Topological Cyclic Homology

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Contents

1.1. Cyclic sets

Let G be a group. The cyclic bar construction of G is the simplicial set with k-simplices

$$N_k^{cy}(G) = \underbrace{G \times \dots \times G}_{k+1} \tag{1.1.1}$$

and simplicial structure maps

$$\begin{aligned} d_i(g_0,\dots,g_k) &= (g_0,\dots,g_i g_{i+1},\dots,g_k) \quad , \ 0 \le i < k \\ &= (g_k g_0, g_1,\dots, g_{k-1}) \quad , \ i = k \end{aligned}$$

$$s_i(g_0,\dots,g_k) = (g_0,\dots,g_i,1,g_{i+1},\dots,g_k)$$

We note that none of the degeneracy maps puts a 1 in the first factor. There is an extra operator given by

$$t_k(g_0,\dots,g_k) = (g_k, g_0, g_1, \dots, g_{k-1})$$

and one easily verifies the formulas

$$
\begin{aligned}
d_i d_j &= d_{j-1} d_i &&, \ i<j \\
s_i s_j &= s_j s_{i-1} &&, \ i>j \\
d_i s_j &= s_{j-1} d_i &&, \ i<j \\
&= id &&, \ i=j, j+1 \\
&= s_j d_{i-1} &&, \ i>j+1 \qquad (1.1.2) \\
d_j t_k &= d_k &&, \ j=0 \\
&= t_{k-1} d_{j-1} &&, \ 1 \leq j \leq k \\
s_j t_k &= t_{k+1}^2 s_k &&, \ j=0 \\
&= t_{k+1} s_{j-1} &&, \ 1 \leq j \leq k \\
t_k^{k+1} &= id
\end{aligned}
$$

__Definition__ 1.1.3 A cyclic set $X_\circ$ consists of sets X_k, $k \geq 0$, together with maps

$$
\begin{aligned}
d_i &: X_k \to X_{k-1} &&, \ 0 \leq i \leq k \\
s_i &: X_k \to X_{k+1} &&, \ 0 \leq i \leq k \\
t_k &: X_k \to X_k
\end{aligned}
$$

such that the formulas (1.1.2) hold.

Let Δ be the category with objects the finite ordered sets

$$[n] = \{0, 1, 2, \ldots, n\} \quad , \quad n \geq 0,$$

and with morphisms the weakly increasing maps, and recall that a simplicial set may be viewed as a functor

$$Y_\bullet : \Delta^{op} \longrightarrow \text{Sets},$$

We may similarly view a cyclic set as a functor

$$X_\bullet : \Lambda^{op} \longrightarrow \text{Sets},$$

where Λ is the cyclic category, which we now describe. We have

$$\text{ob}\, \Lambda = \text{ob}\, \Delta$$

but there are more morphisms: by definition

$$\text{Aut}_{\Lambda}([n]) := C_{n+1}$$

is the set of cyclic permutations of the set $[n] = \{0, 1, 2, .., n\}$, and any morphism may be written <u>uniquely</u> as a composite

$$[n] \xrightarrow{u} [n] \xrightarrow{f} [m]$$

with $u \in Aut_{\mathbb{\Lambda}}([n])$ and $f \in \Delta([n],[m])$. That is,

$$\mathbb{\Lambda}([n],[m]) = \Delta([n],[m]) \times Aut_{\mathbb{\Lambda}}([n]) \quad (1.1.4)$$

Note that in particular there are $n+1$ morphisms in $\mathbb{\Lambda}$ from $[n]$ to $[0]$, so in general morphisms cannot be described as a map of sets $[n] \to [m]$.

The composition

$$[m] \xrightarrow{u} [m] \xrightarrow{f} [k],$$

$$[n] \xrightarrow{v} [n] \xrightarrow{g} [m]$$

is given by the diagram

$$
\begin{tikzcd}
{[n]} \arrow[r, "v"] \arrow[dr] & {[n]} \arrow[r, "g"] \arrow[d, "g^*u"] & {[m]} \arrow[d, "u"] \\
 & {[n]} \arrow[r, "u_*g"] \arrow[dr] & {[m]} \arrow[d, "f"] \\
 & & {[k]}
\end{tikzcd}
\qquad (1.1.5)
$$

where $g^*u \in \mathrm{Aut}_\Delta([n])$ and $u_*g \in \Delta([n],[m])$ are defined as follows: let $A_i = g^{-1}(i)$, $0 \le i \le m$, then

$$[n] = A_0 \amalg A_1 \amalg \dots \amalg A_m .$$

Here some of the A_i's may be empty, some may contain several elements. We let u permute the A_i's: define $B_i = A_{u^{-1}(i)}$, then also

$$[n] = B_0 \amalg B_1 \amalg \dots \amalg B_m$$

but the ordering is changed. The permutation which takes the j'th element in the new ordering to the j'th element in the old ordering is g^*u. Obviously, g^*u is a cyclic permutation of $[n]$. Finally, u_*g is defined by the requirement that the square in (1.1.5) be a commutative square of set maps. It is the unique weakly increasing map determined by

$$\#(u_*g)^{-1}(i) = \# g^{-1}(u^{-1}(i)).$$

<u>Exercise</u> 1.1.5a Show that Λ is a category.

Let

$$d^i : [n-1] \to [n] \quad , 0 \leq i \leq n$$

$$s^i : [n+1] \to [n] \quad , 0 \leq i n$$

denote the unique weakly increasing injective map which leaves out i and the unique weakly increasing surjective map which repeats i, respectively. In addition let

$$I_n : [n] \to [n]$$

denote the cyclic permutation

$$\tau_n(i) = i-1 \qquad (\text{mod } n+1).$$

<u>Lemma</u> 1.1.6 The morphisms in $\Delta\!\!\!\Delta$ are generated by

$$\begin{aligned}
d^i &: [n-1] \to [n] &&, 0 \le i \le n \\
s^i &: [n+1] \to [n] &&, 0 \le i \le n \\
\tau_n &: [n] \to [n] &&
\end{aligned}$$

subject to the relations

$$\begin{aligned}
d^j d^i &= d^i d^{j-1} &&, i<j \\
s^j s^i &= s^{i-1} s^j &&, i>j \\
s^j d^i &= d^i s^{j-1} &&, i<j \\
&= \mathrm{id} &&, i=j,\ j+1 \\
&= d^{i-1} s^j &&, i>j+1 \\
\tau_n d^i &= d^n &&, i=0 \\
&= d^{i-1} \tau_{n-1} &&, i>0 \\
\tau_n s^i &= s^n \tau_{n+1}^2 &&, i=0 \\
&= s^{i-1} \tau_{n+1} &&, i>0 \\
\tau_n^{n+1} &= \mathrm{id}. &&
\end{aligned}$$

Pf The morphisms d^i, s^i and the first 3 relations characterize the morphisms in Δ, see e.g. [MacLane]. So from (1.1.4) it is clear that d^i, s^i, τ_n generate the morphisms in Λ. It is equally clear that the last 3 relations are all the relations needed. We leave it as an exercise to show that they hold. $\square$

Definition 1.1.7 A cyclic object in a category $\mathcal{E}$ is a functor

$$X_\cdot : \Lambda^{op} \longrightarrow \mathcal{E}.$$

Exercise 1.1.8 The generators in 1.1.6 are by no means unique. Define an "extra" codegeneracy

$$s^{n+1} : [n+1] \to [n] \quad , \quad s^{n+1} = s^0 \tau_{n+1}^{-1}$$

and show that the morphisms in Λ are generated by $d^i : [n-1] \to [n]$, $0 \le i \le n$, $s^i : [n+1] \to [n]$, $0 \le i \le n+1$, subject to the first 3 sets of relations in 1.1.6 (which now also involves s^{n+1}) together with the extra relation

$$(s^{n+1} d^0)^{n+1} = \mathrm{id}.$$

$\square$

Lemma 1.1.6 also gives: the opposite cat. Λ^{op} is generated by

$$d_i : [n] \to [n-1] \quad , \quad 0 \le i \le n$$
$$s_i : [n] \to [n+1] \quad , \quad 0 \le i \le n$$
$$t_n : [n] \to [n]$$

subject to the relations (1.1.2). We define a functor

$$(-)^* : \Lambda^{op} \to \Lambda \qquad (1.1.9)$$

to be the identity on objects and to be given on morphisms by

$$(d_i : [n] \to [n-1])^* = (s^i : [n] \to [n-1]) \quad , \quad 0 \le i < n$$
$$= (s^0 \tau_n^{-1} : [n] \to [n-1]) \quad , \quad i = n$$

$$(s_i : [n] \to [n+1])^* = (d^{i+1} : [n] \to [n+1]) \quad , \quad 0 \leq i \leq n$$

$$(t_n : [n] \to [n])^* = (\tau_n^{-1} : [n] \to [n])$$

<u>Proposition</u> 1.1.10 The functor

$$(-)^* : \Lambda^{op} \xrightarrow{\cong} \Lambda$$

is well-defined and an isomorphism of categories, i.e. Λ is self-dual.

<u>Pf</u> It is straight forward to check that $(-)^*$ is well-defined (use lemma 1.1.6). The formulas which defines $(-)^*$ also tell us how to define the inverse except that they don't tell us (immediately) what to do with $d^0 : [n] \to [n+1]$. But now $\tau_{n+1} d^0 = d^{n+1}$ so $d^0 = \tau_{n+1}^{-1} d^{n+1}$ which we send to $t_{n+1} s_n : [n] \to [n+1]$. $\square$

<u>Example</u> 1.1.11 Let A be a ring. We can define a cocyclic abelian groups by

$$C^*(A) = A^{\otimes(n+1)}$$

$$d^i(a_0\otimes\cdots\otimes a_{n-1}) = a_0\otimes\cdots\otimes a_{i-1}\otimes 1\otimes a_i\otimes\cdots\otimes a_{n-1}, \quad 0\le i\le n$$

$$s^i(a_0\otimes\cdots\otimes a_{n+1}) = a_0\otimes\cdots\otimes a_i a_{i+1}\otimes\cdots\otimes a_{n+1}, \quad 0\le i\le n$$

$$\tau_n(a_0\otimes\cdots\otimes a_n) = a_1\otimes\cdots\otimes a_n\otimes a_0$$

If A is an augmented ring with augmentation $\varepsilon : A \to \mathbb{Z}$, then the cx

$$0 \to \mathbb{Z} \xrightarrow{\eta} A \xrightarrow{d} A^{\otimes 2} \xrightarrow{d} A^{\otimes 3} \xrightarrow{d} \cdots,$$

where $d = \sum(-1)^i d^i$, is acyclic, a contracting homotopy is given by

$$s(a_0\otimes\cdots\otimes a_n) = \varepsilon(a_0)\, a_1\otimes\cdots\otimes a_n .$$

<u>Exercise</u> 1.1.12 Describe the cyclic abelian group

$$\Delta^{op} \xrightarrow{(-)^*} \Delta \xrightarrow{C^*(A)} \mathcal{A}b .$$

1.2 Realization

It is well-known that the standard simplices

$$\Delta^n = \{(z_0,..,z_n) \in \mathbb{R}^{n+1} \mid \Sigma z_i = 1,\ z_i \geq 0\}$$

together form a cosimplicial space. In fact, they form a cocyclic space, that is, a functor

$$\Delta^\circ : \Lambda \longrightarrow Top \,. \qquad (1.2.1)$$

The structure maps are

$$d^i(z_0,\dots,z_{n-1}) = (z_0,..,z_{i-1},0,z_i,..,z_{n-1}) \quad , 0 \leq i \leq n$$

$$s^i(z_0,..,z_{n+1}) = (z_0,..,z_i+z_{i+1},..,z_{n+1}) \quad , 0 \leq i \leq n$$

$$\tau_n(z_0,..,z_n) = (z_1,\dots,z_n,z_0) \,.$$

A simplicial set $Y_\bullet$ gives rise to a topological space $|Y_\bullet|$ – the geometric realization – defined by

$$|Y_\bullet| = \Big(\coprod_{n \geq 0} Y_n \times \Delta^n\Big)/\sim$$

where the relation identifies

$$(f^*y, z) \sim (y, f_*z)$$

for every $f \in \Delta([n],[m])$, $y \in Y_m$, $z \in \Delta^m$. The relation is generated by

$$(d_i y, z) \sim (y, d^i z) \qquad , y \in Y_n, z \in \Delta^{n-1}$$
$$(s^i y, z) \sim (y, s^i z) \qquad , y \in Y_n, z \in \Delta^{n+1}$$

and $|Y_\cdot|$ is a CW-complex with one cell for each non-degenerate simplex in $Y_\cdot$. See e.g. [May].

The inclusion functor $\Delta \xrightarrow{K} \Lambda$ gives rise to a forgetful functor

$$U : \mathrm{Sets}^{\Lambda^{op}} \longrightarrow \mathrm{Sets}^{\Delta^{op}} \qquad (1.2.2)$$

which takes a cyclic set $X_\cdot$ to its underlying simplicial set, i.e. we simply forget the cyclic operator t_n. We define

$$|X_\cdot| := |UX_\cdot| \qquad (1.2.3).$$

We shall prove in the rest of this paragraph that in this case $|X_\cdot|$ has a natural action by the circle S^1. First we need to recall some category theory.

Let $\mathcal{E}$ be a category which has all colimits and let $K : I \to J$ be a functor between small categories, i.e. the objects form a set. The functor K determines a forgetful functor

$$U : \mathcal{E}^{J} \to \mathcal{E}^{I}$$

of functor categories. Indeed, if $F : J \to \mathcal{E}$ is a functor, then UF is the composite functor $UF = F \circ K$. It turns out that the functor U has a left adjoint functor

$$L_K : \mathcal{E}^{I} \to \mathcal{E}^{J} \qquad (1.2.4)$$

called the left Kan extension constructed as follows: if $j \in \operatorname{ob} J$, then the comma category $(K \downarrow j)$ has as objects pairs $(i, g : Ki \to j)$, where $i \in \operatorname{ob} I$ and g is a morphism in J. A morphism in $(K \downarrow j)$ from (i, g) to (i', g') is a morphism $\theta : i \to i'$ in I such that the diagram

$$\begin{tikzcd} Ki \arrow[rr, "K\theta"] \arrow[dr, "g"'] & & Ki' \arrow[dl, "g'"] \\ & j & \end{tikzcd}$$

commutes. There is a functor

$$pr_1 : (K \downarrow j) \to I \;; \quad pr_1(i, g) = i .$$

Now if $G : I \to \mathcal{E}$ is a functor then we may precompose G by pr_1 and get a functor

$$(K \downarrow j) \xrightarrow{pr_1} I \xrightarrow{G} \mathcal{E},$$

or what is the same, a diagram in $\mathcal{E}$ indexed by the small categery $(K\downarrow j)$. Now we define

$$L_K G(j) := \varinjlim \left((K\downarrow j) \xrightarrow{pr_1} I \xrightarrow{G} \mathcal{E} \right).$$

We leave it to the reader to define $L_K G$ and L_K on morphisms and to prove the adjunction

$$L_K \dashv U. \tag{1.2.5}$$

If the categery $\mathcal{E}$ has all limits, then one may define a right adjoint - the right Kan extension - of U in a similar fashion. See [MacLane] for details.

We now apply this general construction to the functor (1.2.2).

<u>Definition</u> 1.2.6 Let $Y_\bullet$ by a simplicial set. By the free cyclic set gen. by $Y_\bullet$ we mean the left Kan extension

$$FY_\bullet := L_K Y_\bullet$$

Suppose that $X_\cdot$ is a cyclic set. Then the counit of the adjunction (1.2.5) is a map of cyclic sets $\varepsilon : FX_\cdot \to X_\cdot$ (omiting U) so we get a map of realizations

$$\varepsilon : |F_\cdot X| \longrightarrow |X_\cdot| \qquad (1.2.7).$$

We next describe the cyclic set $F_\cdot Y$ in more detail.

<u>Lemma</u> 1.2.8 Let $C_{n+1} = \{1, t_n, t_n^2, .., t_n^n\}$. Then

$$FY_n = C_{n+1} \times Y_n$$

and the cyclic structure maps are

$$\begin{aligned} d_i(t_n^s, y) &= (t_{n-1}^s, d_{i+s} y) &&, \text{ if } i+s \leq n \\ &= (t_{n-1}^{s-1}, d_{i+s} y) &&, \text{ if } i+s > n \\ s_i(t_n^s, y) &= (t_{n+1}^s, s_{i+s} y) &&, \text{ if } i+s \leq n \\ &= (t_{n+1}^{s+1}, s_{i+s} y) &&, \text{ if } i+s > n \\ t_n(t_n^s, y) &= (t_n^{s-1}, y) \end{aligned}$$

where the indices are understood as their principal representatives modulo $n+1$.

Pf Recall that $\mathrm{Aut}_{\Delta^{op}}([n]) = \{1, t_n, \ldots, t_n^n\}$, where t_n is the opposite of the generator $\tau_n : [n] \to [n]$ in $\mathrm{Aut}_{\Delta}([n])$; $\tau_n(i) = i-1 \pmod{n+1}$. We have an inclusion of categories

$$\mathrm{Aut}_{\Delta^{op}}([n]) \hookrightarrow (K \downarrow [n])$$

$$[n] \xrightarrow{u} [n] \longmapsto ([n], [n] \xrightarrow{u} [n]).$$

We claim that this a cofinal subcategory; see [MacLane] p. 213 for definition and consequences. Indeed, if $([m], [m] \xrightarrow{\theta} [n])$ is any object in $(K \downarrow [n])$ then the morphism θ (in Δ^{op}) decomposes uniquely

$$[m] \xrightarrow{g} [n] \xrightarrow{u} [n]$$

with $g \in \Delta^{op}([m], [n])$ and $u \in \mathrm{Aut}_{\Delta^{op}}([n])$, and hence g induces a map in $(K \downarrow [n])$

$$([m], [m] \xrightarrow{\theta} [n]) \longrightarrow ([n], [n] \xrightarrow{u} [n]).$$

Moreover, this is the only map from $([m], \theta)$ to an object in $\mathrm{Aut}_{\Delta^{op}}([n])$, so the latter is a cofinal subcategory of $(K \downarrow [n])$. It follows that the induced map of colimits

$$\varinjlim\left(\mathrm{Aut}_{\Delta^{op}}([n]) \to \Delta^{op} \xrightarrow{Y_\cdot} \mathrm{Sets}\right)$$

$$\xrightarrow{\cong} \varinjlim\left((K \downarrow [n]) \xrightarrow{p'_1} \Delta^{op} \xrightarrow{Y_\cdot} \mathrm{Sets}\right)$$

is a bijection. But now the upper diagram is constant Y_n so

$$\varinjlim\left(\mathrm{Aut}_{\Delta^{op}}([n]) \to \Delta^{op} \xrightarrow{Y_\cdot} \mathrm{Sets}\right) \cong \mathrm{Aut}_{\Delta^{op}}([n]) \times Y_n .$$

We leave it to the reader to verify that the cyclic structure maps are as claimed. $\square$

Let us also record that under the isomorphism of 1.2.8 the counit is given by

$$\varepsilon : FX_\cdot \to X_\cdot \;; \quad \varepsilon(t_n^s, x) = t_n^{-s} x . \qquad (1.2.9)$$

If we take $Y_\cdot$ to be the trivial simplicial set $*$ with one simplex in each degree, then $FY_\cdot$ is the cyclic set $\mathbb{C}_\cdot$ given by

$$\mathbb{C}_n = C_{n+1} = \{1, t_n, t_n^2, \ldots, t_n^n\} \qquad (1.2.10)$$

$$d_i(t_n^s) = t_{n-1}^s \quad , \text{if } i+s \leq n$$

$$= t_{n-1}^{s-1} \quad , \text{if } i+s > n$$

$$s_i(t_n^s) = t_{n+1}^s \quad , \text{ if } i+s \leq n$$
$$= t_{n+1}^{s+1} \quad , \text{ if } i+s > n$$
$$t_n(t_n^s) = t_n^{s-1}$$

The only non-degenerate simplices in $\mathbb{C}_\circ$ are the 0-simplex 1 and the 1-simplex t_1, so

$$|\mathbb{C}_\circ| = S^1 \qquad (1.2.11)$$

Although $FY_\circ$ and $\mathbb{C}_\circ \times Y_\circ$ have the same n-simplices, for all $n \geq 0$, they are in general different simplicial set. However, we have

<u>Proposition</u> 1.2.12 There is a natural (non-simplicial) homeomorphism

$$h_Y : |FY_\circ| \xrightarrow{\cong} |\mathbb{C}_\circ \times Y_\circ| \xrightarrow{\cong} |\mathbb{C}_\circ| \times |Y_\circ|$$

for every simplicial set Y.

