_{C_{p^n}} \wedge T) \cong \Phi^*((\tilde{E}_{C_{p^n}})^{C_p} \wedge T) \cong \Phi^* T$$

so the statement is equivalent to showing that

$$s: (\tilde{E}_{C_{p^n}} \wedge T)^{C_p} \xrightarrow{\sim} \Phi^*(\tilde{E}_{C_{p^n}} \wedge T)$$

is a weak  $C_{p^n}$ -equivalence. On  $V$ -th spaces,  $V \subset U^G$ ,

this is the map

$$\varinjlim_{W \subset U} (\Omega^{W-V}(\tilde{E}_{G^n} \wedge t_W))^{G^n} \xrightarrow{\text{res}} \varinjlim_{W \subset U} \Omega^{W^{G^n}-V}(\tilde{E}_{G^n} \wedge t_W)^{G^n}$$

induced from the inclusion of the  $G^n$ -fixed set  $W^{G^n}-V \subset W-V$ . The colimits are over closed inclusions so we may conclude that the map of colimits is a weak  $G^n$ -equivalence if it is so for each  $W$ . Now the inclusion  $S^{W^{G^n}-V} \hookrightarrow S^{W-V}$  is a  $G^n$ -fibration so we get a fibration

$$F(S^{W-V}/S^{W^{G^n}-V}, \tilde{E}_{G^n} \wedge t_W)^{G^n} \rightarrow F(S^{W-V}, \tilde{E}_{G^n} \wedge t_W)^{G^n} \\ \downarrow \text{res} \\ F(S^{W^{G^n}-V}, \tilde{E}_{G^n} \wedge t_W)^{G^n}$$

and we wish to show that the fiber is contractible. We may do this by induction over the  $G^n$ -cells in  $S^{W-V}/S^{W^{G^n}-V}$ . But since  $W^{G^n}-V \subset W-V$  is the singular set, i.e. the complement is free, all the  $G^n$ -cells in  $S^{W-V}/S^{W^{G^n}-V}$  are free, i.e. of the form  $G^n \times D^n$ . Hence in the induction step we consider

$$F(V \times S^n \wedge G^n_+, \tilde{E}_{G^n} \wedge t_W)^{G^n} \cong \prod_{\alpha} F(S^n, \tilde{E}_{G^n} \wedge t_W)$$

which is contractible, because  $\tilde{E}_{G^n}$  is non-equivariantly contractible and  $t_W$  is well-pointed.  $\square$

### 5.1 Cyclotomic spectra

Let  $\Gamma$  be a (topological) group and recall the cyclic bar construction  $N_{\bullet}^{cy}(\Gamma)$  of (1.1.1). We write  $N^{cy}(\Gamma)$  for its realization, which we remember, is an  $S^1$ -space, compare (1.2.13). Now let  $G \subset S^1$  be the cyclic subgroup of order  $r$ , let  $p_G: S^1 \rightarrow S^1/G$  be the quotient map and recall from (1.3.18) the  $S^1$ -equivariant homeomorphism

$$|(sdr N_{\bullet}^{cy}(\Gamma))^{G_r}| \xrightarrow[\cong]{D_r} p_{G_r}^* N^{cy}(\Gamma)^{G_r},$$

where on the left  $sdr$  is the  $r$ -fold edgewise subdivision of § 1.3. In the case at hand, the  $r$ -fold diagonal provides an isomorphism of cyclic spaces

$$N_{\bullet}^{cy}(\Gamma) \xrightarrow[\cong]{\Delta_r} (sdr N_{\bullet}^{cy}(\Gamma))^{G_r}; \quad (5.1.1)$$

on  $k$ -simplices this is given by

$$(\gamma_0, \dots, \gamma_k) \mapsto (\gamma_0, \dots, \gamma_k, \dots, \gamma_0, \dots, \gamma_k) \quad (r \text{ times}).$$

Taking realization and composing with  $D_r$  we get an  $S^1$ -equivariant homeomorphism

$$\Gamma_r: p_{G_r}^* N^{cy}(\Gamma)^{G_r} \xrightarrow{\cong} N^{cy}(\Gamma) \quad (5.1.2)$$

If  $G$  is a compact Lie group,  $\mathcal{U}$  a  $G$ -universe and  $X$  a pointed  $G$ -space, then we write

$$\Sigma_{\mathcal{U}}^{\infty} X = L(V \mapsto \Sigma^V X)$$

and call it the suspension G-spectrum of  $X$  indexed on  $\mathcal{U}$ . Now let  $\mathcal{U}$  be the complete  $S^1$ -universe from (4.2.7). Of course (5.1.2) induces an isomorphism in  $S^1\mathcal{U}$ ,

$$\sum_{\mathcal{U}} \rho_{\mathcal{U}}^* N^{\text{cy}}(\Gamma)_+^{\mathcal{G}_r} \xrightarrow{\cong} \sum_{\mathcal{U}} N^{\text{cy}}(\Gamma)_+^{\mathcal{G}_r}. \quad (5.1.3)$$

We want to express the left-hand side as a construction in the  $S^1$ -spectrum  $T = \sum_{\mathcal{U}} N^{\text{cy}}(\Gamma)_+^{\mathcal{G}_r}$ .

Since  $T$  is the spectrification of a good  $S^1$ -prespectrum, we have from (4.3.8) an isomorphism

$$\bigoplus^{\mathcal{G}_r} T \cong \sum_{\mathcal{U}^{\mathcal{G}_r}} N^{\text{cy}}(\Gamma)_+^{\mathcal{G}_r},$$

in  $S^1/\mathcal{G}_r \mathcal{U}^{\mathcal{G}_r}$ . We may now attempt to apply  $\rho_{\mathcal{G}_r}^*$  space-wise. However, this gives us the suspension spectrum

$$\rho_{\mathcal{G}_r}^* \bigoplus^{\mathcal{G}_r} T \cong \sum_{\rho_{\mathcal{G}_r}^* \mathcal{U}^{\mathcal{G}_r}} \rho_{\mathcal{G}_r}^* N^{\text{cy}}(\Gamma)_+^{\mathcal{G}_r}$$

indexed on the universe  $\rho_{\mathcal{G}_r}^* \mathcal{U}^{\mathcal{G}_r}$  rather than  $\mathcal{U}$ . Indeed, if  $D$  is in  $S^1/\mathcal{G}_r \mathcal{U}^{\mathcal{G}_r}$  then the structure maps in  $D$  induces  $S^1$ -maps

$$\sum \rho_{\mathcal{G}_r}^* W - \rho_{\mathcal{G}_r}^* V \quad \rho_{\mathcal{G}_r}^*(D_V) \cong \rho_{\mathcal{G}_r}^*(\Sigma^{W-V} D_V) \rightarrow \rho_{\mathcal{G}_r}^*(D_W)$$

so the  $S^1$ -spaces  $\rho_{\mathcal{G}_r}^*(D_V)$  form an  $S^1$ -spectrum  $\rho_{\mathcal{G}_r}^* D$  indexed on  $\rho_{\mathcal{G}_r}^* \mathcal{U}^{\mathcal{G}_r}$ , i.e.,  $(\rho_{\mathcal{G}_r}^* D)_{\rho_{\mathcal{G}_r}^* V} = \rho_{\mathcal{G}_r}^*(D_V)$ . Therefore to get the left-hand side of (5.1.3) we need an isomorphism of universes

$$f_r : \mathcal{U} \xrightarrow{\cong} \rho_{\mathcal{G}_r}^* \mathcal{U}^{\mathcal{G}_r}. \quad (5.1.4)$$

Remember that

$$\mathcal{U} = \bigoplus_{n, \alpha \geq 0} \mathbb{C}(n)_\alpha$$

where  $\mathbb{C}(n) = \mathbb{C}$  with  $S^1$  acting by  $z \cdot v = z^n v$ . Now

$$\rho_r^* \mathbb{C}(n) = \begin{cases} \mathbb{C}(n) & , r \nmid n \\ 0 & , \text{else} \end{cases}$$

so

$$\rho_r^* \mathcal{U}^G = \bigoplus_{\substack{n, \alpha \geq 0 \\ r \mid n}} \mathbb{C}(n/r)_\alpha$$

and we may therefore choose  $f_r$  to be the isomorphism which maps the summand  $\mathbb{C}(n)_\alpha$ , indexed by  $n, \alpha$  in  $\mathcal{U}$ , to the summand  $\mathbb{C}(n)_\alpha$ , indexed by  $rn, \alpha$  in  $\rho_r^* \mathcal{U}^G$ , by the identity map. Now if  $E$  is in  $S^1 \rho_r^* \mathcal{U}^G$ , we define  $f_r^* E$  in  $S^1 \mathcal{U}$  by

$$(f_r^* E)_V = E_{f_r(V)}$$

with structure maps

$$\sum^{W-V} E_{f_r(V)} \xrightarrow[\cong]{\text{in } f_r} \sum^{f_r(W)-f_r(V)} E_{f_r(V)} \xrightarrow{\sigma} E_{f_r(W)}$$

Finally, if  $D$  is in  $S^1/G \mathcal{U}^G$ , then we define

$$\rho_r^\# D = f_r^*(\rho_r^* D) \tag{5.1.5}$$

which is then an  $S^1$ -spectrum indexed on  $\mathcal{U}$ .

We can now rewrite (5.1.3) as

$$\rho_r^\# \bigoplus^G \left( \sum_n^\infty N^{\text{eq}}(\Gamma)_+ \right) \cong \sum_n^\infty N^{\text{eq}}(\Gamma)_+.$$

Suppose that  $T$  in  $\mathcal{S}^1\mathcal{U}$  is good, i.e., is the spectrification of a good  $\mathcal{S}^1$ -prespectrum  $t$ , cf. p. 94. Then 4.3.8 shows that there is an isomorphism

$$\Phi^{Gr_s/C_s}(\Phi^{C_s}T) \cong \Phi^{Gr_s}T$$

of  $\mathcal{S}^1/Gr_s$ -spectra indexed on  $\mathcal{U}^{Gr_s}$ . It induces an isomorphism

$$\rho_G^{\#} \Phi^{Gr}(\rho_{C_s}^{\#} \Phi^{C_s}T) \cong \rho_{Gr_s}^{\#} \Phi^{Gr_s}T$$

of  $\mathcal{S}^1$ -spectra indexed on  $\mathcal{U}$ . Let  $\mathcal{F}$  be the family of finite subgroups of  $\mathcal{S}^1$ . We call a map  $f: S \rightarrow T$  in  $\mathcal{S}^1\mathcal{U}$  an  $\mathcal{F}$ -equivalence if  $f^C: S^C \rightarrow T^C$  is a weak equivalence, for all  $C \in \mathcal{F}$ .

Definition 5.1.6 A cyclotomic spectrum is a good  $\mathcal{S}^1$ -spectrum indexed on  $\mathcal{U}$  together with an  $\mathcal{F}$ -equivalence

$$r_C: \rho_C^{\#} \Phi^C T \xrightarrow{\sim} T,$$

for every finite  $C \in \mathcal{S}^1$ , such that for every pair of finite subgroups, the diagram

$$\begin{array}{ccc} \rho_G^{\#} \Phi^{Gr}(\rho_{C_s}^{\#} \Phi^{C_s}T) & \cong & \rho_{Gr_s}^{\#} \Phi^{Gr_s}T \\ \downarrow \rho_G^{\#} \Phi^{Gr} r_{C_s} & & \downarrow r_{Gr_s} \\ \rho_G^{\#} \Phi^{Gr} T & \xrightarrow{r_G} & T \end{array}$$

strictly commutes.

We spell out the definition in the following



Lemma 5.1.7 Let  $t$  be a good  $S^1$ -prespectrum indexed on  $\mathcal{U}$ . Then  $T = Lt$  is cyclotomic if for every  $V \in \mathcal{U}$  and every  $C \in S^1$ , finite, there is an  $S^1$ -map

$$r_C: \rho_C^*(t_V)^C \rightarrow t_{f_C^{-1}\rho_C^*V}^C$$

subject to the following conditions:

(i) For every pair  $V \subset W \subset \mathcal{U}$ , the diagram

$$\begin{array}{ccc} \Sigma \rho_C^*(W-V)^C & \xrightarrow{\varepsilon \wedge f_C^{-1}} & \Sigma f_C^{-1}\rho_C^*(W-V)^C \\ \rho_C^*(t_V)^C & \xrightarrow{r_C} & t_{f_C^{-1}\rho_C^*V}^C \\ \downarrow \rho_C^*\sigma^C & & \downarrow \sigma \\ \rho_C^*(t_W)^C & \xrightarrow{r_C} & t_{f_C^{-1}\rho_C^*W}^C \end{array} \quad \text{commutes}$$

(ii) For every pair of subgroups  $C_S \subset C_{rs}$ ,

$$\begin{array}{ccc} \rho_{C_{rs}}^*(t_V)^{C_{rs}} & \xrightarrow{\rho_{C_r}^*(r_{C_S})^{C_r}} & \rho_{C_r}^*(t_{f_{C_S}^{-1}\rho_{C_S}^*V_{C_S}})^{C_r} \\ \downarrow r_{C_{rs}} & & \downarrow r_{C_r} \\ t_{f_{C_{rs}}^{-1}\rho_{C_{rs}}^*V}^{C_{rs}} & = & t_{f_{C_r}^{-1}\rho_{C_r}^*(f_{C_S}^{-1}\rho_{C_S}^*V_{C_S})}^{C_r} \end{array} \quad \text{commutes.}$$

(iii) For every  $V \in \mathcal{U}$ , the map

$$\lim_{W \subset \mathcal{U}} \Omega \rho_C^*W^C - f_C V \rho_C^*(t_W)^C \rightarrow \lim_{W \subset \mathcal{U}} \Omega f_C^{-1}\rho_C^*W^C - V t_{f_C^{-1}\rho_C^*W}^C$$

is an  $F$ -equivalence.

Suppose  $T$  is cyclotomic. Let map of 4.3.7 and the cyclotomic structure map gives rise to a map in  $S^1\mathcal{U}$

$$R_r : \rho_{C_r}^\# T^{C_r} \rightarrow \rho_{C_s}^\# T^{C_s} \quad (5.1.8)$$

which we call the  $r$ 'th restriction map. Concretely, it is the composite

$$\rho_{C_r}^\# T^{C_r} = \rho_{C_s}^\# (\rho_r^\# T^r)^{C_s} \xrightarrow{s_r} \rho_{C_s}^\# (\rho_r^\# \Phi_r T)^{C_s} \xrightarrow{r_r} \rho_{C_s}^\# T^{C_s}.$$

As a corollary of 4.3.22, we get

Theorem 5.1.9 Let  $T$  be a well-pointed cyclotomic spectrum and let  $p$  be a prime. Then there is a fibration sequence

$$H_*(G_p^n; T) \rightarrow T^{G_p^n} \xrightarrow{R_p} T^{G_p^{n-1}}$$

of non-equivariant spectra. □

Now let  $L$  be a symmetric ring spectrum and recall from 4.2.9 the topological Hochschild spectrum

$$T(L) = L t(L)^{\mathbb{Z}},$$

where  $t(L)$  is the  $S^1$ -prespectrum indexed on  $\mathcal{U}$  with

$$t(L)_V = \Delta^* THH(L; S^V),$$

and where  $(-)^{\mathbb{Z}}$  is the thickening functor of (4.2.5). In (3.2.3) we define an  $S^1$ -map  $\rho_C^* THH(L)^C \rightarrow THH(L)$  and it is straight forward to extend this to

give an  $S^1$ -map

$$\tau_c' : \Delta^* \rho_c^* \mathrm{THH}(L; X)^C \rightarrow \Delta^* \mathrm{THH}(L; \rho_c^* X^C)$$

Taking  $X = S^V$  and composing with the map induced from the isomorphism  $f_c^{-1} : \rho_c^* V^C \rightarrow f_c^{-1} \rho_c^* V^C$ , we get

$$\tau_c : \rho_c^* t(L)_V^C \rightarrow t(L)_{f_c^{-1} \rho_c^* V^C}^C \quad (5.1.10)$$

Since

$$\left( \varinjlim_{Z \subset V} \Sigma^{V-Z} t(L)_Z^C \right) \cong \varinjlim_{Z \subset V^C} \Sigma^{V^C-Z^C} t(L)_Z^C$$

by an argument similar to the one on p. 73, we get maps similar to (5.1.10) of the thickened prespectrum, and thus in turn, a map in  $S^1\mathcal{U}$

$$\tau_c : \rho_c^\# \Phi^C T(L) \rightarrow T(L) \quad (5.1.11)$$

Proposition 5.1.12 Let  $L$  be a symmetric ring spectrum and assume that  $L_n$  is well-pointed and  $(n-1)$ -connected. Then  $T(L)$  is a cyclotomic spectrum.

Pf We show that (5.1.11) is an  $F$ -equivalence, leaving it to the reader to verify that the diagram in 5.1.6, or equivalent the diagrams (i) and (ii) in 5.1.7, commute. We must show that (iii) in 5.1.7 holds for the thickened prespectrum  $t(L)^C$ . This will follow if we show that, for every  $C \in S^1$  finite, the connectivity of the maps

$$\left( \Omega_{\rho_c^* W^C}^{\rho_c^* W^C} \rho_c^* t(L)_V^C \right)^C \rightarrow \left( \Omega_{f_c^{-1} \rho_c^* W^C}^{f_c^{-1} \rho_c^* W^C} t(L)_{f_c^{-1} \rho_c^* W^C}^C \right)^C$$

induced from (5.1.10) tends to infinity with  $W$ . We take  $V=0$ , the general case being similar. Recalling the definition of  $\tau'_2$  from page 75, we see that the relevant thing to show is that the connectivity of the map

$$(\Omega_{\mathbb{P}_r}^{*W_r} | (sd_r THH(L; S^W)_*)^{C_r} |)^{C_s} \xrightarrow{res} (\Omega_{\mathbb{P}_r}^{*W_r} | THH(L; S\mathbb{P}_r^{*W_r})_* |)^{C_s}$$

tends to  $\infty$  with  $W$ . We subdivide further and consider the diagram

$$\begin{array}{ccc} | (sd_{rs} THH(L; S^W)_*)^{C_r} | & \xrightarrow{res} & | sd_s THH(L; S\mathbb{P}_r^{*W_r})_* | \\ \cong \downarrow D_r & & \cong \downarrow D_r \\ | (sd_r THH(L; S^W)_*)^{C_r} | & \xrightarrow{res} & | THH(L; S\mathbb{P}_r^{*W_r})_* | \end{array}$$

which commutes by naturality of  $D_r$  and because  $(sd_{rs} X)_*^{C_r} = sd_s (sd_r X)_*^{C_r}$ . Now, in the top row,  $C_s$  acts simplicially and 1.3.7 shows that this action corresponds via  $D_r$  to the action by  $C_s \subset S^1$  in the lower row. Next, 4.1.14 shows that the vertical maps in the diagram

$$(\Omega_{\mathbb{P}_r}^{*W_r} | (sd_{rs} THH(L; S^W)_*)^{C_r} |)^{C_s} \xrightarrow{res} (\Omega_{\mathbb{P}_r}^{*W_r} | sd_s THH(L; S\mathbb{P}_r^{*W_r})_* |)^{C_s}$$

$$\sim \uparrow \gamma \qquad \qquad \sim \uparrow \gamma$$

$$| (\Omega_{\mathbb{P}_r}^{*W_r} (sd_{rs} THH(L; S^W)_*)^{C_r} )^{C_s} | \xrightarrow{res} | (\Omega_{\mathbb{P}_r}^{*W_r} sd_s THH(L; S\mathbb{P}_r^{*W_r})_* )^{C_s} |$$

are weak equivalences, and we are reduced to show that the connectivity of the map of  $k$ -simplices tends to  $\infty$  with  $W$ . Now both the range and target of

$$(sd_{rs} THH(L; S^W)_k)^{C_r} \xrightarrow{res} sd_s THH(L; S\mathbb{P}_r^{*W_r})_k$$

is a homotopy colimit over  $I^{S(k+1)}$ . We want to

replace these by the corresponding homotopy colimits over the diagonally embedded subcategory  $I^{k+1} \subset (I^{k+1})^S$ , because then  $C_S$  will not act on the index category. We would also like to move  $\Omega P_{C_r}^* W^G$  past the homotopy colimit. Both of our wishes comes true up to weak equivalence. Indeed, this follows from the approximation lemma 3.1.15 and the fact the connectivity of the maps of  $C_t$ -fixed sets,  $C_t \subset C_S$ , induced from the maps in the limit systems, tends to  $\infty$  with the indices. This was part of the proof of 4.2.12. We are now reduced to showing that the connectivity of

$$\begin{aligned} & (\Omega P_{C_r}^* W^G F((S^{i_1} \wedge \dots \wedge S^{i_k})^{\wedge rs}, (L_{i_1} \wedge \dots \wedge L_{i_k})^{\wedge rs} \wedge S^W)^{C_r})^{C_S} \\ & \xrightarrow{res} (\Omega P_{C_r}^* W^G F((S^{i_1} \wedge \dots \wedge S^{i_k})^{\wedge rs}, (L_{i_1} \wedge \dots \wedge L_{i_k})^{\wedge rs} \wedge S P_{C_r}^* W^G)^{C_r})^{C_S} \end{aligned}$$

tends to  $\infty$  with  $W$  and  $(i_1, \dots, i_k)$ . Let  $i = i_1 + \dots + i_k$  and let  $R = R_{C_S}$ . We may then rewrite this map as

$$\begin{aligned} & F_{C_{rs}}(S^{W^G} \wedge S^{iR}, (L_{i_1} \wedge \dots \wedge L_{i_k})^{\wedge rs} \wedge S^W) \\ & \xrightarrow{res} F_{C_{rs}}(S^{W^G} \wedge S^{iR^G}, (L_{i_1} \wedge \dots \wedge L_{i_k})^{\wedge rs} \wedge S^W). \end{aligned}$$

This is a Hurewicz fibration with fiber

$$F_{C_{rs}}(S^{W^G} \wedge S^{iR} / S^{iR^G}, (L_{i_1} \wedge \dots \wedge L_{i_k})^{\wedge rs} \wedge S^W),$$

whose connectivity we may estimate by 4.1.7. Let  $K \subset C_{rs}$ . Then we have

$$\begin{aligned} \text{conn}((L_{i_1} \wedge \dots \wedge L_{i_k})^{\wedge rs} \wedge S^W)^K &= i |C_{rs} : K| + \dim(W^K) - 1 \\ &= i \dim(R^K) + \dim(W^K) - 1 \end{aligned}$$

and on the other hand,

$$\dim (S^{W^G} \wedge S^i R / S^i R^G)^K = \begin{cases} 0 & , K \supset G \\ \dim (W^{G-K}) + i \dim (R^K) & , K \not\supset G \end{cases}$$

If  $K \supset G$ , then the difference tends to  $\infty$  with  $i$ . If  $K \not\supset G$ , then the difference is  $\dim (W^K - W^{G-K}) - 1$  and this tends to  $\infty$  as  $W$  runs through the f.d. subrepr. of the complete universe  $U$ . Indeed, we can find a simple  $G_K$ -representation  $U$  which is fixed by  $K$  but not by the bigger group  $G_K$ , and adding copies of  $U$  to  $W$  we can make this difference as big as we like.  $\square$

We show in the appendix that  $T(L)$  is well-pointed such that theorem 5.1.9 applies to  $T(L)$  and give us the fundamental fibration sequence

$$H_1(C_{p^n}; T(L)) \xrightarrow{N} T(L)^{C_{p^n}} \xrightarrow{R} T(L)^{C_{p^{n-1}}}$$

## 5.2 The group of components

Let  $\mathcal{U}$  be our fixed complete  $S^1$ -universe and note that the assignment

$$V \mapsto THH(L; S^V), \quad V \subset \mathcal{U},$$

defines an object in  $S^1\mathcal{P}\mathcal{U}$ ; call this  $t(L)_k$ . Then  $t(L)_k$  is a cyclic object in  $S^1\mathcal{P}\mathcal{U}$ . Hence so is  $t(L)_k^\tau$  and spectrifying, we get  $T(L)_k$  in  $S^1\mathcal{S}\mathcal{U}$ . The realization

$$T'(L) = |T(L)_k|,$$

by which we mean the coequalizer in  $S^1\mathcal{S}\mathcal{U}$ ,

$$\bigvee_{[n] \rightarrow [m]} \Delta_+^n \wedge T(L)_m \rightrightarrows \bigvee_{[n]} \Delta_+^n \wedge T(L)_n \rightarrow |T(L)_k|$$

is then an  $S^1$ -object in  $S^1\mathcal{S}\mathcal{U}$ . We may view  $\mathcal{U}$  as an  $S^1 \times \{1\}$ -trivial  $S^1 \times S^1$ -universe and  $T'(L)$  as an object in  $(S^1 \times S^1)\mathcal{S}\mathcal{U}$ . Viewing  $\mathcal{U}$  as an  $S^1$ -universe via the diagonal  $\Delta: S^1 \rightarrow S^1 \times S^1$  gives us the  $S^1$ -universe we started out with and now

$$T(L) \cong \Delta^* T'(L) \tag{5.2.1}$$

as objects of  $S^1\mathcal{S}\mathcal{U}$ . Indeed, thickening and spectrification both commute with realization; compare 4.2.14.