<u>Pf</u> Recall from (1.2.1) that Δ° has a cocyclic structure and recall also how the realization is defined. Now define a map

$$|FY_\circ| \longrightarrow |\mathbb{C}_\circ \times Y_\circ| \ ;$$

$$(t_n^s, y ; z) \longmapsto (t_n^{-s}, y ; \tau_n^{-s} z) \ .$$

We must check that this is well-defined, i.e. factors thru the equivalence relation $\sim$. For example, suppose $i+s \leq n$. Then in $|FY_\cdot|$

$$(t_n^s, y; d^i z) \sim (t_{n-1}^s, d_{i+s}; y),$$

so we must show that in $|C_\cdot \times Y_\cdot|$

$$(t_n^{-s}, y; \tau_n^{-s} d_i z) \sim (t_{n-1}^{-s}, d_{i+s} y; \tau_{n-1}^{-s} z).$$

That is, we must show that $\tau_n^{-s} d^i = d^{i+s} \tau_{n-1}^{-s}$, when $i+s \leq n$, or

$$(d^i)^* \tau_n^{-s} = \tau_{n-1}^{-s}$$

$$(\tau_n^{-s})_* d^i = d^{i+s}$$

We have

$$[n-1] = \underset{A_0}{\{0\}} \amalg \cdots \amalg \underset{A_{i-1}}{\{i-1\}} \amalg \underset{A_i}{\emptyset} \amalg \underset{A_{i+1}}{\{i\}} \amalg \cdots \amalg \underset{A_n}{\{n-1\}}$$

and $B_0 = A_{n+1-s}$. Since $i+s \leq n$, $n+1-s > i$, so $B_0 = \{n-s\}$,

$$[n-1] = \underset{B_0}{\{n-s\}} \amalg \cdots \amalg \underset{B_{s-1}}{\{n-1\}} \amalg \{0\} \amalg \cdots \amalg \{n-1-s\}$$

Hence $(d^i)^* \tau_n^{-s}(j) = j+n-s = j-s \pmod n$, and this is precisely τ_{n-1}^{-s}. Next

$$\#\left((\tau_n^{-s})_* d^i\right)^{-1}(j) = \#(d^i)^{-1}(\tau_n^s j)$$

which is zero when $\tau_n^s j = i$, i.e. when $j = i+s$. So $(\tau_n^{-s})_* d^i = d^{i+s}$. The other relations are treated similarly. It is clear that the map which results is a homeomorphism, and finally, it is a well-known result that realization commutes with (finite) products, see e.g. [May]. $\square$

Let $X_\bullet$ be a cyclic set. We define the action map

$$\mu : S^1 \times |X_\bullet| \to |X_\bullet| \qquad (1.2.13)$$

as the composite

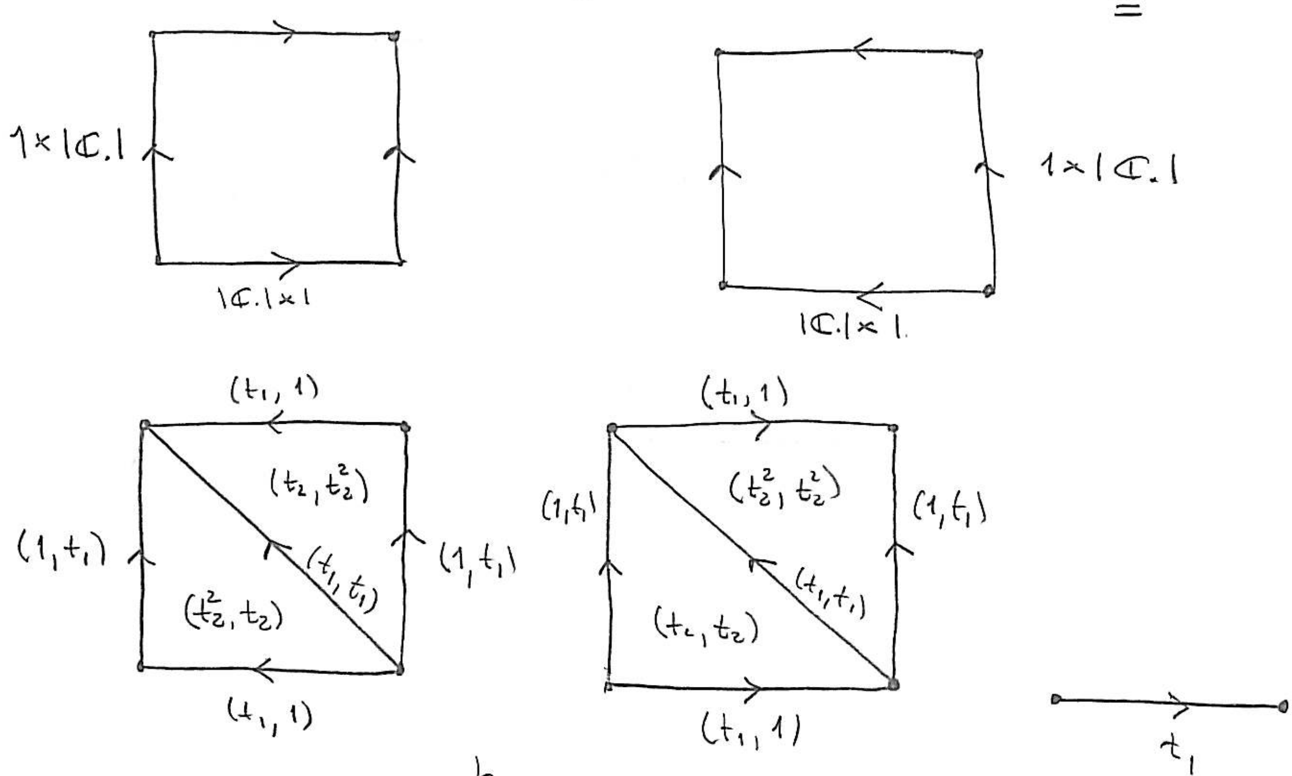
$$|\mathbb{C}_\bullet| \times |X_\bullet| \xrightarrow{h_{pt} \times id} |\mathbb{C}_\bullet| \times |X_\bullet| \xrightarrow{h_X^{-1}} |F.X| \xrightarrow{\varepsilon} |X_\bullet|$$

where h is the homeomorphism of 1.2.12 and ε is the counit of the adjunction $F \dashv U$.

<u>Lemma</u> 1.2.14 The map $\mu : |\mathbb{C}.| \times |\mathbb{C}.| \to |\mathbb{C}.|$ is the usual group structure on S^1.

Pf There are two non-degenerate 2-simplices in $F\mathbb{C}.$, namely, (t_2, t_2) and (t_2^2, t_2^2). They assemble to give $S^1 \times S^1$ as illustrated below. Similarly, the non-degenerate 2-simplices in $\mathbb{C}. \times \mathbb{C}.$ are (t_2^2, t_2) and (t_2, t_2^2). The map μ is given by the following cartoon, which is made such that the homeomorphisms involved preserves the position of points in the squares

$$|\mathbb{C}.| \times |\mathbb{C}.| \xrightarrow[\cong]{h_{pt} \times id} |\mathbb{C}.| \times |\mathbb{C}.| \xleftarrow[\cong]{pr_1 \times pr_2}$$



$$|\mathbb{C}. \times \mathbb{C}.| \xleftarrow{h_{\mathbb{C}.}} |F\mathbb{C}.| \xrightarrow{\varepsilon} |\mathbb{C}.|$$

The map ε projects along the edge (t_1, t_1). I.e. If we identify $|C_\cdot| = \mathbb{R}/\mathbb{Z}$ then we get

$$\mu : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, \quad \mu(x,y) = x+y,$$

which is indeed the usual group structure. $\square$

1.3 Edgewise subdivision

The concatenation of the ordered sets $A_1, \dots, A_r$ is the disjoint union $A_1 \amalg \dots \amalg A_r$ where each A_i has its original ordering and where in addition $a<b$ if $a \in A_i$ and $b \in A_j$ and $i<j$. We may also concatenate maps: if $f_i : A_i \to B_i$, $i = 1, \dots, r$, are order preserving so is $f_1 \amalg \dots \amalg f_r : A_1 \amalg \dots \amalg A_r \to B_1 \amalg \dots \amalg B_r$. It follows that r-fold concatenation defines a functor

$$sd_r : \Delta \longrightarrow \Delta \tag{1.3.1}$$

with

$$sd_r([n-1]) = [n-1] \amalg \dots \amalg [n-1] = [rn-1]$$

$$sd_r(f) = f \amalg \dots \amalg f$$

Let us spell out that if $f : [n-1] \to [m-1]$ then $sd_r(f) : [rn-1] \to [rm-1]$ is the map given by $an+b \mapsto am + f(b)$, $0 \le a < r$, $0 \le b < n$.

Definition 1.3.2 The r-fold edgewise subdivision of the simplicial set $Y_\cdot$ is the composite

$$\Delta^{op} \xrightarrow{sd_r^{op}} \Delta^{op} \xrightarrow{Y_\cdot} Sets$$

which we denote $sd_r Y_\cdot$.

So $sd_r Y.$ has n-simplices $sd_r Y_n = Y_{r(n+1)-1}$ and simplicial structure maps

$$\bar{d}_i : sd_r Y_n \to sd_r Y_{n-1} \quad , 0 \le i \le n$$

$$\bar{s}_i : sd_r Y_n \to sd_r Y_{n+1} \quad , 0 \le i \le n$$

given by

$$\bar{d}_i = d_i \circ d_{i+(n+1)} \circ \cdots \circ d_{i+(r-1)(n+1)}$$

$$\bar{s}_i = s_{i+(r-1)(n+2)} \circ \cdots \circ s_{i+(n+2)} \circ s_i$$

<u>Example</u> 1.3.3 Suppose $Y. = \Delta^1.$ then an n-simplex of $sd_r \Delta^1.$ is a map in Δ

$$\underbrace{[n] \amalg \cdots \amalg [n]}_{r} \to [1]$$

There are $r+1$ 0-simplices, namely,

$$\underbrace{0 \amalg \cdots \amalg 0}_{i} \amalg \underbrace{1 \amalg \cdots \amalg 1}_{r-i} \quad , 0 \le i \le r$$

where $\varepsilon : [0] \to [1]$ is the constant map ε.
Similarly, there are $2r+1$ 1-simplices

$$\underbrace{0 \amalg \cdots \amalg 0}_{i} \amalg \underbrace{1 \amalg \cdots \amalg 1}_{r-i} \quad , 0 \le i \le r$$

$$\underbrace{0 \amalg \cdots \amalg 0}_{i} \amalg id \amalg 1 \amalg \cdots \amalg 1 \quad , 0 \le i \le r-1$$

of which the first $r+1$ are degenerate. Indeed,

$\bar{s}_0 : sd_r \Delta^1_0 \to sd_r \Delta^1_1$ is takes $f : [0]\amalg\cdots\amalg[0] \to [1]$ to

$$\underbrace{[1]\amalg\cdots\amalg[1]}_{r} \xrightarrow{s^0\amalg\cdots\amalg s^0} \underbrace{[0]\amalg\cdots\amalg[0]}_{r} \xrightarrow{f} [1]$$

which are exactly the maps which are constant on each component. One sees similarly that all n-simplices for $n \geq 2$ are degenerate and that

$$\bar{d}_0(\underbrace{0\amalg\cdots\amalg 0}_{i}\amalg id \amalg \underbrace{1\amalg\cdots\amalg 1}_{r-1-i}) = \underbrace{0\amalg\cdots\amalg 0}_{i}\amalg\underbrace{1\amalg\cdots\amalg 1}_{r-i}$$

$$\bar{d}_1(\underbrace{0\amalg\cdots\amalg 0}_{i}\amalg id \amalg \underbrace{1\amalg\cdots\amalg 1}_{r-1-i}) = \underbrace{0\amalg\cdots\amalg 0}_{i+1}\amalg\underbrace{1\amalg\cdots\amalg 1}_{r-i-1}.$$

so the realization is the interval

$$|sd_r \Delta^1_\cdot| = [0, r]$$

with the vertex $\underbrace{0\amalg\cdots\amalg 0}_{i}\amalg\underbrace{1\amalg\cdots\amalg 1}_{r-i}$ corresponding to $i \in [0,r]$. $\square$

We recall that $\Delta^{n-1} \subset \mathbb{R}^n$ and let

$$d_r : \Delta^{n-1} \hookrightarrow \Delta^{rn-1} \quad ; \quad d_r(z) = \tfrac{1}{r}z \oplus \cdots \oplus \tfrac{1}{r}z$$

be the diagonal embedding. We then have a non-simplicial map

$$D_r : |sd_r Y_\cdot| \longrightarrow |Y_\cdot| \qquad (1.3.4)$$

induced from $1 \times d_r : Y_{rn-1} \times \Delta^{n-1} \to Y_{rn-1} \times \Delta^{rn-1}$

<u>Proposition</u> 1.3.5 The map $D_r : |sd_r Y.| \to |Y.|$ is a homeomorphism.

Pf We first prove the case $Y. = \Delta^1_.$ Let

$$\sigma_i = \underbrace{0 \amalg \cdots \amalg 0}_{i} \amalg id \amalg \underbrace{1 \amalg \cdots \amalg 1}_{r-1-i} : [2r-1] \to [1]$$

be the i'th non-degenerate 1-simplex of $sd_r Y.$
As a $2r-1$ simplex of $Y.$ this is degenerate, $\sigma_i = \sigma_i^* id$, where $id : [1] \to [1]$ is the non-degenerate 1-simplex.
So

$$D_r(\sigma_i, z) = (\sigma_i, \tfrac{1}{r}z \oplus \cdots \oplus \tfrac{1}{r}z) \sim (id, \sigma_{i*}(\tfrac{1}{r}z \oplus \cdots \oplus \tfrac{1}{r}z))$$

Now $\sigma_{i*} : \Delta^{2r-1} \to \Delta^1$ is the map

$$\sigma_{i*}(x_1, \dots, x_r) = (x_1 + \dots + x_{2i+1}, x_{2i} + \dots + x_{2r})$$

so if $z = (x_1, x_2) \in \Delta^1$ with $d_r(z) = \frac{1}{r}(x_1, x_2, \dots, x_1, x_2) \in \Delta^{2r-1}$ then

$$\sigma_{i*} d_r(z) = \left(\frac{i}{r} + \frac{x_1}{r}, \frac{r-i}{r} + \frac{x_2}{r}\right)$$

Hence $D_r : |sd_r \Delta^1_.| \to |\Delta^1_.|$ is a homeomorphism.

Next, we consider $Y. = \Delta^n_.$ There is a retraction

$$\Delta^n_. \hookrightarrow (\Delta^1_.)^n \xrightarrow{a} \Delta^n_.$$

The first map is induced from the n non-constant maps $[n] \to [1]$ in Δ and second is given by addition: If $f_i : [k] \to [1]$, $i = 1, \dots, n$, are order preserving maps, then $a(f_1, \dots, f_n) : [k] \to [n]$ given by $a(f_1, \dots, f_n)(j) = f_1(j) + \dots + f_n(j)$ is also order preserving. Now the diagram

$$\begin{tikzcd}
{|sd_r \Delta^n_\cdot|} \arrow[r, hook] \arrow[d, "D_r"] & {|sd_r (\Delta^1_\cdot)^n|} \arrow[r] \arrow[d, "D_r", "\cong"'] & {|sd_r \Delta^n_\cdot|} \arrow[d, "D_r"] \\
{|\Delta^n_\cdot|} \arrow[r, hook] & {|(\Delta^1_\cdot)^n|} \arrow[r] & {|\Delta^n_\cdot|}
\end{tikzcd}$$

commutes by naturality of D_r and the middle map is a homeomorphism since sd_r and $|-|$ commutes with finite products. It follows that $D_r : |sd_r \Delta^n_\cdot| \to |\Delta^n_\cdot|$ is a continuous bijection and hence a homeomorphism.

Finally, in the general case we use the isomorphism of simplicial sets

$$\Big(\coprod_{n \geq 0} Y_n \times \Delta^n_\cdot \Big) / \sim \;\xrightarrow{\cong}\; Y.$$

which sends $(y, id : [n] \to [n])$ to $y \in Y_n$. (The Yoneda lemma shows this defines a map.) The relation identifies

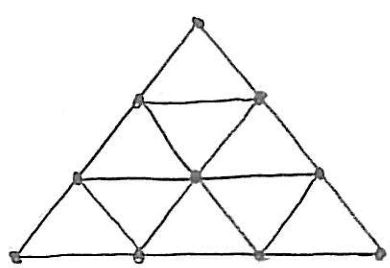
$$(f^* y, \sigma) \sim (y, f_* \sigma)$$

whenever $y \in Y_n$, $\sigma \in \Delta^m_k$, $f : [m] \to [n]$. Again by naturality of D_r and since realization commutes with colimits, we get

$$\begin{array}{ccc}
\left(\coprod_{n \geq 0} Y_n \times |sd_r \Delta^n_\cdot|\right)/\sim & \xrightarrow{\cong} & |sd_r Y_\cdot| \\
\cong \downarrow D_r & & \downarrow D_r \\
\left(\coprod_{n \geq 0} Y_n \times |\Delta^n_\cdot|\right)/\sim & \xrightarrow{\cong} & |Y_\cdot|
\end{array}$$

which proves the general case. $\square$

The proof shows that in passing from $|X_\cdot|$ to the homeomorphic space $|sd_r X_\cdot|$ all edges get subdivided into r pieces - hence the name. For example $|sd_3 \Delta^2_\cdot|$ is the simplicial cx



Note that this is not the barycentric subdivision. We also note that

$$sd_r\, sd_s\, Y_\cdot \;=\; sd_{rs}\, Y_\cdot \tag{1.3.6}$$

and that the diagram

$$\begin{tikzcd} |sd_{rs} Y_\cdot| \arrow[rr, "D_r"] \arrow[dr, "D_{rs}"'] & & |sd_s Y_\cdot| \arrow[dl, "D_s"] \\ & |Y_\cdot| & \end{tikzcd}$$

commutes.