Of course the  $G$ -fixed set of  $T'(L)$  is the realization of  $T(L)_k^G$ . However, this is of little interest to us; it is spectrum  $T(L)_k^G$  we are after. However, non-equivariantly  $T(L) = T'(L)$  and so the

skeleton filtration gives

Lemma 5.2.2 Suppose that each  $L_n$  is well-printed. Then there is a strongly convergent right half plane homology type spectral sequence

$$E_{s,t}^2 = H_s(\pi_t T(L)_0; d = \sum (-1)^i d_i) \Rightarrow \pi_{s+t} T(L). \quad \square$$

Let  $A$  be a unital associative ring and recall from §3.1 that  $A$  determines a symmetric ring spectrum  $\tilde{A}$  whose  $n$ th space  $\tilde{A}_n$  is an Eilenberg-MacLane space for  $A$  concentrated in degree  $n$ . An alternative and smaller model for  $\tilde{A}$  is given by

$$\tilde{A}_n = | A \langle \underbrace{S^1 \wedge \dots \wedge S^1}_n \rangle / A \langle * \rangle |,$$

where  $S^1 = \Delta^1 / \partial \Delta^1$  is the simplicial circle and  $A \langle - \rangle$  is the free  $A$ -module. In any case, we have the topological Hochschild spectrum  $T(\tilde{A})$  which we also write  $T(A)$ .

Lemma 5.2.3 There are isomorphisms

$$\pi_i T(A) \cong HH_i(A), \quad i=0,1,$$

and  $\pi_2 T(A)$  surjects onto  $HH_2(A)$ .

Prf By 4.2.3 and 3.1.15 we have

$$\pi_t T(A)_k \cong \varinjlim_V \pi_{t+|V|} THH(A; S^V) \cong \pi_t THH(A)_k,$$

and again by the approximation lemma 3.1.15, there



exists  $N$  such that if  $i_s \geq N$ ,  $s=0, 1, \dots, k$ , then

$$\pi_t \mathrm{THH}(A)_k \cong \pi_{t+i_0+\dots+i_k} (\tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_k}).$$

In particular,  $\pi_0 \mathrm{THH}(A)_k = A^{\otimes k+1}$  and  $\pi_t \mathrm{THH}(A)_0 = 0$ , for  $t > 0$ . Hence the  $E^1$ -term of the sp. seq. of 5.2.2 takes the form

|   |                 |                 |                 |    |
|---|-----------------|-----------------|-----------------|----|
|   |                 |                 |                 |    |
| 0 | ?               | ?               | ?               | ?  |
| A | $A^{\otimes 2}$ | $A^{\otimes 3}$ | $A^{\otimes 4}$ | .. |

with the  $d^1$ -differential in the bottom row being given by the usual Hochschild boundary. So  $E^2$  looks like

|                 |                 |                 |                 |     |
|-----------------|-----------------|-----------------|-----------------|-----|
|                 |                 |                 |                 |     |
| 0               | ?               | ?               | ?               | ..  |
| $\mathrm{HH}_0$ | $\mathrm{HH}_1$ | $\mathrm{HH}_2$ | $\mathrm{HH}_3$ | ... |

We read off that  $\pi_i T(A) = \mathrm{HH}_i(A)$ ,  $i=0, 1$ , and since the  $d^2$ -differential on  $E_{2,0}^2$  must vanish, that  $\pi_2 T(A)$  surjects onto  $\mathrm{HH}_2(A)$ .  $\square$

We next consider the spectral sequence of 4.3.17

$$E_{k,l}^2 = H_k(G_n; \pi_l T(A)) \Rightarrow \pi_{k+l} H_*(G_n; T(A)).$$

The group homology may be calculated using the standard resolution

$$0 \leftarrow \mathbb{Z} \xleftarrow{\varepsilon} \mathbb{Z}G_n \xleftarrow{1-t} \mathbb{Z}G_n \xleftarrow{N} \mathbb{Z}G_n \xleftarrow{1-t} \mathbb{Z}G_n \xleftarrow{N} \dots$$

where  $N = 1 + t + \dots + t^{p^n - 1}$  is the norm element. We find that for a  $G_n$ -module  $M$ ,  $H_*(G_n; M)$  is the homology of

$$0 \leftarrow M \xleftarrow{1-t} M \xleftarrow{N} M \xleftarrow{1-t} M \xleftarrow{N} \dots$$

Note that since the  $G_n$ -action on  $T(A)$  extends to an  $S^1$ -action,  $\pi_* T(A)$  is a trivial  $G_n$ -module so

$$H_k(G_n; \pi_* T(A)) \cong \begin{cases} \pi_* T(A) & , k=0 \\ \pi_* T(A)/p^n & , k \text{ odd} \\ p^n \pi_* T(A) & , k > 0, \text{ even.} \end{cases}$$

By lemma 5.2.3, the  $E^2$ -term of our sp. seq. starts out

|        |            |            |         |
|--------|------------|------------|---------|
| ?      | ?          | ?          |         |
| $HH_1$ | $HH_1/p^n$ | $p^n HH_1$ | $\dots$ |
| $HH_0$ | $HH_0/p^n$ | $p^n HH_0$ | $\dots$ |

so we have a possible differential  $d^2: E_{2,0}^2 \rightarrow E_{0,1}^2$ . It follows that the edge homomorphism

$$HH_0(A) \xrightarrow{\cong} \pi_0 H_0(G_n; T(A)) \quad (5.2.4)$$

and that there is an exact sequence

$$HH_1(A) \xrightarrow{\hookrightarrow} \pi_1 H_0(G_n; T(A)) \rightarrow HH_0/p^n \rightarrow 0.$$

In particular, the fundamental fibration sequence of 5.1.9 starts out

$$\dots \xrightarrow{\partial} HH_0(A) \xrightarrow{N} \pi_0 T(A)^{G_n} \rightarrow \pi_0 T(A)^{G_{n-1}} \rightarrow 0.$$

We consider  $N$  in a more general frame work.

Lemma 5.2.5 Let  $G$  be a finite group and let  $T$  be a well-pointed  $G$ -spectrum indexed on a complete universe  $\mathcal{U}$ . Then the composite

$$\pi_* T \xrightarrow{\iota} \pi_* H_*(G; T) \xrightarrow{N} \pi_* T^G \xrightarrow{F} \pi_* T,$$

where  $\iota$  is the edge-homomorphism of 4.3.17 and  $F$  is induced from the inclusion, is given by multiplication by the norm element  $N \in \mathbb{Z}G$ .

Pr The map in question is the composite

$$\begin{array}{ccccccc} |T| \xrightarrow[\cong]{\tilde{\iota}} F(G_+, T)^G & \xrightarrow{\tau} & (G_+ \wedge T)^G & \longrightarrow & (EG_+ \wedge T)^G \\ & & \downarrow \text{pr} \sim 1 & & \downarrow \text{pr} \sim 1 \\ & & T^G & = & T^G \hookrightarrow i^* T, \end{array}$$

where  $\tau$  is the equivalence of 4.1.16 and  $i: \mathcal{U}^G \hookrightarrow \mathcal{U}$  is the inclusion of the trivial universe. We recall that  $T^G$  really means  $(i^* T)^G$  such that our map is the composite of the maps in the top row and right hand column in the diagram

$$\begin{array}{ccccccc} |T| \xrightarrow{\tilde{\iota}} (i^* F(G_+, T))^G & \xrightarrow{\tau} & (i^* (G_+ \wedge T))^G & \longrightarrow & (i^* T)^G \\ \parallel & & \downarrow & & \downarrow \\ |T| \longrightarrow i^* F(G_+, T) & \xrightarrow{\tau} & i^* (G_+ \wedge T) & \xrightarrow{\text{pr} \sim 1} & i^* T \end{array}$$

We may instead evaluate the composite of the maps in the bottom row. Consider the adjoint

$$i_* |T| \longrightarrow F(G_+, T) \xrightarrow{\tau} G_+ \wedge T \xrightarrow{\text{pr} \sim 1} T,$$

which is a map in  $G\&\mathcal{U}$ . Let  $t: S^V \rightarrow G_+ \wedge S^V$  be

Thom-Pontryagin map from p. 110. Then the latter composite is the weak map

$$i_* |T| \xrightarrow{\sim} \Omega^V \Sigma^V i_* |T| \xrightarrow{\Omega^V f} \Omega^V \Sigma^V T \xleftarrow{\sim} T,$$

where  $f$  is the spectrification of the map in  $G.P.U$  which on  $W$ -th spaces is given by

$$\begin{aligned} |T_W G| \wedge S^{W-W^G} \wedge S^V &\xrightarrow{\mu \wedge 1} F(G_+, T_W G) \wedge S^{W-W^G} \wedge S^V \\ &\xrightarrow{\sigma \wedge 1} F(G_+, T_W) \wedge S^V \xrightarrow{1 \wedge t} F(G_+, T_W) \wedge G_+ \wedge S^V \\ &\xrightarrow{(ev, 1) \wedge 1} T_W \wedge G_+ \wedge S^V \xrightarrow{1 \wedge \rho \wedge 1} T_W \wedge S^V. \end{aligned}$$

This composite is equal to

$$\begin{aligned} |T_W G| \wedge S^{W-W^G} \wedge S^V &\xrightarrow{1 \wedge t} |T_W G| \wedge S^{W-W^G} \wedge G_+ \wedge S^V \\ &\xrightarrow{1 \wedge \tau \wedge 1} |T_W G| \wedge G_+ \wedge S^{W-W^G} \wedge S^V \xrightarrow{\mu \wedge 1} T_W G \wedge S^{W-W^G} \wedge S^V \\ &\xrightarrow{\sigma \wedge 1} T_W \wedge S^V \end{aligned}$$

which is the map in  $G.S.U$

$$\Sigma^V i_* |T| \xrightarrow{t \wedge 1} \Sigma^V i_* (G_+ \wedge |T|) \xrightarrow{\mu} \Sigma^V i_* i^* T \xrightarrow{\varepsilon} \Sigma^V T,$$

and the map we were interested in is then

$$\begin{aligned} |T| &\xrightarrow{\eta} i^* i_* |T| \xrightarrow{\sim} i^* \Omega^V \Sigma^V i_* |T| \xrightarrow{t \wedge 1} i^* \Omega^V \Sigma^V i_* (G_+ \wedge |T|) \\ &\xrightarrow{\mu} i^* \Omega^V \Sigma^V i_* i^* T \xrightarrow{\varepsilon} i^* \Omega^V \Sigma^V T \xleftarrow{\sim} i^* T. \end{aligned}$$

We are interested in the induced map on homotopy groups, so we consider this <sup>as</sup> a map of non-equivariant

spectra. But if  $\delta_g: G_+ \rightarrow S^0$  is the map which send  $g$  to the non-basepoint and the remaining group elements to the basepoint, then the composite  $S^V \xrightarrow{t} G_+ \wedge S^V \xrightarrow{\delta_g \wedge 1} S^V$  is clearly homotopic to the identity. Hence the map of homotopy groups becomes

$$\pi_* T \xrightarrow{N \otimes \text{id}} \mathbb{Z}G \otimes \pi_* T \xrightarrow{\mu} \pi_* T$$

or multiplication by  $N \in \mathbb{Z}G$  as claimed.  $\square$

Proposition 5.2.6 Let  $A$  be a unital associative ring. Then the sequence

$$0 \rightarrow \pi_0 H_1(G_{p^n}; T(A)) \xrightarrow{N} \pi_0 T(A)^{G_{p^n}} \xrightarrow{F} \pi_0 T(A)^{G_{p^{n-1}}} \rightarrow 0$$

is exact and the left hand group is isomorphic to  $HH_0(A)$  via the edge homomorphism of 4.3.17.

Prf We must show that  $N$  is injective. By the lemma the composite

$$HH_0(A) \xrightarrow[\cong]{\hookrightarrow} \pi_0 H_1(G_{p^n}; T(A)) \xrightarrow{N} \pi_0 T(A)^{G_{p^n}} \xrightarrow{F} \pi_0 T(A)$$

is multiplication by the norm element  $N \in \mathbb{Z}G_{p^n}$ , and since  $\pi_0 T(A)$  is a trivial  $G_{p^n}$ -module this is simply multiplication by  $p^n$ . This shows that  $N$  is 1-1 when  $HH_0(A)$  has no  $p$ -torsion.

In general, we choose a ring homomorphism  $B \rightarrow A$  as in the lemma below and recall that by 5.2.4 we have an exact sequence

$$HH_1(B) \rightarrow \pi_1 H_1(G_{p^n}; T(B)) \rightarrow HH_0(B)/p^n \rightarrow 0$$

and similar for  $A$ . It follows that

$$\pi_1 H_*(G_p^n; T(B)) \twoheadrightarrow \pi_1 H_*(G_p^n; T(A))$$

is onto. The obvious induction argument based on the diagram

$$\begin{array}{ccccc} \pi_1 H_*(G_p^n; T(B)) & \xrightarrow{N} & \pi_1 T(B)^{G_p^n} & \xrightarrow{R} & \pi_1 T(B)^{G_p^{n-1}} \xrightarrow{\partial} 0 \\ \downarrow & & \downarrow & & \downarrow \\ \pi_1 H_*(G_p^n; T(A)) & \xrightarrow{N} & \pi_1 T(A)^{G_p^n} & \xrightarrow{R} & \pi_1 T(A)^{G_p^{n-1}} \end{array}$$

shows that  $\pi_1 T(B)^{G_p^n} \rightarrow \pi_1 T(A)^{G_p^n}$  is surjective, for all  $n \geq 0$ . Here the map  $R$  in the top row is surjective by the special case. It follows that the same holds in the bottom row.

□

In the proof we used the following

Lemma 5.2.7 For any unital ring  $A$ , there exists a unital ring  $B$  and a ring homomorphism  $B \rightarrow A$  such that

$$HH_i^*(B) \twoheadrightarrow HH_i^*(A) \quad , \quad i=0,1,$$

is onto and such that  $HH_0(B) = B/[B, B]$  is a free abelian group.

Prf Let  $V = \mathbb{Z}\{A\}$  be the free associative ring gen. by the point set  $A$ . Then  $V/[V, V]$  is a free abelian group and  $HH_0(V) \rightarrow HH_0(A)$  is onto. However,  $HH_1(V) \rightarrow HH_1(A)$  need not be so. Suppose  $z \in A \otimes A$

is a Hochschild cycle,  $z = \sum_{i=1}^{n_z} a_i \otimes b_i$ , and let  $U_z$  be the associative unital ring on generators  $x_{i,z}, y_{i,z}$ ,  $i=1, \dots, n_z$ , subject to the relation

$$\sum_{i=1}^{n_z} x_{i,z} y_{i,z} - y_{i,z} x_{i,z} = 0.$$

Then there is a unique ring homomorphism  $U_z \rightarrow A$  which sends  $x_{i,z}$  to  $a_i$  and  $y_{i,z}$  to  $b_i$ . Now the element  $z' = \sum x_{i,z} \otimes y_{i,z} \in U_z \otimes U_z$  is a Hochschild cycle which is mapped to  $z$  under  $U_z \rightarrow A$ . We let  $B$  be the associative ring on generators  $a \in A$  and  $x_{i,z}, y_{i,z}$ , where  $z$  runs through all cycles in  $A \otimes A$  and  $i=1, \dots, n_z$ , subject to the relations above. We have ring homomorphisms  $B \rightarrow A$ ,  $V \rightarrow B$  and  $U_z \rightarrow B$  and the composites  $V \rightarrow B \rightarrow A$  and  $U_z \rightarrow B \rightarrow A$  are the maps first considered. Hence  $HH_i(B) \rightarrow HH_i(A)$  is onto for  $i=0, 1$ . We leave it to the reader to exhibit an additive basis for  $B/[B, B]$ .  $\square$

### 5.3 Witt vectors

Recall that an element  $x \in \mathbb{Z}_p$  in the ring of  $p$ -adic integers may be written uniquely as an infinite sum

$$x = \sum_{i=0}^{\infty} x_i p^i, \quad 0 \leq x_i < p,$$

with addition and multiplication given by the usual rules. Note that the only thing we use about our set of digits  $\{0, 1, \dots, p-1\} \subset \mathbb{Z}_p$  is that they form a set of representatives of the residue classes modulo  $p$ , i.e. are the image of a set section of the projection  $\mathbb{Z}_p \rightarrow \mathbb{F}_p$ . It is natural to ask if there exists a less ad hoc set of representatives. The answer is yes: recall that a commutative  $\mathbb{F}_p$ -algebra  $k$  is called perfect if the Frobenius endomorphism  $x \mapsto x^p$  is an automorphism. The terminology stems from the fact that for a field  $k$ , being perfect is equivalent to all algebraic extensions of  $k$  being separable.

Lemma 5.3.1 (Teichmüller) Let  $R$  be a commutative ring which is complete and Hausdorff in the  $p$ -adic topology and assume that  $k = R/pR$  is perfect. Then there is a unique multiplicative section

$$w: k \rightarrow R$$

of the projection  $\pi: R \rightarrow k$ .

pf We define  $w$  as follows: let  $\bar{x} \in k$  be a residue class mod  $p$ . Since  $k$  is perfect, there will be another



residue class  $\bar{y}$  such that  $\bar{y}p^n = \bar{x}$ . If  $y \in \bar{y}$  then of course  $yp^n \in \bar{x}$ . But more is true. If also  $z \in \bar{y}$  such that  $y \equiv z \pmod{p}$  then by an easy application of the binomial formula  $yp^n \equiv zp^n \pmod{p^{n+1}}$ . We have shown that  $\bar{x}$  contains an element, well-defined modulo  $p^{n+1}$ , which is a  $p^n$ 'th power. Since the  $p$ -adic topology is complete and Hausdorff it follows that there is a unique element  $w(\bar{x}) \in \bar{x}$  which is a  $p^n$ th power for all  $n \geq 0$ . Clearly  $w$  is multiplicative. For if  $x$  and  $y$  are  $p^n$ th powers, for all  $n$ , then so is  $xy$ . By definition  $w$  commutes with taking  $p^n$ th powers, for all  $n$ , and is the only such map.  $\square$

Assume  $R$  has characteristic 0. We can use the set of representatives  $w(b) \in R$  as our digits and write

$$x = \sum_{i=0}^{\infty} w(a_i) p^i, \quad a_i \in k.$$

The next question is then: how do we add and multiply? To answer this question it is more convenient instead to write  $x \in R$  as

$$x = \sum_{i=0}^{\infty} w(a_i p^{-i}) p^i, \quad a_i \in k. \quad (5.3.2)$$

The question may be restated: how do we add and multiply the "vectors"  $(a_0, a_1, a_2, \dots)$ ,  $a_i \in k$ ?

Let  $A$  be a commutative ring. The ring of  $p$ -typical Witt vectors in  $A$  is the set

$$W(A) = A^{\mathbb{N}_0}$$

with a new addition and multiplication characterized by the requirement that the ghost maps

$$w: W(A) \rightarrow A^{\mathbb{N}_0} \quad (5.3.3)$$

$$w_0 = a_0$$

$$w_1 = a_0^p + pa_1$$

⋮

$$w_n = a_0^{p^n} + pa_1^{p^{n-1}} + \dots + p^n a_n$$

⋮

be a natural transformation of functors from the category of commutative rings to itself. Here the right hand side of (5.3.3) is given the coordinate-wise ring structure. We proceed to show that this requirement defines a ring structure on  $W(A)$ .

Lemma 5.3.4 (Dwork) Suppose that  $A$  is equipped with an endomorphism  $\phi: A \rightarrow A$  such that  $\phi(x) \equiv x^p$  modulo  $pA$ . Then a sequence  $(x_0, x_1, \dots)$  is in the image of the ghost map  $\text{if and only if}$

$$x_n \equiv \phi(x_{n-1}) \pmod{p^n A}.$$

$\square$  Again note that by the binomial formula,  $\text{if } x \equiv y \pmod{pA}$  then  $x^{p^n} \equiv y^{p^n} \pmod{p^{n+1}A}$ . Calculate:

$$\begin{aligned}\phi(w_{n-1}) &= \phi(a_0^{p^{n-1}} + \dots + p^{n-1}a_{n-1}) = \phi(a_0)^{p^{n-1}} + p\phi(a_1)^{p^{n-2}} + \dots + p^{n-1}\phi(a_{n-1}) \\ &\equiv a_0^{p^n} + pa_1^{p^{n-1}} + \dots + p^{n-1}a_{n-1}^p \equiv w_n \pmod{p^n A}.\end{aligned}$$

Conversely, choose  $a_n$  s.t.  $p^n a_n = x_n - \phi(x_{n-1})$ .  $\square$

In the case of a polynomial algebra  $A = \mathbb{Z}[X_\alpha]$ , we get such a map  $\phi$  by letting  $\phi(f(X_\alpha)) = f(X_\alpha^p)$ . We let  $A = \mathbb{Z}[a_0, a_1, \dots; b_0, b_1, \dots]$  and note that since  $A$  has no  $p$ -torsion, the ghost map

$$W(A) \rightarrow A^{\mathbb{N}_0}$$

is injective, its image given by the lemma. Now the requirement  $x_n \equiv \phi(x_{n-1}) \pmod{p^n A}$  is stable under the ring operations, so the ghost map identifies  $W(A)$  with a subring of  $A^{\mathbb{N}_0}$ . Let  $s_i, p_i \in A$  be the unique polynomials such that

$$\begin{aligned}(a_0, a_1, \dots) + (b_0, b_1, \dots) &= (s_0, s_1, \dots) \\ (a_0, a_1, \dots) \cdot (b_0, b_1, \dots) &= (p_0, p_1, \dots)\end{aligned} \tag{5.3.5}$$

in this subring structure. Then if  $A$  is any ring, the requirement that  $W(-)$  be a functor, forces us to define the sum and product of two elements  $(a_0, a_1, \dots), (b_0, b_1, \dots) \in W(A)$  by substituting the elements  $a_0, a_1, \dots; b_0, b_1, \dots \in A$  for the variables in (5.3.5). The additive inverse is defined similarly; note:  $1 = (1, 0, 0, \dots)$ .