Suppose $X_\cdot$ is a cyclic set. Then $sd_r X_\cdot$ is a simplicial set with an extra structure which we now clarify. There is a unique (up to isomorphism) category Λ_r and functor $sd_r : \Lambda_r \to \Lambda$ such that the square

$$\begin{tikzcd} \Delta \arrow[r, "sd_r"] \arrow[d] & \Delta \arrow[d] \\ \Lambda_r \arrow[r, "sd_r"] & \Lambda \end{tikzcd} \tag{1.3.7}$$

is a pull-back of categories. More concretely, $ob\, \Lambda_r = ob\, \Delta$ and on objects $sd_r : \Lambda_r \to \Lambda$ is given by $sd_r([n]) = [n] \amalg \cdots \amalg [n]$. A morphism in Λ_r from $[n]$ to $[m]$ decomposes uniquely as

$$[n] \xrightarrow{u} [n] \xrightarrow{f} [m]$$

with $u \in Aut_\Lambda(sd_r([n]))$ and $f \in \Delta([n],[m])$

and $sd_r : \Delta_r \to \Delta$ is given on morphisms by $sd_r(f, u) = (sd_r f, u)$. This determines the composition law in Δ_r: in the square

$$\begin{array}{ccc} [n] & \xrightarrow{g} & [m] \\ \downarrow g^* u & & \downarrow u \\ [n] & \xrightarrow{u_* g} & [m] \end{array}$$

where $u \in \mathrm{Aut}_\Delta(sd_r[m])$ and $g \in \Delta([n], [m])$, we must have

$$g^* u = sd_r(g)^* u \quad , \quad sd_r(u_* g) = u_* sd_r(g).$$

Alternatively, the morphisms in Δ_r are generated by

$$d^i : [n-1] \to [n] \quad , \quad 0 \le i \le n$$

$$s^i : [n+1] \to [n] \quad , \quad 0 \le i \le n$$

$$\tau_n : [n] \to [n]$$

subject to the relations of lemma 1.1.6 except that the last relation is replaced by

$$\tau_n^{r(n+1)} = id \qquad (1.3.8)$$

The functor $sd_r : \Delta_r \to \Delta$ is given by r-fold concatenation on $\Delta \subset \Lambda_r$ and $sd_r(\tau_{n-1}) = \tau_{rn-1}$.

<u>Definition</u> 1.3.9 A Λ_r^{op}-object in a category $\mathcal{E}$ is a functor $X_\cdot : \Lambda_r^{op} \to \mathcal{E}$.

<u>Remark</u> 1.3.10 Let $X_\cdot$ be a Λ_r^{op}-set and note that by 1.1.6 the operator $t_n^{n+1} : X_n \to X_n$ satisfies

$$d_i t_n^{n+1} = t_{n-1}^{n} d_i \,, \qquad t_n^{n+1} s_i = s_i t_{n+1}^{n+2} \,.$$

Hence we get a simplicial action by the cyclic group C_r of order r on $X_\cdot$ by letting the generator act through $t_n^{n+1} : X_n \to X_n$ in simplicial degree n. The simplicial C_r-action induces a C_r-action on $|X_\cdot|$. We shall see below that this action extends to an S^1-action.

The inclusion $\Delta^{op} \overset{K}{\hookrightarrow} \Lambda_r^{op}$ induces a forgetful functor

$$U_r : \mathrm{Sets}^{\Lambda_r^{op}} \longrightarrow \mathrm{Set}^{\Delta^{op}}$$

and we let

$$F_r : \mathrm{Sets}^{\Delta^{op}} \longrightarrow \mathrm{Sets}^{\Lambda_r^{op}}$$

be the left adjoint given by a left Kan extension. We described F_1 in 1.2.8 and now give a (more conceptual) description of F_r.

<u>Proposition</u> 1.3.11 Let $Y_\cdot$ be a simplicial set. Then

$$F_r Y_n \cong \mathrm{Aut}_{\Lambda_r^{op}}([n]) \times Y_n$$

and if $\theta = g \circ u$ is the unique decomposition of $\theta \in \Lambda_r([n],[m])$ with $u \in \mathrm{Aut}_{\Lambda}([n])$ and $g \in \Delta([n],[m])$ and if $(v^{op}, y) \in F_r Y_m$ then

$$\theta^{op}(v^{op}, y) = ((g^* v \circ u)^{op}, (v_* g)^{op} y)$$

in $F_r Y_n$.

Pf We recall from the proof of 1-2-8 that

$$\mathrm{Aut}_{\Lambda_r^{op}}([n]) \hookrightarrow (K \downarrow [n])$$

which sends the automorphism $[n] \xrightarrow{v^{op}} [n]$ to $([n], v^{op})$ is a cofinal subcategory so induces a bijection

$$\mathrm{Aut}_{\Lambda_r^{op}}([n]) \times Y_n \xrightarrow{\cong} \varinjlim\left((K\downarrow[n]) \xrightarrow{pr_1} \Delta^{op} \xrightarrow{Y_\cdot} Sets\right) =: F_r Y_n .$$

Now let $\theta = g \circ u$ be as in the statement of the proposition. Then the composite functor

$$\mathrm{Aut}_{\Lambda_r^{op}}([m]) \hookrightarrow (K\downarrow[m]) \xrightarrow{\theta^{op}_*} (K \downarrow [n])$$

takes $v^{op} \in \mathrm{Aut}_{\Lambda_r^{op}}([m])$ to

$$([m], [m] \xrightarrow{v^{op}} [m] \xrightarrow{\theta^{op}} [n])$$

in $(K\downarrow[n])$. Now $\theta^{op} = u^{op} \circ g^{op}$ so this in turn is is equal to

$$([m], [m] \xrightarrow{v^{op}} [m] \xrightarrow{g^{op}} [n] \xrightarrow{u^{op}} [n])$$

$$= ([m], [m] \xrightarrow{(v_* g)^{op}} [n] \xrightarrow{(g^* v \circ u)^{op}} [n]).$$

The induced map of colimits,

$$\varinjlim (\mathrm{Aut}_{\Lambda_r^{op}}([m]) \to \Delta^{op} \xrightarrow{Y.} \mathrm{Sets})$$

$$\Big\downarrow \cong$$

$$\varinjlim ((K \downarrow [m]) \xrightarrow{pr_1} \Delta^{op} \xrightarrow{Y.} \mathrm{Sets})$$

$$\Big\downarrow \theta^{op}$$

$$\varinjlim ((K \downarrow [n]) \xrightarrow{pr_1} \Delta^{op} \xrightarrow{Y.} \mathrm{Sets})$$

takes (v^{op}, y) to

$$(([m], [m] \xrightarrow{(v_* g)^{op}} [n] \xrightarrow{(g^* v \circ u)^{op}} [n]), y)$$

$$\sim (([n], [n] \xrightarrow{(g^* v \circ u)^{op}} [n]), (v_* g)^{op} y)$$

in the lower colimit. This proves the claimed formula. □

<u>Lemma</u> 1.3.12 The canonical map

$$F_r \, sd_r Y. \longrightarrow sd_r F Y.$$

is an isomorphism.

Pf The canonical map is induced from the functor $sd_r : (K_r \downarrow [n]) \to (K \downarrow sd_r[n])$ which takes the object $([m], \theta: [m] \to [n])$ to $(sd_r[m], sd_r\theta : sd_r[m] \to sd_r[n])$. This is the inclusion of a full subcategory and maps the cofinal subcategory $\mathrm{Aut}_{\Lambda_r^{op}}([n])$ of $(K_r \downarrow [n])$ to the cofinal subcategory $\mathrm{Aut}_{\Lambda^{op}}(sd_r[n])$ of $(K \downarrow sd_r[n])$ isomorphically. Hence the induced map of colimits is a bijection. □

Analogously to 1.2.10 we define

$$\mathbb{C}(r)_\bullet = F_r * = sd_r \mathbb{C}_\bullet \qquad (1.3.13)$$

In particular

$$|\mathbb{C}(r)_\bullet| = |sd_r \mathbb{C}_\bullet| = \mathbb{R}/r\mathbb{Z} \; ;$$

we fix the identification by requirering that $1 + r\mathbb{Z}$ corresponds to the vertex which is dual to the permutation $i \mapsto i-1 \pmod r$ of $[r]$.

<u>Proposition</u> 1.3.14 There is a (non-simplicial) homeomorphism

$$h_r : |F_r Y_\bullet| \xrightarrow{\cong} |\mathbb{C}(r)_\bullet \times Y_\bullet| \xrightarrow{\cong} |\mathbb{C}(r)_\bullet| \times |Y_\bullet|$$

Pf There is a more naive functor $p_r : \Lambda_r \to \Lambda$. This the identity on the subcategory Δ and

$$\mathrm{Aut}_{\Lambda_r}([n]) \xrightarrow{p_r} \mathrm{Aut}_{\Lambda}([n])$$

is the surjection given by $p_r(\tau_n) = \tau_n$, Recall from 1.2.1 the cocyclic space $\Delta^{\cdot}$. We view this as a Δ_r-space via $pr: \Delta_r \to \Delta$. By abuse of notation we shall also write this $\Delta^{\cdot}$. Now we define a map

$$l_r : |F_r Y_\cdot| \to |\mathbb{C}(r)_\cdot \times Y_\cdot|$$

by

$$l_r(v^{op}, y; z) = ((v^{-1})^{op}, y; pr(v) z).$$

Here $v^{op} \in Aut_{\Delta_r^{op}}([n])$, $y \in Y_n$ and $z \in \Delta^n$. We check that l_r is a well-defined map: let $g: [m] \to [n]$ be in Δ. Then we must show

$$l_r(v^{op}, y; pr(g) z) = l(g^{op}(v^{op}, y); z)$$

We note that the following formula holds in Δ_r

$$g^*(u \circ v) = (v_* g)^* u \circ g^* v.$$

In effect, this is what one has to show to verify that the composition law in Δ_r is associative. Taking $u = v^{-1}$ we get

$$(v_* g)^*(v^{-1}) = (g^* v)^{-1}$$

Now we calculate using 1.3.11

$$l_r(v^{op}, y; pr(g) z) = ((v^{-1})^{op}, y; pr(v \circ g) z)$$

$$= ((v^{-1})^{op}, y; pr(v_* g \circ g^* v) z)$$

$$= ((v_*g)^{op}(v^{-1})^{op}, (v_*g)^{op}y \,;\, p_r(g^*v)z)$$

$$= (((v_*g)^*(v^{-1}))^{op}, (v_*g)^{op}y \,;\, p_r(g^*v)z)$$

$$= (((g^*v)^{-1})^{op}, (v_*g)^{op}y \,;\, p_r(g^*v)z)$$

$$= l_r((g^*v)^{op}, (v_*g)^{op}y \,;\, z)$$

$$= l_r(g^{op}(v^{op}, y) \,;\, z).$$

It is obvious that l_r is a homeomorphism. $\square$

We now define an action map

$$\mu_r : \mathbb{R}/r\mathbb{Z} \times |X.| \longrightarrow |X.| \qquad (1.3.15)$$

for any Λ_r^{op}-set $X.$ as the composition

$$|\mathbb{C}(r).| \times |X.| \xrightarrow{h_r^{-1}\times id} |\mathbb{C}(r).| \times |X.| \xrightarrow{h_r} |F_rX.| \xrightarrow{\varepsilon_r} |X.|$$

where ε_r is the counit of $F_r \dashv \mathcal{U}$. It is given, under the isomorphism of 1.3.11, by

$$\varepsilon_r(v^{op}, x) = v^{op}x$$

Now recall from 1.3.10 that a Λ_r^{op}-set has a simplicial C_r-action.

<u>Lemma</u> 1.3.16 The simplicial C_r-action is equal to the action by the subgroup $\mathbb{Z}/r\mathbb{Z} \subset \mathbb{R}/r\mathbb{Z}$ via μ_r. The generator of C_r acts like $1+r\mathbb{Z}$.

Pf Let $K_\cdot \subset \mathbb{C}(r)_\cdot$ denote the simplicial subset gen. by the vertices and note that in simplicial degree n, $K_n \subset \mathbb{C}(r)_n$ is equal to the kernel of the map $p_r : \mathrm{Aut}_{\Delta_r^{op}}([n]) \to \mathrm{Aut}_{\Delta^{op}}([n])$. When restricted to $|K_\cdot| \times |X_\cdot|$, the map h_r^{-1} of 1.3.14 is simplicial:

$$h_r^{-1}(v^{op}, x; z) = ((\bar{v}^{-1})^{op}, x; p_r(\bar{v}^{-1})z) = ((\bar{v}^{-1})^{op}, x; z)$$

which by ε_r is mapped to $((\bar{v}^{-1})^{op}x; z) \in |X_\cdot|$. Finally, $h : |\mathbb{C}(r)_\cdot| \to |\mathbb{C}(r_\cdot)|$ takes $v \in K_\cdot$ to $\bar{v}^{-1} \in K_\cdot$. □

<u>Proposition</u> 1.3.17 Let $X_\cdot$ be a cyclic set. Then the following diagram commutes

$$\begin{tikzcd}
\mathbb{R}/r\mathbb{Z} \times |sd_r X_\cdot| \arrow[r, "\mu_r"] \arrow[d, "\frac{1}{r} \times D_r"] & |sd_r X_\cdot| \arrow[d, "D_r"] \\
\mathbb{R}/\mathbb{Z} \times |X_\cdot| \arrow[r, "\mu_1"] & |X_\cdot|
\end{tikzcd}$$

Pf We must prove that the outer rectangle in diagram

$$\begin{tikzcd}
|\mathbb{C}(r)_\cdot \times sd_r X_\cdot| \arrow[d, equal] & |F_r\, sd_r X_\cdot| \arrow[l, "l_r"'] \arrow[r, "\varepsilon_r"] \arrow[d, "can"] & |sd_r X_\cdot| \arrow[dd, "D_r"] \\
|sd_r(\mathbb{C}_\cdot \times X_\cdot)| \arrow[d, "D_r"] & |sd_r FX_\cdot| \arrow[ur, "sd_r \varepsilon_1"'] \arrow[d, "D_r"] & \\
|\mathbb{C}_\cdot \times X_\cdot| & |FX_\cdot| \arrow[l, "l_1"'] \arrow[r, "\varepsilon_1"] & |X_\cdot|
\end{tikzcd}$$

commutes. The upper right hand triangle commutes by category theory and the lower right quadrilateral commutes by naturality of D_r. To see that the left hand square commutes, suppose $(v^{op}, x) \in (F_r \, sd_r X)_{n-1} = \mathrm{Aut}_{\Delta^{op}}([rn-1]) \times X_{rn-1}$ and $z \in \Delta^{n-1}$. We get

$$D_r \ell_r (v^{op}, x; z) = D_r ((v^{-1})^{op}, x; p_r(v) z)$$

$$= ((v^{-1})^{op}, x; d_r(p_r(v) z))$$

and

$$\ell_1 D_r (v^{op}, x; z) = \ell_1 (v^{op}, x; d_r(z))$$

$$= ((v^{-1})^{op}, x; v \, d_r(z)).$$

But $d_r : \Delta^{n-1} \hookrightarrow \Delta^{rn-1}$ embeds Δ^{n-1} as the subset of Δ^{rn-1} which is fixed by elements in the kernel of p_r, and hence $v d_r(z) = d_r(p_r(v) z)$. $\square$

It follows from 1.2.14 and 1.3.17 that the action map $\mu_r : |\mathbb{C}(r).| \times |\mathbb{C}(r).| \to |\mathbb{C}(r).|$ is the usual group structure on $\mathbb{R}/r\mathbb{Z}$. We leave it as an exercise to show that μ_r is in fact an action.

Proposition 1.3.17 shows that we can analyse the action by the subgroup $C_r \subset S^1$ on $|X.|$ by simplicial means. In particular, we get a simplicial model for the C_r-fixed points $|X.|^{C_r}$, namely $(sd_r X.)^{C_r}$. The latter is a cyclic set and we can use 1.3.17 to compare the S^1-action on its realization to that on $|X.|^{C_r}$. We first introduce some notation.

Let $S^1 \subset \mathbb{C}$ be the unit circle and identify $\mathbb{R}/\mathbb{Z} \cong S^1$ via the exponential map. Then $C_r \subset S^1$ is the r'th roots of unity. The r'th root of a complex number is well-defined up to an r'th root of unity and provides a group isomorphism

$$\rho_{C_r} : S^1 \xrightarrow{\cong} S^1/C_r \quad ; \quad z \mapsto z^{1/r} C_r \, ;$$

the inverse is given by raising to the r'th power. If Z is an S^1/C_r-space, then we write $\rho_{C_r}^* Z$ for Z view as an S^1-space via ρ_r.

Corollary 1.3.18 Let $X.$ be a cyclic set. Then D_r induces an S^1-equivariant homeomorphism

$$|(\mathrm{sd}_r X.)^{C_r}| \underset{\cong}{\xrightarrow{D_r}} \rho_{C_r}^* |X.|^{C_r}$$

2.1 Hochschild homology

Let k be a commutative ground ring, let A be a unital associative k-algebra and let M be an A-A bimodule. The Hochschild complex of A over k with coefficients in M is the simplicial k-module with n-simplices ($\otimes = \otimes_k$)

$$N^{cy}_{\otimes,n}(A;M) = M \otimes \underbrace{A \otimes \cdots \otimes A}_{n} \qquad (2.1.1)$$

and

$$\begin{aligned} d_i(m\otimes a_1\otimes\cdots\otimes a_n) &= ma_1\otimes a_2\otimes\cdots\otimes a_n &&, i=0\\ &= m\otimes\cdots\otimes a_i a_{i+1}\otimes\cdots\otimes a_n &&, 0<i<n\\ &= a_n m\otimes a_1\otimes\cdots\otimes a_{n-1} &&, i=n\end{aligned}$$

$$s_i(m\otimes a_1\otimes\cdots\otimes a_n) = m\otimes\cdots\otimes a_i\otimes 1\otimes a_{i+1}\otimes\cdots\otimes a_n, \quad 0\le i\le n$$

When $M=A$ we write $N^{cy}_{\otimes}(A)$ or $N^{cy}_{\otimes}(A/k)$ instead of $N^{cy}_{\otimes}(A;A)$. This is a cyclic k-module with

$$t_n(a_0\otimes\cdots\otimes a_n) = a_n\otimes a_0\otimes\cdots\otimes a_{n-1}.$$

In effect, $N^{cy}_{\otimes}(A)$ is the dual of the cobar construction which we considered in 1.1.11. The Hochschild homology of A over k with coefficients in M is the homology of the associated chain complex of $N^{cy}_{\otimes}(A;M)$,

$$HH_*(A;M) := H_*(N^{cy}_{\otimes}(A;M)\,;\, d = \Sigma(-1)^i d_i) \qquad (2.1.2)$$

Again, we write $HH_*(A)$ or $HH_*(A/k)$ for $HH_*(A;A)$.