To verify the ring axioms, we again place ourselves in the universal case. For example, to show that

addition is associative, we let  $A = \mathbb{Z}[a_0, a_1, \dots; b_0, b_1, \dots; c_0, c_1, \dots]$  and consider the vectors  $a = (a_0, a_1, \dots)$ ,  $b = (b_0, b_1, \dots)$ ,  $c = (c_0, c_1, \dots)$ . Write  $s(a, b) = (s_0(a, b), s_1(a, b), \dots)$ . Then the ghost map  $W(A) \rightarrow A^{\mathbb{N}_0}$  maps the equation

$$s(s(a, b), c) = s(a, s(b, c))$$

to the equation

$$(w(a) + w(b)) + w(c) = w(a) + (w(b) + w(c))$$

which certainly holds. Since  $w$  is 1-1, so does the first equation, and now functoriality shows that the addition in  $W(A)$  is associative. Similar for the other axioms. (This may of course be formalized.) It is easy to work out the first few sum polynomials from 5.3.3; we find

$$s_0 = a_0 + b_0, \quad s_1 = a_1 + b_1 + \frac{a_0^p + b_0^p - (a_0 + b_0)^p}{p}$$

and urge the reader to try to find  $s_2$ . Also note that since  $w_n$  only depends on  $a_0, \dots, a_n$ , the same holds for  $s_n, p_n$ :

$$s_n = s_n(a_0, \dots, a_n; b_0, \dots, b_n)$$

$$p_n = p_n(a_0, \dots, a_n; b_0, \dots, b_n).$$

Let again  $R$  be as in 5.3.1 and assume  $\text{char}(R) = 0$ . We are ready to answer the question of how to do arithmetic with the digits  $w(k) \in R$ .

Theorem 5.3.6 The bijection

$$W(k) \xrightarrow{\cong} R, \quad (a_0, a_1, \dots) \mapsto \sum_{i=0}^{\infty} \omega(a_i^{p^{-i}}) p^i,$$

is an isomorphism of rings.

PP We must show:

$$\sum_{i=0}^{\infty} \omega(a_i^{p^{-i}}) p^i + \sum_{i=0}^{\infty} \omega(b_i^{p^{-i}}) p^i = \sum_{i=0}^{\infty} \omega(s_i(a, b)^{p^{-i}}) p^i.$$

It is enough to show that this holds modulo  $p^{n+1}$  for all  $n \geq 0$ . So fix  $n$  and let  $x_i = a_i^{p^{-i}}$ ,  $y_i = b_i^{p^{-i}}$ . Using that  $\omega(-)$  commutes with  $p^i$ th powers, we want to show that modulo  $p^{n+1}R$

$$\sum_{i=0}^n \omega(x_i)^{p^{n-i}} p^i + \sum_{i=0}^n \omega(y_i)^{p^{n-i}} p^i \equiv \sum_{i=0}^n \omega(s_i(x, y))^{p^{n-i}} p^i$$

or

$$w_n(\omega(x)) + w_n(\omega(y)) \equiv w_n(\omega(s(x, y)))$$

Since  $W(R) \xrightarrow{\omega} R^{\text{add}}_{\text{add}}$  is ~~additive~~, the left hand side is equal to  $w_n(s(\omega(x), \omega(y)))$ , so we are reduced to showing

$$w_n(s(\omega(x, y))) \equiv w_n(\omega(s(x, y))) \pmod{p^{n+1}R}$$

Now we saw in the proof of lemma 5.3.4 that this holds if

$$s(\omega(x, y)) \equiv \omega(s(x, y)) \pmod{pR}$$

Finally, this holds since  $\omega$  was a section of  $R^{\pi}k$ . Multiplicativity is proved analogously.  $\square$

Corollary 5.3.7  $W(\mathbb{F}_p) \cong \mathbb{Z}_p$

A discrete valuation ring (DVR) is a PID  $A$  with a unique non-zero prime ideal  $m \subset A$ . The field  $k = A/m$  is called the residue field. For every  $x \in A - \{0\}$ , there exists a unique natural number  $n$ , called the valuation of  $x$  and written  $v(x)$ , such that  $x \in m^n - m^{n+1}$ . We set  $v(0) = \infty$ . The units  $A^\times$  are the elements of valuation zero. The valuation gives rise to a (non-Archimedean) absolute value,

$$|x| = a^{-v(x)},$$

where  $a > 1$  is any real number, and  $A$  is called complete if the absolute value is complete, i.e. if Cauchy sequences converge. Alternatively,  $A$  is complete if the canonical map

$$A \rightarrow \varprojlim_n A/m^n$$

is an isomorphism. Suppose that  $A$  and  $k$  have unequal characteristic, i.e.  $\text{char}(A) = 0$ ,  $\text{char}(k) = p > 0$ . Then  $\mathbb{Z} \subset A$  and the number  $e = v(p)$  is called the absolute ramification index;  $A$  is called absolutely unramified, if  $e = 1$ , i.e.  $m = (p)$ .

Corollary 5.3.8 Let  $A$  be an absolutely unramified complete discrete valuation ring with perfect residue field  $k$ . Then  $A \cong W(k)$ .  $\square$

See [Serre] for a fuller discussion.

We return to  $W(A)$  for  $A$  a general commutative ring. Let  $a = (a_0, a_1, \dots) \in W(A)$  be a Witt vector. The  $a_i$ 's are called the Witt coordinates of  $a$  and we call the  $w_i$ 's the ghost coordinates of  $a$ . We have two operators, Frobenius and Verschiebung, on  $W(A)$  characterized by

$$F(w_0, w_1, w_2, \dots) = (w_1, w_2, \dots) \quad (5.3.9)$$

$$V(a_0, a_1, a_2, \dots) = (0, a_0, a_1, \dots).$$

To define  $F$ , we place ourselves in the universal case  $A = \mathbb{Z}[a_0, a_1, \dots]$ . We want  $F: W(A) \rightarrow W(A)$  to satisfy  $w_n(Fa) = w_{n+1}(a)$ . Let  $a = (a_0, a_1, \dots)$ . Then

$$w_n(Fa) = w_{n+1}(a) \equiv \phi(w_n(a)) = \phi(w_{n-1}(Fa)) \pmod{p^{n+1}A}$$

so by lemma 5.3.4 there exist polynomials  $f_i(a) \in A$  such that  $w_n(f_0, \dots, f_n) = w_{n+1}(a_0, \dots, a_{n+1})$ . We then define  $F$  for any ring  $A$  by

$$F(a_0, a_1, \dots) = (f_0(a), f_1(a), \dots). \quad (5.3.10)$$

Note that  $f_n$  only depends on the variables  $a_0, \dots, a_{n+1}$ . We may rewrite the defining equation 5.3.9 as

$$f_0^{p^n} + p f_1^{p^{n-1}} + \dots + p^n f_n = a_0^{p^{n+1}} + p a_1^{p^n} + \dots + p^n a_n^{p^1} + p^{n+1} a_{n+1}$$

from which we get by induction that

$$f_n(a) \equiv a_n^p \pmod{p}. \quad (5.3.11)$$

Lemma 5.3.11 For any ring  $A$ ,  $F: W(A) \rightarrow W(A)$  is a ring homomorphism,  $V: W(A) \rightarrow W(A)$  is additive and in addition

$$x \cdot Vy = V(Fx \cdot y) \quad (\text{Frobenius reciprocity})$$

$$FVx = p \cdot x$$

$$VFx = V(1) \cdot x$$

Moreover, if  $A$  is an  $\mathbb{F}_p$ -algebra, then  $F = W(\sigma)$ , where  $\sigma: A \rightarrow A$  is the Frobenius on  $A$ , and  $V(1) = p$ .

PF By considering the universal case it is enough to show that the claims hold in ghost coordinates. Let us prove Frobenius reciprocity. We calculate  $V$  in ghost coordinates:

$$w_n Va = 0 + pa_0^{p^{n-1}} + \dots + p^n a_{n-1} = p w_{n-1} a$$

or

$$V(w_0, w_1, \dots) = (0, pw_0, pw_1, \dots).$$

Note by the way that this is clearly additive showing that  $V$  is additive. Now let  $(w_0, w_1, \dots)$  be the ghost coordinates of  $x$  and  $(v_0, v_1, \dots)$  those of  $y$ .

Then  $x \cdot Vy$  has ghost coordinates

$$(0, w_1 \cdot pv_0, w_2 \cdot pv_1, \dots),$$

and  $Fx \cdot y$  has ghost coordinates  $(w_1 \cdot v_0, w_2 \cdot v_1, \dots)$ , so  $V(Fx \cdot y)$  has ghost coordinates  $(0, pw_1 \cdot v_0, pw_2 \cdot v_1, \dots)$ .

This proves that  $x \cdot Vy = V(Fx \cdot y)$ . The other claims are proved similarly.  $\square$



The Teichmüller map is defined by

$$\omega: A \rightarrow W(A), \quad \omega(x) = (x, 0, 0, \dots) \quad (5.3.12)$$

An argument similar to the lemma shows that  $\omega$  is multiplicative but not additive, and that every element  $a = (a_0, a_1, \dots) \in W(A)$  may be written

$$a = \sum_{i=0}^{\infty} V^i(\omega(a_i)). \quad (5.3.13)$$

This is a non-trivial statement because addition in  $W(A)$  is usually not componentwise. Finally, it follows from Frobenius reciprocity that the additive subgroup  $V^n W(A) \subset W(A)$  is an ideal. We call the quotient

$$W_n(A) = W(A) / V^n W(A) \quad (5.3.14)$$

the ring of ( $p$ -typical) Witt vectors of length  $n$  in  $A$ . It may be identified with the set of sequences  $(a_0, a_1, \dots, a_{n-1})$  of length  $n$  with addition and multiplication given by the first  $n$  sum and product polynomials. It follows  $W(A)$  is the limit over the restriction maps  $R: W_n(A) \rightarrow W_{n-1}(A)$

$$R(a_0, a_1, \dots, a_{n-1}) = (a_0, \dots, a_{n-2}). \quad (5.3.15)$$

The Frobenius and Verschiebung operators restrict to give  $F: W_n(A) \rightarrow W_{n-1}(A)$ ,  $V: W_n(A) \rightarrow W_{n+1}(A)$  and one easily calculates that

$$w_i = R^{n-i} F^i: W_n(A) \rightarrow A. \quad (5.3.16)$$

## 5.4 Mackey functors

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. By a bifunctor from  $\mathcal{C}$  to  $\mathcal{D}$  we mean a pair  $M = (M^*, M_*)$  of functors from  $\mathcal{C}$  to  $\mathcal{D}$  with  $M^*$  contravariant and  $M_*$  covariant and such that  $M^*(c) = M_*(c)$ , for all  $c \in \text{ob } \mathcal{C}$ . We write  $M(c)$  for the common value. Let  $G$  be a finite group.

Definition 5.4.1 (Dress) A bifunctor  $M$  from the category of finite  $G$ -sets to the category of abelian groups is called a Mackey functor if

i) for every  $X, Y$ , the inclusions  $X \xrightarrow{i} X \amalg Y \xleftarrow{j} Y$  induces a sum diagram

$$M(X) \begin{matrix} \xrightarrow{i_*} \\ \xleftarrow{i^*} \end{matrix} M(X \amalg Y) \begin{matrix} \xleftarrow{j^*} \\ \xrightarrow{j_*} \end{matrix} M(Y)$$

ii) for every pull-back diagram

$$\begin{array}{ccccc} X' & & \xrightarrow{g'} & & X \\ f' \downarrow & \text{p.b.} & & & \downarrow f \\ Y' & & \xrightarrow{g} & & Y \end{array}$$

one has  $f^* g_* = g'_* f'^*$

A Mackey functor  $U$  is called a Green functor if for every  $X$ ,  $U(X)$  is a ring, and for every  $G$ -map  $f: X \rightarrow Y$ ,  $f^*: U(Y) \rightarrow U(X)$  is a ring homomorphism and  $f_*: U(X) \rightarrow U(Y)$  a map of  $U(X)$ -bimodules.

Example 5.4.2 Let  $A(S)$  be the Grothendieck construction on the commutative monoid of finite  $G$ -sets over  $S$  under disjoint union. Product over  $S$  makes  $A(S)$  a ring. If  $f: S \rightarrow T$  is a  $G$ -map then pull-back along  $f$  defines a ring homomorphism  $f^*: A(T) \rightarrow A(S)$  and composition with  $f$  defines an additive map  $f_*: A(S) \rightarrow A(T)$ . One easily verifies that this makes  $A$  into a Green functor. We call this the Burnside ring functor. It is the initial example of a Green functor: If  $U$  is another Green functor then the maps

$$A(S) \rightarrow U(S), \quad (x \xrightarrow{f} S) \mapsto f_* f^* 1_S$$

form a map of Green functors and it is the unique such map.

We may of course replace the category of abelian groups in 5.4.1 by any additive category. Now a bifunctor  $M: G\text{-sets} \rightarrow \mathcal{A}$  determines a bifunctor

$$M^{\text{op}}: G\text{-sets} \rightarrow \mathcal{A}^{\text{op}}, \quad (M^{\text{op}})^* = (M_*)^{\text{op}}, \quad (M^{\text{op}})_* = (M^*)^{\text{op}}.$$

Clearly the notion of a Mackay functor is self-dual such that  $M$  is a Mackay functor if and only if  $M^{\text{op}}$  is one.

Let  $G$  be a finite group. The orbit category  $Or(G)$  is the full subcategory of the category of  $G$ -sets and  $G$ -maps whose objects are the canonical orbits  $G/H$ ,  $H \subset G$  a subgroup. Suppose  $f: G/H \rightarrow G/K$  is a  $G$ -map; then  $f(H) = aK$  for some  $a \in G$ , and we then have  $f(gH) = gaK$ ,  $g \in G$ . In particular, we must have  $haK = aK$  or

$$a^{-1}Ha \subset K.$$

Conversely, for every such  $a \in G$ , we define  $\hat{a}: G/H \rightarrow G/K$  to be the  $G$ -map  $\hat{a}(gH) = gaK$ . We note that  $\hat{a}$  decomposes

$$\begin{array}{ccc} G/H & \xrightarrow{\pi_H^{aKa^{-1}}} & G/aKa^{-1} \\ \downarrow \Gamma_a & \searrow \hat{a} & \downarrow \Gamma_a \\ G/\hat{a}^{-1}Ha & \xrightarrow{\pi_{\hat{a}^{-1}Ha}^K} & G/K \end{array}$$

where  $\pi$  is the proj. and  $\Gamma_a$  right multiplication.

Lemma 5.4.3 The morphisms in  $Or(G)$  are generated by the canonical projections  $\pi_H^K: G/H \rightarrow G/K$ ,  $H \subset K$ , and by right multiplication  $\Gamma_a: G/H \rightarrow G/\hat{a}^{-1}Ha$ ,  $\Gamma_a(gH) = gHa = ga(\hat{a}^{-1}Ha)$ .  $\square$

In particular,  $\text{End}_{Or(G)}(G/H) \cong WH$ ;  $\Gamma_a \mapsto \hat{a}^{-1}H$ .

Lemma 5.4.4 A Mackey functor determines and is determined by a bifunctor  $M: \text{Or}(G) \rightarrow \mathcal{A}b$  such that

i) for every isomorphism  $i$ ,  $i_* i^* = \text{id}$

ii) the double coset formula holds: If  $H, H' \subset K$  and  $K = \sqcup_i H x_i H'$  then

$$\text{res}_H^K \circ \text{ind}_{H'}^K = \sum_i \text{ind}_{H \cap x_i H' x_i^{-1}}^H \circ \text{res}_{H \cap x_i H' x_i^{-1}}^{x_i H' x_i^{-1}} \circ \Gamma_{x_i}^*$$

where  $\text{res}_H^K = (\pi_H^K)^*$ ,  $\text{ind}_H^K = (\pi_H^K)_*$ .

Prf Suppose  $M$  is the restriction of a Mackey functor to  $\text{Or}(G)$ . Then i) above follows from 5.4.1 ii) applied to the pullback diagram

$$\begin{array}{ccc} G/H & = & G/H \\ \parallel & & \perp i \\ G/H & \xrightarrow{i} & G/H \end{array}$$

In order to prove ii) above we consider the pull-back diagram

$$\begin{array}{ccc} X & \xrightarrow{p_1} & G/H \\ p_2 \downarrow & & \downarrow \pi_H^K \\ G/H' & \xrightarrow{\pi_{H'}^K} & G/K \end{array} \quad \text{pull-back}$$

where

$$X = \{ (gH, g'H') \in G/H \times G/H' \mid gK = g'K \}.$$

The map  $K/H' \xrightarrow{i} X$ ;  $kH' \mapsto (H, kH')$  induces a sur-

jection  $K/H' \rightarrow G \backslash X$  onto the orbit space of the left diagonal  $G$ -action and clearly  $(H, kH')$  and  $(H, k'H')$  are in the orbit of and only of  $k^{-1}k' \in H$  such that we have a bijection

$$H \backslash K/H' \cong G \backslash X.$$

The equation  $(H, kH') = (gH, gkH')$  implies that  $g \in H$  and  $k^{-1}gk \in H'$  or  $g \in kH'k^{-1}$  such that

$$G_{(H, kH')} = Hn(kH'k^{-1}).$$

We have shown that as a  $G$ -set

$$X \cong \coprod_i G/Hn(x_i H' x_i^{-1}),$$

where the  $x_i$  are a set of representatives for the double cosets  $H \backslash K/H'$ . Now the double coset formula follows easily from 5.4.1 i) and ii).

Conversely, for every finite  $G$ -set  $X$ , we choose an isomorphism  $X \cong \coprod_{\alpha} G/H_{\alpha}$  and set  $M(X) = \bigoplus_{\alpha} M(G/H_{\alpha})$ . If also  $Y = \coprod_{\beta} G/K_{\beta}$  then a  $G$ -map  $f: X \rightarrow Y$  is the same as  $G$ -maps  $f_{\alpha\beta}: G/H_{\alpha} \rightarrow G/K_{\beta}$  where  $\beta$  is determined uniquely by  $\alpha$ . Now define

$$M(X) = \bigoplus_{\alpha} M(G/H_{\alpha}) \hookrightarrow M(G/H_{\alpha})$$

$$f^* \downarrow \uparrow f^*$$

$$f_{\alpha\beta}^* \downarrow \uparrow f_{\alpha\beta}^*$$

$$M(Y) = \bigoplus_{\beta} M(G/K_{\beta}) \hookrightarrow M(G/K_{\beta}).$$

By 5.4.4 i) this gives a bifunctor  $M: G\text{-sets} \rightarrow \mathcal{A}b$  well-defined up to a unique isomorphism.  $\square$

Example 5.4.5 Let  $A$  be the Burnside Green functor from 5.4.2. The category of  $G$ -sets over  $G/H$  is equivalent to the category of  $H$ -sets: If  $f: X \rightarrow G/H$  is a  $G$ -set over  $G/H$  then  $f^{-1}(H)$  is an  $H$ -set. Conversely, if  $Y$  is an  $H$ -set then  $G \times_H Y \rightarrow G/H$  is a  $G$ -set over  $G/H$ . Hence  $A(G/H)$  is the Grothendieck construction on finite  $H$ -sets under disjoint union. We also write  $A(H)$ .

Let  $k$  be a commutative ground ring and let  $k[X]$  be the permutation module generated by the  $G$ -set  $X$ . A  $G$ -map  $f: X \rightarrow Y$  induces  $f_*: k[X] \rightarrow k[Y]$  by functoriality, and we define

$$f^*: k[Y] \rightarrow k[X]; \quad f^*(y) = \sum_{x \in f^{-1}(y)} x.$$

Then  $k[-]$  is a Mackey functor. Let  $M$  be a  $kG$ -module. Then the functors

$$\mathrm{Tor}_*^{kG}(k[-], M), \quad \mathrm{Ext}_{kG}^*(k[-], M)$$

are again Mackey functors by functoriality and by the remark on p. 148. Evaluating these on the canonical orbit  $G/H$  we get group (co)homology

$$G/H \mapsto H_*(H; M), \quad G/H \mapsto H^*(H; M),$$

and for  $\pi_H^K: G/K \rightarrow G/H$ , the map going in the "wrong" direction is usually called the transfer or corestriction. We want to replace the role of  $k$  by the sphere spectrum.

Let  $\mathcal{U}$  be a complete  $G$ -universe and let  $X$  be a finite  $G$ -set. We recall the equivariant suspension spectrum

$$\Sigma_{\mathcal{U}}^{\infty} X_+ = L(S^V \mapsto X_+ \wedge S^V),$$

which we also will write  $\Sigma_G^{\infty} X_+$ . A  $G$ -map  $f: X \rightarrow Y$  of course induces a map of  $G$ -spectra

$$f_*: \Sigma_{\mathcal{U}}^{\infty} X_+ \rightarrow \Sigma_{\mathcal{U}}^{\infty} Y_+; \quad (5.4.6)$$

we define a transfer map in the opposite direction. Choose  $V \in \mathcal{U}$  and a  $G$ -map  $\phi: X \rightarrow V$  such that the product map

$$\begin{array}{ccc} X & \xrightarrow{f \times \phi} & Y \times V \\ f \searrow & & \swarrow \text{pr}_1 \\ & Y & \end{array}$$

is an embedding. Trivializing the normal bundle using the frame given by the zero section of  $\text{pr}_1$ , the Thom-Pontryagin construction yields a  $G$ -map

$$f^{\phi}: X_+ \wedge S^V \rightarrow Y_+ \wedge S^V$$

Suppose  $\psi: X \rightarrow W$  is another such map which yields a map  $f^{\psi}: X_+ \wedge S^V \rightarrow Y_+ \wedge S^V$ . Then so are all the maps

$h_t: X \rightarrow W \oplus V$ ,  $h_t = t\psi \oplus (1-t)\phi$ ,  $0 \leq t \leq 1$   
and we get a  $G$ -homotopy

$$\begin{array}{ccc} Y_+ \wedge S^{W \oplus V} & \xrightarrow{\Sigma^W f^{\phi}} & X_+ \wedge S^{W \oplus V} \\ & \text{///} & \\ & \Sigma^V f^{\psi} & \end{array}$$



It follows that we have a well-defined  $G$ -homotopy class

$$f^* : \Sigma_n^\infty Y_+ \rightarrow \Sigma_n^\infty X_+ \quad (5.4.7)$$

Lemma 5.4.8 For a pull-back diagram as in 5.4.1. ii), the diagram

$$\begin{array}{ccc} \Sigma_n^\infty X'_+ & \xrightarrow{g'_*} & \Sigma_n^\infty X_+ \\ f'^* \uparrow & & \uparrow f^* \\ \Sigma_n^\infty Y'_+ & \xrightarrow{g_*} & \Sigma_n^\infty Y_+ \end{array}$$

$G$ -homotopy commutes.

Pf Let  $\phi : X \hookrightarrow V$  be an embedding and note that also the map  $f' \times g' : X' \hookrightarrow Y' \times X$  is an embedding. Hence if we  $\phi' = \phi \circ g' : X' \rightarrow V$ , then  $g' \times \phi' : X' \hookrightarrow Y' \times V$  is an embedding such that  $f' \phi' : Y'_+ \wedge S^V \rightarrow X'_+ \wedge S^V$  represents  $f'^*$ . Now the diagram

$$\begin{array}{ccccc} X' & \xrightarrow{f' \times g'} & Y' \times X & \xrightarrow{1 \times \phi} & Y' \times V \\ \downarrow g' & & \downarrow g \times 1 & & \downarrow g \times 1 \\ X & \xrightarrow{f \times 1} & Y \times X & \xrightarrow{1 \times \phi} & Y \times V \end{array}$$

commutes and gives a commutative diagram

$$\begin{array}{ccc} X'_+ \wedge S^V & \xleftarrow{f' \phi'} & Y'_+ \wedge S^V \\ \downarrow g'^{-1} & & \downarrow g^{-1} \\ X_+ \wedge S^V & \xleftarrow{f \phi} & Y_+ \wedge S^V \end{array}$$

Since  $f \phi$  and  $f' \phi'$  represents  $f^*$  and  $f'^*$  we are done.  $\square$

Let  $T$  be a  $G$ -spectrum indexed on  $\mathcal{U}$  and note that by various adjunctions

$$\pi_* T^H \cong [\Sigma_{\mathcal{U}}^{\infty} G/H_+, T]_{G,*}, \quad (5.4.9)$$

the  $G$ -homotopy classes of maps in  $G\mathcal{U}$ . If  $f: G/H \rightarrow G/K$  is a  $G$ -map then the maps

$$\Sigma_{\mathcal{U}}^{\infty} G/H_+ \begin{matrix} \xrightarrow{f_*} \\ \xleftarrow{f^*} \end{matrix} \Sigma_{\mathcal{U}}^{\infty} G/K_+$$

induces maps

$$\pi_* T^H \begin{matrix} \xrightarrow{f^!} \\ \xleftarrow{f_!} \end{matrix} \pi_* T^K,$$

respectively. For example,  $(\pi_H^K)_!$  is the map induced from the inclusion, and if  $\text{att} \in WH$ , then  $(\tau_{\text{att}}^{-1})_!$  is the map induced from the action by  $\text{att}$  on  $T^H$ .

Corollary 5.4.10 If  $T$  is a  $G$ -spectrum indexed on a complete  $G$ -universe, then the bifunctor

$$G/H \mapsto \pi_* T^H$$

is a Mackey functor. Moreover, if  $T$  is a  $G$ -ring spectrum, then this is a Green functor.

Pf An argument similar to the one which shows that the homotopy class of  $f^*$  is well-defined shows that  $(f \circ g)^*$  is  $G$ -homotopic to  $g^* \circ f^*$ . Hence we have a bifunctor. It is a Mackey functor by 5.4.8. We leave it to reader to make sense of and to prove the last statement.  $\square$

Example 5.4.11 Let  $S_u^0$  be the sphere spectrum. Then one may use a jazzed up version of 5.1.9 to show that the canonical map

$$A(H) \xrightarrow{\cong} \pi_0(S_u^0)^H$$

from the Burnside Green functor is an isomorphism. We shall have occasion to do so when  $G = C_p$ .

Let  $T$  be a well-pointed  $G$ -spectrum indexed on  $u$ . Then we have canonical weak  $G$ -equivalences

$$X_+ \wedge T \xrightarrow{\sim} \Omega^V \Sigma^V X_+ \wedge T, \quad F(X_+, T) \xrightarrow{\sim} F(\Sigma^V X_+, \Sigma^V T),$$

and hence two well-defined weak  $G$ -homotopy types which we shall denote

$$\Sigma_u^\infty X_+ \wedge T, \quad F(\Sigma_u^\infty X_+, T),$$

respectively. These are functorial w.r.t.  $G$ -spectrum maps in both factors. By 5.4.8, the bifunctors

$$X \mapsto \pi_* (\Sigma_u^\infty X_+ \wedge T)^G, \quad X \mapsto \pi_* F(\Sigma_u^\infty X_+, T)^G \quad (5.4.12)$$

with maps induced from 5.4.6 and 5.4.7 are Mackey functors. The latter is the Mackey functor of 5.4.9.

Lemma 5.4.13 The weak  $G$ -equivalence of 4.3.15,

$$F(\Sigma_u^\infty X_+, T) \xrightarrow{\sim} \Sigma_u^\infty X_+ \wedge T$$

induces an isomorphism of the Mackey functors in 5.4.12.

Pf Let  $f: X \rightarrow Y$  be a  $G$ -map. We show that the

diagram

$$\begin{array}{ccc} F(\Sigma_n^\infty X_+, T) & \xrightarrow{\tau} & \Sigma_n^\infty X_+ \wedge T \\ \downarrow F(f^*, T) & & \downarrow f_* \wedge T \\ F(\Sigma_n^\infty Y_+, T) & \xrightarrow{\tau} & \Sigma_n^\infty Y_+ \wedge T \end{array}$$

is G-homotopy commutative leaving the dual case as an exercise. Let  $\pi_X: X \rightarrow \text{pt}$  be the only map. Then the equivalence  $\tau$  is given by the composite

$$\Sigma_n^\infty \text{pt}_+ \wedge F(\Sigma_n^\infty X_+, T) \xrightarrow{\pi_X^* \wedge 1} \Sigma_n^\infty X_+ \wedge F(\Sigma_n^\infty X_+, T) \xrightarrow{(1, \text{ev})} \Sigma_n^\infty X_+ \wedge T.$$

Since  $\pi_X = \pi_Y \circ f$ ,  $\pi_X^* \simeq f^* \circ \pi_Y^*$  so it is enough to show that the diagram

$$\begin{array}{ccc} \Sigma_n^\infty X_+ \wedge F(\Sigma_n^\infty X_+, T) & \xrightarrow{(1, \text{ev})} & \Sigma_n^\infty X_+ \wedge T \\ \uparrow f^* \wedge 1 & & \\ \Sigma_n^\infty Y_+ \wedge F(\Sigma_n^\infty X_+, T) & & \downarrow f_* \wedge T \\ \downarrow 1 \wedge F(f^*, T) & & \\ \Sigma_n^\infty Y_+ \wedge F(\Sigma_n^\infty Y_+, T) & \xrightarrow{(1, \text{ev})} & \Sigma_n^\infty Y_+ \wedge T \end{array}$$

G-homotopy commutes. We claim that the diagram

$$\begin{array}{ccc} \Sigma_n^\infty X_+ & \xrightarrow{(f_*, 1)} & \Sigma_n^\infty Y_+ \wedge \Sigma_n^\infty X_+ \\ \uparrow f^* & & \uparrow 1 \wedge f^* \\ \Sigma_n^\infty Y_+ & \xrightarrow{\Delta} & \Sigma_n^\infty Y_+ \wedge \Sigma_n^\infty Y_+ \end{array} \quad (5.4.14)$$

G-homotopy commutes. Indeed, this follows from 5.4.8

applied to the pull-back square

$$\begin{array}{ccc} X & \xrightarrow{(f,1)} & Y \times X \\ \downarrow f & & \downarrow 1 \times f \\ Y & \xrightarrow{\Delta} & Y \times Y \end{array}$$

Now we may identify the diagram we want to show is  $G$ -homotopy commutative with the outer hexagon in the diagram

$$\begin{array}{ccccc} \Sigma_{\mathcal{U}}^{\infty} X_+ \wedge F(\Sigma_{\mathcal{U}}^{\infty} X_+, T) & \xrightarrow{(f_*, 1) \wedge 1} & \Sigma_{\mathcal{U}}^{\infty} Y_+ \wedge \Sigma_{\mathcal{U}}^{\infty} X_+ \wedge F(\Sigma_{\mathcal{U}}^{\infty} X_+, T) & & \\ \uparrow f^* \wedge 1 & & \uparrow 1 \wedge f^* \wedge 1 & \searrow 1 \wedge \text{ev} & \\ \Sigma_{\mathcal{U}}^{\infty} Y_+ \wedge F(\Sigma_{\mathcal{U}}^{\infty} X_+, T) & \xrightarrow{\Delta \wedge 1} & \Sigma_{\mathcal{U}}^{\infty} Y_+ \wedge \Sigma_{\mathcal{U}}^{\infty} Y_+ \wedge F(\Sigma_{\mathcal{U}}^{\infty} X_+, T) & & \Sigma_{\mathcal{U}}^{\infty} Y_+ \wedge T \\ \downarrow F(f^*, T) & & & \nearrow 1 \wedge \text{ev} & \\ \Sigma_{\mathcal{U}}^{\infty} Y_+ \wedge F(\Sigma_{\mathcal{U}}^{\infty} Y_+, T) & \xrightarrow{\Delta \wedge 1} & \Sigma_{\mathcal{U}}^{\infty} Y_+ \wedge \Sigma_{\mathcal{U}}^{\infty} Y_+ \wedge F(\Sigma_{\mathcal{U}}^{\infty} Y_+, T) & & \end{array}$$

The upper left hand square  $G$ -htpy commutes by 5.4.14; the lower hexagonal diagram by definitions.  $\square$

Remark 5.4.15 In the proof of 4.3.18 we used without proof that the diagram

$$\begin{array}{ccc} F(G_+, T) & \xrightarrow{\tau} & G_+ \wedge T \\ \downarrow F(r_{g-1}, T) & & \downarrow r_g \wedge T \\ F(G_+, T) & \xrightarrow{\tau} & G_+ \wedge T \end{array}$$

$G$ -homotopy commutes. This is hereby established.

### 5.5 The component group II

Let  $A$  be a unital associative ring and let  $T(A)$  be the topological Hochschild spectrum of  $A$ . This, we recall, is an  $S^1$ -spectrum indexed on the complete  $S^1$ -universe  $\mathcal{U}$  of 4.2.7. Let  $G_N \subset S^1$  be a finite subgroup. Then  $\mathcal{U}$  is also a complete  $G_N$ -universe, and by neglect of structure,  $T(A)$  is a  $G_N$ -spectrum indexed on  $\mathcal{U}$ . By 5.4.10, we get a Mackey functor

$$G_N/G_N \mapsto \pi_* T(A)^{G_N}.$$

If  $\pi_S^{rs} : G_N/G_S \rightarrow G_N/G_{rs}$  is the projection, then we write

$$\pi_* T(A)^{G_{rs}} \begin{matrix} \xrightarrow{F_r} \\ \xleftarrow{V_r} \end{matrix} \pi_* T(A)^{G_S} \quad (5.5.1)$$

for the maps  $(\pi_S^{rs})^*$  and  $(\pi_S^{rs})_*$ , respectively, and call them the  $r$ 'th Frobenius and Verschiebung. We note that  $F_r$  is induced from the inclusion, and so in particular, does not depend on the choice of  $G_N \subset S^1$ . We ask the reader to verify that the same holds for the map  $V_r$ . Recall the restriction maps of 5.1.8.

Lemma 5.5.2 The following relations hold:

- i)  $R_r R_s = R_{rs}$ ,  $F_r F_s = F_{rs}$ ,  $V_r V_s = V_{rs}$
- ii)  $R_r F_s = F_s R_r$ ,  $R_r V_s = V_s R_r$
- iii)  $F_r V_r = r$ .

Pr The relations i) are obvious. To verify ii) recall that  $R_r$  is induced from a map  $\rho_r^\# T(A)^{C_r} \rightarrow T(A)$  in  $S^1 S U$ . It induces a map of Mackey functors

$$\pi_* T(A)^{C_{rt}} = \pi_*(\rho_r^\# T(A)^{C_r})^{C_t} \xrightarrow{R_r} \pi_* T(A)^{C_t};$$

whence ii). Finally, iii) follows from the double coset formula: to evaluate

$$\pi_* T(A)^{C_s} \xrightarrow{V_r} \pi_* T(A)^{C_{rs}} \xrightarrow{F_r} \pi_* T(A)^{C_s}$$

we use 5.4.4 ii) with  $K = C_{rs}$ ,  $H = H' = C_s$ . Then  $C_{rs} = \coprod_{i=0}^{r-1} C_s t^i C_s$ , where  $t \in C_{rs}$  is a generator, so we get

$$F_r V_r = \sum_{i=0}^{r-1} 1 \cdot t^i : \pi_* T(A)^{C_s} \rightarrow \pi_* T(A)^{C_s},$$

that is, multiplication by the relative norm element  $1+t+\dots+t^{r-1}$ . But  $t$  acts like the identity on  $\pi_* T(A)^{C_s}$  and iii) follows.  $\square$

Recall the maps  $\iota$  and  $\nu$  of 5.2.5. The previous lemma suggests the following

Lemma 5.5.3. The following diagram commutes:

$$\begin{array}{ccc} \pi_* H_*(C_{p^n}; T(A)) & \xrightarrow{N} & \pi_* T(A)^{C_{p^n}} \\ \nwarrow \iota & & \nearrow V_{p^n} \\ & \pi_* T(A) & \end{array}$$

Pr The claim is that the diagram below is homotopy commutative. This is precisely the statement of 5.4.13.

$$(\sum_n G_{p^{n+1}} \wedge T(A))^{G_p} \xrightarrow{p \wedge 1} (\sum_n G_p^n / G_{p^{n+1}} \wedge T(A))^{G_p}$$

$\sim \uparrow \tau$

$\sim \uparrow \tau$

$$F(\sum_n G_{p^{n+1}}, T(A))^{G_p} \xrightarrow{d_f^*} F(\sum_n G_p^n / G_{p^{n+1}}, T(A))^{G_p}$$

□

We may restate 5.2.6 as saying that we have exact sequences

$$0 \rightarrow \pi_0 T(A) \xrightarrow{V^n} \pi_0 T(A)^{G_p^n} \xrightarrow{R} \pi_0 T(A)^{G_p^{n-1}} \rightarrow 0,$$

where  $V = V_p$  and  $R = R_p$ . By 5.2.3,  $\pi_0 T(A) = \text{THH}_0(A)$ , which is equal to  $A$ , provided that  $A$  is commutative. We want to compare this sequence to the sequence

$$0 \rightarrow A \xrightarrow{V^n} W_{n+1}(A) \xrightarrow{R} W_n(A) \rightarrow 0.$$

To this end we need an analogue of the Teichmüller map  $w: A \rightarrow W_{n+1}(A)$ . Define a set map

$$\Delta_r: A \rightarrow \pi_0 T(A)^{G_r} \quad (5.5.4)$$

as the map of components induced by the composite

$$\text{THH}(A)_0 \xrightarrow{\Sigma_r} (\text{Set THH}(A)_0)^{G_r} \hookrightarrow |(\text{Set THH}(A)_0)^{G_r}| \xrightarrow[\cong]{D_r} \text{THH}(A)^{G_r}$$

and the w.e.  $\text{THH}(A)^{G_r} \simeq T(A)_0^{G_r}$  (zeroth space) of 4.2.12.

Here the map  $\Sigma_r$  sends  $f: S^0 \rightarrow L_0$  to the  $r$ -fold smash product

$$f \wedge \dots \wedge f: S^0 \wedge \dots \wedge S^0 \rightarrow L_0 \wedge \dots \wedge L_0,$$

and the middle map is the inclusion of the zero-skeleton.



Lemma 5.5.5 Let  $\pi: A \rightarrow \pi_0 T(A)$  be the projection. Then

$$R_r \circ \Delta_r a = \pi a, \quad F_r \circ \Delta_r a = \pi a^r.$$

Prf The diagram

$$\begin{array}{ccc} THH(A)_0 & \xrightarrow{\delta_r} & (sd_r THH(A)_0)^{h_r} \hookrightarrow |(sd_r THH(A)_0)^{h_r}| \\ & \searrow \text{id} & \downarrow \text{res} \qquad \qquad \downarrow \text{res} \\ & & THH(A)_0 \hookrightarrow |THH(A)_0| \end{array}$$

identifies  $R_r \circ \Delta_r$  with the inclusion of the 0-skeleton, and by 5.2.3 this is  $\pi$ . Next, we consider

$$\begin{array}{ccccc} THH(A)_0 & \xrightarrow{\delta_r} & (sd_r THH(A)_0)^{h_r} \hookrightarrow |(sd_r THH(A)_0)^{h_r}| & \xrightarrow[\cong]{D_r} & |THH(A)_0|^{h_r} \\ & \downarrow & \downarrow & & \downarrow \\ & sd_r THH(A)_0 \hookrightarrow |sd_r THH(A)_0| & \xrightarrow[\cong]{D_r} & & |THH(A)_0| \end{array}$$

We claim that  $D_r$  is homotopic to the simplicial map

$$D'_r: sd_r THH(A)_{k-1} \rightarrow THH(A)_{k-1}; \quad D'_r = d_0^{k(r-1)}$$

Granting this for the moment, we see that  $F_r \circ \Delta_r$  is homotopic to the composite

$$THH(A)_0 \xrightarrow{\delta_r} THH(A)_{r-1} \xrightarrow{d_0^{r-1}} THH(A)_0 \hookrightarrow |THH(A)_0|.$$

The composition of the first two maps induces the  $r$ 'th power map on  $\pi_0$ ; the inclusion of the 0-skeleton induces  $\pi$ . To verify the claim, we recall from 1.3.4 that

$$D_r: |sd_r Y| \rightarrow |Y| \text{ is given by } 1 \times d_r: Y_{rk-1} \times \Delta^{k-1} \rightarrow Y_{rk-1} \times \Delta^{rk-1}$$

where  $d_r: \Delta^{k-1} \rightarrow \Delta^{rk-1}$  is  $d_r(z) = \frac{1}{r}z \oplus \dots \oplus \frac{1}{r}z$  ( $r$  times),  $\Delta^{n-1} \subset \mathbb{R}^n$ . Consider the homotopy

$$d_{r,t}(z) = \frac{t}{r}z \oplus \dots \oplus \frac{t}{r}z \oplus \left( \frac{t}{r}z + (1-t)z \right) ; \quad 0 \leq t \leq 1,$$

- r - 1 -

Then  $d_{r,1} = d_r$  and  $d_{r,0} = (d_0)^{k(r-1)}: \Delta^{k-1} \rightarrow \Delta^{r(k-1)}$ . It follows that  $D_r$  is homotopic to the map  $D_r'$  given by  $1 \times d_{r,0}: Y_{rk-1} \times \Delta^{k-1} \rightarrow Y_{rk-1} \times \Delta^{rk-1}$ . Finally, in the realization this is equal to  $d_0^{k(r-1)} \times 1: Y_{rk-1} \times \Delta^{k-1} \rightarrow Y_{k-1} \times \Delta^{k-1}$ .  $\square$

Theorem 5.5.6 Let  $A$  be a commutative ring. Then the map

$$W_{n+1}(A) \xrightarrow[\cong]{I} \pi_0 T(A)^{G_p^n}$$

$$(a_0, \dots, a_n) \mapsto \sum_{i=0}^n \bigvee^i \Delta_{p^{n-i}} a_i$$

is an isomorphism of abelian groups.

Prf It follows from 5.2.6 and 5.5.3 that the sequence

$$0 \rightarrow \pi_0 T(A)^{G_p^{n-1}} \xrightarrow{V} \pi_0 T(A)^{G_p^n} \xrightarrow{R^n} \pi_0 T(A) \rightarrow 0$$

is exact, and 5.5.4 shows that  $\Delta_{p^n}$  is a set section of  $\mathbb{R}^n$ . Hence  $I$  is a set bijection and we must show that it is additive. We define

$$\bar{w}: \pi_0 T(A)^{G_p^n} \rightarrow \prod_{i=0}^n A, \quad \bar{w}_i = R^{n-i} F^i.$$

Then 5.5.2 and 5.5.5 shows that  $\bar{w} \circ I = w$ , the ghost

map. Indeed,

$$\begin{aligned} R^{n-i} F^i \left( \sum_{j=0}^n V^j \Delta_{p^{n-j}} a_j \right) &= \sum_{j=0}^i p^j R^{n-i} F^{i-j} \Delta_{p^{n-j}} a_j \\ &= \sum_{j=0}^i p^j F^{i-j} R^{n-i} \Delta_{p^{n-j}} a_j = \sum_{j=0}^i p^j F^{i-j} \Delta_{p^{i-j}} a_j = \sum_{j=0}^i p^j a_j^{p^{i-j}}. \end{aligned}$$

It follows that  $\bar{w}$  is a homomorphism when  $A$  has no  $p$ -torsion. For in this case,  $w = \bar{w} \circ I$  is injective and  $w$  and  $\bar{w}$  are homomorphisms. For general  $A$ , we choose a surjective ring homomorphism  $\tilde{A} \rightarrow A$  with  $\tilde{A}$   $p$ -torsion free and consider the diagram

$$\begin{array}{ccc} W_{n+1}(\tilde{A}) & \xrightarrow{I} & \pi_0 T(\tilde{A}) \mathbb{G}^n \\ \downarrow & & \downarrow \\ W_{n+1}(A) & \xrightarrow{I} & \pi_0 T(A) \mathbb{G}^n \end{array}$$

The vertical maps are both surjective. For as a set both functors are just an  $n+1$  fold product of the set underlying the ring. They are also homomorphisms. Now the top horizontal map is a homomorphism by the special case. It follows that the lower horizontal map is a homomorphism, too.  $\square$

Corollary 5.5.7  $\pi_0 T(\mathbb{F}_p) \mathbb{G}^n \cong \mathbb{Z}/p^{n+1} \mathbb{Z}.$

For any cyclotomic spectrum  $T$ , we define  $S^1$ -spectra indexed on  $\mathcal{U}$

$$TR(T) = \varprojlim_{\mathbb{R}} p_{\mathbb{G}}^{\#} T^{\mathbb{G}}$$

$$TR(T; p) = \varprojlim_{\mathbb{R}} p_{\mathbb{G}^n}^{\#} T^{\mathbb{G}^n},$$

(5.5.8)

the homotopy limit over the restriction maps, cf. 5.1.8 and p.75. Let  $i: \mathcal{U}^{S^1} \rightarrow \mathcal{U}$  be the inclusion of the trivial universe. The inclusion maps

$$F_r: i^* \rho_{Grs}^\# T^{Grs} \rightarrow i^* \rho_G^\# T^G$$

again satisfies  $F_r(z \cdot t) = z \cdot F_r(t)$ ,  $z \in S^1$ , and commutes strictly with the restriction maps. As in 3.2.5, the Frobenius maps define an action by the multiplicative monoid of natural numbers on the underlying non-equivariant spectrum of  $TR(T)$  and we define a spectrum indexed on  $\mathcal{U}^{S^1}$

$$TC(T) = |TR(T)|^{h\{F_r \mid r \in \mathbb{N}\}} \quad (5.5.9)$$

The map  $F = F_p$  induces a similar map on  $|TR(T;p)|$  and we define

$$TC(T;p) = |TR(T;p)|^{h\{F^n \mid n \in \mathbb{N}_0\}} \quad (5.5.10)$$

If  $T = T(L)$  for a symmetric ring spectrum  $L$ , we write  $TC(L)$  and  $TC(L;p)$  for these spectra.

Lemma 5.5.11 The diagram

$$TC(T;p) \rightarrow TR(T;p) \begin{array}{c} \xrightarrow{F} \\ \xrightarrow{id} \end{array} TR(T;p)$$

is a homotopy equalizer.

Pf Let  $\underline{\mathbb{N}}_0$  be the category with a single object whose monoid of endomorphisms is  $\mathbb{N}_0$ . We get a functor

from  $\underline{N}_0$  to spectra indexed on  $\mathcal{U}^{S^1}$  sending the object to  $TR(T;p)$  and the generator of the morphisms to  $F$ , and by definition  $TC(T;p)$  is the homotopy limit of this functor. In particular, there is a canonical projection  $pr$  from  $TC(T;p)$  to  $TR(T;p)$  and a canonical homotopy from  $pr$  to  $F \circ pr$ . This specifies a map

$$TC(T;p) \rightarrow \varprojlim_{\leftarrow} (TR(T;p) \xrightarrow[F]{F} TR(T;p))$$

and the statement of the lemma is that this is the spectral sequence of BKT for the homotopy groups of  $TC(T;p)$  takes the form

$$E_{s,t}^2 = \text{Ext}_{\mathbb{Z}[x]}^{-s}(\mathbb{Z}, \pi_t TR(T;p)) \Rightarrow \pi_{s+t} TC(T;p),$$

where  $x$  acts on  $\pi_* TR(T;p)$  through  $F$ . Indeed, the  $E^1$ -term may be identified with the cosimplicial abelian group

$$\text{Hom}_{\mathbb{Z}[x]}(B.(\mathbb{Z}[x], \mathbb{Z}[x], \mathbb{Z}), \pi_* TR(T;p)).$$

Hence the spectral sequence collapses to give a long-exact sequence

$$\cdots \xrightarrow{\partial} \pi_n TC(T;p) \rightarrow \pi_n TR(T;p) \xrightarrow{F-1} \pi_n TR(T;p) \xrightarrow{\partial} \cdots \quad (5.5.12)$$

The map in the lemma maps this to the corresponding sequence for the homotopy equalizer. Use 5-lemma.  $\square$

Corollary 5.5.13 Let  $A$  be a commutative ring. Then there is an isomorphism

$$I : W(A) \xrightarrow{\cong} \pi_0 TR(A;p)$$

such that  $IF = FI$ .

FF Recall Milner's exact sequence :

$$0 \rightarrow \varprojlim_n^{(1)} \pi_1 T(A)^{G_n} \rightarrow \pi_2 TR(A; p) \rightarrow \varprojlim_n \pi_2 T(A)^{G_n} \rightarrow 0.$$

We must show that the derived limit vanishes. But all the maps in the limit system are surjections by 5.2.6, so this follows from the Mittag-Leffler condition, c.f. [BK] p. 256. It is also not hard to give a direct argument in this case.  $\square$

Theorem 5.5.14 There is an isomorphism

$$\pi_{-1} TC(A; p) \cong W(A)_F,$$

the co-invariants of the Frobenius  $F: W(A) \rightarrow W(A)$ .