<u>Example</u> 2.1.3 i) Suppose that $A = M = k$. Then the Hochschild complex becomes

$$\cdots \xrightarrow{1} k \xrightarrow{0} k \xrightarrow{1} k \xrightarrow{0} k \xrightarrow{1} k \to 0$$

so $HH_0(k/k) = k$, $HH_i(k/k) = 0$ for $i > 0$.

ii) Let G be a discrete group and recall the cyclic bar construction $N^{cy}_{\cdot}(G)$ from (1.1.1). Then we have an isomorphism of cyclic k-modules

$$k\langle N^{cy}_{\cdot}(G)\rangle \cong N^{cy}_{\otimes}(k[G])$$

between the free k-module gen. by $N^{cy}_{\cdot}(G)$ and the Hochschild complex of the group algebra $k[G]$. The right-hand side is the cellular complex for the CW-complex $|N^{cy}_{\cdot}(G)| \simeq \mathcal{L}BG$ with coeff. in k, so we get

$$H_*(\mathcal{L}BG\,;\,k) \cong HH_*(k[G]).$$

$\square$

Let $[M,A) \subset M$ denote the k-submodule of M generated by element of the form $ma - am$,

$m \in M$, $a \in A$. Then

$$HH_0(A; M) = M/[M, A], \tag{2.1.4}$$

Let A^{op} be the opposite algebra, i.e. $A^{op} = A$ but with multiplication $a * b = ba$. Then every right A-module is a left A^{op}-module and vice versa. An A-A bimodule may be viewed both as a left or right $A \otimes A^{op}$-module: in the left structure

$$(a \otimes b) \cdot m := amb$$

and in the right structure

$$m \cdot (a \otimes b) := bma.$$

We may then rewrite (2.1.4) as

$$HH_0(A; M) \cong M \underset{A \otimes A^{op}}{\otimes} A.$$

Next, we recall the two-sided bar construction;

$$B_n(A, A, A) = \underbrace{A \otimes \cdots \otimes A}_{n+2} \tag{2.1.5}$$

$$d_i(a_0 \otimes \cdots \otimes a_{n+1}) = a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}, \quad 0 \leq i \leq n$$

$$s_i(a_0 \otimes \cdots \otimes a_{n+1}) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_{n+1}, \quad 0 \leq i \leq n$$

It is a simplicial A-A bimodule where A multiplies in the first and last tensor factor. We define an augmentation $\varepsilon : B_0(A,A,A) \to A$; $\varepsilon(a_0 \otimes a_1) = a_0 a_1$.

Lemma 2.1.6 The augmented complex

$$\cdots \xrightarrow{d} B_n(A,A,A) \xrightarrow{d} \cdots \xrightarrow{d} B_0(A,A,A) \xrightarrow{\varepsilon} A \to 0$$

is acyclic.

Pf The "extra degeneracy", $s_{-1} : B_n(A,A,A) \to B_{n+1}(A,A,A)$ given by $s_{-1}(a_0 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes \cdots \otimes a_{n+1}$, is a chain contraction ; $d s_{-1} + s_{-1} d = id$. □

If M is an A-A bimodule, then we have an isomorphism of simplicial k-modules

$$M \underset{A \otimes A^{op}}{\otimes} B_{\cdot}(A,A,A) \xrightarrow{e} N^{cy}_{\otimes, \cdot}(A; M)$$

given by

$$e\left(m \underset{A \otimes A^{op}}{\otimes} (a_0 \otimes \cdots \otimes a_{n+1})\right) = a_{n+1} m a_0 \otimes a_1 \otimes \cdots \otimes a_n .$$

If A is flat over k, then $B_{\cdot}(A,A,A)$ is a flat resolution of A by A-A bimodules. Hence

Corollary 2.1.7 If A is k-flat then

$$HH_*(A;M) \cong \operatorname{Tor}_*^{A \otimes A^{op}}(M, A).$$ □

Example 2.1.8 If $A = k[x]$ then we may use the following resolution of A by free A-A bimodules:

$$0 \to A \otimes A \xrightarrow{x\otimes 1 - 1 \otimes x} A \otimes A \xrightarrow{\mu} A \to 0.$$

This shows that $HH_i(k[x]) = k[x]$, for $i = 0, 1$, and 0 else. More generally, if $A = k[x_1, \dots, x_n]$, then we can tensor the resolution above with itself n times and get a resolution

$$A \otimes \Lambda_k^*\{x_1, \dots, x_n\} \otimes A \xrightarrow{\varepsilon} A.$$

It follows that

$$HH_*(k[x_1, \dots, x_n]) \cong k[x_1, \dots, x_n] \otimes \Lambda_k^*\{x_1, \dots, x_n\}. \quad \square$$

Definition 2.1.9 Two k-algebras A and B are called Morita equivalent if there exist bimodules ${}_A P_B$ and ${}_B Q_A$ s.t. $Q \otimes_A P \cong B$ and $P \otimes_B Q \cong A$.

It follows that the categories of (left) A- and B-modules are equivalent:

$$\begin{array}{ccc} {}_A\mathcal{M} & \cong & {}_B\mathcal{M} \\ M & \longmapsto & Q \otimes_A M \\ P \otimes_B N & \longleftarrow & N \end{array}$$

In fact, this statement is equivalent to 2.1.9 and is usually taken as the definition of Morita equivalence. See [Bass] for a proof of this.

Lemma 2.1.10 P is projective both as a left A-module and as a right B-module. Q is projective as a right A-module and as a left B-module.

Pf By category th.: suppose $M \to P$ is an epi of A-modules, then $Q \otimes_A M \to Q \otimes_A P$ is an epi of B-modules. For suppose two composites

$$Q \otimes_A M \to Q \otimes_A P \rightrightarrows N$$

are equal. Then so are

$$\underset{\cong M}{P \otimes_B Q \otimes_A M} \to \underset{\cong P}{P \otimes_B Q \otimes_A P} \rightrightarrows P \otimes_B N$$

But then the two maps on the right are equal and hence so are

$$Q \otimes_A P \rightrightarrows Q \otimes_A P \otimes_B N \cong N.$$

Now $Q \otimes_A M \to Q \otimes_A P = B$ splits and therefore so does $M \to P$. $\square$

<u>Theorem 2.1.11</u> Let A and B be Morita equivalent k-algebras and let ${}_AP_B$, ${}_BQ_A$ be as above. Then for every A-A bimodule M,

$$HH_*(A;M) \cong HH_*(B; Q \otimes_A M \otimes_A P).$$

Pf We shall assume that A and B are k-flat; for a proof of the general case see [Loday]. Let $N = Q \otimes_A M$. Then we must show

$$HH_*(A; P \otimes_B N) \cong HH_*(B; N \otimes_A P).$$

The idea, due to Waldhausen, is to splice the two two-sided bar constructions

$$B.(N,A,P) := N \otimes_A B.(A,A,A) \otimes_A P \qquad \text{and}$$

$$B.(P,B,N) := P \otimes_B B.(B,B,B) \otimes_B N$$

together to form a bisimplicial k-module with p,q-simplices

$$C_{p,q} = \begin{array}{ccccc} & \otimes & \overbrace{A \otimes \cdots \otimes A}^{p} & & \\ P & & & \otimes & N \\ & \otimes & \underbrace{B \otimes \cdots \otimes B}_{q} & \otimes & \end{array}$$

More precisely, $B.(N,A,P)$ is a simplicial B-B

bimodule, $B_\bullet(P,B,N)$ is a simplicial A-A bimodule and

$$C_{\bullet,\bullet} \cong N^{cy}_{\otimes,\bullet}(B; B_\bullet(N,A,P)) \cong N^{cy}_{\otimes,\bullet}(A; B_\bullet(P,B,N)).$$

We may filter the total cx. $\mathrm{Tot}(C_{\bullet,\bullet})$ in two different ways and get two spectral sequences. Because we assumed that A and B are k-flat their E^2-terms will take the form

$${}'E^2_{p,q} = HH_p(B; \mathrm{Tor}^A_q(N,P)) \qquad \text{, and}$$

$${}''E^2_{q,p} = HH_q(A; \mathrm{Tor}^B_p(P,N)) \ .$$

Now P is right A-projective and left B-projective, so both spectral sequences collapses and yield the desired isomorphism. $\square$

Suppose that Q is a finitely generated projective right A-module and define

$$B = \mathrm{End}_A(Q)$$

$$P = \mathrm{Hom}_A(Q,A).$$

Then Q is a B-A bimodule and P is an A-B bimodule. The B-module structures are given by $f \cdot q = f(q)$ and $\phi \cdot f = \phi \circ f$. The evaluation map defines an isomorphism

$$P \otimes_B Q = \mathrm{Hom}_A(Q, A) \otimes_B Q \xrightarrow[\cong]{ev} A$$

and we also have an isomorphism

$$Q \otimes_A P = Q \otimes_A \mathrm{Hom}_A(P, A) \xrightarrow{\cong} \mathrm{Hom}_A(Q, Q) = B$$

given by the adjoint of the map of right A-modules

$$Q \otimes_A \mathrm{Hom}_A(Q, A) \otimes_k Q \xrightarrow{id \otimes ev} Q \otimes_A A \xrightarrow[\cong]{\mu} Q ,$$

We need that Q be f.g. proj. for these maps to be iso's, see e.g. [MacLane].

<u>Corollary</u> 2.1.12 Let Q be a finitely generated projective (right) A-module. Then

$$HH_*(\mathrm{End}_A(Q)) \cong HH_*(A).$$

□

We can use 2.1.12 to define transfer maps in Hochschild homology. Suppose that

$$\phi : A \to B$$

is a (not necessarily unital) homomorphism of k-algebras s.t. B becomes a f.g. projective right A-module via ϕ. Then we define

$$\phi^! : HH_*(B) \to HH_*(A) \qquad (2.1.13)$$

as the composite

$$HH_*(B) \to HH_*(\mathrm{End}_A(B,B)) \cong HH_*(A).$$

The first map is induced from the k-algebra homomorphism, which takes $b \in B$ to left multiplication by b on B.

<u>Remark</u> 2.1.14 The isomorphism is called the trace. If Q is the free A-module on n generators, then (2.1.13) reads $HH_*(M_n(A)) \cong HH_*(A)$. The isomorphism is given by the simplicial, in fact cyclic, map which in simplicial degree n is , c.f. (2,1,1),

$$tr(\phi^0 \otimes \cdots \otimes \phi^n) = \sum_{(i_0,\dots,i_n)} \phi^0_{i_0 i_1} \otimes \phi^1_{i_1 i_2} \otimes \cdots \otimes \phi^n_{i_n i_0},$$

i.e. formally multiply the matrices $\phi^0, \dots, \phi^n$ together writing a tensor sign instead of multiplying, then take the trace. For a proof that this map induces the iso. see [Loday].

2.2 Connes' B-operator

Let again A be an algebra over a commutative ground ring k. We shall use the structure of cyclic k-module on $N^{cy}_{\otimes}(A)$ to define a map

$$\mu_* : H_*(S^1; k) \otimes_k HH_*(A) \to HH_*(A) \qquad (2.2.1)$$

similar to the action map of (1.2.13). If $Y.$ is a simplicial abelian group then we denote by Y_* the associated chain complex with differential $d = \Sigma (-1)^i d_i$. The Eilenberg-Zilber map defines a quasi-isomorphism

$$\theta : Y_* \otimes Z_* \xrightarrow{\simeq} (Y. \otimes Z.)_* \qquad (2.2.2)$$

If $y \in Y_p$, $z \in Z_q$, $p+q = n$, then

$$\theta(y \otimes z) = \sum_{\sigma \in S(p,q)} \operatorname{sgn}(\sigma)\, s_{\sigma(p+q)} \cdots s_{\sigma(p+1)} y \otimes s_{\sigma(p)} \cdots s_{\sigma(1)} z,$$

where the sum runs over all (p,q)-shuffles.

The forgetful functor from cyclic k-modules to simplicial k-modules has a left adjoint, which we denote F. Recall $\mathbb{C}.$ from (1.2.10) and let $k[\mathbb{C}.]$ be the (degree wise) group algebra. Then the proof of lemma 1.2.8 shows that

$$FY_n = k[\mathbb{C}_n] \otimes_k Y_n \qquad (2.2.3)$$

with cyclic structure maps

$$
\begin{aligned}
d_i(t_n^s \otimes y) &= t_{n-1}^s \otimes d_{i+s} y &&, \text{ if } i+s \leq n \\
&= t_{n-1}^{s-1} \otimes d_{i+s} y &&, \text{ if } i+s > n \\
s_i(t_n^s \otimes y) &= t_{n+1}^s \otimes s_{i+s} y &&, \text{ if } i+s \leq n \\
&= t_{n+1}^{s+1} \otimes s_{i+s} y &&, \text{ if } i+s > n \\
t_n(t_n^s \otimes y) &= t_n^{s-1} \otimes y .
\end{aligned}
$$

As a simplicial k-module $FY.$ is usually not isomorphic to $k[\mathbb{C}.] \otimes Y.$ However, their associated complexes are canonically isomorphic: In analogue with 1.2.12, we have the isomorphism

$$\ell : FY_* \xrightarrow{\cong} (k[\mathbb{C}.] \times Y.)_* \quad ; \tag{2.2.4}$$

$$\ell(t_n^s \otimes y) = (-1)^{ns} t_n^{-s} \otimes y .$$

If $X.$ is a cyclic k-module then we have the counit $\varepsilon : FX. \to X.$; $\varepsilon(t_n^s \otimes x) = t_n^{-s} x$, and we define a chain map

$$\mu_* : k[\mathbb{C}.]_* \otimes X_* \longrightarrow X_* \tag{2.2.5}$$

as the composite

$$k[\mathbb{C}.]_* \otimes X_* \xrightarrow{\ell \otimes id} k[\mathbb{C}.]_* \otimes X_* \xrightarrow{\Theta} (k[\mathbb{C}.] \otimes X.)_*$$

$$\xrightarrow{\ell^{-1}} FX_* \xrightarrow{\varepsilon} X_*$$

Now $k[\mathbb{C}.]$ is the cellular complex for S^1, so on homology we get

$$\mu_* : H_*(S^1;k) \otimes H_*(X_*) \longrightarrow H_*(X_*)$$

which when $X.$ is $N^{cy}_{\otimes}(A)$ is the wanted map.

<u>Definition</u> 2.2.6 Let $X.$ be a cyclic k-module. Connes' B-operator is the map

$$B : H_n(X_*) \to H_{n+1}(X_*) \; ; \qquad B(x) = \mu_*(t_1 \otimes x) .$$

<u>Exercise</u> 2.2.7 Show that μ_* in (2.2.5) makes X_* a DG-module over $k[\mathbb{C}.]_*$ and conclude that $B \circ B = 0$.

In the case of Hochschild homology, it is straight forward to calculate that

$$B : HH_n(A) \longrightarrow HH_{n+1}(A) \tag{2.2.8}$$

is given on the chain level by

$$B(a_0 \otimes \dots \otimes a_n) = \sum_{i=0}^{n} (-1)^{ni} \, 1 \otimes a_i \otimes \dots \otimes a_n \otimes a_0 \otimes \dots \otimes a_{i-1} .$$

In the rest of this paragraph we shall assume that A is <u>commutative</u>. In that case we have a cyclic map

$$\tilde{m} : N^{cy}_{\otimes,\cdot}(A) \otimes N^{cy}_{\otimes,\cdot}(A) \to N^{cy}_{\otimes,\cdot}(A) \quad ;$$

$$(a_0 \otimes \dots \otimes a_n) \otimes (b_0 \otimes \dots \otimes b_n) \mapsto a_0 b_0 \otimes \dots \otimes a_n b_n .$$

We need A to commutative in order for $\tilde{m}$ to commute with the cyclic structure maps. For instance $\tilde{m} d_1 = d_1 \tilde{m}$ requires that $a_0 a_1 b_0 b_1 = a_0 b_0 a_1 b_1$. Composing $\tilde{m}$ with θ we get

$$m : N^{cy}_{\otimes}(A)_* \otimes N^{cy}_{\otimes}(A)_* \longrightarrow N^{cy}_{\otimes}(A)_*$$

and passing to homology a product

$$m : HH_*(A) \otimes HH_*(A) \longrightarrow HH_*(A). \qquad (2.2.9)$$

Since $\tilde{m}$ is obviously associative so is m, and since $\tilde{m}$ is commutative, m makes $HH_*(A)$ into a graded commutative k-algebra, i.e.

$$a \cdot b = (-1)^{|a||b|} b \cdot a$$

Moreover, B becomes a derivation for the product in the sense that

$$B(a \cdot b) = B(a) \cdot b + (-1)^{|a|} a \cdot B(b).$$

This follows by standard arguments from the fact that by naturality of μ, the following diagram commutes

$$\begin{array}{ccc}
k[\mathbb{C}.]_* \otimes (N^{cy}_{\otimes,\cdot}(A) \otimes N^{cy}_{\otimes,\cdot}(A))_* & \xrightarrow{1\otimes\tilde{m}} & k[\mathbb{C}.]_* \otimes N^{cy}_{\otimes}(A)_* \\
\downarrow \mu & & \downarrow \mu \\
(N^{cy}_{\otimes,\cdot}(A) \otimes N^{cy}_{\otimes,\cdot}(A))_* & \xrightarrow{\tilde{m}} & N^{cy}_{\otimes}(A)_* .
\end{array}$$

In other words, $HH_*(A)$ with the differential B is a differential-graded-algebra (DGA) over k whose 0th term is A. We shall now see that there is a universal example of such a structure.

A k-linear <u>derivation</u> from A to an A-module N is a k-linear map $D: A \to N$ such that

$$D(ab) = D(a)b + aD(b).$$

The module of differentials of A over k is the A-module

$$\Omega_{A/k} = I/I^2,$$

where I is the kernel of the multiplication $A \otimes_k A \to A$. We may view I as an A-module by multiplication in either the first or second factor; the two agree modulo I^2. Define

$$d: A \to \Omega_{A/k}; \quad d(a) = 1\otimes a - a\otimes 1.$$

Then d is a k-linear derivation because

$$1\otimes ab - ab\otimes 1 = (1\otimes a - a\otimes 1)\, 1\otimes b + a\otimes 1\,(1\otimes b - b\otimes 1)$$

Lemma 2.1.10 The derivation $d: A \to \Omega_{A/k}$ is universal, i.e. if $D: A \to N$ is a k-linear derivation then there is a unique A-linear map $\phi: \Omega_{A/k} \to N$ such that $D = \phi \circ d$.

Pf The map $\phi'': A\otimes_k A \to N$, $\phi'(a\otimes b) = a\,Db$, is A-linear when we view $A\otimes A$ as an A-module via multiplication in the first factor. Hence so is the restriction $\phi': I \to N$ of ϕ'' to I. Now the calculation

$$\sum a_i\otimes b_i = \sum a_i\otimes 1(1\otimes b_i - b_i\otimes 1) + \sum a_i b_i\otimes 1$$

shows that I as an A-module is generated by elements of the form da, $a\in A$. Hence I^2 is gen. by

$$\begin{aligned} da\cdot db &= (1\otimes a - a\otimes 1)(1\otimes b - b\otimes 1) \\ &= 1\otimes ab - b\otimes a - a\otimes b + ab\otimes 1 \end{aligned}$$

which by ϕ' is mapped to

$$D(ab) - b\,Da - a\,Db + ab\,D1 = 0$$

So ϕ' factors to $\phi: \Omega_{A/k} \to N$. Uniqueness is clear. $\square$

If $\mathrm{Der}_k(A,N)$ denotes the k-module of k-linear derivations from A to N, then we can also state the lemma as $\mathrm{Der}_k(A,N) \cong \mathrm{Hom}_A(\Omega_{A/k}, N)$. The derivation $B : A \to HH_1(A)$ takes $b \in A$ to $B(b)$ which is represented by the 1-cycle $1 \otimes b - b \otimes 1$, so we get

$$\phi : \Omega_{A/k} \xrightarrow{\cong} HH_1(A) \qquad (2.1.11)$$

$$a\,db \mapsto a \otimes b - ab \otimes 1$$

which is an isomorphism with inverse given by $\phi^{-1}(a_0 \otimes a_1) = a_0 da_1$.

We now define the de Rham-complex of A over k as the exterior algebra

$$\Omega^*_{A/k} := \Lambda^*_A \Omega_{A/k} \quad ;$$

$$d(a_0 da_1 \wedge \dots \wedge da_n) := da_0 \wedge da_1 \wedge \dots \wedge da_n .$$

Then $d \circ d = 0$ and $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{|\omega|} \omega \wedge d\eta$ so that $\Omega^*_{A/k}$ is a graded commutative DGA with zero'th term A. Moreover, it is universal with these properties. Now $HH_*(A)$, B is another example, so we get a canonical map

$$\gamma : \Omega^*_{A/k} \longrightarrow HH_*(A). \qquad (2.2.12)$$

<u>Example</u> 2.2.13 It follows from (2.1.8) and (2.2.11) that γ is an isomorphism if $A = k[x_1, \dots, x_n]$.

3.1 Topological Hochschild homology.

A symmetric spectrum E consists of a sequence of spaces E_n, $n \geq 0$, together with, for every $n \geq 0$, an action by the symmetric group Σ_n on E_n, and for each pair $m, n \geq 0$, maps

$$\lambda_{m,n} : S^m \wedge E_n \longrightarrow E_{m+n}, \qquad (3.1.1)$$

which are $\Sigma_m \times \Sigma_n$-equivariant when $\Sigma_m \times \Sigma_n$ acts diagonally on the left and via the inclusion $\Sigma_m \times \Sigma_n \subset \Sigma_{m+n}$ on the right. Here Σ_m acts on $S^m = S^1 \wedge \dots \wedge S^1$ by permuting the smash factors. Let $\sigma_{n,m} \in \Sigma_{m+n}$ be the permutation $\sigma_{n,m}(i) = i+n \pmod{m+n+1}$ and define

$$\rho_{n,m} : E_n \wedge S^m \longrightarrow E_{n+m}; \qquad (3.1.2)$$

$$\rho_{n,m} = \sigma_{n,m} \circ \lambda_{m,n} \circ tw_{S^m, E_n}.$$

We then require that

i) $\lambda_{0,n} = E_n$

ii) $\lambda_{k+m,n} = \lambda_{k,m+n} \circ (S^k \wedge \lambda_{m,n})$

iii) $\rho_{k+m,n} \circ (\lambda_{k,m} \wedge S^n) = \lambda_{k,m+n} \circ (S^k \wedge \rho_{m,n})$,

Here for a space X, we also have written X for the identity map on X.

Suppose that L is a pointed continuous endo-functor on the category of pointed topological spaces (in the compactly generated, weak Hausdorff topology). That is, for every pair of spaces X, Y, the function

$$F(X,Y) \xrightarrow{L} F(L(X), L(Y))$$

is a continuous basepoint preserving map. Here $F(X,Y)$ denotes the space of pointed cts. maps from X to Y. The topology on $F(X,Y)$ is obtained from the compact-open topology by making it a compactly generated weak Hausdorff topology. This topology has the advantage that, for every triple of spaces X, Y and Z, one has an adjunktion homeomorphism

$$F(X \wedge Y, Z) \cong F(X, F(Y,Z)) .$$

Taking $Z = X \wedge Y$, the identity map on the left gives a map

$$X \to F(Y, X \wedge Y)$$

which composed with

$$F(Y, X\wedge Y) \xrightarrow{L} F(L(Y), L(X \wedge Y))$$

gives $X \to F(L(Y), L(X\wedge Y))$. Adjoining back we obtain

$$\lambda_{X,Y} : X \wedge L(Y) \to L(X \wedge Y) \qquad (3.1.3)$$

Exercise 3.1.4 Define $\rho_{X,Y} : L(X) \wedge Y \to L(X \wedge Y)$ similarly and show that up to coherent nat'l homeomorphisms

1) $\lambda_{S^0, X} = L(X)$

2) $\lambda_{X, Y\wedge Z} \circ (X \wedge \lambda_{Y,Z}) = \lambda_{X\wedge Y, Z}$

3) $\lambda_{X, Y\wedge Z} \circ (X \wedge \rho_{Y,Z}) = \rho_{X\wedge Y, Z} \circ (\lambda_{X,Y} \wedge Z)$.

Conclude that the spaces $L(S^n)$, $n \geq 0$, with the Σ_n-action induced from that on S^n by the functoriality of L, is a symmetric spectrum.

Example 3.1.5 1) For any space Z, the functor

$$L_Z(X) = X \wedge Z$$

is a pointed continuous functor, so defines a symmetric spectrum $\underline{\underline{Z}}$ - the suspension spectrum of Z.

2) Let A be an abelian group and let X be a pointed space. The configuration space of particles in X with labels in A is defined as follows: let

$$\tilde{C}^n(X) = \{ (x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j \}$$

be the complement of the fat diagonal in X^n. It has a free Σ_n-action and we let

$$A(X) = \Big(\coprod_{n \geq 0} \tilde{C}^n(X) \times_{\Sigma_n} A^n \Big) / \sim$$

where the relation identifies

$$(x_1, \ldots, x_n ; a_1, \ldots, a_n) \sim (x_1, \ldots, x_{n-1} ; a_1, \ldots, a_{n-1})$$

if $x_n = *$ or $a_n = 0$. So $A(X)$ consists of formal linear combinations $\Sigma\, a_x \cdot x$, where all but finitely many $a_x = 0$. A theorem of Dold-Thom states that

$$\pi_* A(X) = \tilde{H}(X; A).$$

Hence the symmetric spectrum associated with the functor $A(-)$ is an Eilenberg-MacLane spectrum for the group A.

3) If $\mathcal{E}$ is a category with cofibrations and weak equivalences, then Waldhausen's $S_\cdot$-construction gives rise to a symmetric spectrum $K(\mathcal{E})$ with

$$K(\mathcal{E})_n = |w \underbrace{S_\cdot \ldots S_\cdot}_{n} \mathcal{E}|. \qquad \square$$

<u>Definition</u> 3.1.6 A symmetric <u>ring</u> spectrum is a symmetric spectrum E together with, for every $n \geq 0$, a Σ_n-equivariant map

$$1_n : S^n \to E_n ,$$

and for every pair $m, n \geq 0$, a $\Sigma_m \times \Sigma_n$-equivariant map

$$\mu_{m,n} : E_m \wedge E_n \to E_{m+n}$$

such that

i) $\mu_{m,n} \circ (1_m \wedge E_n) = \lambda_{m,n}$

$\mu_{m,n} \circ (E_m \wedge 1_n) = \rho_{m,n}$

ii) $\mu_{k+m,n} \circ (\mu_{k,m} \wedge E_n) = \mu_{k,m+n} \circ (E_k \wedge \mu_{m,n})$

It is commutative if $\mu_{n,m} \circ tw_{E_m,E_n} = \sigma_{m,n} \circ \mu_{m,n}$.

<u>Definition</u> 3.1.7 A functor with smash product is a continuous pointed endofunctor L on pointed spaces together with two natural transformations

$$1_X : X \to L(X), \qquad \mu_{X,Y} : L(X) \wedge L(Y) \to L(X \wedge Y)$$

s.t.

i) $\mu_{X,Y} \circ (1_X \wedge L(Y)) = \lambda_{X,Y}$

$\mu_{X,Y} \circ (L(X) \wedge 1_Y) = \rho_{X,Y}$

ii) $\mu_{X \wedge Y, Z} \circ (\mu_{X,Y} \wedge L(Z)) = \mu_{X, Y \wedge Z} \circ (L(X) \wedge \mu_{Y,Z})$.

Commutative if $\mu_{Y,X} \circ tw_{L(X),L(Y)} = L(tw_{X,Y}) \circ \mu_{X,Y}$.

<u>Exercise</u> 3.1.8 Show that the axioms for a symmetric ring spectrum imply that $1_{m+n} = \mu_{m,n} \circ (1_m \wedge 1_n)$.

<u>Example</u> 3.1.9 1) Let Γ be a (topological) monoid. Then the functor $X \mapsto X \wedge \Gamma_+$ is an FSP with

$$1_X : X \cong X \wedge S^0 \xrightarrow{X \wedge 1_\Gamma} X \wedge \Gamma_+$$

$$\mu_{X,Y} : (X \wedge \Gamma_+) \wedge (Y \wedge \Gamma_+) \cong X \wedge Y \wedge (\Gamma \times \Gamma)_+ \xrightarrow{X \wedge Y \wedge \mu_\Gamma} X \wedge Y \wedge \Gamma_+ ;$$

it is commutative iff Γ is.

2) Let A be a ring and let $A(X)$ be as in 3.1.5. Then $A(-)$ is an FSP with

$$1_X : X \to A(X) \quad ; \quad x \longmapsto x$$

$$\mu_{X,Y} : A(X) \wedge A(Y) \to A(X \wedge Y) \; ; (\Sigma a_x x , \Sigma a_y y) \mapsto \Sigma a_x a_y (x,y) .$$

Again $A(-)$ is commutative iff A is.

3) Let $\mathcal{E}$ be a category with cofibrations and weak equivalences which is also a monoidal category s.t. the product $(A,B) \mapsto A \wedge B$ is bi-exact, i.e. the functors $A \wedge -$ and $- \wedge B$ preserves cofibrations and if $A \rightarrowtail A'$ and $B \rightarrowtail B'$ are cofibrations then so is $A' \wedge B \cup_{A \wedge B} A \wedge B' \to A' \wedge B'$. Then the symmetric spectrum $K(\mathcal{E})$ from 3.1.5 becomes a symmetric ring spectrum, commutative if $\mathcal{E}$ is symmetric monoidal.

Let L be a symmetric ring spectrum. We define a cyclic space $THH(L)$. mimicking the definition of $N^{cy}_{\otimes,\cdot}(A)$. Consider the category I:

$$\text{ob } I \quad : \quad \underline{n} = \{1,2,\dots,n\} \ , \ n \geq 1 \ , \quad \underline{0} = \emptyset$$

$$\text{mor } I \quad : \quad \underline{\text{all}} \text{ injective set maps}$$

We define a functor

$$G_0 : I \longrightarrow \{\text{pointed spaces}\} \qquad (3.1.10)$$

by on objects:

$$G_0(\underline{n}) = F(S^n, L_n) = \Omega^n L_n$$

and on morphism: write $f : \underline{n} \to \underline{m}$ as $f = \sigma \circ \iota$, where $\iota : \underline{n} \to \underline{m}$ is the standard inclusion, $\iota(i) = i$, and $\sigma \in \Sigma_m$. Then $G_0(f) : G_0(\underline{n}) \to G_0(\underline{m})$ is the composite

$$F(S^n, L_n) \xrightarrow{-\wedge S^{m-n}} F(S^n \wedge S^{m-n}, L_n \wedge S^{m-n})$$
$$\xrightarrow{F(S^m, \rho_{n,m-n})} F(S^m, L_m) \xrightarrow{F(\sigma^{-1}, \sigma)} F(S^m, L_m).$$

Note that $G_0(f)$ is well-defined, i.e. independent of the choice of σ with $f = \sigma \circ \iota$, because (3.1.2) is $\Sigma_n \times \Sigma_m$-equivariant. More generally, we define

$$G_k : I^{k+1} \longrightarrow \{\text{pointed spaces}\} \qquad (3.1.11)$$

with

$$G_k(\underline{n_0},\dots,\underline{n_k}) = F(S^{n_0}\wedge\dots\wedge S^{n_k}, L_{n_0}\wedge\dots\wedge L_{n_k})$$

similarly, and then

$$THH(L)_k := \underset{I^{k+1}}{\underrightarrow{\mathrm{holim}}}\; G_k . \tag{3.1.12}$$

To define the cyclic structure maps, we first note that concatenation defines a functor

$$I \times I \xrightarrow{\amalg} I \quad ; \quad (\underline{m}, \underline{n}) \longmapsto \underline{m+n}$$

and if $f: \underline{m} \to \underline{m}'$ and $g: \underline{n} \to \underline{n}'$ then

$$f \amalg g = \begin{cases} f(i) & , \text{ if } 1 \le i \le m \\ g(i-m)+m' & , \text{ if } m < i \le m+n \end{cases}$$

We then define functors

$$\partial_i : I^{k+1} \longrightarrow I^k \quad , \quad 0 \le i \le k$$

$$s_i : I^{k+1} \longrightarrow I^{k+2} \quad , \quad 0 \le i \le k$$

$$t : I^{k+1} \longrightarrow I^{k+1}$$

by

$$\partial_i(\underline{n_0},\dots,\underline{n_k}) = (\underline{n_0},\dots,\underline{n_i}\amalg\underline{n_{i+1}},\dots,\underline{n_k}) \quad , \quad 0 \le i < k$$

$$= (\underline{n_k}\amalg\underline{n_0}, \underline{n_1},\dots,\underline{n_{k-1}}) \quad , \quad i = k$$

$$s_i(\underline{n_0},\dots,\underline{n_k}) = (\underline{n_0},\dots,\underline{n_i},\underline{0},\underline{n_{i+1}},\dots,\underline{n_k}) \quad , \quad 0 \le i \le k$$

$$t(\underline{n_0},\dots,\underline{n_k}) = (\underline{n_k},\underline{n_0},\underline{n_1},\dots,\underline{n_{k-1}})$$

and natural transformations

$$d_i' : G_k \to G_{k-1} \circ \partial_i \quad , \ 0 \le i \le k$$

$$s_i' : G_k \to G_{k+1} \circ \delta_i \quad , \ 0 \le i \le k$$

$$t_k' : G_k \to G_k \circ t$$

by

$$d_i' = F(S^{n_0} \wedge \ldots \wedge S^{n_k}, L_{n_0} \wedge \ldots \wedge \mu_{n_i, n_{i+1}} \wedge \ldots \wedge L_{n_k}) \quad , \ 0 \le i < k$$

$$= F(S^{n_k} \wedge S^{n_0} \wedge \ldots \wedge S^{n_{k-1}}, \mu_{n_k, n_0} \wedge L_{n_1} \wedge \ldots \wedge L_{n_{k-1}}) \circ \tau^{\#} \quad , \ i = k$$

$$s_i' = F(S^{n_0} \wedge \ldots \wedge S^0 \wedge \ldots \wedge S^{n_k}, L_{n_0} \wedge \ldots \wedge L_{n_i} \wedge 1_0 \wedge L_{n_{i+1}} \wedge \ldots \wedge L_{n_k}) , \ 0 \le i \le k$$

$$t_k' = \tau^{\#} ,$$

where $\tau^{\#} = F(\tau^{-1}, \tau)$ and τ cyclically permutes the smash factors as indicated. Now we define the cyclic structure maps as the composites

$$d_i : \underset{\overrightarrow{I^{k+1}}}{\mathrm{holim}}\, G_k \xrightarrow{d_i'} \underset{\overrightarrow{I^{k+1}}}{\mathrm{holim}}\, G_{k-1} \circ \partial_i \xrightarrow{can} \underset{\overrightarrow{I^{k}}}{\mathrm{holim}}\, G_{k-1}$$

$$s_i : \underset{\overrightarrow{I^{k+1}}}{\mathrm{holim}}\, G_k \xrightarrow{s_i'} \underset{\overrightarrow{I^{k+1}}}{\mathrm{holim}}\, G_{k+1} \circ \delta_i \xrightarrow{can} \underset{\overrightarrow{I^{k+2}}}{\mathrm{holim}}\, G_{k+1}$$

$$t_k : \underset{\overrightarrow{I^{k+1}}}{\mathrm{holim}}\, G_k \xrightarrow{t_k'} \underset{\overrightarrow{I^{k+1}}}{\mathrm{holim}}\, G_{k+1} \circ t \xrightarrow{can} \underset{\overrightarrow{I^{k+1}}}{\mathrm{holim}}\, G_{k+1} ,$$

where can is the canonical map of homotopy colimits, see [BK].

Lemma 3.1.13 $THH(L)_\bullet$ is a well-defined cyclic space.

Pf We must verify that the formulas for d_i', s_i' and t_i' actually yield natural transformations of functors and show that the cyclic identities are satisfyed.

Let us show that $d_1 : G_1 \to G_0 \circ \partial_1$ is natural. So let $f : (n_0, n_1) \to (n_0', n_1')$ be a morphism in $I \times I$. We can write $f = (id, f_2) \circ (f_1, id)$ and treat the two cases separately. So suppose $f = (f_1, id)$ and write $f = \sigma \circ \iota$. We must show that the following diagram commute

$$\begin{tikzcd}
F(S^{n_0}\wedge S^{n_1}, L_{n_0}\wedge L_{n_1}) \arrow[r, "\mu_{n_0,n_1}"] \arrow[d, "-\wedge S^d"] & F(S^{n_0}\wedge S^{n_1}, L_{n_0+n_1}) \arrow[d, "-\wedge S^d"] & d = n_0' - n_0 \\
F(S^{n_0}\wedge S^{n_1}\wedge S^d, L_{n_0}\wedge L_{n_1}\wedge S^d) \arrow[r, "\mu_{n_0,n_1}"] \arrow[d, "F(tw^{-1}, tw)"] & F(S^{n_0}\wedge S^{n_1}\wedge S^d, L_{n_0+n_1}\wedge S^d) \arrow[d, "\rho_{n_0+n_1,d}"] & \\
F(S^{n_0}\wedge S^d\wedge S^{n_1}, L_{n_0}\wedge S^d\wedge L_{n_1}) \arrow[d, "\rho_{n_0,d}\wedge L_{n_1}"] & F(S^{n_0}\wedge S^{n_1}\wedge S^d, L_{n_0+n_1+d}) \arrow[d, "F(S^{n_0}\wedge tw,\, \underline{n_0}\amalg \sigma_{n_1,d})"] & \\
F(S^{n_0'}\wedge S^{n_1}, L_{n_0'}\wedge L_{n_1}) \arrow[r, "\mu_{n_0',n_1}"] \arrow[d, "F(\sigma^{-1}\wedge S^{n_1}, \sigma\wedge L_{n_1})"] & F(S^{n_0}\wedge S^d\wedge S^{n_1}, L_{n_0+d+n_1}) \arrow[d, "F(\sigma^{-1}\wedge S^{n_1}, \sigma\amalg \underline{n_1})"] & \\
F(S^{n_0'}\wedge S^{n_1}, L_{n_0'}\wedge L_{n_1}) \arrow[r, "\mu_{n_0',n_1}"] & F(S^{n_0'}\wedge S^{n_1}, L_{n_0'+n_1}) &
\end{tikzcd}$$

where, we remember, $\sigma_{n_1,d}$ is the permutation which interchanges the first n_1 and last d elements but preserves the order of each block. The top square commutes by naturality of smash product and the bottom square because μ_{n_0',n_1} is $\Sigma_{n_0'} \times \Sigma_{n_1}$-equivariant. So it remains to show that the outer square in the following diagram commutes

$$
\begin{tikzcd}
L_{n_0} \wedge L_{n_1} \wedge S^d \arrow[rr, "\mu_{n_0,n_1} \wedge S^d"] \arrow[drr, "L_{n_0} \wedge L_{n_1} \wedge 1_d"'] \arrow[dd, "tw"'] & & L_{n_0+n_1} \wedge S^d \arrow[rr, "L_{n_0+n_1} \wedge 1_d"] \arrow[d, phantom, "a"] & & L_{n_0+n_1} \wedge L_d \arrow[dd, "\mu_{n_0+n_1,d}"] \\
 & & L_{n_0} \wedge L_{n_1} \wedge L_d \arrow[urr, "\mu_{n_0,n_1} \wedge L_d"] \arrow[d, "L_{n_0} \wedge \mu_{n_1,d}"] \arrow[rr, phantom, "b"] & & \\
L_{n_0} \wedge S^d \wedge L_{n_1} \arrow[dd, "L_{n_0} \wedge 1_d \wedge L_{n_1}"'] \arrow[rr, phantom, "d"] & & L_{n_0} \wedge L_{n_1+d} \arrow[rr, "\mu_{n_0,n_1+d}"] \arrow[d, "L_{n_0} \wedge \sigma_{n_1,d}"'] \arrow[rr, phantom, "c", shift right=4ex] & & L_{n_0+n_1+d} \arrow[dd, "n_0 \amalg \sigma_{n_1,d}"] \\
 & & L_{n_0} \wedge L_{d+n_1} \arrow[drr, bend left, "\mu_{n_0,d+n_1}"] \arrow[d, phantom, "b"] & & \\
L_{n_0} \wedge L_d \wedge L_{n_1} \arrow[urr, bend left] \arrow[rr, "\mu_{n_0,d} \wedge L_{n_1}"] & & L_{n_0+d} \wedge L_{n_1} \arrow[rr, "\mu_{n_0',n_1}"] & & L_{n_0+d+n_1}
\end{tikzcd}
$$

Here we have used (3.1.7 i). Now the diagram labeled a commutes for trivial reasons, the two labeled b by associativity of μ and the diagram c by naturality of μ. Finally, the diagram d is obtained from the following diagram using (3.1.7 i) and smashing with L_{n_0}:

$$
\begin{tikzcd}
L_{n_1} \wedge S^d \arrow[r, "\rho_{n_1,d}"] \arrow[d, "tw"] & L_{n_1+d} \arrow[d, "\sigma_{n_1,d}"] \\
S^d \wedge L_{n_1} \arrow[r, "\lambda_{d,n_1}"] & L_{d+n_1}
\end{tikzcd}
.
$$

This commutes by definition of ρ.