We end this paragraph by giving the group  $W(A)_F$  a different description when  $A$  is a field  $k$  of characteristic  $p$ . Let  $\bar{k}$  be a separable closure of  $k$  and let  $G = \text{Gal}(\bar{k}/k)$  be the Galois group. Then we have an exact sequence of  $G$ -modules

$$0 \rightarrow \mathbb{F}_p \rightarrow \bar{k} \xrightarrow{\bar{\varphi}-1} \bar{k} \rightarrow 0,$$

where  $\bar{\varphi}$  is the Frobenius of  $\bar{k}$ . Here  $\bar{\varphi}-1$  is onto because the equation  $x^p - x = a$  is separable. We get an exact sequence of coh. groups

$$\begin{array}{ccccccc} H^0(G; \bar{k}) & \xrightarrow{\bar{\varphi}-1} & H^0(G; \bar{k}) & \xrightarrow{2} & H^1(G; \mathbb{F}_p) & \rightarrow & H^1(G; \bar{k}) \\ \parallel & & \parallel & & \parallel & & \parallel \\ k & \xrightarrow{\varphi-1} & k & \rightarrow & H^1(G; \mathbb{F}_p) & \rightarrow & 0 \end{array}$$

The vanishing of the group on the right follows from the normal basis theorem, cf. [Lang], p. 344.

Taking Witt vectors, we get an exact sequence of  $G$ -modules

$$0 \rightarrow W_n(\mathbb{F}_p) \rightarrow W_n(\bar{k}) \xrightarrow{F-1} W_n(\bar{k}) \rightarrow 0.$$

Since the fixed set of a group acting on a module is equal to the fixed set of the underlying set we get

$$\begin{array}{ccccccc} H^0(G; W_n(\bar{k})) & \xrightarrow{F-1} & H^0(G; W_n(\bar{k})) & \rightarrow & H^1(G; \mathbb{Z}/p^n\mathbb{Z}) & \rightarrow & H^1(G; W_n(\bar{k})) \\ \parallel & & \parallel & & \parallel & & \parallel \\ W_n(k) & \xrightarrow{F-1} & W_n(k) & \rightarrow & H^1(G; \mathbb{Z}/p^n\mathbb{Z}) & \rightarrow & 0 \end{array}$$

The vanishing of the group on the right follows by induction using the sequence

$$0 \rightarrow \bar{k} \xrightarrow{V^{n-1}} W_n(\bar{k}) \rightarrow W_{n-1}(\bar{k}) \rightarrow 0.$$

Let us write  $H^*(G; \mathbb{Z}_p) := \varprojlim_n H^*(G; \mathbb{Z}/p^n\mathbb{Z})$ . We have shown

Corollary 5.5.15 For a field  $k$  of characteristic  $p$  we have

$$\pi_{-1} TC(k; p) \cong H^1(\text{Gal}(\bar{k}/k); \mathbb{Z}_p).$$

## 6.1 Calculation of $\mathrm{THH}$

We begin by defining products. To this end we modify the definition of  $\mathrm{THH}(L)$  given §3.1 as follows: let  $P$  be a finite ordered set and let  $I^P$  be the product indexed by  $P$  of copies of the category  $I$ . Iterated concatenation specifies a functor

$$\cup_P : I^P \rightarrow I$$

and we let

$$G_k^P(L) = G_k(L) \circ \cup_P : (I^P)^{k+1} \rightarrow \{\text{pointed spaces}\}$$

be the composition of  $\cup_P$  with the functor  $G_k(L)$  of 3.1.11. Componentwise concatenation gives a functor

$$\cup^P : I^P \times I^P \rightarrow I^P$$

and hence the spaces

$$\mathrm{THH}^P(L)_k := \varinjlim_{(I^P)^{k+1}} G_k^P(L) \quad (6.1.1)$$

form a cyclic space  $\mathrm{THH}^P(L)$ , with structure maps given by the same formulas as in the case of  $\mathrm{THH}(L)$ . We let

$$\mathrm{THH}^P(L) = |\mathrm{THH}^P(L), 1|$$

be the realization. The canonical map



$$\varinjlim_{(IP)^{k+1}} G_k(L) \circ \mathbb{L}^P \xrightarrow{\text{can}} \varinjlim_{IP^{k+1}} G_k(L)$$

gives a map  $\mathrm{THH}^P(L)_k \rightarrow \mathrm{THH}(L)_k$ , for all  $k$ . However this is not a simplicial map. If it were, we would not have to go through this.

Let  $P \hookrightarrow P'$  be an injective but not necessarily order preserving map. Extending by zeros, we get a functor  $IP \rightarrow IP'$  and in turn an  $S^1$ -map

$$\mathrm{THH}^P(L) \rightarrow \mathrm{THH}^{P'}(L). \quad (6.1.2)$$

In particular the automorphism group  $\Sigma_P$  acts on  $\mathrm{THH}^P(L)$ . The approximation lemma 3.1.15 shows that 6.1.2 is an  $F$ -equivalence, provided  $P \neq \emptyset$ . So we do not change the weak homotopy type.

Suppose  $L$  is a commutative symmetric ring spectrum. Then there are cyclic product maps

$$\delta(\mathrm{THH}^P(L), \wedge \mathrm{THH}^{P'}(L)) \rightarrow \mathrm{THH}^{P \amalg P'}(L). \quad (6.1.3)$$

defined as follows: the multiplication in  $L$  gives rise to a natural transformation

$$G_k^P(L) \wedge G_k^{P'}(L) \xrightarrow{\lambda} G_k^{P \amalg P'}(L) \circ \text{tw}$$

where

$$\text{tw}: (IP)^{k+1} \times (IP')^{k+1} \rightarrow (IP \amalg P')^{k+1}$$

is the obvious shuffle functor. Then 6.1.3 is given

as the composite

$$\begin{aligned} \varinjlim_{(IP)^{k+1}} G_k^P(L) \wedge \varinjlim_{(IP')^{k+1}} G_k^{P'}(L) &\xleftarrow[\cong]{\text{can}} \varinjlim_{(IP)^{k+1} \times (IP')^{k+1}} G_k^P(L) \wedge G_k^{P'}(L) \\ &\xrightarrow{\lambda} \varinjlim_{(IP)^{k+1} \times (IP')^{k+1}} G_k^{P \sqcup P'}(L) \xrightarrow[\text{can}]{\text{can}} \varinjlim_{(IP \sqcup IP')^{k+1}} G_k^{P \sqcup P'}(L) \end{aligned}$$

This is a cyclic map because  $L$  is commutative (try to show this). Realizing 6.1.3 we get an  $S^1$ -map

$$T\mathbb{H}^P(L) \wedge T\mathbb{H}^{P'}(L) \rightarrow T\mathbb{H}^{P \sqcup P'}(L) \quad (6.1.4)$$

which is functorial in  $P$  and  $P'$ . Here  $S^1$  acts diagonally on the left.

Let  $X$  be a pointed space. Then we have the functor  $G_k^X(L)$  from p. 95 and using this in the construction above we get a cyclic space  $T\mathbb{H}^P(L; X)$ . which is functorial w.r.t. injections  $P \hookrightarrow P'$  and pointed maps  $X \rightarrow Y$ . The product 6.1.3 gives

$$S(T\mathbb{H}^P(L; X) \wedge T\mathbb{H}^{P'}(L; Y)) \rightarrow T\mathbb{H}^{P \sqcup P'}(L; X \wedge Y). \quad (6.1.5)$$

functorial in  $P, P'$  and  $X, Y$ . We take  $X$  to be the  $m$ -sphere  $S^m = (S^1)^{\wedge m}$ , with the obvious  $\Sigma_m$ -action, and let  $P = \{1, 2, \dots, m\}$ . Then the cyclic space  $T\mathbb{H}^P(L; X)$  has two commuting  $\Sigma_m$ -actions, that is, a  $\Sigma_m \times \Sigma_m$ -action. We define

$$t_*(L)_m = \Delta^* T\mathbb{H}^P(L; S^m)$$

where  $\Delta: \Sigma_m \rightarrow \Sigma_m \times \Sigma_m$  is the diagonal, and let

$$t(L)_m = |t_*(L)_m| \quad (6.1.6)$$

be the realization  $\Sigma_m \times S^1$ -space. The product 6.1.5 gives a  $\Sigma_m \times \Sigma_n$ -equivariant cyclic product map

$$S(t_*(L)_m \wedge t_*(L)_n) \rightarrow t_*(L)_{m+n}$$

and hence a  $\Sigma_m \times \Sigma_n \times S^1$ -equivariant map

$$\mu_{m,n}: t(L)_m \wedge t(L)_n \rightarrow t(L)_{m+n} \quad (6.1.7)$$

where  $S^1$  acts diagonally on the left. Smashing with the unit  $S^0 \rightarrow L_0$  defines a map  $S^n \rightarrow F(S^0, L_0 \wedge S^n)$  and the composite

$$S^n \rightarrow F(S^0, L_0 \wedge S^n) \rightarrow THH^P(L; S^n)_0 \hookrightarrow |THH^P(L; S^n)|$$

defines a  $\Sigma_n \times S^1$ -equivariant map

$$1_n: S^n \rightarrow t(L)_n \quad (6.1.8)$$

One may show that

Proposition 6.1.9 Let  $L$  be a commutative symmetric ring spectrum. Then  $t(L) = \{t(L)_n\}_n$  is a commutative symmetric ring spectrum with an  $S^1$ -action.

Remark 6.1.10 Let  $t(L)_{\mathbb{P}^n}$  be as in 4.2.8. Then by 6.1.2 a choice of point in  $P = \{1, 2, \dots, n\}$  specifies

an  $F$ -equivalence  $t(L)_{\mathbb{R}^n} \xrightarrow{\sim} t(L)_n$ . This justifies the notation.

We define the spectrum homology of a symmetric spectrum  $E$  with coeff. in a ring  $k$  by

$$H_*^{\text{spec}}(E; k) := \varinjlim_n \tilde{H}_{*+n}(E_n; k) \quad (6.1.11)$$

where the limit runs over the maps  $\mathcal{B}_{1,n}$ . If  $E$  is a (commutative) symmetric ring spectrum then this is a (graded commutative) graded ring. Let  $t(L)$  be as in 6.1.9. Then the action  $S_+^1 \wedge t(L)_n \rightarrow t(L)_n$  gives rise to a multiplication map

$$H_*(S_+^1; k) \otimes H_*^{\text{spec}}(t(L); k) \Rightarrow H_*^{\text{spec}}(t(L); k);$$

compare 2.2.1. In analogue with 2.2.6, multiplication with the fund. class  $[S_+^1]$  defines

$$B: H_*^{\text{spec}}(t(L); k) \rightarrow H_{*+1}^{\text{spec}}(t(L); k). \quad (6.1.12)$$

and one may show that this is a derivation.

Proposition 6.1.13 Let  $L$  be a symmetric ring spectrum and let  $k$  be a field. Then there is a first quadrant homology type spectral sequence

$$E^2 = HH_*(H_*^{\text{spec}}(L; k)/k) \Rightarrow H_*^{\text{spec}}(t(L); k),$$

compatible with  $B$ -operators. Moreover, if  $L$  is commutative, this spectral seq. is multiplicative.

PP I forget to say in the statement that I want each  $L_n$  to be well-pointed and  $(n-1)$ -connected. Then we get a spectral sequence for each  $n$

$$E^2 = H_* (\tilde{H}_{*+n}(t_*(L)_n; k); \Sigma(-1)^i di) \Rightarrow \tilde{H}_{*+n}(t(L)_n; k).$$

Taking the colimit we get a spectral sequence for  $H_*^{\text{spec}}(t(L); k)$ ; here we use that taking the colimit of a string of maps is an exact functor. We identify the  $E^1$ -term:

$$E_{s,t}^1 = \varinjlim_n \tilde{H}_{t+n}(t_s(L)_n; k),$$

where

$$t_s(L)_n = \varinjlim_{(I^{n+1})^{s+1}} G_s^{n+1}(L).$$

By the approximation lemma 3.1.15

$$\tilde{H}_{t+n}(t_s(L)_n; k) \cong \tilde{H}_{t+n}(F(S^{i_0} \wedge \dots \wedge S^{i_s}; L_{i_0} \wedge \dots \wedge L_{i_s} \wedge S^n); k)$$

if  $i_0, \dots, i_s$  are sufficiently large. And since  $L_n$  is  $(n-1)$ -conn the suspension theorem shows  $(i = i_0 + \dots + i_s)$

$$\tilde{H}_{t+n}(t_s(L)_n; k) \cong \tilde{H}_{t+i+n}(L_{i_0} \wedge \dots \wedge L_{i_s} \wedge S^n; k)$$

If  $n$  is sufficiently large. Since we have field coeff. this in turn is isomorphic to the degree  $t+i$  part of

$$\tilde{H}_*(L_{i_0}; k) \otimes \dots \otimes \tilde{H}_*(L_{i_s}; k).$$

This group is stable and equal to the degree  $t$

part of

$$H_*^{\text{spec}}(L; k) \otimes \cdots \otimes H_*^{\text{spec}}(L; k) \\ \quad \quad \quad -s+1$$

In fact,

$$E^1 = N_{\otimes}^{\text{co}}(H_*^{\text{spec}}(L; k))$$

and hence  $E^2$  is Hochschild homology of  $H_*^{\text{spec}}(L; k)$ .

To see that the sequence is compatible with B-operators recall from 1.2.13 that if  $X_0$  is a pointed cyclic space then the  $S^1$ -action  $|X_0|$  is given by

$$|C_+| \wedge |X_0| \xrightarrow{h \times 1} |C_+| \wedge |X_0| \xrightarrow{h^{-1}} |FX_0| \xrightarrow{\varepsilon} |X_0|.$$

Here the maps  $h$  are not simplicial. But they are cellular and induces the map  $l$  of 2.2.4 on cellular complexes. They are also filtration preserving, so the spectral sequence for  $\tilde{H}_*(|FX_0|; k)$  becomes the tensor product of the spectral sequences for  $H_*(|C_+|; k)$  and for  $\tilde{H}_*(|X_0|; k)$ . But the former is concentrated in bidegrees  $(0,0)$  and  $(0,1)$ , so there are no differentials. Hence  $[s']$  survives to  $E^\infty$  in the spectral sequence for  $\tilde{H}_*(|FX_0|; k)$ . Finally, we use that by naturality,  $\varepsilon$  induces a map of sp. sequences.

The multiplicativity of the spectral sequence, when  $L$  is commutative, follows from the product 6.1.7 being induced from a simplicial map 6.1.5 via the standard argument, see [CE] p. 337.  $\square$

$$\mathcal{A} = S_{\mathbb{F}_p} \{ \xi_1, \xi_2, \dots \} \otimes \bigwedge_{\mathbb{F}_p} \{ \tau_0, \tau_1, \dots \}$$

$$\deg \xi_i = 2p^i - 2, \quad \deg \tau_i = 2p^i - 1$$

Let

$$\mathcal{A}' = S_{\mathbb{F}_p} \{ \xi_1 \otimes 1 - 1 \otimes \xi_1, \xi_2 \otimes 1 - 1 \otimes \xi_2, \dots \}$$

$\otimes$

$\subset \mathcal{A} \otimes \mathcal{A}$

$$\bigwedge_{\mathbb{F}_p} \{ \tau_0 \otimes 1 - 1 \otimes \tau_0, \tau_1 \otimes 1 - 1 \otimes \tau_1, \dots \}$$

Also let  $\mathcal{A} \hookrightarrow \mathcal{A} \otimes \mathcal{A}$ ,  $a \mapsto a \otimes 1$ . Then

$$\mathcal{A} \otimes \mathcal{A}' = \mathcal{A} \otimes \mathcal{A}$$

so

$$\mathrm{Tor}^{\mathcal{A} \otimes \mathcal{A}}(\mathcal{A}, \mathcal{A}) = \mathrm{Tor}^{\mathcal{A} \otimes \mathcal{A}'}(\mathcal{A} \otimes \mathbb{F}_p, \mathcal{A} \otimes \mathbb{F}_p)$$

$$\cong \mathrm{Tor}^{\mathcal{A}}(\mathcal{A}, \mathcal{A}) \otimes \mathrm{Tor}^{\mathcal{A}'}(\mathbb{F}_p, \mathbb{F}_p)$$

$$\cong \mathcal{A} \otimes \mathrm{Tor}^{\mathcal{A}'}(\mathbb{F}_p, \mathbb{F}_p)$$

Let  $\bar{\xi}_i = \xi_i \otimes 1 - 1 \otimes \xi_i$ ,  $\bar{\tau}_i = \tau_i \otimes 1 - 1 \otimes \tau_i$  so that

$$\mathcal{A}' = S_{\mathbb{F}_p} \{ \bar{\xi}_1, \bar{\xi}_2, \dots \} \otimes \Lambda_{\mathbb{F}_p} \{ \bar{\sigma}, \bar{\tau}_1, \dots \}$$

A resolution of  $\mathbb{F}_p$  by free  $S_{\mathbb{F}_p} \{ \bar{\xi}_1, \dots \}$ -modules  
 $\square$  given by

$$S_{\mathbb{F}_p} \{ \bar{\xi}_1, \bar{\xi}_2, \dots \} \otimes \Lambda_{\mathbb{F}_p} \{ \sigma \bar{\xi}_1, \sigma \bar{\xi}_2, \dots \} \xrightarrow{\varepsilon} \mathbb{F}_p$$

where  $\deg(\sigma \bar{\xi}_i) = (1, 2p^i - 2)$  and the

$$d(1 \otimes \sigma \bar{\xi}_i) = \bar{\xi}_i \otimes 1.$$

Compare with bar-resolution :

$$\begin{array}{ccc} S_{\mathbb{F}_p} \{ \bar{\xi}_i \} \otimes \Lambda_{\mathbb{F}_p} \{ \sigma \bar{\xi}_i \} & \xrightarrow{f} & B(S_{\mathbb{F}_p} \{ \bar{\xi}_i \}, S_{\mathbb{F}_p} \{ \bar{\xi}_i \}, \mathbb{F}_p) \\ \downarrow \varepsilon & & \downarrow \cong \\ \mathbb{F}_p & \xlongequal{\quad} & \mathbb{F}_p \end{array}$$

$$f_n(1 \otimes x) = s_{-1} f_{n-1} d(1 \otimes x)$$

$$f_0(1 \otimes x) = s_{-1} \varepsilon(1 \otimes x)$$

So

$$f_0 = \text{id}, \quad f_1(1 \otimes \bar{\xi}_i) = s_{-1}(\bar{\xi}_i) = 1 \otimes \bar{\xi}_i.$$



$$1 \otimes \sigma \bar{\xi}_i \longmapsto 1 \otimes \bar{\xi}_i \otimes 1 \in B(A', A', \mathbb{F}_p)$$

$$\longmapsto 1 \otimes \bar{\xi}_i \otimes 1$$

$$S_{\mathbb{F}_p} \{ \bar{\xi}_i \} \otimes \wedge_{\mathbb{F}_p} \{ \sigma \bar{\xi}_i \} \rightarrow B(S_{\mathbb{F}_p} \{ \bar{\xi}_i \}, S_{\mathbb{F}_p} \{ \bar{\xi}_i \}, \mathbb{F}_p)$$

$$0: \quad 1 \otimes 1 \longmapsto 1 \otimes 1$$

$$1: \quad 1 \otimes \sigma \bar{\xi}_i \longmapsto 1 \otimes \bar{\xi}_i \otimes 1$$

$\downarrow$

$\downarrow S_1$

$$\bar{\xi}_i \otimes 1 \longmapsto \bar{\xi}_i \otimes 1$$

$$B(S_{\mathbb{F}_p} \{ \bar{\xi}_i \}, S_{\mathbb{F}_p} \{ \bar{\xi}_i \}; \mathbb{F}_p) \rightarrow B(A \otimes A, A \otimes A, A)$$

$$1 \otimes \bar{\xi}_i \otimes 1 \longmapsto (1 \otimes 1) \otimes \bar{\xi}_i \otimes 1$$

$$\begin{array}{ccc} B(A \otimes A, A \otimes A, A) & \xrightarrow{\quad} & B(A, A, A) \\ \downarrow \phi & & \downarrow \phi \\ 0: & (1 \otimes 1) \otimes x & \longmapsto 1 \otimes x \\ 1: & (1 \otimes 1) \otimes (x \otimes y) \otimes 1 & \\ & \downarrow \downarrow & \\ & (x \otimes y) \otimes 1 = (1 \otimes 1) \otimes xy & \longmapsto \end{array}$$

$$B(A \otimes A, A \otimes A, A) \rightarrow B(A, A, A)$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ A & \xlongequal{\quad} & A \end{array}$$

$$\begin{array}{ccc} 0: & (1 \otimes 1) \otimes x & \xrightarrow{\quad} 1 \otimes x \\ & \downarrow \varepsilon & \\ & x & \xrightarrow{\quad} \frac{1}{1} s_{-1} x \end{array}$$

$$\begin{array}{ccc} 1: & (1 \otimes 1) \otimes (x \otimes y) \otimes z & \xrightarrow{\quad} 1 \otimes x \otimes zy - 1 \otimes 1 \otimes xzy \\ & \downarrow & \\ & & \downarrow s_{-1} \end{array}$$

$$(x \otimes y) \otimes z - (1 \otimes 1) \otimes xzy \xrightarrow{\quad} x \otimes zy - 1 \otimes xzy$$

In our case:

$$\begin{array}{ccc} (1 \otimes 1) \otimes (\xi_i \otimes 1) \otimes 1 & & 1 \otimes \xi_i \otimes 1 - 1 \otimes 1 \otimes \xi_i \\ - (1 \otimes 1) \otimes (1 \otimes \xi_i) \otimes 1 & \xrightarrow{\quad} & - 1 \otimes 1 \otimes \xi_i + 1 \otimes 1 \otimes \xi_i \end{array}$$

From 120. on p. 43:

$$A \otimes_{A \otimes A} B(A, A, A) \xrightarrow{c} N_{\otimes}^{cy}(A)$$

$$\begin{array}{ccc} 1 \otimes (1 \otimes \xi_i \otimes 1 - 1 \otimes 1 \otimes \xi_i) & \xrightarrow{\quad} & 1 \otimes \xi_i - \xi_i \otimes 1 \\ \otimes_{A \otimes A} & & \parallel \quad \neq \\ & & B \xi_i \quad \text{degenerate.} \end{array}$$

$$\varinjlim_n H_{*+n}(\mathrm{THH}(L^{(n)}; S^n); \mathbb{F}_p) =: H_*^{\mathrm{spec}}(T(L); \mathbb{F}_p)$$

$$[k] \mapsto \mathrm{THH}(L^{(n)}; S^n)_k \quad \text{simplicial pre-spectrum.}$$

$$|[k]| \mapsto \mathrm{THH}(L^{(n)}; S^n)_k \quad \text{pre-spectrum.}$$

$$\varinjlim_n H_{n+i}(|[k]| \mapsto \mathrm{THH}(L^{(n)}; S^n)_k; \mathbb{F}_p)$$

Spectral seq.

$$E^1 = N_{\otimes}^{\mathrm{cy}}(H_*^{\mathrm{spec}}(L; \mathbb{F}_p)) \Rightarrow H_*^{\mathrm{spec}}(T(L); \mathbb{F}_p).$$

$$X_k \times 2\Delta^k \Rightarrow \|X\|^{(k-1)} \quad \cong$$

$\downarrow$

$\downarrow$

$$X_k \times \Delta^k \Rightarrow \|X\|^{(k)}$$

$\downarrow$

$\downarrow$

$$X_{k+1} \times \Delta^k \Rightarrow \|X\|^{(k)} / \|X\|^{(k-1)}$$

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## 6.2 The Tate spectrum

Let  $k$  be a commutative ground ring, let  $G$  be a finite group and let  $M$  be a (left)  $G$ -module. Multiplication by the norm element  $N = \sum_{g \in G} g \in kG$  induces a  $k$ -linear map

$$H_0(G; M) = M_G \xrightarrow{N} M^G = H^0(G; M) \quad (6.2.1)$$

called the norm homomorphism. By a complete resolution of  $k$  we mean a long-exact sequence of f.g. projective  $kG$ -modules

$$\begin{array}{ccccccc} \cdots & \rightarrow & X_2 & \xrightarrow{d_2} & X_1 & \xrightarrow{d_1} & X_0 & \xrightarrow{d_0} & X_{-1} & \xrightarrow{d_{-1}} & X_{-2} & \rightarrow \cdots \\ & & & & & & \varepsilon \searrow_k \nearrow \eta & & & & & \end{array} \quad (6.2.2)$$

where  $d_0$  factors as indicated. The dual of a left  $kG$ -module  $X$  is the right  $kG$ -module

$$X^* = \text{Hom}_k(X, k)$$

which we may view as a left  $kG$ -module via the antipode  $c: kG \rightarrow kG^{\text{op}}, g \mapsto g^{-1}$ . If  $X$  is f.g. proj. then so is  $X^*$ . Therefore, a resolution  $X_* \xrightarrow{\varepsilon} k$  of  $k$  by f.g. projective  $k$ -modules induces a complete res:

$$\begin{array}{ccccccc} \cdots & \rightarrow & X_2 & \rightarrow & X_1 & \rightarrow & X_0 & \xrightarrow{\varepsilon^* \circ \varepsilon} & X_0^* & \rightarrow & X_1^* & \rightarrow \cdots \\ & & & & & & \varepsilon \searrow_k \nearrow \varepsilon^* & & & & & \end{array} \quad (6.2.3)$$

So we can always find a complete resolution. We now

define the Tate (co)homology of  $G$  with coefficients in  $M$  by

$$\hat{H}^*(G; M) = H^*(\text{Hom}_{kG}(X_0, M))$$

$$\hat{H}_*(G; M) = H_*(X_0 \otimes_{kG} M) \quad (6.2.4)$$

Lemma 6.2.5 There is a canonical isomorphism

$$\hat{H}^{-i}(G; M) \cong \hat{H}_{i-1}(G; M).$$

Pr If  $X_0$  is a complete resolution, then so is  $X_0^*$ .

Moreover the  $k$ -linear map

$$X_0^* \otimes_{kG} M \xrightarrow{\tau} \text{Hom}_{kG}(X_0, M),$$

$$\tau(\phi \otimes m)(x) = \sum_{g \in G} \phi(g^{-1}x) gm$$

is an isomorphism, provided  $X$  is f.g. proj.  $\square$

Lemma 6.2.6 For  $i \geq 1$ ,  $\hat{H}^i(G; M) = H^i(G; M)$ , for  $i \leq -2$ ,  $\hat{H}^i(G; M) = H_{-i-1}(G; M)$ , and there is an exact seq.

$$0 \rightarrow \hat{H}^{-1}(G; M) \rightarrow M_G \xrightarrow{N} M^G \rightarrow \hat{H}^0(G; M) \rightarrow 0.$$

Pr The first two statements follow from the definitions and 6.2.5. To prove the last statement we use that  $X_{-1} \rightarrow X_0$  factors  $X_{-1} \rightarrow k \rightarrow X_0$  to get a diagram

$$\begin{array}{ccc}
 X_{-1}^* \otimes_{kG} M & \xrightarrow[\cong]{\tau} & \text{Hom}_{kG}(X_{-1}, M) \rightarrow \text{Hom}_{kG}(X_0, M) \\
 \downarrow \varepsilon \otimes 1 & & \downarrow \\
 k \otimes_{kG} M & \xrightarrow{\tau} & \text{Hom}_{kG}(k, M) \\
 \parallel & \xrightarrow{\sim} & \parallel \\
 M_G & \xrightarrow{\sim} & M^G
 \end{array}
 \quad \nearrow \varepsilon^*$$

The claim follows.  $\square$

One may show that  $\hat{H}^*(G; k)$  is a  $\mathbb{Z}$ -graded  $k$ -alg. and that  $\hat{H}^*(G; M)$  is a graded module over this  $k$ -algebra. Moreover,  $G/H \mapsto \hat{H}^*(H; k)$  is a Green functor and  $G/H \mapsto \hat{H}^*(G; M)$  is a Mackey functor which is a module over that Green functor, compare 5.4.5 and [CE], p 242, 256.

Example 6.2.7 For the cyclic group  $C_n$  we have the periodic standard resolution  $W$ :

$$\begin{array}{ccccccc}
 \cdots \rightarrow kC_n & \xrightarrow{N} & kC_n & \xrightarrow{1-t} & kC_n & \xrightarrow{N} & kC_n \xrightarrow{1-t} kC_n \rightarrow \cdots \\
 & & & \searrow \varepsilon & \nearrow \tau N & & \\
 & & & k & & & 
 \end{array}$$

Applying  $\text{Hom}_{kG}(-; M)$ , we get

$$\cdots \leftarrow M \xleftarrow{N} M \xleftarrow{1-t} M \xleftarrow{N} M \xleftarrow{1-t} M \leftarrow \cdots$$

so

$$\hat{H}^i(C_n; M) = \begin{cases} M^G / NM & , \text{ for } i \text{ even} \\ NM / IM & , \text{ for } i \text{ odd} \end{cases}$$

where  $I \subset kC_n$  is the augmentation ideal gen. by  $1-t$ .

To evaluate the product on  $\hat{H}^*(G; k)$ , we use the map  $\Phi : W. \rightarrow W. \otimes W.$  with

$$\Phi_{p,q} : W_{p+q} \rightarrow W_p \otimes W_q$$

$$\Phi_{p,q}(1) = \begin{cases} 1 \otimes 1 & , p \text{ even} \\ 1 \otimes t & , p \text{ odd}, q \text{ even} \\ \sum_{0 \leq m < n \leq r-1} t^m \otimes t^n & , p, q \text{ odd} \end{cases}$$

The formula

$$(t \otimes t - 1 \otimes 1) \sum_{0 \leq m < n \leq r-1} t^m \otimes t^n = N \otimes 1 - 1 \otimes N$$

shows that  $\Phi$  is a map of DG  $kG$ -modules. The map  $\Phi$  gives rise to a product

$$\hat{H}^p(G; k) \otimes \hat{H}^q(G; k) \xrightarrow{\mu} \hat{H}^{p+q}(G; k)$$

given by

$$\mu(1 \otimes 1) = \begin{cases} 1 \otimes 1 & , p \text{ or } q \text{ even} \\ \frac{r(r-1)}{2} 1 \otimes 1 & , p \text{ and } q \text{ odd.} \end{cases}$$

In particular, we get

$$\hat{H}^*(G^n; \mathbb{F}_p) \cong \begin{cases} S_{\mathbb{F}_p} \{t, t^{-1}\} \otimes \Delta \{u\} & , p \text{ odd or } p=2, n>1 \\ S_{\mathbb{F}_2} \{u, \bar{u}\} & , p=2 \text{ and } n=1 \end{cases}$$

where  $\deg t = 2$ ,  $\deg u = 1$ .

□

Let  $T$  be a  $G$ -spectrum indexed on a complete  $G$ -universe  $U$ . Before we define the Tate spectrum  $\hat{H}^*(G; T)$ , we need to deal with the problem that the function spectrum  $F(EG_+, T)$  usually is not well-pointed even though  $T$  is. We define a CW-approximation functor

$$(-)^w: G\mathcal{S}U \rightarrow G\mathcal{S}U \quad (6.2.8)$$

as follows: Suppose  $D$  is in  $G\mathcal{S}U$  and define maps  $D_n^w \rightarrow D$  in  $G\mathcal{S}U$  recursively by  $D_{-1}^w = * \rightarrow D$  and given  $D_{n-1}^w \rightarrow D$ , we let  $\mathcal{A}_n$  be the set of all diagrams of the form

$$\begin{array}{ccc} \sum_{\alpha}^{\infty} G/H_{\alpha+} \wedge S^{j_{\alpha}} & \longrightarrow & D_{n-1}^w \\ \downarrow & & \downarrow \\ \sum_{\alpha}^{\infty} G/H_{\alpha+} \wedge D^{j_{\alpha}} & \longrightarrow & D \end{array}$$

and define  $D_n^w \rightarrow D$  as the push out in  $G\mathcal{S}U$

$$\begin{array}{ccc} \bigvee_{\alpha \in \mathcal{A}_n} \sum_{\alpha}^{\infty} G/H_{\alpha+} \wedge S^{j_{\alpha}} & \longrightarrow & D_{n-1}^w \\ \downarrow & & \downarrow \\ \bigvee_{\alpha \in \mathcal{A}_n} \sum_{\alpha}^{\infty} G/H_{\alpha+} \wedge D^{j_{\alpha}} & \longrightarrow & D_n^w \end{array}$$

Then  $D^w = \varinjlim_n D_n^w \rightarrow D$  and the small object



argument of [Quillen] shows that  $D^w \xrightarrow{\sim} D$  is a weak  $G$ -equivalence.

Recall the cofibration seq. of pointed  $G$ -spaces from 4.3.22:

$$EG_+ \rightarrow S^0 \rightarrow \tilde{E}G$$

For a  $G$ -spectrum  $T$  in  $G\mathcal{S}\mathcal{U}$ , we smash this seq. with  $F(EG_+, T)^w$  and get a cofibration sequence in  $G\mathcal{S}\mathcal{U}$ . This is then also a fibration sequence, since the spectra are well-pointed. Taking fix-point spectra we get the following fibration sequence in  $\mathcal{S}\mathcal{U}^G$

$$(EG_+ \wedge F(EG_+, T)^w)^G \rightarrow (F(EG_+, T)^w)^G \rightarrow (\tilde{E}G \wedge F(EG_+, T)^w)^G.$$

The left hand term is  $H_*(G; F(EG_+, T)^w)$  and the spectral sequence of 4.3.17 shows that we have canonical equivalences

$$H_*(G; F(EG_+, T)^w) \xleftarrow{\sim} H_*(G; T^w) \xrightarrow{\sim} H_*(G; T),$$

where the second requires that  $T$  be well-pointed. We shall therefore use these interchangeably. We also write

$$\begin{aligned} H^*(G; T) &= (F(EG_+, T)^w)^G \\ \hat{H}^*(G; T) &= (\tilde{E}G \wedge F(EG_+, T)^w)^G \end{aligned} \tag{6.3.9}$$

and call them the Borel cohomology (or htpy fixed point) spectrum and Tate spectrum of  $G$  with coefficients in  $T$ . The projection  $EG_+ \rightarrow S^0$  induces

a map  $T \rightarrow F(EG_+, T)$  in  $G\mathcal{S}\mathcal{U}$  and we get a map of fibration sequences

$$\begin{array}{ccccc} H_*(G; T) & \xrightarrow{N} & (T^w)^G & \rightarrow & (\tilde{E}G \wedge T^w)^G \\ \parallel & & \downarrow \Gamma & & \downarrow \hat{\Gamma} \\ H_*(G; T) & \xrightarrow{N^h} & H^*(G; T) & \rightarrow & \hat{H}^*(G; T) \end{array} \quad (6.2.10)$$

Suppose now that  $G$  is finite and consider the skeleton filtration of  $\tilde{E}G$ . We note that

$$\tilde{E}_{k+1}G / \tilde{E}_kG \cong \Sigma E_kG / E_{k-1}G \cong \bigvee S^{k+1} \wedge G_+.$$

On homology we get that

$$0 \rightarrow \mathbb{Z} \rightarrow H_0(\tilde{E}_0G, S^0) \rightarrow H_1(\tilde{E}_1G, \tilde{E}_0G) \rightarrow H_2(\tilde{E}_2G, \tilde{E}_1G) \rightarrow \dots$$

is an exact sequence of f.g. free  $\mathbb{Z}G$ -modules. It is exact because  $\tilde{E}G$  is contractible. Define spectra  $X_k$ ,  $k \in \mathbb{Z}$ , in  $G\mathcal{S}\mathcal{U}$  by

$$X_k = \begin{cases} \Sigma_{\mathbb{N}}^{\infty} \tilde{E}_kG & , k \geq 0 \\ F(\tilde{E}_{-k}G; S_{\mathbb{N}}^0)^w & , k < 0 \end{cases}$$

where  $S_{\mathbb{N}}$  is the sphere spectrum on  $\mathbb{N}$ . We get Greenlees' 'filtration' of  $\tilde{E}G$  (or rather  $\Sigma_{\mathbb{N}}^{\infty} \tilde{E}G$ ),

$$\dots \rightarrow X_{-2} \rightarrow X_{-1} \rightarrow X_0 \rightarrow X_1 \rightarrow \dots \quad (6.2.11)$$

$\uparrow$   
 $S_{\mathbb{N}}$

We have

$$X_k \cup_{X_{k-1}} CX_{k-1} \underset{G}{\simeq} \begin{cases} \Sigma_{\infty}^{\infty} \tilde{E}_k G / \tilde{E}_{k-1} G & , k \geq 0 \\ \Sigma F(\tilde{E}_{-k+1} G / \tilde{E}_{-k} G, \mathcal{S}_{\mathcal{U}}^{\infty})^w & , k < 0 \end{cases}$$

$$\underset{G}{\simeq} \bigvee S_{\mathcal{U}}^k \wedge G_+ \quad , k \in \mathbb{Z},$$

where for  $k < 0$ , the latter spectrum is that of 4.2.6. We have used 4.3.15 in the identification. Let

$$P_k = H_k(X_k \cup_{X_{k-1}} CX_{k-1}).$$

Then it follows from 5.4.15 that the sequence

$$\begin{array}{ccccccc} \cdots & \rightarrow & P_{-2} & \rightarrow & P_{-1} & \rightarrow & P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow \cdots \\ & & & & \searrow & & \\ & & & & \mathbb{Z} & & \end{array} \quad (6.2.12)$$

is a complete resolution of  $\mathbb{Z}$  by f.g. free  $\mathbb{Z}G$ -modules. We have an exact complex

$$\begin{aligned} \hat{E}_{s,t} &= \pi_{s+t} (X_s \cup_{X_{s-1}} CX_{s-1} \wedge F(EG_+, T)^w)^G \\ \hat{D}_{s,t} &= \pi_{s+t} (X_s \wedge F(EG_+, T)^w)^G \end{aligned} \quad (6.2.13)$$

and from 6.2.12 we see that  $E_{s,t} = P_s \otimes_{\mathbb{Z}G} \pi_t^T$ . Finally, an argument similar to 4.3.17 shows:

$$\hat{E}_{s,t}^2 = \hat{H}^{-s}(G; \pi_t^T) \Rightarrow \pi_{s+t} \hat{H}^*(G; T). \quad (6.2.14)$$

We call this the Tate spectral sequence.

It is fair to remark that in (6.2.13) we have used a smash product of spectra. These exist and have the properties we need to get (6.2.14). In another incarnation, these notes might offer a treatment.

Remark 6.2.15 The map  $\tilde{\Gamma}$  in 6.2.10 is in general not an equivalence. It is a deep theorem, the Segal conjecture, that it is (after completion at  $\mathbb{I}\mathbb{G}$ ) for  $T = S^0_{\mathbb{I}\mathbb{G}}$ . On the other hand, we can replace  $F(EG_+, T)^w$  by  $T^w$  in 6.2.13 and get a sp. seq. for  $\pi_*(\tilde{E}\mathbb{G} \wedge T^w)$ . Now the map  $F(EG_+, T)^w \rightarrow T^w$  induces an isomorphism of the  $E^1$ -terms of these sp. sequences. They can therefore necessarily not both converge.  $\square$

Lemma 6.2.16 The Tate spectral sequence is conditional convergent in the sense of [Boardman].

Pf (Sketch) We must show that  $\varprojlim_{\mathbb{S}} \hat{D}_{s,t} = \varprojlim_{\mathbb{S}}^{(1)} \hat{D}_{s,t} = 0$ .  
Calculate:

$$\begin{aligned} \varprojlim_{\mathbb{S}} (X_s \wedge F(EG_+, T)^w)^G &= \varprojlim_{\mathbb{S}} (F(\tilde{E}_{-s}G, S^0_{\mathbb{I}\mathbb{G}})^w \wedge F(EG_+, T)^w)^G \\ &\cong \varprojlim_{\mathbb{S}} (F(\tilde{E}_{-s}G \wedge EG_+, T)^w)^G \cong (F(\tilde{E}\mathbb{G} \wedge EG_+, T)^w)^G. \end{aligned}$$

But  $\tilde{E}\mathbb{G} \wedge EG_+$  is equivariantly contractible.  $\square$

For a discussion of convergence see also [HM2], §3.

The spectra  $\hat{H}^*(G; T)$  and  $H^*(G; T)$  make sense for any compact Lie group, but obviously this is not the case for the Tate spectral sequence. We can, however, define a similar spectral sequence when  $G = S^1$ . In this case, we may take as our model for the fundamental cofibration sequence 4.3.22 the cofibration sequence (we write  $n\mathbb{C}$  for  $\mathbb{C}^n$ )

$$\bigcup_{n \geq 0} S(n\mathbb{C}) \rightarrow S^0 \rightarrow \bigcup_{n \geq 0} S^{n\mathbb{C}} \quad (6.2.17)$$

In place of Greenlees' filtration 6.2.11, we take

$$\cdots \rightarrow S_n^{-\mathbb{C}} \rightarrow S_n^{-\mathbb{C}} \xrightarrow{\quad} S_n^0 \xrightarrow{\quad} S_n^0 \rightarrow S_n^{\mathbb{C}} \rightarrow S_n^{\mathbb{C}} \rightarrow S_n^{2\mathbb{C}} \rightarrow \cdots$$

$\uparrow$   
 $S_n^0$

where  $S_n^{-n\mathbb{C}} = F(S_n^{n\mathbb{C}}, S_n^0)$ . To calculate the filtration quotients, we recall from 4.1.10 that  $S^V = S(\mathbb{R} \oplus V)$ . In the basic case  $S^0 \xrightarrow{\hookrightarrow} S^{\mathbb{C}}$  we use

$$D(\mathbb{C}) \times S(\mathbb{R}) \hookrightarrow D(\mathbb{C}) \times S(\mathbb{R}) \cup S(\mathbb{C}) \times D(\mathbb{R}) \rightarrow \frac{S(\mathbb{C}) \times D(\mathbb{R})}{S(\mathbb{C}) \times S(\mathbb{R})}$$

to see that the cofiber is  $S_+^1 \wedge S^1$ . For  $n \geq 0$ , the inclusion  $S^{(n-1)\mathbb{C}} \hookrightarrow S^{n\mathbb{C}}$  is  $\hookrightarrow \wedge S^{(n-1)\mathbb{C}}$ , so the cofiber is  $S_+^1 \wedge S^{2n-1}$ . It follows that we have cofiber seq. in  $G\mathcal{A}u$

$$S_n^{(n-1)\mathbb{C}} \rightarrow S_n^{n\mathbb{C}} \rightarrow S_+^1 \wedge S_n^{2n-1}$$

where for  $n \leq 0$ , we use 4.3.15 in the identification.

For  $T$  in  $S^1 \mathcal{S}U$  we have, again by 4.3.15:

$$\begin{aligned} (S^1_+ \wedge S^{2n-1}_n \wedge F(ES^1_+, T)^w)^{S^1} &\xleftarrow{\sim} (S^1_+ \wedge S^{2n-1}_n \wedge T^w)^{S^1} \\ &\xleftarrow{\tau} \Sigma F(S^1_+, S^{2n-1}_n \wedge T^w)^{S^1} \xrightarrow{w_1} \Sigma^{2n} T \end{aligned}$$

so we get a spectral sequence,  $\deg t = (-2, 0)$ :

$$\hat{E}^2 = S\{t, t^{-1}\} \otimes \pi_* T \Rightarrow \pi_* \hat{H}^*(S^1; T). \quad (6.2.18)$$

Let us write  $\hat{E}^*(G; T)$  for either of the sequences 6.2.14 or 6.2.18, where  $G$  is the relevant group. An  $S^1$ -spectrum is also a  $G$ -spectrum for every  $G \subset S^1$ , and  $\tilde{E}S^1 := \bigcup S^n \mathbb{C}$  is a model for  $\tilde{E}G$ . Hence

$$(\tilde{E}S^1 \wedge F(ES^1_+, T)^w)^{S^1} \hookrightarrow (\tilde{E}S^1 \wedge F(ES^1_+, T)^w)^{G}$$

is a map of spectra  $\hat{H}(S^1; T) \rightarrow \hat{H}(G; T)$ .

Lemma 6.2.19 Let  $T$  be in  $S^1 \mathcal{S}U$ . Then there is a map of spectral sequences  $\hat{E}^*(S^1; T) \rightarrow \hat{E}^*(G; T)$  such that the map on  $E^2_{2s, *}$  is the reduction

$$\pi_* T \rightarrow \pi_* T / r \pi_* T.$$

Pf The  $S^1$ -space  $ES^1 = \bigcup S(n\mathbb{C})$  has a  $G$ -CW-structure with one free cell in each dimension. The skeleton filtration is

$$\begin{array}{ccccccc} & & & S(\mathbb{C}) * S(\mathbb{C}) & & & \\ & & & \parallel & & & \\ \emptyset & \rightarrow & C_r & \rightarrow & S(\mathbb{C}) & \rightarrow & C_r * S(\mathbb{C}) \rightarrow S(2\mathbb{C}) \rightarrow C_r * S(2\mathbb{C}) \rightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & C_{r+} & & C_{r+} \wedge S^1 & & C_{r+} \wedge S^2 & & C_{r+} \wedge S^3 & & C_{r+} \wedge S^4 \end{array}$$

and the cellular complex is precisely the standard resolution, cf. 6.2.7. Now  $\tilde{E}S^1 = ES^1 * S^0$  so is a  $G$ -CW-complex with one free cell in each positive dimension. The skeleton filtration is

$$\begin{array}{ccccccccc} S^0 & \rightarrow & C_+ * S^0 & \rightarrow & S^1 & \rightarrow & C_+ * S^1 & \rightarrow & S^2 & \rightarrow & C_+ * S^2 & \rightarrow \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ C_+ \wedge S^1 & & C_+ \wedge S^2 & & C_+ \wedge S^3 & & C_+ \wedge S^4 & & C_+ \wedge S^5 & & & \end{array}$$

Again the cellular  $\alpha$  is the standard resolution  $W$ , but shifted up one degree. We have a  $G$ -map of filtrations of  $\tilde{E}S^1$ .

$$\begin{array}{ccccccccccc} S^0 & \rightarrow & S^0 & \rightarrow & S^1 & \rightarrow & S^1 & \rightarrow & S^2 & \rightarrow & S^2 & \rightarrow & S^3 & \rightarrow \dots \\ \parallel & & \downarrow & & \parallel & & \downarrow & & \parallel & & \downarrow & & \parallel & \\ S^0 & \rightarrow & C_+ * S^0 & \rightarrow & S^1 & \rightarrow & C_+ * S^1 & \rightarrow & S^2 & \rightarrow & C_+ * S^2 & \rightarrow & S^3 & \rightarrow \dots \end{array}$$

The map of filtration quotients in even degree  $2s$  is given by

$$S_+^1 \wedge S^{2s-1} \xrightarrow{\pi \wedge id} C_+ \wedge S_+^1 / C_+ \wedge S^{2s-1} \xrightarrow{\text{no action}} C_+ \wedge S^{2s}$$

where  $\pi$  is the projection from  $S_+^1$  to  $S_+^1$  with the subspace  $C_+$  collapsed to a point. Let  $\psi: Y \rightarrow X$  be the map of 'complete resolutions' corresponding to the map of filtrations above. The map of sp. seq. is then given by the composite

$$(\psi_* \wedge F(ES_+^1, T))^w)^{S^1} \rightarrow (\psi_* \wedge F(ES_+^1, T))^w)^{C_+} \rightarrow (X_* \wedge F(ES_+^1, T))^w)^{C_+}$$

and we claim the map  $\hat{E}_{2s, *}^1(S^1; T) \rightarrow \hat{E}_{2s, *}^1(C_+; T)$  is an isomorphism. To show this we must show that for  $D$  in  $G\mathcal{U}$ , the following composite is a w.e.

$$(S_+^1 \wedge D)^{S^1} \hookrightarrow (S_+^1 \wedge D)^{C_+} \xrightarrow{\pi \wedge 1} (C_+ \wedge S_+^1 / C_+ \wedge D)^{C_+} \quad (6.2.20)$$

We shall use 4.3.15 to rewrite this. Let  $i: \mathcal{U}^{S^1} \hookrightarrow \mathcal{U}$  be the inclusion of the trivial universe. Then we identify the left hand using the equivalence

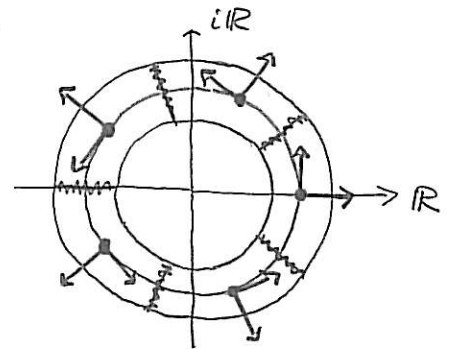
$$\Sigma |D| \xrightarrow[\sim]{\tau} i^*(Q^{\mathbb{F}}(S'_+ \wedge S^{\mathbb{F}} \wedge D))^{S^1} \xleftarrow{\sim} i^*(S'_+ \wedge D)^{S^1}.$$

Here  $|D|$  is  $i^*D$  with the  $S^1$ -action forgotten and  $\tau$  is the adjoint of the composite

$$i_* S^{\mathbb{F}} \wedge \Sigma |D| \xrightarrow{TP} i_* S'_+ \wedge S^2 \wedge |D| \xrightarrow[\cong]{\cong} S'_+ \wedge S^{\mathbb{F}} \wedge i_* i^* D \xrightarrow{\varepsilon} S'_+ \wedge S^{\mathbb{F}} \wedge D,$$

where  $TP: S^{\mathbb{F}} \rightarrow S'_+ \wedge S^1$  is the Thom-Pontryagin collapse map associated with the standard inclusion  $S^1 \hookrightarrow \mathbb{C}$ .