Next, we verify the simplicial identity $d_i' s_j' = s_{j-1}' d_i'$ if $i < j$. Note that $\partial_i s_j = s_{j-1} \partial_i$. We must show that the outer square below commutes

$$
\begin{tikzcd}
\operatorname{holim}\limits_{\overrightarrow{I^{k+1}}} G_k \arrow[r, "s_j'"] \arrow[d, "d_i'"] \arrow[dr, phantom, "c"] & \operatorname{holim}\limits_{\overrightarrow{I^{k+1}}} G_{k+1} \circ s_j \arrow[r, "can"] \arrow[d, "d_i' \circ id"] \arrow[dr, phantom, "b"] & \operatorname{holim}\limits_{\overrightarrow{I^{k+2}}} G_{k+1} \arrow[d, "d_i'"] \\
\operatorname{holim}\limits_{\overrightarrow{I^{k+1}}} G_{k-1} \circ \partial_i \arrow[dr, "s_{j-1}' \circ id"] \arrow[dd, "can"] \arrow[ddr, phantom, "b"] & \operatorname{holim}\limits_{\overrightarrow{I^{k+1}}} G_k \circ \partial_i \circ s_j \arrow[r, "can"] \arrow[d, equal] \arrow[ddr, phantom, "a"] & \operatorname{holim}\limits_{\overrightarrow{I^{k+2}}} G_k \circ \partial_i \arrow[dd, "can"] \\
& \operatorname{holim}\limits_{\overrightarrow{I^{k+1}}} G_k \circ s_{j-1} \circ \partial_i \arrow[d, "can"] & \\
\operatorname{holim}\limits_{\overrightarrow{I^{k}}} G_{k-1} \arrow[r, "s_{j-1}'"] & \operatorname{holim}\limits_{\overrightarrow{I^{k}}} G_k \circ s_{j-1} \arrow[r, "can"] & \operatorname{holim}\limits_{\overrightarrow{I^{k+1}}} G_k
\end{tikzcd}
$$

Now the diagram a commutes by functoriality of the homotopy colimit, the diagrams b by naturality of can and the diagram c from the calculation that $(d_i' \circ id) s_j' = (s_{j-1}' \circ id) d_i'$. The other identities are proved similarly. Some of the involves the axioms (3.1.7).

$\square$

Our next task is to show that $THH(L)_k$ has the "correct" homotopy type. To this end we need the theory of quasi-fibrations which we now recall. The original paper by [Dold-Thom] is an excellent resource for details.

A map $p: E \to B$ is called a quasi-fibration if, for every $x \in B$, the canonical map from $p^{-1}(x)$ to the homotopy fiber of p over x is a weak equivalence. For example, a Serre fibration is a q.f. but not the other way around. We recall the following results from [Dold-Thom]:

- let $U, V \subset B$ be open with $B = U \cup V$ and let $p: E \to B$ be a map which restricts to a q.f. over U, V and $U \cap V$. Then $p: E \to B$ is a q.f.

- suppose $B = \varinjlim B_n$ where $B_0 \subset B_1 \subset B_2 \subset \ldots \subset B$ are T_1-spaces. Then a map $p: E \to B$ which restricts to a q.f. over each B_i is itself a q.f.

- let $p: E \to B$ be a surjection with a fiber preserving deformation into $p: E' \to B'$, i.e. $B' \subset B$, $E' = p^{-1}(B')$ and there are maps

$$D: E \times [0,1] \to E, \qquad d: B \times [0,1] \to B$$

s.t.

$$\begin{array}{llll} D_0 = id & D_t(E') \subset E' & D_1(E) \subset E' & \\ d_0 = id & d_t(B') \subset B' & d_1(B) \subset B' & pD_1 = d_1 p \end{array}$$

Assume that for all $x \in B$, $D_1: p^{-1}(x) \to p^{-1}(d_1(x))$ is a weak equivalence. Then if $p: E' \to B'$ is a q.f. then so is $p: E \to B$.

The following lemma, taken from D. Quillen: Higher Algebraic K-theory I, Proc. Batelle, LNM 341, shows how q.f.'s arise in nature:

<u>Lemma.</u> <u>Let</u> $i \mapsto X_i$ <u>be a functor from a small category</u> I <u>to topological spaces, and let</u> $g : X_I \to BI$ <u>be the space over</u> BI <u>obtained by realizing the simplicial space</u>

$$p \mapsto \coprod_{i_0 \to \cdots \to i_p} X_{i_0} .$$

<u>If</u> $X_i \to X_{i'}$ <u>is a homotopy equivalence for every arrow</u> $i \to i'$ <u>in</u> I, <u>then</u> g <u>is a quasi-fibration.</u>

Proof. It suffices by Lemma 1.5 of [Dold-Lashof] to show that the restriction of g to the p-skeleton F_p of BI is a quasi-fibration for all p. We have a map of cocartesian squares

$$\begin{array}{ccccccc} \coprod \partial\Delta^p & \subset & \coprod \Delta^p & & \coprod X_{i_0} \times \partial\Delta^p & \subset & \coprod X_{i_0} \times \Delta^p \\ \downarrow & & \downarrow & \xleftarrow{g} & \downarrow & & \downarrow \\ F_{p-1} & \subset & F_p & & g^{-1}(F_{p-1}) & \subset & g^{-1}(F_p) \end{array}$$

where the disjoint unions are taken over the nondegenerated p-simplices $i_0 \to \cdots \to i_p$ of NI. Let U be the open set of F_p obtained by removing the barycenters of the p-cells, and let $V = F_p - F_{p-1}$. It suffices by Lemma 1.4 of <u>loc. cit.</u> to show the restrictions of g to U, V and $U \cap V$ are quasi-fibrations. This is clear for V and $U \cap V$, since over each p-cell g is a product map.

We will apply Lemma 1.3 of <u>loc. cit.</u> to $g|U$, assuming as we may by induction that $g|F_{p-1}$ is a quasi-fibration, and using the evident fibre-preserving deformation D of $g|U$ into $g|F_{p-1}$ provided by the radial deformation of Δ^p minus barycenter onto $\partial\Delta^p$. We have only to check that if D carries $x \in U$ into $x' \in F_{p-1}$, then the map $g^{-1}(x) \to g^{-1}(x')$ induced by D induces isomorphisms of homotopy groups. Supposing $x \notin F_{p-1}$ as we may, let x come from an interior point z of the copy of Δ^p corresponding to the simplex $s = (i_0 \to \cdots \to i_p)$, and let the radial deformation push z into the open face of Δ^p with vertices $j_0 < \cdots < j_q$. Then it is easy to see that $g^{-1}(x) = X_{i_0}$ and $g^{-1}(x') = X_k$, $k = i_{j_0}$, and that the map in question is the one $X_{i_0} \to X_k$ induced by the face $i_0 \to k$ of s. As these induced maps are homotopy equivalences by hypothesis, the proof of the lemma is complete.

The conclusion is that $\underset{I}{\underrightarrow{\mathrm{holim}}}\, X_i \to BI$ is a q.f.
Note that $g^{-1}(i) = X_i$, for all objects $i \in I$.

From now on we assume that our symmetric spectrum satisfies that adjoint

$$\tilde{\lambda}_{m,n} : L_n \longrightarrow \Omega^m L_{m+n} \qquad (3.1.14)$$

is $n+\ell(n)$ - connected, such that $\ell(n) \to \infty$ as $n \to \infty$.

<u>Lemma</u> 3.1.15 (Approximation) Given $s \geq 0$ there exists $N \geq 0$ s.t. the canonical inclusion

$$G_k(\underline{n_0}, \dots, \underline{n_k}) \hookrightarrow \underset{\overrightarrow{I^{k+1}}}{\mathrm{holim}}\, G_k = THH(L)_k$$

is s-connected, whenever $n_i \geq N$, $i = 0, \dots, k$.

Pf We consider the case $k=0$; the general case is similar. By assumption there exists N s.t. for all $f : \underline{n} \to \underline{m}$ with $n \geq N$, $G_0(f) : G_0(\underline{n}) \to G_0(\underline{m})$ is s-connected. Let $I_{\geq N}$ be the full sub-category of I with objects $\{\underline{n} \mid n \geq N\}$. Then

$$\underset{\overrightarrow{I_{\geq N}}}{\mathrm{holim}}\, G_0 \xhookrightarrow{\sim} \underset{\overrightarrow{I}}{\mathrm{holim}}\, G_0$$

induced from the inclusion functor $i : I_{\geq N} \to I$ is a homotopy equivalence. A homotopy inverse is provided by the map of homotopy colimits induced from the functor $r : I \to I_{\geq N}$ given by, say, right concatenation by $\underline{N}$; $r(\underline{n}) = \underline{n} + \underline{N}$.

The required homotopies are induced from the natural transformations $h: id_I \to i\circ r$, $g: id_{I_{\geq N}} \to r\circ i$ given by the standard inclusion $\underline{n} \hookrightarrow \underline{n} \amalg \underline{N}$.

When we restrict the functor G_0 to $I_{\geq N}$ we get a diagram of spaces in which all maps are s-connected. It follows from the lemma p. 70 that the map

$$\underset{\substack{\longrightarrow \\ I_{\geq N}}}{\mathrm{holim}}\, G_0 \xrightarrow{p} BI_{\geq N}$$

has the property that, for all $x \in BI_{\geq N}$, the map from $p^{-1}(x)$ to the homotopy fiber of p over x is s-connected. For example one can functorially replace each $G_0(\underline{n})$ by the coskeleton $G_0(\underline{n})[0,s)$. If we take x to be the vertex $\underline{n}$, then $p^{-1}(x) = G_0(\underline{n})$. So the lemma follows once we show $BI_{\geq N} \simeq *$. Now by the first argument $BI_{\geq N} \simeq BI$. Moreover, there are natural transformations from each of the projections $pr_i: I\times I \to I$ to the concatenation functor $\amalg: I\times I \to I$. It follows that

$$pr_1 \simeq pr_2 : BI\times BI \to BI$$

but then $BI \simeq *$. $\square$

<u>Definition</u> 3.1.16 $\quad THH(L) = |THH(L)_\cdot|$.

Remark 3.1.17 We recall the functorial coskeleton construction used in the proof of 3.1.15. Given a space X, we let FX be the pushout

$$\begin{tikzcd} \coprod\limits_{S^k \to X,\, k \geq n} S^k \arrow[r] \arrow[d, hook] & X \arrow[d, hook] \\ \coprod\limits_{S^k \to X,\, k \geq n} D^{k+1} \arrow[r] & FX \end{tikzcd},$$

where the coproduct is over all continuous maps from sheres of dimension $\geq n$ to X (note that we have not chosen a base point), and define

$$X[0,n) = \varinjlim_i F^i X .$$

The map $X \to FX$ being a closed Hurewicz cofibration is in particular a closed inclusion, so the homotopy groups of the colimit is the colimit of the homotopy groups. It follows that for every choice of basepoint in X, the map

$$X \to X[0,n)$$

induces an isomorphism of $\pi_i(-,x_0)$, for $i<n$, and that $\pi_i(X[0,n),x_0)=0$. for $i \geq n$.

3.2 The space $TC(L)$

Let $C_r \subset S^1$ be the (cyclic) subgroups of order r and recall the root isomorphism $\rho_{C_r}: S^1 \to S^1/C_r$ from §1.3. We consider the S^1-space

$$\rho_{C_r}^* THH(L)^{C_r} \underset{S^1}{\cong} |(sd_r THH(L)_\bullet)^{C_r}| \qquad (3.2.1)$$

with the homeomorphism given by 1.3.18. We have

$$sd_r THH(L)_{k-1} = THH(L)_{rk-1} = \underset{\underset{I^{rk}}{\longrightarrow}}{\mathrm{holim}}\, G_{rk-1}$$

and the simplicial C_r-action is given by the operator t_{rk-1}^k. We may write the homotopy colimit as

$$\underset{\underset{I^{rk}}{\longrightarrow}}{\mathrm{holim}}\, G_{rk-1} = \Big| [q] \mapsto \bigvee_{i_0 \to \cdots \to i_q} G_{rk-1}(i_0) \Big|$$

where the disjoint union is over sequences of q composable arrows in I^{rk}. The operator t_{rk-1}^k is then induced from the simplicial operator which takes the summand indexed by $i_0 \to \cdots \to i_q$ to the one indexed by $t^k i_0 \to \cdots \to t^k i_q$ via the map $(t'_{rk-1})^k : G_{rk-1}(i_0) \to G_0(t^k i_0)$. In particular, if a q-simplex is fixed under the operator t_{rk-1}^k, then it must belong to a summand whose index is fixed by $t^k: I^{rk} \to I^{rk}$,

$$t^k(\underline{n}_0, \ldots, \underline{n}_{rk-1}) = (\underline{n}_{r(k-1)}, \ldots, \underline{n}_{rk-1}, \underline{n}_0, \ldots, \underline{n}_{r(k-1)-1}).$$

Obviously, the subcategory of I^{rk} fixed by f^k is equal to the image of the diagonal functor

$$\Delta_r : I^k \to I^{rk} \quad ;$$

$$(\underline{n}_0, \ldots, \underline{n}_{k-1}) \mapsto (\underline{n}_0, \ldots, \underline{n}_{k-1}, \ldots \ldots, \underline{n}_0, \ldots, \underline{n}_{k-1}) .$$

Now the natural transformation $(t'_{rk-1})^k$ restricts to a natural transformation from the functor $G_{rk-1} \circ \Delta_r$ to itself, and hence the cyclic group C_r acts on the functor $G_{rk-1} \circ \Delta_r$ through natural transformations. We have shown that the canonical map induces a homeomorphism

$$\underset{\overrightarrow{I^k}}{\mathrm{holim}} \, (G_{rk-1} \circ \Delta_r)^{C_r} \xrightarrow{\cong} \Big(\underset{\overrightarrow{I^{rk}}}{\mathrm{holim}} \, G_{rk-1} \Big)^{C_r} .$$

Let us spell out that the left hand side is the homotopy colimit of equivariant mappings spaces

$$\underset{\overrightarrow{I^k}}{\mathrm{holim}} \, F\big((S^{n_0} \wedge \ldots \wedge S^{n_{k-1}})^{\wedge r}, (L_{n_0} \wedge \ldots \wedge L_{n_{k-1}})^{\wedge r} \big)^{C_r} \qquad (3.2.2)$$

where C_r acts by cyclically permuting the smash factors in both variables and by conjugation on the mapping space: the fixed set is the subspace of C_r-equivariant maps. Now for any C_r-spaces X, Y, we get a map

$$\mathrm{res} : F(X, Y)^{C_r} \to F(X^{C_r}, Y^{C_r})$$

by restricting a C_r-equivariant map to the fixed set $X^{C_r} \subset X$. In the case of (3.2.2) we get a map

$$\operatorname*{holim}_{\overrightarrow{I^k}} F((S^{n_0}\wedge\ldots\wedge S^{n_{k-1}})^{\wedge r}, (L_{n_0}\wedge\ldots\wedge L_{n_{k-1}})^{\wedge r})^{C_r}$$

$$\xrightarrow{\text{res}} \operatorname*{holim}_{\overrightarrow{I^k}} F(S^{n_0}\wedge\ldots\wedge S^{n_{k-1}}, L_{n_0}\wedge\ldots\wedge L_{n_{k-1}}),$$

and one readily checks that this commutes with the cyclic structure maps such that we have a map of cyclic spaces

$$r_{C_r,\cdot} : (\mathrm{sd}_r \mathrm{THH}(L)_\cdot)^{C_r} \longrightarrow \mathrm{THH}(L)_\cdot$$

Upon realizing and composing with (3.2.1) we get an S^1-equivariant map

$$r_{C_r} : \rho_{C_r}^* \mathrm{THH}(L)^{C_r} \longrightarrow \mathrm{THH}(L) \tag{3.2.3}$$

which we call the <u>restriction</u> maps. It induces

$$R_r : \rho_{C_{rs}}^* \mathrm{THH}(L)^{C_{rs}} \longrightarrow \rho_{C_s}^* \mathrm{THH}(L)^{C_s},$$

also called restriction. We define new S^1-space valued functors

$$\mathrm{TR}(L) = \operatorname*{holim}_{\overleftarrow{R}} \rho_{C_r}^* \mathrm{THH}(L)^{C_r},$$

$$\mathrm{TR}(L;p) = \operatorname*{holim}_{\overleftarrow{R}} \rho_{C_{p^n}}^* \mathrm{THH}(L)^{C_{p^n}}.$$