We have  $C_r \subset S^1$  and choose a  $C_r$ -invariant division of the normal ball  $\nu$  of the embedding  $S^1 \hookrightarrow \mathbb{C}$  as indicated in the figure to the right. Collapsing the complement of  $\nu$  and the division lines to a point we get a  $C_r$ -map



$$tp: S^{\mathbb{F}} \rightarrow C_r \wedge S^2.$$

Here in the identification of the range we have used the ( $C_r$ -invariant) framing of the normal ball of  $C_r \subset \mathbb{C}$  inherited from the canonical framing of  $\nu$ , see the figure. The following diagram commutes

$$\begin{array}{ccc} S^{\mathbb{F}} & \xrightarrow{TP} & S'_+ \wedge S^1 \\ tp \downarrow \wr & & \downarrow \pi \wr \\ C_r \wedge S^2 / C_r & \xrightarrow{\quad} & S^1 \end{array}$$

We identify the range of the map 6.2.20 using the equivalence of 4.3.15:



$$\Sigma |D| \xrightarrow[\sim]{\tilde{\varepsilon}} i^*(\Omega^{\mathbb{C}}(G_+ \wedge S^{\mathbb{C}} \wedge \Sigma D))^{G_+} \xleftarrow[\sim]{\tilde{\varepsilon}} i^*(G_+ \wedge \Sigma D)^{G_+}$$

where  $\tilde{\varepsilon}$  is the adjoint of

$$i_* S^{\mathbb{C}} \wedge \Sigma |D| \xrightarrow{+p} i_* G_+ \wedge S^2 \wedge \Sigma |D| \xrightarrow{\tilde{\varepsilon}} G_+ \wedge S^{\mathbb{C}} \wedge \Sigma i_* i^* D \xrightarrow{\tilde{\varepsilon}} G_+ \wedge S^{\mathbb{C}} \wedge \Sigma D.$$

The commutativity of the triangular diagram above implies that the following diagram commutes

$$\begin{array}{ccc} \Sigma |D| & \xrightarrow[\sim]{\tilde{\varepsilon}} & i^*(\Omega^{\mathbb{C}}(S_+^1 \wedge S^{\mathbb{C}} \wedge D))^{S^1} \\ & & \downarrow \\ & & i^*(\Omega^{\mathbb{C}}(S_+^1 \wedge S^{\mathbb{C}} \wedge D))^{G_+} \\ & & \downarrow \pi \\ & & i^*(\Omega^{\mathbb{C}}(G_+ \wedge S^{\mathbb{R}} \wedge S^{\mathbb{C}} \wedge D))^{G_+} \\ & & \cong \downarrow \xi \\ \Sigma |D| & \xrightarrow[\sim]{\tilde{\varepsilon}} & i^*(\Omega^{\mathbb{C}}(G_+ \wedge S^{\mathbb{C}} \wedge S^{\mathbb{R}} \wedge D))^{G_+} \end{array}$$

$\downarrow \text{id}$

where  $\xi$  is induced from the composite

$$G_+ \wedge S^{\mathbb{R}} \wedge S^{\mathbb{C}} \xrightarrow{\tilde{\varepsilon}_{23}^{-1}} G_+ \wedge S^{\mathbb{R}^3} \xrightarrow{\tilde{\varepsilon}_{12}} G_+ \wedge S^{\mathbb{C}} \wedge S^{\mathbb{R}}.$$

This map does not permute the coordinates in  $\mathbb{R}^3$ .

But it is  $G_+$ -homotopic to the twist map which permutes the smash factors  $S^{\mathbb{R}}$  and  $S^{\mathbb{C}}$ . Indeed,

$$\begin{array}{ccc} G_+ \wedge S^{\mathbb{R}} \wedge S^{\mathbb{C}} & \xleftarrow{\tilde{\varepsilon}_{23}} & G_+ \wedge S^{\mathbb{R}^3} \\ \downarrow \downarrow \text{tw} & & \downarrow \downarrow \text{tw}_{1,23} \\ G_+ \wedge S^{\mathbb{C}} \wedge S^{\mathbb{R}} & \xleftarrow{\tilde{\varepsilon}_{12}} & G_+ \wedge S^{\mathbb{R}^3} \end{array}$$

commutes and  $tw_{123} \simeq id: S^{\mathbb{R}^3} \rightarrow S^{\mathbb{R}^3}$ . Finally, the diagram

$$\begin{array}{ccc}
 i^*(\Omega^{\mathbb{C}}(S'_+ \wedge S^{\mathbb{C}} \wedge D))^{S^1} & \xleftarrow{\sim} & i^*(S'_+ \wedge D)^{S^1} \\
 \downarrow & & \downarrow \\
 i^*(\Omega^{\mathbb{C}}(S'_+ \wedge S^{\mathbb{C}} \wedge D))^{G_r} & \xleftarrow{\sim} & i^*(S'_+ \wedge D)^{G_r} \\
 \downarrow \pi & & \downarrow \pi \\
 i^*(\Omega^{\mathbb{C}}(G_r \wedge S^{\mathbb{R}} \wedge S^{\mathbb{C}} \wedge D))^{G_r} & & \\
 \downarrow tw_{123} & & \\
 i^*(\Omega^{\mathbb{C}}(G_r \wedge S^{\mathbb{C}} \wedge S^{\mathbb{R}} \wedge D))^{G_r} & \xleftarrow{\sim} & i^*(G_r \wedge S^{\mathbb{R}} \wedge D)^{G_r}
 \end{array}$$

commutes. In all, we see that 6.2.20 is homotopic to the identity map. This concludes the proof.  $\square$

Let  $\tilde{\sigma}: S^{n+k} \rightarrow S^n \wedge X$  represent  $\sigma \in \pi_k^S(X)$  and let  $D$  be a spectrum. Then smashing with  $\tilde{\sigma}$  gives rise to a map

$$\pi_* D \xrightarrow{\sigma} \pi_{*+k}(X \wedge D)$$

which does not depend on the choice of  $\tilde{\sigma}$ . We call this exterior multiplication by  $\sigma$ . Define  $\sigma, \eta$  to be the elements of  $\pi_1^S(S'_+)$  which corresponds to  $(id, \sigma)$  and  $(\alpha, \eta)$ , respectively, under the isomorphism

$$\pi_1^S(S'_+) \xrightarrow{\cong} \pi_1^S(S') \oplus \pi_1^S(S^0)$$

induced by the obvious collapse maps.

**Lemma 6.2.21.** In the Tate spectral sequence  $\hat{E}^*(S^1; T)$  the differential  $d^2: \hat{E}_{2s,t}^2 \rightarrow \hat{E}_{2s-2,t}^2$  is given by

$$\pi_t T \xrightarrow{\sigma + s\eta} \pi_{t+1}(S_+^1 \wedge j^* T) \xrightarrow{\mu} \pi_{t+1} T,$$

where the first map is exterior multiplication and the second map is induced from the  $S^1$ -action on  $j^* T$ .

*Proof.* In order to evaluate the  $d^2$ -differential we consider the following diagram of non-equivariant spectra

$$\begin{array}{ccccc} j^*[\Sigma G_+ \wedge S^{(n-1)\mathbb{C}} \wedge T]^G & \xrightarrow{\partial} & j^*[\Sigma S^{(n-1)\mathbb{C}} \wedge T]^G & \xrightarrow{\text{pr}} & j^*[\Sigma^2 G_+ \wedge S^{(n-2)\mathbb{C}} \wedge T]^G \\ \simeq \uparrow \tau & & & & \simeq \uparrow \tau \\ \Sigma^2 G_+ \wedge_G S^{(n-1)\mathbb{C}} \wedge j^* T & \xrightarrow{\partial} & (\Sigma^2 S^{(n-1)\mathbb{C}} \wedge j^* T)/G & \xrightarrow{\text{pr}} & \Sigma^3 G_+ \wedge_G S^{(n-2)\mathbb{C}} \wedge j^* T \\ \cong \downarrow \bar{\mu} & & & & \cong \downarrow \bar{\mu} \\ \Sigma^2 |S^{(n-1)\mathbb{C}} \wedge T| & & & & \Sigma^3 |S^{(n-2)\mathbb{C}} \wedge T| \end{array}$$

Here  $G$  is the circle group and  $j: \mathcal{U}^G \rightarrow \mathcal{U}$  is the inclusion of the trivial universe. If we apply  $\pi_{m+n}(-)$ , then  $d^2: \hat{E}_{2m,n}^2 \rightarrow \hat{E}_{2m-2,n+1}^2$  is the composite of the maps from the lower left hand corner to the lower right hand corner. The unit map  $S^0 \rightarrow G_+$  induces a map of non-equivariant spectra

$$\iota: \Sigma^2 |S^{(n-1)\mathbb{C}} \wedge T| \rightarrow \Sigma^2 G_+ \wedge S^{(n-1)\mathbb{C}} \wedge j^* T$$

which composed with the projection onto the orbit spectrum is the inverse of  $\bar{\mu}$ . Moreover, the composite

$$\Sigma^2 |S^{(n-1)\mathbb{C}} \wedge T| \xrightarrow{\iota} \Sigma^2 G_+ \wedge S^{(n-1)\mathbb{C}} \wedge j^* T \xrightarrow{\partial} \Sigma^2 S^{(n-1)\mathbb{C}} \wedge j^* T \xrightarrow{\text{pr}} \Sigma^3 G_+ \wedge S^{(n-2)\mathbb{C}} \wedge j^* T$$

represents exterior multiplication by  $\sigma \in \pi_1^S(G_+)$ . We may write the action map  $\mu$  as the composite

$$G_+ \wedge S^{(n-2)\mathbb{C}} \wedge j^* T \xrightarrow{\xi \wedge 1} G_+ \wedge S^{(n-2)\mathbb{C}} \wedge j^* T \xrightarrow{\text{tw} \wedge 1} S^{(n-2)\mathbb{C}} \wedge G_+ \wedge j^* T \xrightarrow{1 \wedge \mu} S^{(n-2)\mathbb{C}} \wedge j^* T,$$

where  $\xi$  is given by

$$\xi: G_+ \wedge S^{(n-2)\mathbb{C}} \xrightarrow{\Delta \wedge 1} G_+ \wedge G_+ \wedge S^{(n-2)\mathbb{C}} \xrightarrow{1 \wedge \mu} G_+ \wedge S^{(n-2)\mathbb{C}}$$

We claim that under the isomorphism  $\pi^S(S_+^1) \cong \pi_1^S(S^1) \oplus \pi_1^S(S^0)$

$$\xi = \begin{pmatrix} 1 & 0 \\ (n-2)\eta & 1 \end{pmatrix},$$

where the matrix multiplies from the right. It suffices to consider the case  $n = 3$  where the representation is  $\mathbb{C}$ . For the case  $n > 3$  follows by composition and the case  $n < 3$  follows by smashing with  $S^{N\mathbb{C}}$  for some  $N > 3 - n$ . Moreover, when  $n = 3$  we may identify  $\xi$  with the composite

$$D(\mathbb{C}) \times S(\mathbb{C}) \cup S(\mathbb{C}) \times D(\mathbb{C}) \xrightarrow{\text{pr}} (S(\mathbb{C}) \times D(\mathbb{C})) / (S(\mathbb{C}) \times S(\mathbb{C})) \xrightarrow{\mu} D(\mathbb{C})/S(\mathbb{C}),$$

and finally, it is well-known that this map represents  $\eta$ . □

Let again  $G$  be a finite group and let  $T$  be a  $G$ -spectrum indexed on a complete universe. The homotopy groups of the terms in the Tate fibration sequence (p.185)

$$H_*(G; T) \xrightarrow{N^h} H^*(G; T) \xrightarrow{B^h} \hat{H}^*(G; T) \quad (6.2.22)$$

are all approximated by spectral sequences:

$$E_{s,t}^2(H_*(G; T)) = H_s(G; \pi_t T) \Rightarrow \pi_{s+t} H_*(G; T), \quad s \geq 0$$

$$E_{s,t}^2(H^*(G; T)) = H^{-s}(G; \pi_t T) \Rightarrow \pi_{s+t} H^*(G; T), \quad s \leq 0$$

$$\hat{E}_{s,t}^2(G; T) = \hat{H}^{-s}(G; \pi_t T) \Rightarrow \pi_{s+t} \hat{H}^*(G; T)$$

The first sequence is that of 4.3.17 and the last is the Tate sequence. The middle sequence is obtained like the Tate sequence using the negatively graded half of 6.2.11 as a 'filtration' of  $S_n^0$ . Therefore, we have a map of spectral sequences

$$R_{s,t}^h : E_{s,t}^r(H^*(G; T)) \rightarrow \hat{E}_{s,t}^r(G; T)$$

which is an isomorphism for  $r=2$  and  $s < 0$  and an epimorphism for  $r > 2$  and  $s < 0$ . Similarly, the homeomorphism  $\sum_{k \in \mathbb{Z}} E_k G_+ \cong \tilde{E}_{k+1} G$  shows that there is a degree -1 map of spectral sequences

$$\partial_{s,t}^r : \hat{E}_{s,t}^r(G; T) \rightarrow E_{s-1,t}^r(H_*(G; T))$$

This is an isomorphism for  $r=2$  and  $s \geq 2$  and a monomorphism for  $r > 2$  and  $s \geq 2$ . For  $r=2$  and

$s=0,1$  the sequences are connected by the exact sequence

$$0 \rightarrow \hat{E}_{1,t}^2(G;T) \xrightarrow{d^2} E_{0,t}^2(H_*(G;T)) \xrightarrow{N} E_{0,t}^2(H^*(G;T)) \xrightarrow{P^h} \hat{E}_{0,t}^2(G;T) \rightarrow 0$$

where  $N$  is the norm map of 6.2.1. So possibly for the line  $s=0$ , the  $E^2$ -term in  $E^*(H^*(G;T))$  is the left half plane of that in the Tate spectral seq. and the differentials in  $E^*(H^*(G;T))$  are those differentials in  $\hat{E}^*(G;T)$  which originate (and end) in the left half plane. Similarly, the  $E^2$ -term in  $E^*(H_*(G;T))$  is, with the possible exception of the line  $s=1$ , part of  $\hat{E}^2(G;T)$  lying to the right of or on the line  $s=1$  shifted down in base degree by 1. The differentials are the those differentials in  $\hat{E}^*(G;T)$  which end (and originate) in the half plane right of or on  $s=1$ . Hence the Tate sp seq.  $\hat{E}^*(G;T)$  'contains' the other two. The differentials in the Tate seq. which crosses the line  $s=1/2$  give information about the map  $N^h$  in 6.3.22. For a precise statement see [BM] 2.15.

Finally, we mention without proof that if  $T$  is a  $G$ -ring spectrum then  $H_*(G;T)$  and  $H^*(G;T)$  are ring spectra and the corresponding spectral sequences multiplicative.

### 6.3 $TC(\mathbb{F}_p)$

Let  $T = TC(\mathbb{F}_p)$  be the topological Hochschild spectrum of  $\mathbb{F}_p$ ; cf. 4.2.9. It is cyclotomic by 5.1.12. We shall evaluate the fixed point spectra  $TC(\mathbb{F}_p)^{G_p^n}$ . To this end, we use the map of fibration sequences 6.2.10, which since  $TC(\mathbb{F}_p)$  is cyclotomic takes the form

$$\begin{array}{ccccc} H_0(G_p^n; TC(\mathbb{F}_p)) & \xrightarrow{\sim} & TC(\mathbb{F}_p)^{G_p^n} & \xrightarrow{R} & TC(\mathbb{F}_p)^{G_p^{n-1}} \\ \parallel & & \downarrow \Gamma_n & & \downarrow \hat{\Gamma}_n \\ H_1(G_p^n; TC(\mathbb{F}_p)) & \xrightarrow{N^n} & H^1(G_p^n; TC(\mathbb{F}_p)) & \xrightarrow{R^n} & \hat{H}^1(G_p^n; TC(\mathbb{F}_p)) \end{array} \quad (6.3.1)$$

and the spectral sequences for the homotopy groups in the lower sequence. We prove inductively in 6.3. that the maps  $\Gamma_n$  and  $\hat{\Gamma}_n$  induce isomorphisms of homotopy groups in non-negative dimensions.

Let  $S/p$  be the mod  $p$  Moore spectrum and set  $T/p = T \wedge S/p$ . Since  $S/p$  is finite  $\hat{H}^1(G_p^n; T/p) \simeq \hat{H}^1(G_p^n; T)/p$  so the Tate spectral seq.  $\hat{E}^r(G_p^n; T/p)$  approximates the mod  $p$  homotopy groups  $\pi_*(\hat{H}^1(G_p^n; T); \mathbb{F}_p)$ . We have maps of spectral sequences

$$\begin{array}{ccc} \hat{E}^r(S'; T) & \longrightarrow & \hat{E}^r(G_p^n; T) \\ \downarrow & & \downarrow \\ \hat{E}^r(S'; T/p) & \longrightarrow & \hat{E}^r(G_p^n; T/p), \end{array}$$

the upper spectral sequences are multiplicative and

the lower sp. seq. are module sp. seq. over the upper spectral sequences. The  $E^2$ -terms are

$$\hat{E}^2(S'; T) = S_{\mathbb{F}_p} \{t, t^{-1}\} \otimes S_{\mathbb{F}_p} \{\sigma\}$$

$$\hat{E}^2(G^n; T) = \Lambda_{\mathbb{F}_p} \{u_n\} \otimes S_{\mathbb{F}_p} \{t, t^{-1}\} \otimes S_{\mathbb{F}_p} \{\sigma\}$$

$$\hat{E}^2(S'; T/p) = S_{\mathbb{F}_p} \{t, t^{-1}\} \otimes \Lambda_{\mathbb{F}_p} \{e_1\} \otimes S_{\mathbb{F}_p} \{\bar{\sigma}\} \quad (6.3.2)$$

$$\hat{E}^2(G^n; T/p) = \Lambda_{\mathbb{F}_p} \{u_n\} \otimes S_{\mathbb{F}_p} \{t, t^{-1}\} \otimes \Lambda_{\mathbb{F}_p} \{e_1\} \otimes S_{\mathbb{F}_p} \{\bar{\sigma}\}$$

where  $\deg t = (-2, 0)$ ,  $\deg \sigma = (0, 2)$ ,  $\deg u_n = (-1, 0)$  and  $\deg e_1 = (0, 1)$ . Moreover, the Bockstein on  $e_1$  is 1. The identifications of the first two sequences are as bi-graded  $\mathbb{F}_p$ -algebras except for the particular case

$$\hat{E}^2(C_2; T) = S_{\mathbb{F}_p} \{u_1, u_1^{-1}\} \otimes S_{\mathbb{F}_p} \{\sigma\} \quad ; \quad \deg u_1 = (-1, 0);$$

compare 6.2.7.  $\hat{E}^2(S'; T/p)$  and  $\hat{E}^2(G^n; T/p)$  are free modules over  $\hat{E}^2(S'; T)$  and  $\hat{E}^2(G^n; T/p)$ , respectively, on generators 1 and  $e_1$ .

Lemma 6.3.3 The differentials in  $\hat{E}^r(G^n; T/p)$  are generated in the  $\hat{E}^r(G^n; T)$ -module structure by  $d^2 e_1 = t\bar{\sigma}$ .  
Hence

$$\pi_*(H^1(G^n; T/\mathbb{F}_p); \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p} \{u_n\} \otimes S_{\mathbb{F}_p} \{t, t^{-1}\}$$

as a graded abelian group.

¶ In  $\hat{E}^2(S'; T)$  all elements are concentrated in even

total degree, so there are no differentials in this sp. seq. It follows from 6.2.16 that  $t$  and  $\sigma$  are permanent cycles in  $\hat{E}^r(G^n; T)$ . We also note that for degree reasons  $d^2 u_n = 0$  in that sequence. Bökestedt's calculation of  $\pi_* T(\mathbb{F}_p)$  shows that the action map

$$\pi_1^S(S_+^1) \otimes \pi_1(T(\mathbb{F}_p); \mathbb{F}_p) \xrightarrow{\mu} \pi_2(T(\mathbb{F}_p); \mathbb{F}_p)$$

maps  $\sigma \otimes e_1$  to  $\bar{\sigma}$ . By 6.2.21 this implies  $d^2 e_1 = t\bar{\sigma}$  in  $\hat{E}^2(S_+^1; T/p)$  and by 6.2.16 also in  $\hat{E}^2(G^n; T/p)$ . The module structure over  $\hat{E}^2(G^n; T)$  now gives all the  $d^2$ -differentials and we find

$$\hat{E}^3(G^n; T/p) = \Delta_{\mathbb{F}_p} \{u_n\} \otimes S_{\mathbb{F}_p} \{t, t^{-1}\}$$

and there is no room for further differentials.  $\square$

The proof above also shows that in  $\hat{E}^*(G^n; T)$  the differentials are multiplicatively generated from

$$d^{2r+1} u_n = t^{r+1} \sigma^r, \quad (6.3.4)$$

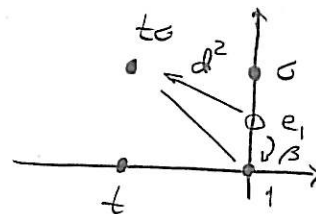
for some  $r \geq 3$ , or there are no differentials at all. We shall determine  $r$  later on. First we solve the extension problems in passing from  $\hat{E}^\infty$  to the actual homotopy groups:

Lemma 6.3.5 The groups  $\pi_* \hat{H}(G^n; \pi(\mathbb{F}_p))$  are cyclic  $\mathbb{Z}_p$ -modules.

Pf The differential 6.3.4, if it is there, will kill the



part of the spectral sequence lying on or above the line  $t=2r$ . So in total degree  $2n$  in  $\hat{E}^\infty = \hat{E}^{2r+2}$  there are the  $r+1$  non-zero groups  $\hat{E}_{2(n-i), 2i}^\infty$ ,  $0 \leq i \leq r$ , all of which are isomorphic to  $\mathbb{F}_p$ . We claim that these assemble to give a copy of  $\mathbb{Z}/p^{r+1}\mathbb{Z}$ , or  $\mathbb{Z}_p$  if  $r = \infty$ . We show that the groups  $\hat{E}_{0,0}^\infty$  and  $\hat{E}_{-2,2}^\infty$  assemble to give  $\mathbb{Z}/p^2\mathbb{Z}$ ; the general case is complete analogues. We write out the exact complex which gives the spectral sequence:



$$\begin{array}{cccccccc}
 \vdots & & \vdots & & \vdots & & \vdots & \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 \cdots \rightarrow D_{-1,2} \rightarrow E_{-1,2} \rightarrow D_{-2,2} \rightarrow E_{-2,2} \rightarrow D_{-3,2} \rightarrow E_{-3,2} \rightarrow D_{-4,2} \rightarrow \cdots \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 \cdots \rightarrow D_{0,1} \rightarrow E_{0,1} \rightarrow D_{-1,1} \rightarrow E_{-1,1} \rightarrow D_{-2,1} \rightarrow E_{-2,1} \rightarrow D_{-3,1} \rightarrow \cdots \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 \cdots \rightarrow D_{1,0} \rightarrow E_{1,0} \rightarrow D_{0,0} \rightarrow E_{0,0} \rightarrow D_{-1,0} \rightarrow E_{-1,0} \rightarrow D_{-2,0} \rightarrow \cdots \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 \pi_1 \hat{H}(\mathbb{G}^n; T) & & \pi_0 \hat{H}(\mathbb{G}^n; T) & & \pi_{-1} \hat{H}(\mathbb{G}^n; T) & & \pi_{-2} \hat{H}(\mathbb{G}^n; T) & 
 \end{array}$$

$\tilde{t}_0 \downarrow d^2 \rightarrow t_0$   
 $e_1 \nearrow$

Let  $1 \in E_{0,0}$ ,  $t_0 \in E_{-2,2}$  be representatives of the classes with the same names in  $E^\infty$ . Since these are permanent cycles, there are elements  $\tilde{1} \in D_{0,0}$  and  $\tilde{t}_0 \in D_{-2,2}$  mapping to  $1 \in E_{0,0}$  and  $t_0 \in E_{-2,2}$ , respectively, and since 1 and  $t_0$  are not boundaries in the

spectral sequence, the classes  $\tilde{\tau}$  and  $\tilde{\sigma}$  maps non-trivially to  $\pi_0 \hat{H}(G^n; T)$ . We want show that  $p \tilde{\tau} \neq 0$  in  $D_{0,0}$ . To this end we consider the exact couple which gives the sp. seq.  $\hat{E}^*(G^n; T/p)$ . We write this  $\bar{E}_{s,t}$ ,  $\bar{D}_{s,t}$ . We have the exact sequence

$$\bar{D}_{0,1} \xrightarrow{\beta} D_{0,0} \xrightarrow{p} D_{0,0} \rightarrow \bar{D}_{0,0}.$$

Suppose  $\tilde{\tau} \in D_{0,0}$  is killed by  $p$ . Then there is an element  $\tilde{q} \in \bar{D}_{0,1}$  whose Bockstein is  $\tilde{\tau}$ . This element would map to an element  $q \in \bar{E}_{0,1}$  with Bockstein  $1 \in \bar{E}_{0,0}$ . Hence  $q$  would map to  $0 \in \bar{D}_{-1,1}$  and therefore represent a permanent cycle in  $\hat{E}_{0,1}^*(G^n; T/p)$ . But we know from 6.3.3 that  $d^2 q \neq 0$  and hence  $p \tilde{\tau} \neq 0$ .  $\square$

Corollary 6.3.6  $\pi_* \hat{H}(G; T(\mathbb{F}_p)) \cong S_{\mathbb{F}_p} \{\bar{\sigma}, \bar{\sigma}^{-1}\}; \deg \bar{\sigma} = 2.$

Pf The map  $\hat{\Gamma}_1: T(\mathbb{F}_p) \rightarrow \hat{H}(G; T(\mathbb{F}_p))$  is a map of ring spectra, so in particular it preserves the unit. It follows that  $\pi_0 T(\mathbb{F}_p) \rightarrow \pi_0 \hat{H}(G; T(\mathbb{F}_p))$  maps the generator to the generator and hence that  $\pi_0 \hat{H}(G; T(\mathbb{F}_p)) = \mathbb{F}_p \mathbb{Z}$ . This can only happen if in 6.3.4,  $r = 1$ .  $\square$

Lemma 6.3.7 The map  $\hat{\Gamma}_1: T(\mathbb{F}_p) \rightarrow \hat{H}(G; T(\mathbb{F}_p))$  induces a weak equivalence of connective covers.

Pf On homotopy groups,  $\hat{\Gamma}_1$  induces a ring homomorphism

$$\hat{\Gamma}_{1*}: S_{\mathbb{F}_p} \{\sigma\} \rightarrow S_{\mathbb{F}_p} \{\bar{\sigma}, \bar{\sigma}^{-1}\}.$$

It suffices to show that  $\hat{\Gamma}_{1*} \sigma \neq 0$ . Consider the

diagram

$$\begin{array}{ccc}
 \pi_2 T(\mathbb{F}_p) & \xrightarrow{\partial_1} & \pi_1 H_1(G; T(\mathbb{F}_p)) \\
 \downarrow \hat{\Gamma}_1 & & \parallel \\
 \pi_2 H_1(G; T(\mathbb{F}_p)) & \rightarrow & \pi_1 H_1(G; T(\mathbb{F}_p))
 \end{array}$$

We may instead show that  $\partial_1$  is 1-1. It will then be an isomorphism since we already know from 52.4 that  $\pi_i H_1(G; T(\mathbb{F}_p)) \cong \mathbb{F}_p$ ,  $i=0, 1$ . The map  $\partial_1$  is induced from a map of  $S^1$ -spectra

$$T(\mathbb{F}_p) \xleftarrow{\sim} \rho_p^\#(\tilde{E}S^1 \wedge T(\mathbb{F}_p))^{G_p} \xrightarrow{\partial} \rho_p^\#(\Sigma E S_+^1 \wedge T(\mathbb{F}_p))^{G_p}$$

and therefore commutes with the action maps. We use the diagram

$$\begin{array}{ccccc}
 \pi_1^S(S_+^1) \otimes \pi_1(T(\mathbb{F}_p); \mathbb{F}_p) & \xrightarrow{\mu_1} & \pi_2(T(\mathbb{F}_p); \mathbb{F}_p) & \xleftarrow{\sim} & \pi_2 T(\mathbb{F}_p) \\
 \downarrow 1 \otimes \bar{\partial}_0 & & \downarrow \bar{\partial}_1 & & \\
 \pi_1^S(S_+^1) \otimes \pi_0(H_1(G; T(\mathbb{F}_p)); \mathbb{F}_p) & \rightarrow & \pi_1(H_1(G; T(\mathbb{F}_p)); \mathbb{F}_p) & & \\
 \uparrow \cong & & \uparrow & \nearrow \partial_1 & \\
 \pi_1^S(S_+^1) \otimes \pi_0 H_1(G; T(\mathbb{F}_p)) & \xrightarrow{\mu_0} & \pi_1 H_1(G; T(\mathbb{F}_p)) & & 
 \end{array}$$

Let  $\sigma \in \pi_0 H_1(G; T(\mathbb{F}_p))$  be a generator. We claim that  $\mu_0(\sigma \otimes \sigma)$  is a generator of  $\pi_1 H_1(G; T(\mathbb{F}_p))$  and that the map  $\bar{\partial}_0$  above is an isomorphism. Granting this, a diagram chase shows  $\partial_1(\sigma) \neq 0$  and hence the lemma.

To prove the first claim we note that the map induced from the inclusion of the 1-skeleton,

$$\rho_{G_p}^{\#}(S_+^1 \wedge T(\mathbb{F}_p))^{G_p} \rightarrow \rho_{G_p}^{\#}(ES_+^1 \wedge T(\mathbb{F}_p))^{G_p}$$

induces an isomorphism on  $\pi_i$ ,  $i=0,1$ . This is because the group  $E_{0,1}^2(H_1(G; T(\mathbb{F}_p))) \cong HH_1(\mathbb{F}_p)$  vanishes, c.f. 5.2.4. From 4.3.15 we have a weak  $S^1$ -equivalence in  $S^1\mathcal{U}$

$$\Sigma F(S_+^1, T(\mathbb{F}_p)) \xrightarrow{\sim} S_+^1 \wedge T(\mathbb{F}_p)$$

and restricting to the trivial universe

$$i^*F(S_+^1, T(\mathbb{F}_p)) = F(S_+^1, i^*T(\mathbb{F}_p)) \cong F(S_+^1, |T(\mathbb{F}_p)|).$$

Taking  $G_p$ -fixed sets,  $F(S_+^1, |T(\mathbb{F}_p)|)^{G_p} = F(S_+^1/G_p, |T(\mathbb{F}_p)|)$  so in all we get a w.e. in  $S^1\mathcal{U}^{S^1}$

$$i^*\rho_{G_p}^{\#}(S_+^1 \wedge T(\mathbb{F}_p))^{G_p} \xleftarrow{\sim} \Sigma F(S_+^1, |T(\mathbb{F}_p)|).$$

We want to show that the action map, which we may write as the composite

$$S_+^1 \wedge \Sigma F(S_+^1, |T(\mathbb{F}_p)|) \xrightarrow{1 \times u^*} S_+^1 \wedge \Sigma F(S_+^1 \times S_+^1, |T(\mathbb{F}_p)|)$$

$$\cong S_+^1 \wedge \Sigma F(S_+^1, F(S_+^1, |T(\mathbb{F}_p)|)) \xrightarrow{ev} \Sigma F(S_+^1, |T(\mathbb{F}_p)|)$$

takes the element  $\sigma \in \pi_1^S(S_+^1)$  tensor the generator in  $\pi_0 \Sigma F(S_+^1, |T(\mathbb{F}_p)|)$  to one in  $\pi_1 \Sigma F(S_+^1, |T(\mathbb{F}_p)|)$ .

Now the inclusion of the zero-skeleton  $H\mathbb{F}_p \rightarrow |T(\mathbb{F}_p)|$  induces an isomorphism in the relevant degrees and the question translates to showing that

the composite

$$H_1(S^1) \otimes H^1(S^1; \mathbb{F}_p) \xrightarrow{1 \otimes \mu^*} H_1(S^1) \otimes H^1(S^1 \times S^1; \mathbb{F}_p) \xrightarrow{\quad} H^0(S^1; \mathbb{F}_p)$$

is an isomorphism. This is easy to see.

To prove the second claim we consider the exact sequence

$$\pi_1(T(\mathbb{F}_p); \mathbb{F}_p) \xrightarrow{\bar{\partial}_0} \pi_0(H^1(\mathbb{C}_p; T(\mathbb{F}_p)); \mathbb{F}_p) \xrightarrow{N} \pi_0(T(\mathbb{F}_p)^{\mathbb{C}_p}; \mathbb{F}_p) \xrightarrow{R} \pi_0(T(\mathbb{F}_p); \mathbb{F}_p) \rightarrow 0$$

We know from 5.5.7 that  $\pi_0 T(\mathbb{F}_p)^{\mathbb{C}_p} \cong \mathbb{Z}/p^2\mathbb{Z}$  so the groups in the above sequence are all a copy of  $\mathbb{F}_p$ . Hence  $R$  and  $\bar{\partial}_0$  must be isomorphisms and  $N = 0$ .  $\square$

Proposition 6.3.8 The maps  $\hat{\Gamma}_n: T(\mathbb{F}_p)^{\mathbb{C}_p^{n-1}} \rightarrow \hat{H}^1(\mathbb{C}_p^n; T(\mathbb{F}_p))$  and  $\Gamma_n: T(\mathbb{F}_p)^{\mathbb{C}_p^n} \rightarrow H^1(\mathbb{C}_p^n; T(\mathbb{F}_p))$  induce weak equivalences of connective covers, for all  $n \geq 1$ .

Pf The proof is by induction with the case  $n=1$  given by the lemma and 6.3.1. The spectra involved are all  $p$ -complete so it is enough to show that  $\hat{\Gamma}_n$  and  $\Gamma_n$  induce isomorphism on mod  $p$  homotopy groups in non-negative dimensions. We shall write  $\hat{T}(\mathbb{F}_p) = \hat{A}_p^{\#} \hat{H}^1(\mathbb{C}_p; T(\mathbb{F}_p))$  and by abuse of notation we write the maps

$$\hat{A}_p^{\#}(\tilde{E}S^1 \wedge T(\mathbb{F}_p)^w)^{\mathbb{C}_p} \longrightarrow \hat{A}_p^{\#}(\tilde{E}S^1 \wedge F(E S^1_+, T(\mathbb{F}_p))^w)^{\mathbb{C}_p}$$

as

$$\hat{\gamma}: T(\mathbb{F}_p) \rightarrow \hat{T}(\mathbb{F}_p).$$

Then  $\hat{\Gamma}_n = \hat{\gamma}^{\mathbb{C}_p^{n-1}}$ . We consider the diagram

$$\begin{array}{ccc}
 T(\mathbb{F}_p)G^{n-1} & \xrightarrow{\hat{\gamma}^{G^{n-1}}} & \hat{T}(\mathbb{F}_p)G^{n-1} \\
 \downarrow \Gamma_{n-1} & & \downarrow G_{n-1} \\
 H^*(G^{n-1}; T(\mathbb{F}_p)) & \xrightarrow{\hat{\gamma}^{H^*G^{n-1}}} & H^*(G^{n-1}; \hat{T}(\mathbb{F}_p))
 \end{array} \quad (6.3.9)$$

We want to show that  $\hat{\gamma}^{G^{n-1}} = \hat{\Gamma}_n$  induces an equivalence of connective covers assuming that this holds for  $\Gamma_{n-1}$ . By lemma 6.3.7,  $\hat{\gamma}$  induce isomorphisms of homotopy groups in non-negative dimensions and the spectral sequence 6.2.22 then shows that the same holds for the lower horizontal map above. We show that  $G_{n-1}$  induce isomorphisms of mod  $p$  homotopy groups. From 6.3.6

$$\pi_* \hat{T}(\mathbb{F}_p) \cong S_{\mathbb{F}_p} \{ \hat{\sigma}, \hat{\sigma}^{-1} \}.$$

We consider the Tate spectral sequences  $\hat{E}^*(G^{n-1}; \hat{T}(\mathbb{F}_p))$   $\hat{E}^*(G^{n-1}; \hat{T}(\mathbb{F}_p)/p)$ . The former is multiplicative and the latter is a module spectral seq over that. We have

$$\hat{E}^2(G^{n-1}; \hat{T}(\mathbb{F}_p)/p) \cong \Delta_{\mathbb{F}_p} \{ u, t \} \otimes S_{\mathbb{F}_p} \{ \hat{\sigma}, \hat{\sigma}^{-1} \} \otimes \Delta_{\mathbb{F}_p} \{ \hat{e}_1 \} \otimes S_{\mathbb{F}_p} \{ \hat{\sigma}, \hat{\sigma}^{-1} \}.$$

The map  $\hat{\gamma}: T(\mathbb{F}_p) \rightarrow \hat{T}(\mathbb{F}_p)$  induces a map of spectral sequences which injects the  $E^2$ -term. It follows that  $d^2 \hat{e}_1 = t \hat{\sigma}$  and that  $\hat{E}^*(G^{n-1}; \hat{T}(\mathbb{F}_p))$  has no  $d^2$ -differentials. This gives all the  $d^2$ -differentials in  $\hat{E}(G^{n-1}; T(\mathbb{F}_p)/p)$ . We see that

$$d^2(\hat{e}_1 t^{-1} \hat{\sigma}^{-1}) = 1.$$

such that  $\hat{E}^3 = 0$ . Hence  $\pi_*(H^1(G^{n-1}; \hat{T}(\mathbb{F}_p)); \mathbb{F}_p) = 0$  and

and the spectrum being  $p$ -complete, we get

$$\hat{H}^1(\mathbb{G}^{n-1}; \hat{T}(\mathbb{F}_p)) \sim *.$$

Restricting to the left half plane, we also get the  $d^2$ -differentials in  $E^*(H^1(\mathbb{G}^{n-1}; \hat{T}(\mathbb{F}_p)) / p)$ . We get

$$E^3 = \Lambda_{\mathbb{F}_p} \{u_{n-1}\} \otimes S_{\mathbb{F}_p} \{\bar{\sigma}, \bar{\sigma}^{-1}\}$$

and there is no room for further differentials, so

$$\pi_*(H^1(\mathbb{G}^{n-1}; \hat{T}(\mathbb{F}_p); \mathbb{F}_p)) \cong \Lambda_{\mathbb{F}_p} \{u_{n-1}\} \otimes S_{\mathbb{F}_p} \{\bar{\sigma}, \bar{\sigma}^{-1}\}.$$

Hence the domain and the range of the map  $G_{n-1}$  in 6.3.9 have abstractly isomorphic mod  $p$  homotopy groups. Therefore, it is enough to show that  $\pi_*(G_{n-1}; \mathbb{F}_p)$  is a surjection. This follows from the diagram

$$\begin{array}{ccccc} H_1(\mathbb{G}^{n-1}; \hat{T}(\mathbb{F}_p)) & \xrightarrow{N} & \hat{T}(\mathbb{F}_p)^{\mathbb{G}^{n-1}} & & \\ \parallel & & \downarrow G_{n-1} & & \\ H_0(\mathbb{G}^{n-1}; \hat{T}(\mathbb{F}_p)) & \xrightarrow[\sim]{N^h} & H^0(\mathbb{G}^{n-1}; \hat{T}(\mathbb{F}_p)) & \xrightarrow[\sim]{R^h} & \hat{H}^1(\mathbb{G}^{n-1}; \hat{T}(\mathbb{F}_p)) \xrightarrow[\sim]{*} \end{array}$$

and the fact that  $\hat{H}^1(\mathbb{G}^{n-1}; \hat{T}(\mathbb{F}_p))$  is w.e. to a point.  $\square$

Remark 6.3.10 A theorem of S. Tsalidis states that for every connective cyclotomic spectrum  $T$  satisfying a certain mild hypothesis, if  $\hat{\Gamma}_1: T \rightarrow \hat{H}^1(\mathbb{G}; T)$  induces an equivalence of connective covers, then it does

$$\hat{\Gamma}_n: T^{\mathbb{G}^{n-1}} \rightarrow \hat{H}^1(\mathbb{G}^n; T)$$

for all  $n \geq 1$ . This gives a different proof of 6.3.8 assuming 6.3.7.

Proposition 6.3.11  $\pi_* \hat{H}(G_n; T(\mathbb{F}_p)) \cong S_{\mathbb{Z}/p^n\mathbb{Z}} \{\hat{\sigma}, \hat{\sigma}^{-1}\}.$

Pf We have just shown that

$$\Gamma_{n*} : \pi_0 T(\mathbb{F}_p)^{G_{p^{n-1}}} \xrightarrow{\cong} \pi_0 \hat{H}(G_n; T(\mathbb{F}_p))$$

and the domain is isomorphic to  $\mathbb{Z}/p^n\mathbb{Z}$  by 5.5.7. It follows from 6.3.5 that the differential 6.3.4 is

$$d^{2n+1} u_n = t^{n+1} \sigma^n.$$

This determines the spectral sequence  $\hat{E}^*(G_n; T(\mathbb{F}_p))$ .  $\square$

Theorem 6.3.12 As a graded ring

$$\pi_* T(\mathbb{F}_p)^{G_{p^{n-1}}} \cong S_{\mathbb{Z}/p^n\mathbb{Z}} \{\sigma_n\} \quad ; \quad \deg \sigma_n = 2.$$

Moreover,  $F\sigma_n = \sigma_{n-1}$ ,  $V\sigma_n = p\sigma_{n+1}$  and  $R\sigma_n = p\lambda_n\sigma_{n-1}$ , where  $\lambda_n \in (\mathbb{Z}/p^n\mathbb{Z})^\times$  is a unit.

Pf The first claim follows from 6.3.9 and 6.3.11. To prove the claim for  $F$ , we use the diagram

$$\begin{array}{ccc} T(\mathbb{F}_p)^{G_{p^{n-1}}} & \xrightarrow{F} & T(\mathbb{F}_p)^{G_{p^{n-2}}} \\ \downarrow \hat{\Gamma}_n & & \downarrow \hat{\Gamma}_{n-1} \\ \hat{H}(G_n; T(\mathbb{F}_p)) & \xrightarrow{F^n} & \hat{H}(G_{n-1}; T(\mathbb{F}_p)) \end{array}$$

The vertical maps induce isomorphism of  $H^q$  groups in non-negative dimensions and the map  $F^n$  induces the restriction maps in Tate cohomology

$$\text{res} : \hat{H}^*(G_n; \pi_* T(\mathbb{F}_p)) \rightarrow \hat{H}^*(G_{n-1}; \pi_* T(\mathbb{F}_p))$$

on the  $E^2$ -terms of the spectral sequences. This  $\square$



isomorphism in even degrees. Alternatively, the map  $\hat{E}^2(S'; T(\mathbb{F}_p)) \rightarrow \hat{E}^2(\mathbb{G}_p^n; T(\mathbb{F}_p))$  is an isomorphism in even base degree, for all  $n \geq 1$ . In any case, it follows that we may pick our generators  $\sigma_n$  s.t.  $F\sigma_n = \sigma_{n-1}$ . For  $V$  we use Frobenius reciprocity:  $V\sigma_{n-1} = VF\sigma_n = p\sigma_n$ . Finally, the claim for  $R$  follows from the exact sequence

$$\pi_2 T(\mathbb{F}_p)^{\mathbb{G}_p^{n-1}} \xrightarrow{R} \pi_2 T(\mathbb{F}_p)^{\mathbb{G}_p^{n-2}} \rightarrow \pi_1 H_1(\mathbb{G}_p^{n-1}; T(\mathbb{F}_p)) \rightarrow \pi_1 T(\mathbb{F}_p)^{\mathbb{G}_p^{n-1}}$$

since  $\pi_1 H_1(\mathbb{G}_p^{n-1}; T(\mathbb{F}_p)) \cong \mathbb{F}_p$  and  $\pi_1 T(\mathbb{F}_p)^{\mathbb{G}_p^{n-1}} = 0$ .  $\square$

Corollary 6.3.13 The topological cyclic homology of  $\mathbb{F}_p$  is the generalized Eilenberg-MacLane spectrum

$$TC(\mathbb{F}_p; p) \simeq H(\mathbb{Z}_p; 0) \vee H(\mathbb{Z}_p, -1).$$

Pf Recall from 5.5.11 that we have a homotopy equalizer

$$TC(\mathbb{F}_p; p) \rightarrow TR(\mathbb{F}_p; p) \begin{matrix} \xrightarrow{F} \\ \xrightarrow{\text{rel}} \end{matrix} TR(\mathbb{F}_p; p).$$

Now 6.3.13 and Milnor's sequence

$$0 \rightarrow \varprojlim_{\leftarrow R}^{(1)} \pi_{s-1} T(\mathbb{F}_p)^{\mathbb{G}_p^n} \rightarrow \pi_s TR(\mathbb{F}_p; p) \rightarrow \varprojlim_{\leftarrow R} \pi_s T(\mathbb{F}_p)^{\mathbb{G}_p^n} \rightarrow 0$$

shows that  $\pi_0 TR(\mathbb{F}_p; p) = \mathbb{Z}_p$  is the only non-zero group. Hence

$$TR(\mathbb{F}_p; p) \simeq H(\mathbb{Z}_p; 0)$$

and  $F \simeq \text{rel}$ . The claim follows.  $\square$