Here the first homotopy limit runs over the natural numbers ordered after division. The second runs over the non-negative integers ordered as usual. There is another map from the C_{rs}-fixed set to the C_s-fixed set, namely, the inclusion

$$F_r : \rho^*_{C_{rs}} THH(L)^{C_{rs}} \to \rho^*_{C_s} THH(L)^{C_s}, \qquad (3.2.4)$$

which we call the <u>Frobenius</u>. It is <u>not</u> equivariant: for $z \in S^1$

$$F_r(z^r \cdot x) = z \cdot F_r(x).$$

The following relations are clear:

$$R_r R_s = R_{rs}, \quad F_r F_s = F_{rs}, \quad F_r R_s = R_s F_r.$$

The Frobenius maps induce selfmaps

$$F_r : TR(L) \to TR(L) \qquad (3.2.5)$$

given by the composite

$$\underset{\underset{R}{\leftarrow}}{\mathrm{holim}}\, \rho^*_{C_s} THH(L)^{C_s} \xrightarrow{can} \underset{\underset{R}{\leftarrow}}{\mathrm{holim}}\, \rho^*_{C_{rt}} THH(L)^{C_{rt}} \xrightarrow{F_r} \underset{\underset{R}{\leftarrow}}{\mathrm{holim}}\, \rho^*_{C_t} THH(L)^{C_t}$$

where the first map is the canonical map which restricts the homotopy limit to the numbers divisible by r. Similar F_p induces a selfmap

$$F : TR(L;p) \to TR(L;p) \qquad (3.2.6)$$

In the case of (3.2.5), $F_r F_s = F_{rs}$ such that we get an action by the multiplicative monoid of natural numbers on $TR(L)$. In the case of (3.2.6), F induces an action by the additive monoid of non-negative integers. We define topological cyclic homology as the homotopy fixed sets of these actions,

$$
\begin{aligned}
TC(L) &= TR(L)^{h\{F_r \mid r \geq 0\}} \\
TC(L;p) &= TR(L;p)^{h\{F^n \mid n \geq 0\}}
\end{aligned}
\tag{3.2.7}
$$

4.1 Equivariant homotopy theory

Let G be a compact Lie group. We shall here assume that our G-spaces have base points and that this base point is fixed under the action of G. If $f, g: X \to Y$ are two pointed (base point preserving) G-maps then a pointed G-homotopy from f to G is a pointed G-map

$$h : X \wedge [0,1]_+ \to Y$$

such that $h(x,0) = f(x)$, $h(x,1) = g(x)$. Here G acts on $X \wedge [0,1]_+$ in the first variable. We denote by $[X,Y]_G$ the set of pointed G-homotopy classes of pointed G-maps from X to Y. If $F_G(X,Y)$ is the space of pointed G-maps, then

$$[X,Y]_G = \pi_0 F_G(X,Y).$$

We note that if we let G act on the space $F(X,Y)$ of all pointed maps from X to Y by conjugation, $(gf)(x) = gf(g^{-1}x)$, then $F_G(X,Y)$ is the fixed set,

$$F_G(X,Y) = F(X,Y)^G \subset F(X,Y).$$

Suppose that $i: H \hookrightarrow G$ is the inclusion of a closed subgroup. Then we can view any pointed

G-space Y as an H-space i^*Y via i.
Conversely, if X is a pointed (left) H-space, then

$$G_+ \wedge_H X := G_+ \wedge X / \sim ,$$

where $(gh, x) \sim (g, hx)$, with G acting by left multiplication in the first variable is a pointed G-space, and we have the adjunction

$$F_G(G_+ \wedge_H X, Y) \cong F_H(X, i^*Y). \qquad (4.1.1)$$

If $X = i^*Y$ for a pointed G-space Y, then we get an isomorphism of pointed G-spaces

$$\begin{array}{ccc} G_+ \wedge_H i^*Y & \cong & G/H_+ \wedge Y \\ (g, y) & \mapsto & (g, gy) \\ (g, g^{-1}y) & \leftarrow & (g, y) \end{array} \qquad (4.1.2)$$

Let again $H \subset G$ be a closed subgroup and let Y be a pointed left G-space. Then mult. by g gives a homeomorphism of fixed sets

$$Y^H \xrightarrow{\cong} Y^{gHg^{-1}} , \quad y \mapsto gy$$

In particular, the normalizer $N_G(H) \subset G$ acts on Y^H, and since $H \subset N_G(H)$ acts trivially this action factors to an action by the Weyl group

$$W_G(H) := N_G(H)/H.$$

If $N \triangleleft G$ is a closed normal subgroup, then Y^N is a G/N-space. Conversely, we may view a G/N-space X as a G-space $\varepsilon^* X$ via the projection $\varepsilon : G \to G/N$, and we have the adjunction

$$F_G(\varepsilon^* X, Y) \cong F_{G/N}(X, Y^N). \qquad (4.1.3)$$

We define a (Hurewicz) G-fibration to be a pointed G-map $p : E \to B$ such that for any diagram of pointed G-maps

$$\begin{tikzcd} X \times \{0\} \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ X \wedge [0,1]_+ \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

the dotted arrow exists. Dually, a (Hurewicz) G-cofibration is a pointed G-map $i : A \to X$ such that for every diagram of pointed G-spaces

$$\begin{tikzcd} Y & X \arrow[l] \arrow[dl, dashed] \\ F([0,1]_+, Y) \arrow[u, "ev_0"] & A \arrow[l] \arrow[u, "i"] \end{tikzcd}$$

the dotted arrow exists.

If $A \overset{i}{\rightarrowtail} X$ is a G-cofibration, then for any pointed G-space Y, the map

$$F_G(X, Y) \xrightarrow{i^*} F_G(A, Y) \tag{4.1.4}$$

is a fibration with fiber $F_G(X/A, Y)$. We wish to estimate its connectivity.

We now turn to G-CW-complexes. Let $H \subset G$ be a closed subgroup. We call the unbased G-space $G/H \times D^n$, with G acting by left multiplication in the first variable, a G-cell of dimension n and orbit type G/H. By a relative G-CW-complex we mean a pair of G-spaces (X, A) such that

$$X = \varinjlim_n X_n$$

where $X_{-1} = A$, and for every $n \geq 0$, there is a push out diagram of (unbased) G-spaces

$$\begin{tikzcd} \coprod_\alpha G/H_\alpha \times S^{n-1} \arrow[r] \arrow[d, hook] & X_{n-1} \arrow[d, hook] \\ \coprod_\alpha G/H_\alpha \times D^n \arrow[r] & X_n \end{tikzcd} \tag{4.1.5}$$

Here, by convention, $S^{-1} = \emptyset$. If $A = \emptyset$, then we call X a G-CW-complex, and if $A = *$, a pointed G-CW-complex. We call the map $G/H_\alpha \times S^{n-1} \to X_{n-1}$ the attaching map of the cell $G/H_\alpha \times D^n$. By (4.1.1) and (4.1.3) this is the same as a non-equivariant map $S^{n-1} \to X_{n-1}^{H_\alpha}$.

<u>Lemma</u> 4.1.6 If X is a G-CW-complex and $H \subset G$ a closed subgroup, then X^H is a $W_G(H)$-CW-complex and X_n^H is the n-skeleton.

<u>Pf</u> It is easy to see that $X^H = \varinjlim_n X_n^H$. Taking H-fixed sets in (4.1.5) we obtain a push out square of $W_G(H)$-spaces

$$\begin{array}{ccc} \coprod_\alpha (G/H_\alpha)^H \times S^{n-1} & \longrightarrow & X_{n-1}^H \\ \cap & & \cap \\ \coprod_\alpha (G/H_\alpha)^H \times D^n & \longrightarrow & X_n^H . \end{array}$$

Finally, we decompose $(G/H_\alpha)^H$ into $W_G(H)$-orbits to see that X_n^H is obtained from X_{n-1}^H by attaching orbits of the form $W_G(H)/K \times D^n$. We need that $(G/H_\alpha)^H$ is a disjoint union of its $W_G(H)$-orbits. If G is finite this is trivial. In general, this follows from the theorem that $(G/H_\alpha)^H$ has only a finite number of $W_G(H)$-orbits. See for example [tom Dieck] p. 41 for a proof. $\square$

Recall that the connectivity of a space X is the largest integer, $\mathrm{conn}(X)$, such that $\pi_i X = 0$, for all $i \leq \mathrm{conn}(X)$. For example $\mathrm{conn}(S^n) = n-1$. By the dimension of a G-CW-complex X, we mean the maximal dimension of a G-cell in X, or ∞ if no such maximum exists.

<u>Lemma</u> 4.1.7 Let X be a pointed G-CW-complex and let Y be any pointed G-space. Then

$$\mathrm{conn}(F_G(X,Y)) \geq \min_{H \subset G} \left(\mathrm{conn}(Y^H) - \dim(X^H)\right),$$

where H runs through all closed subgroups of G.

Pf By induction: let $X_{-1} = *$, then $F_G(X_{-1}, Y) = *$, so the inequality is trivially satisfied. We assume then that $\pi_m F_G(X_{n-1}, Y) = 0$ and want to show that $\pi_m F_G(X_n, Y) = 0$. The inclusion $X_{n-1} \rightarrowtail X_n$ is a G-cofibration so induces a fibration $F_G(X_n, Y) \to F_G(X_{n-1}, Y)$ with fiber

$$F_G(X_n/X_{n-1}, Y) \cong \prod_\alpha F_G(G/H_{\alpha+} \wedge S^n, Y) \cong \prod_\alpha \Omega^n(Y^{H_\alpha}).$$

Now $\pi_m \Omega^n(Y^{H_\alpha}) = 0$ if $m \leq \mathrm{conn}(Y^{H_\alpha}) - n$. Now $(G/H_\alpha)^{H_\alpha} = W_G(H_\alpha)$ so $W_G(H_\alpha) \times D^n$ is a cell in X^{H_α}. It follows that $n \leq \dim X^{H_\alpha}$. Finally, a limit argument finishes the proof. $\square$

If $f: X \to Y$ is a pointed G-homotopy equivalence, then for all $H \subset G$ closed, $f^H: X^H \to Y^H$ is a pointed homotopy equivalence. This motivates us to define a (pointed) G-map $f: X \to Y$ to be a <u>weak G-equivalence</u> if $f^H: X^H \to Y^H$ is a weak equivalence, for all $H \subset G$.

<u>Exercise</u> 4.1.8 Show that a weak G-equivalence $f: X \to Y$ between (pointed) G-CW-complexes is a (pointed) G-homotopy equivalence.

Let V be a finite dimensional orthogonal G-representation and let $S(V)$ be the unit sphere in V.

<u>Proposition</u> 4.1.9 $S(V)$ is a G-CW-complex.

Pf We only prove this when G is finite. Then we can choose a basis $e_1, \dots, e_n \in S(V)$ for V such that $Ge_i \cap Ge_j = \emptyset$ if $i \neq j$. Then radial projection defines a G-homeomorphism between $S(V)$ and the boundary of the polytope

$$P = \operatorname{Conv}\{ge_i \mid g \in G,\ i = 1, \dots, n\}.$$

A simplex in the boundary ∂P has the form

$$\Delta = \operatorname{Conv}\{g_1 e_{i_1}, \dots, g_n e_{i_n}\}$$

It follows that G acts on ∂P thru cellular maps and that if $g\Delta = \Delta$ then multiplication by g on Δ is the identity map. Finally, it is easy to see that, for G finite, a CW-complex with a cellular G-action such that, if multiplication by $g \in G$ maps a cell to itself, then it does so by the identity map, is a G-CW-complex. In the compact Lie case, the proposition is a special case of the theorem that any smooth manifold with a smooth G-action is a G-CW-complex, see [Illman].

$\square$

It follows that the unit disc $D(V)$ and the one-point compactification $S^V \cong D(V)/S(V)$ are G-CW-complexes. There are canonical G-homeomorphism

$$\begin{aligned} S(V \oplus W) &\cong S(V) * S(W) \\ S(\mathbb{R} \oplus W) &\cong S^V \\ S^{V \oplus W} &\cong S^V \wedge S^W. \end{aligned} \tag{4.1.10}$$

<u>Remark</u> 4.1.11 If G is finite and $i : H \hookrightarrow G$ a subgroup, then for a G-CW-complex X, $i^* X$ is an H-CW-complex, although in a non-canonical way. When G is compact Lie this is not at all obvious. However, one may show that $i^* X$ has the H-homotopy type of an H-CW-complex. We refer to [Waner] for a proof of this.

We next recall some facts about simplicial spaces. Following [Segal], we call a simplicial space $X.$ good if all degeneracies $s_i : X_{n-1} \to X_n$ are closed (Hurewicz) cofibrations. This implies, by a theorem of [Lillig] that the inclusion $\bigcup_{i=0}^{n-1} s_i X_{n-1} \to X_n$ of the subspace of degenerate simplices is a cofibration. A simplicial space with this property is said to be proper by [May]. This in turn implies that the inclusion $|X.|^{(n-1)} \rightarrowtail |X.|^{(n)}$ of the $n-1$ skeleton in the n-skeleton is a cofibration with cofiber

$$|X.|^{(n)}/|X.|^{(n-1)} \cong \Sigma^n \left(X_n / \bigcup_{i=0}^{n-1} s_i X_{n-1} \right).$$

One may use this to show that, if E_* is a homology theory, then there is strongly convergent right half plane spectral sequence.

$$E^2_{s,t} = H_s\left(E_t(X.);\ d = \Sigma(-1)^i d_i\right) \Rightarrow E_{s+t}(|X.|).$$

It also follows that if $f : X. \to Y.$ is a map between good simplicial spaces such that, for every $n \geq 0$, $f_n : X_n \xrightarrow{\sim} Y_n$ is a weak equivalence then so is

$$f : |X.| \xrightarrow{\sim} |Y.| .$$

Given any simplicial space $X.$, we define the thickits thickening $\tau X_\cdot$ by

$$\tau X_\cdot = \underset{\rightarrow}{\mathrm{holim}} \left((\Delta^{op} \downarrow [\cdot]) \xrightarrow{pr} \Delta^{op} \xrightarrow{X_\cdot} \mathrm{Top} \right) \qquad (4.1.12)$$

where $(\Delta^{op} \downarrow [n])$ is the category over $[n]$. It has $[n] \xrightarrow{id} [n]$ as terminal object, so the canonical map $\tau X_n \to X_n$ is a weak equivalence. If $[n] \xrightarrow{\theta} [m]$ in Δ is a surjection then $(\Delta^{op} \downarrow [m]) \xrightarrow{\theta^*} (\Delta^{op} \downarrow [n])$ is an inclusion functor, and an inclusion of index categories always induces a closed cofibration on homotopy colimits. Hence $\tau X_\cdot$ is good. If $X_\cdot$ is a cyclic space, then we can replace Δ^{op} by Λ^{op} in (4.1.12) and get a cyclic space $\tau X_\cdot$ whose underlying simplicial space is good.

<u>Lemma</u> 4.1.13 Let $X_\cdot$ be proper and suppose that, for every $n \geq 0$, X_n is <u>connected</u>. Then

$$|PX_\cdot| \xrightarrow{p} |X_\cdot|$$

is a quasi-fibration with fiber $|\Omega X_\cdot|$, and since $|PX_\cdot|$ is contractible, the canonical map

$$|\Omega X_\cdot| \xrightarrow{\sim} \Omega |X_\cdot|$$

is a weak equivalence.

Pf The proof that p is a q.f. is similar to the proof of lemma on page 70; see [May] for details. $\square$

Proposition 4.1.14 Let G be a compact Lie group and let $Y_\cdot$ be a based simplicial G-space such that Y_k^H is $n(H)$-connected, for all $k \geq 0$, and assume that $n(H) \geq 0$, for all $H \subset G$ closed. Suppose X is a based G-CW-complex with finitely many orbit types, and such that $\dim X^H \leq n(H)$, for all $H \subset G$. If $Y_\cdot^H$ is proper for the occuring orbit types, then the canonical map

$$\gamma : |F(X, Y_\cdot)| \longrightarrow F(X, |Y_\cdot|)$$

is a weak G-equivalence.

Pf We first prove that γ^G is a w.e. By induction we may assume that this is so if we replace X by its $n-1$ skeleton X_{n-1}. Now the G-cofibration seq.

$$X_{n-1} \rightarrowtail X_n \longrightarrow \bigvee_\alpha G/H_{\alpha +} \wedge S^n$$

induces a simplicial Hurewicz fibration

$$\prod_\alpha \Omega^n(Y_\cdot^{H_\alpha}) \longrightarrow F_G(X_n, Y_\cdot) \longrightarrow F_G(X_{n-1}, Y_\cdot)$$

Here $n \leq \operatorname{conn}(Y^{H_\alpha})$, so by induction the base is a connected space and therefore, after realization we get a q.f., see [May], p. 120. We now consider the diagram:

$$\begin{tikzcd}
{|\prod_\alpha \Omega^n(Y_\cdot^{H_\alpha})|} \arrow[r] \arrow[d] & {|F_G(X_n, Y_\cdot)|} \arrow[r] \arrow[d, "\gamma^G"] & {|F_G(X_{n-1}, Y_\cdot)|} \arrow[d, "\gamma^G"] \\
{\prod_\alpha \Omega^n |Y_\cdot^{H_\alpha}|} \arrow[r] & {F_G(X_n, |Y_\cdot|)} \arrow[r] & {F_G(X_{n-1}, |Y_\cdot|)}
\end{tikzcd}$$

The left hand map is a w.e. by 4.1.3 and the fact that realization commutes with products. This proves the induction step. Finally, to see that γ^H is a w.e., for $H \subset G$, we use that $Y_\cdot$ is H-homotopic to an H-CW-complex, c.f. 4.1.11, and repeat the argument above. □

Recall that a map $f: Y \to Z$ is called n-connected if, for all $z \in Z$, the homotopy fiber of f over z is an $n-1$ connected space.

<u>Lemma</u> 4.1.15 Let $f: Y \to Z$ be a map of based G-spaces and let X be a G-CW-complex. Then the induced map

$$F_G(X, Y) \longrightarrow F_G(X, Z)$$

is n-connected with $n \geq \min\{\mathrm{conn}(f^H) - \dim(X^H) \mid H \subseteq G\}$.

<u>Pf</u> If $F_G(X, Z)$ is connected this follow from 4.1.7 by considering the homotopy fiber of f over the basept. of Z. In the applications I have in mind, X will be a G-H-cogroup, so that $F_G(X, f)$ is a map of H-spaces; we are done in that case. The general case is left to the reader; see also [tom Dieck], p 106. □

4.2 G-spectra

Let G be cpt. Lie and let V be a f.d. orthogonal G-representation. Then S^V denotes the one-point cpt'n of V, and if X is a based G-space then we also write

$$\Sigma^V X := X \wedge S^V, \qquad \Omega^V X = F(S^V, X)$$

where G acts diagonally (from the left) on the smash product and by conjugation on the mapping space. If $V \subset W$ are two f.d. orthogonal G-repr.'s then $W-V$ stands for the orthogonal complement of V in W. The coherent natural isomorphisms

$$(V-U) \oplus (W-V) \xrightarrow{\cong} W-U \qquad , \quad x \oplus y \mapsto x+y,$$

for $U \subset V \subset W$, induce coherent natural G-homeomorphisms

$$\Sigma^{W-V} \Sigma^{V-U} X \xrightarrow{\cong} \Sigma^{W-U} X.$$

Coherent means that every diagram we can write down will commute, see [MacLane]. For example the homeomorphism for the pair $U \subset V \subset W$ and that for $U \subset W-(V-U) \subset W$ will give a commutative diagram

$$\begin{tikzcd}
\Sigma^{W-V}\Sigma^{V-U}X \arrow[d, "X \wedge tw"] \arrow[dr, "\cong"] & \\
& \Sigma^{W-U}X \\
\Sigma^{V-U}\Sigma^{W-V}X \arrow[ur, "\cong"'] &
\end{tikzcd}$$

By a G-*universe* we mean a countably dim'l orthogonal G-representation $\mathcal{U}$ with the property that $\mathcal{U} \supset \mathbb{R}$, the trivial representation, and that if a f.d. orthogonal G-repr. V embeds in $\mathcal{U}$ then so does a countable sum of copies of V. We call $\mathcal{U}$ *complete* if every f.d. orthogonal G-representation embeds in $\mathcal{U}$. In the following when we write $V \subset \mathcal{U}$, it will always be understood that V is a f.d. G-subrepresentation. We topologize $\mathcal{U}$ as the colimit of $V \subset \mathcal{U}$ f.d.

Definition 4.2.1 A G-*prespectrum* t indexed on $\mathcal{U}$ consists of for every $V \subset \mathcal{U}$, a based G-space t_V, and for every $V \subset W \subset \mathcal{U}$, a based G-map

$$\sigma : \Sigma^{W-V} t_V \to t_W$$

such that for every $V \subset \mathcal{U}$, $\sigma : \Sigma^0 t_V \to t_V$ is the canonical homeomorphism, and such that for every triple $U \subset V \subset W \subset \mathcal{U}$, the square

$$\begin{array}{ccc} \Sigma^{W-V}\Sigma^{V-U} t_U & \xrightarrow{\cong} & \Sigma^{W-U} t_U \\ \downarrow \Sigma^{W-V}\sigma & & \downarrow \sigma \\ \Sigma^{W-V} t_V & \xrightarrow{\sigma} & t_W \end{array}$$

commutes.

A map $f: s \to t$ of G-prespectra indexed on $\mathcal{U}$ is a collection of based G-maps $f_V: s_V \to t_V$, one for every $V \subset \mathcal{U}$, such that for every pair $V \subset W \subset \mathcal{U}$, the diagram

$$\begin{array}{ccc} \Sigma^{W-V} s_V & \xrightarrow{\Sigma^{W-V} f_V} & \Sigma^{W-V} t_V \\ \downarrow \sigma & & \downarrow \sigma \\ s_W & \xrightarrow{f_W} & t_W \end{array}$$

commutes.

A G-__spectrum__ T indexed on $\mathcal{U}$ is a G-prespectrum indexed on $\mathcal{U}$ such that for every pair $V \subset W \subset \mathcal{U}$, the adjoint of the structure map

$$\tilde{\sigma} : T_V \xrightarrow{\cong} \Omega^{W-V} T_W$$

is a homeomorphism. We write $G\mathcal{P}\mathcal{U}$ for the category of G-prespectra indexed on $\mathcal{U}$ and let $G\mathcal{S}\mathcal{U} \subset G\mathcal{P}\mathcal{U}$ be the full subcategory of G-spectra. One may use Freyd's adjoint functor theorem to show that the forgetful functor

$$\ell : G\mathcal{S}\mathcal{U} \to G\mathcal{P}\mathcal{U}$$

has a left adjoint

$$L : G\mathcal{P}\mathcal{U} \to G\mathcal{S}\mathcal{U},$$

see [LMS], appendix. The categories $G\mathcal{P}\mathcal{U}$ and $G\mathcal{S}\mathcal{U}$ are topological categories in an obvious way. It is clear that ℓ is continuous and this implies that L is, too.

Consider the functor $L' : G\mathcal{P}\mathcal{U} \to G\mathcal{P}\mathcal{U}$ given by

$$(L't)_V = \varinjlim_{Z \supset V} \Omega^{Z-V} t_Z \qquad (4.2.2)$$

with $\tilde{\sigma} : (L't)_V \to \Omega^{W-V}(L't)_W$ given by

$$\varinjlim_{Z \supset V} \Omega^{Z-V} t_Z \xleftarrow{\cong} \varinjlim_{Z \supset W} \Omega^{Z-V} t_Z \to \Omega^{W-V} \varinjlim_{Z \supset W} \Omega^{Z-W} t_Z,$$

where the first map is the cofinality homeomorphism and the second is the canonical homeomorphism $\Omega^{Z-V} \cong \Omega^{W-V}\Omega^{Z-W}$ followed by the map which permutes Ω^{W-V} and $\varinjlim$.

<u>Lemma</u> 4.2.3 Let t be a G-prospectrum indexed on $\mathcal{U}$ and suppose $L't$ happens to be a G-spectrum. Then $Lt \cong L't$.

<u>Pf</u> Remember that $L \dashv \ell$. So we must show that $L't$ has the universal property that if $t \to T$ is a map in $G\mathcal{P}\mathcal{U}$ and T is in $G\mathcal{S}\mathcal{U}$, then the map factors uniquely through $t \to L't$. But this is obvious from the def'n of $L't$. $\square$

<u>Remark</u> 4.2.4 Spheres being compact are small w.r.t. closed inclusions, so if the maps $\tilde{\sigma} : t_V \to \Omega^{W-V} t_W$ are closed inclusions, for all $V \subset W \subset \mathcal{U}$, then the structure maps in 4.2.2 are homeomorphisms, and hence in this case $Lt \cong L't$.

We call a G-prespectrum t good if $\forall\, V \subset W \subset \mathcal{U}$, $\sigma : \Sigma^{W-V} t_V \to t_W$ is a closed inclusion. This implies that the adjoints $\tilde{\sigma} : t_V \to \Omega^{W-V} t_W$ are closed inclusions too and hence that $L't \cong Lt$.

If t in $G\mathcal{P}\mathcal{U}$ is any prespectrum, then we define the thickening t^{τ} in $G\mathcal{P}\mathcal{U}$ by

$$t^{\tau}_V = \underset{Z \subset V}{\underrightarrow{\mathrm{holim}}}\ \Sigma^{V-Z} t_Z \qquad (4.2.5)$$

where for $Z_1 \subset Z_2$ the maps in the system are

$$\Sigma^{V-Z_1} t_{Z_1} \xleftarrow{\cong} \Sigma^{V-Z_2} \Sigma^{Z_2 - Z_1} t_{Z_1} \xrightarrow{\Sigma^{V-Z_2}\sigma} \Sigma^{V-Z_2} t_{Z_2}.$$

The structure maps in t^{τ} are given by

$$\Sigma^{W-V} \underset{Z \subset V}{\underrightarrow{\mathrm{holim}}}\ \Sigma^{V-Z} t_Z \xleftarrow{\cong} \underset{Z \subset V}{\underrightarrow{\mathrm{holim}}}\ \Sigma^{W-Z} t_Z \rightarrowtail \underset{Z \subset W}{\underrightarrow{\mathrm{holim}}}\ \Sigma^{W-Z} t_Z,$$

where the last map is induced from the inclusion of a subcategory and is therefore a closed G-cofibration. In particular, t^{τ} is good. The index cat. $\{Z \subset V\}$ has V as terminal object. This gives us a map in $G\mathcal{P}\mathcal{U}$

$$\pi : t^{\tau} \longrightarrow t$$

such that for all $V \subset \mathcal{U}$, $\pi_V : t^{\tau}_V \to t_V$ is a G-homotopy equivalence. The homotopy inverses do not form a map of prespectra, though.

Example 4.2.6 For $V \subset \mathcal{U}$, let S^{-V} denote the spectrification of the (good) G-prespectrum

$$W \mapsto \begin{cases} S^{W-V} & , \text{ if } W \supset V \\ * & , \text{ else ,} \end{cases}$$

and note that S^{-V} corepresents the Vth space functor in that we have a natural G-homeomorphism

$$G\mathcal{S}\mathcal{U}(S^{-V}, T) \cong T_V \ .$$

In particular, S^0 is the sphere G-spectrum for the universe $\mathcal{U}$. We sometimes write $S^0_{\mathcal{U}}$. $\square$

We shall now define the topological Hochschild spectrum $T(L)$ which will be an S^1-spectrum indexed on the complete S^1-universe

$$\mathcal{U} = \bigoplus_{n, \alpha \geq 0} \mathbb{C}(n)_\alpha \qquad (4.2.7)$$

where $\mathbb{C}(n) = \mathbb{C}$, with S^1 acting by $z \cdot v = z^n v$, and α is a dummy. For a pointed space X, we may define an S^1-space $THH(L; X)$ similar to (3.1.16) by replacing the functor G_k in (3.1.12) by

$$G_k^X(\underline{n}_0, \ldots, \underline{n}_k) = F(S^{n_0} \wedge \ldots \wedge S^{n_k}, L_{n_0} \wedge \ldots \wedge L_{n_k} \wedge X)$$

with X acting as a dummy as well for maps in I^{k+1} as for the cyclic structure. If Y is another pointed space, then we get an S^1-map

$$\rho_{X,Y} : THH(L;X) \wedge Y \longrightarrow THH(L; X \wedge Y)$$

defined as the composite

$$|THH(L;X)_\bullet| \wedge Y \xrightarrow{\cong} |THH(L;X_\bullet) \wedge Y| \longrightarrow |THH(L;X \wedge Y)_\bullet|$$

where the first map is the canonical map and the second is induced from the obvious natural transformation $G_k^X(\underline{n}_0, \dots, \underline{n}_k) \wedge Y \to G_k^{X \wedge Y}(\underline{n}_0, \dots, \underline{n}_k)$.
If X is a based S^1-space, then we get an additional S^1-action on $THH(L;X)$ which commutes with the one given by the cyclic structure, i.e. $THH(L;X)$ becomes an $S^1 \times S^1$-space and $\rho_{X,Y}$ becomes $S^1 \times S^1$-equivariant. Now for $V \subset \mathcal{U}$, we define

$$t(L)_V := \Delta^* THH(L; S^V), \tag{4.2.8}$$

where $\Delta : S^1 \to S^1 \times S^1$ is the diagonal, and let

$$\sigma : \Sigma^{W-V} t(L)_V \longrightarrow t(L)_W$$

be ρ_{S^V, S^W}. Then $t(L)$ is an S^1-prespectrum indexed on $\mathcal{U}$, and we define the topological Hochschild spectrum by

$$T(L) = L\, t(L)^{\tau}, \tag{4.2.9}$$

the spectrification of the thickening of $t(L)$.
We next show that passing from $t(L)$ to $T(L)$ we didn't change much in the way of homotopy type. We start with the

Lemma 4.2.10 If $L_0 = S^0$ and each L_n well-pointed, then $THH(L)_\cdot$ is a good simplicial space.

Pf Indeed, $s_i : THH(L;X)_k \to THH(L;X)_{k+1}$ is then simply the map of homotopy colimits induced by the inclusion functor $s_i : I^{k+1} \to I^{k+2}$ and hence a closed cofibration; compare p. 65. $\square$

We always can and will assume that $L_0 = S^0$. For If $\tilde{L}$ is obtained from L by replacing L_0 by S^0 then $THH(\tilde{L};X)_k \to THH(L;X)$ is a w.e. by (3.1.15).

Lemma 4.2.11 Let $f : X \to Y$ be an $n+k$-connected map of well-pointed spaces with X n-connected. Then

$$f^{\wedge t} : X^{\wedge t} \to Y^{\wedge t}$$

is $tn+k$ - connected. $\square$

Proposition 4.2.12 Let L be a symmetric ring spectrum and assume that L_n is well-pointed and $n-1$-connected. Then for every $V \subset \mathcal{U}$ and $C \subset S^1$ finite, the map

$$t(L)^C_V \longrightarrow T(L)_V$$

is a weak C-equivalence.

Pf It suffices to show that, $\forall\, V \subset W \subset \mathcal{U}$, $C_r \subset S^1$

$$t(L)^{C_r}_V \to (\Omega^{W-V} t(L)_W)^{C_r}$$

is a w.e. Can assume $W - V = lR$, where $R = \mathbb{R}C_r$ is the reguler representation. After r-fold edgewise subdivision the map in question becomes

$$|\operatorname{sd}_r THH(L;S^V)_\cdot^{C_r}| \xrightarrow{\tilde{\sigma}} (\Omega^{\ell R}|\operatorname{sd}_r THH(L;S^{\ell R}\wedge S^V)_\cdot|)^{C_r}$$

To show that this map is an equivalence we consider the diagram on page 100, where $\tilde{\sigma}$ is the top map. The maps labeled Δ are homeomorphisms by the argument on page 74. Let us for convinience assume that $V=0$, the general case is analogous. The map η_* is induced from the adjunction map

$$(L_{i_0}\wedge\ldots\wedge L_{i_k})^{\wedge r} \xrightarrow{\eta} \Omega^{\ell R}(L_{i_0}\wedge\ldots\wedge L_{i_k}\wedge S^\ell)^{\wedge r}.$$

We use the equivariant suspension theorem, see [Adams]. By 4.2.11

$$\operatorname{conn}(L_{i_0}\wedge\ldots\wedge L_{i_k})^{\wedge(r/s)} \geq (r/s)(i-1) \quad ; \quad i=i_0+\ldots+i_k;$$

so we must have $n(C_s) \leq (r/t)(i-1)$, for all proper subgroups $C_t \subset C_s$. This is satisfied by the fact $n(C_s) = 2(r/s)(i-1)$. and so the equivariant suspension theorem shows that

$$\operatorname{conn}(\eta^{C_s}) \geq 2(r/s)(i-1).$$

Now lemma 4.1.15 shows that the map induced from η and the obvious adjunction homeomorphism

$$F((S^i)^{\wedge r},(L_{i_0}\wedge\ldots\wedge L_{i_k})^{\wedge r})^{C_r} \to F((S^i\wedge S^\ell)^{\wedge r},(L_{i_0}\wedge\ldots\wedge L_{i_k}\wedge S^\ell)^{\wedge r})^{C_r}$$

is n-connected, where

$$n \geq \min_{s|r} \left(2(r/s)(i-1)-1 - (r/s)(i-1) \right) = i-2 .$$

Note that this tends to ∞ as $(i_0, \dots, i_k)$ runs through I^{k+1}. We want to conclude that the induced map of homotopy colimits is a w.e. This follows from (the proof of) lemma 3.1.15 once we know that the ~~maps in~~ the connectivity of the maps in the limit system grows to ∞ with i. Now the maps in the limit system from the space indexed by $(i_0, \dots, i_k)$ to that with index $(i_0, \dots, i_k + l)$ is given by the composition of the map we have just show to be $i-2$ connected, and the map

$$F((S^{i+l})^{\wedge r}, (L_{i_0} \wedge \dots \wedge L_{i_k} \wedge S^l)^{\wedge r})^{C_r} \xrightarrow{\rho_*} F((S^{i+l})^{\wedge r}, (L_{i_0} \wedge \dots \wedge L_{i_k+l})^{\wedge r})$$

induced from $\rho : L_{i_k} \wedge S^l \to L_{i_k+l}$. It follows from the assumptions made here and in (3.1.14) that ρ is $i_k + l + l(i_k)$ - connected, where $l(i_k) \to \infty$ as $i_k \to \infty$. Therefore, by 4.2.11 and 4.1.15, ρ_* is m-connected,

$$m \geq \min_{s|r} \left((r/s)(i+l-1) + l(i_k) - (r/s)(i+l-1) \right) = l(i_k) .$$

But this tends to ∞ with $(i_0, \dots, i_k)$ as was to be shown. Hence we get a w.e. on homotopy colimits. Finally, the simplicial spaces in question are good by (the proof of) lemma 4.1.10, and so η_* is a w.e. Similar arguments show that can is a w.e., and finally, γ^{C_r} is a w.e. by proposition 4.1.14. $\square$

$$
\begin{tikzcd}
\left|[k] \mapsto \underset{I^{r(k+1)}}{\underrightarrow{\mathrm{holim}}}\, F(S^{i_0}\wedge\cdots\wedge S^{i_{r(k+1)-1}}, L_{i_0}\wedge\cdots\wedge L_{i_{r(k+1)-1}}\wedge S^V)\right|^{G_r} \arrow[r, "\tilde{\sigma}"] & \left(\Omega^{\ell R}\left|[k] \mapsto \underset{I^{r(k+1)}}{\underrightarrow{\mathrm{holim}}}\, F(S^{i_0}\wedge\cdots\wedge S^{i_{r(k+1)-1}}, L_{i_0}\wedge\cdots\wedge L_{i_{r(k+1)-1}}\wedge S^{\ell R}\wedge S^V)\right|\right)^{G_r} \\
& \left|[k] \mapsto \left(\Omega^{\ell R}\underset{I^{r(k+1)}}{\underrightarrow{\mathrm{holim}}}\, F(S^{i_0}\wedge\cdots\wedge S^{i_{r(k+1)-1}}, L_{i_0}\wedge\cdots\wedge L_{i_{r(k+1)-1}}\wedge S^{\ell R}\wedge S^V)\right)^{G_r}\right| \arrow[u, "\gamma^{G_r}", "\sim"'] \\
& \left|[k] \mapsto \left(\underset{I^{r(k+1)}}{\underrightarrow{\mathrm{holim}}}\, \Omega^{\ell R} F(S^{i_0}\wedge\cdots\wedge S^{i_{r(k+1)-1}}, L_{i_0}\wedge\cdots\wedge L_{i_{r(k+1)-1}}\wedge S^{\ell R}\wedge S^V)\right)^{G_r}\right| \arrow[u, "\mathrm{can}", "\sim"'] \\
\left|[k] \mapsto \underset{I^{k+1}}{\underrightarrow{\mathrm{holim}}}\, F((S^{i_0}\wedge\cdots\wedge S^{i_k})^{\wedge r}, (L_{i_0}\wedge\cdots\wedge L_{i_k})^{\wedge r}\wedge S^V)^{G_r}\right| \arrow[r, "\eta_*", "\sim"'] \arrow[uuu, "\Delta", "\cong"'] & \left|[k] \mapsto \underset{I^{k}}{\underrightarrow{\mathrm{holim}}}\, F((S^{i_0}\wedge\cdots\wedge S^{i_k}\wedge S^{\ell})^{\wedge r}, (L_{i_0}\wedge\cdots\wedge L_{i_k}\wedge S^{\ell})^{\wedge r}\wedge S^V)^{G_r}\right| \arrow[u, "\Delta", "\cong"']
\end{tikzcd}
$$

Diagram used in proof of proposition 4.2.12

Let now again G be any compact Lie group and let t be a G-prespectrum indexed on a G-universe $\mathcal{U}$. For X a based G-space, we define $X \wedge t$ and $F(X,t)$ in $G\mathcal{P}\mathcal{U}$ spacewise, i.e.

$$(X \wedge t)_V = X \wedge t_V$$
$$F(X,t)_V = F(X, t_V)$$

with the obvious structure maps. If T is a G-spectrum, then so is $F(X,T)$, but for smash products we must spectrify:

$$X \wedge T := L(X \wedge \ell T) \qquad (4.2.13)$$

We note the adjunction

$$X \wedge - \dashv F(X,-)$$

valid in both $G\mathcal{P}\mathcal{U}$ and $G\mathcal{S}\mathcal{U}$.

<u>Exercise</u> 4.2.14 Show, by category theory, that the obvious isomorphism $F(X, \ell T) \cong \ell F(X,T)$ implies

$$X \wedge Lt \cong L(X \wedge t). \qquad \square$$

Colimits and homotopy colimits are defined spacewise in $G\mathcal{P}\mathcal{U}$ but for spectra we must again spectrify like in 4.2.13. Limits and homotopy limits, however, are defined spacewise in both $G\mathcal{P}\mathcal{U}$ and $G\mathcal{S}\mathcal{U}$.