18.917: *Topics in algebraic topology*

In this course, I plan to explain and prove the semi-stable conjecture of Fontaine-Jannsen following [1] and [2].

To briefly explain the statement, let V be a complete discrete valuation ring with fraction field K and perfect residue field k of mixed characteristic $(p, 0)$, and let X be a scheme over $\operatorname{Spec} V$ with generic fiber X_K and special fiber X_k,

$$\begin{tikzcd} X_k \arrow[r, "i"] \arrow[d] & X \arrow[d] & X_K \arrow[l, "j"'] \arrow[d] \\ \operatorname{Spec} k \arrow[r, "i"] & \operatorname{Spec} V & \operatorname{Spec} K. \arrow[l, "j"'] \end{tikzcd}$$

Then X is said to have semi-stable reduction if, in the étale topology, X can be covered by schemes of the form

$$U = \operatorname{Spec} V[x_1, \dots, x_d]/(x_1 \dots x_r - \pi).$$

Here $1 \leq r \leq d$ and π is a generator of the maximal ideal in V. (It is expected that for any regular scheme X over $\operatorname{Spec} V$, one can find a finite extension V'/V such that $X_{V'} \to \operatorname{Spec} V'$ has semi-stable reduction.) One has the following cohomology theories

(i) the étale cohomology $H^*(X_{\bar{K}}, \mathbb{Z}_p)$, where $X_{\bar{K}} = X \times_{\operatorname{Spec} K} \operatorname{Spec} \bar{K}$; it is a $\mathbb{Z}_p$-module with a continuous $\operatorname{Gal}(\bar{K}/K)$-action;

(ii) the log-crystalline cohomology $H^*((X_k, M_k)/W)$; it is a $W(k)$-module with two operators, Frobenius and monodromy, and with a natural filtration after extending scalars to K.

The conjecture relates these objects: for $X \to \operatorname{Spec} V$ proper with semi-stable reduction, there is a natural isomorphism

$$\alpha_K : B_{\mathrm{st}} \otimes_{W(k)} H^q((X_k, M_k)/W) \xrightarrow{\sim} B_{\mathrm{st}} \otimes_{\mathbb{Z}_p} H^q(X_{\bar{K}}, \mathbb{Z}_p),$$

where B_{st} is a ring of "p-adic periodes" defined by Fontaine. In fact, up to torsion, one can retrieve (i) and (ii) from this common object. Time permitting, I also will explain this.

After an introduction, here is a tentative route:

(i) log schemes;

(ii) crystalline cohomology;

(iii) the Fontaine ring B_{st} and its crystalline interpretation;

(iv) syntomic cohomology and p-adic nearby cycles;

(v) proof of the conjecture.

References

[1] K. Kato, *Semi-stable reduction and p-adic étale cohomology*, Périodes p-adiques (Séminaire de Bures, 1988), Asterisque, vol. 223, 1994, pp. 269–293.

[2] T. Tsuji, *p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case*, Invent. Math. **137** (1999), 233–411.

$$\begin{tikzcd} X_k \arrow[r, hook, "i"] \arrow[d] & X \arrow[d] & X_K \arrow[l, hook', "j"'] \arrow[d] \\ \mathrm{Spec}\, k \arrow[r, hook] & \mathrm{Spec}\, V & \mathrm{Spec}\, K \arrow[l, hook'] \end{tikzcd}$$

semi-stable reduction, proper;

$$V^q = H^q(X_{\bar{K}}, \mathbb{Q}_p)$$

$\mathrm{Gal}(\bar{K}/K)$-action;

$$D^q = K_0 \otimes_W H^q((X_k, M_k)/W) \qquad [\, V = W[\pi]/(\phi(\pi)).\,]$$

$\varphi, N : D^q \to D^q$ frob., monodromy

$$\rho_\pi : K \otimes_{K_0} D^q \xrightarrow{\sim} H^q_{dR}(X_K/K).$$

$$\dim_{\mathbb{Q}_p} V^q = \dim_{K_0} D^q.$$

$B_{st,\pi}$ $\quad \mathbb{Q}_p, K_0$ algebra;

$\mathrm{Gal}(\bar{K}/K_0)$-action

$\varphi, N : B_{st,\pi} \to B_{st,\pi}$

$B_{st,\pi} \hookrightarrow B_{dR}$; B_{dR} CDV field

Thm $\quad B_{st} \otimes_{\mathbb{Q}_p} V^q \xrightarrow{\sim} B_{st} \otimes_{K_0} D^q$

$\varphi \otimes 1,\ N \otimes 1,\ \sigma \otimes \sigma \qquad \varphi \otimes \varphi,\ N \otimes 1 + 1 \otimes N,\ \sigma \otimes 1$

$(\mathrm{Fil}^\cdot B_{dR}) \otimes_{\mathbb{Q}_p} V^q \qquad \mathrm{Fil}^\cdot (B_{dR} \otimes_{K_0} D^q)$

$$D^q = (B_{st} \otimes_{\mathbb{Q}_p} V^q)^{Gal(\bar{K}/K)}$$

$$V^q = (B_{st} \otimes_{K_0} D^q)^{\varphi=1,\, N=0} \cap Fil^0(B_{dR} \otimes_{K_0} D^q)$$

Define map

$$V^q \longrightarrow (B_{st} \otimes_{K_0} D^q)^{N=0,\, \varphi=1};$$

extend scalars,

Thm $(B^+_{st} \otimes_{K_0} D^q)^{N=0} \cong \mathbb{Q} \otimes \lim_n H^q((\bar{X}_n, \bar{M}_n)/W_n)$. //

$$\begin{tikzcd} X \arrow[r, hook] & Z \\ X_n \arrow[r, hook] & Z_n \\ J_{D_n} \arrow[r, phantom, "\subset"] & D_n \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[loop right, "f"] \arrow[d, "sm."] \\ \mathrm{Spec}\, W \end{tikzcd}$$

$$D_n = D_{X_n}(Z_n)$$

$$H^q(X_n, \underbrace{\mathcal{O}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \Omega^\bullet_{Z_n}}_{j_n(0)}) \cong H^q(X_n/W_n)$$

$$j_n(r): \; J^{[r]}_{D_n} \xrightarrow{d} J^{[r-1]}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \Omega^1_{Z_n} \xrightarrow{d} J^{[r-2]}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \Omega^2_{Z_n} \longrightarrow \cdots$$

If $0 \le r < p$, $f(J^{[r]}_{D_n}) \subset p^r \mathcal{O}_{D_n}$, so can define

$$\underset{\text{def}}{s_n^{\log}(r)} \longrightarrow j_n^{\log}(r) \xrightarrow{1-\frac{\varphi}{p^r}} j_n^{\log}(0) \qquad \text{mapping fiber}$$

$$H^q(\bar{X}_n, s_n^{\log}(r)) \longrightarrow H^q(\bar{X}_n, j_n^{\log}(r))^{\varphi = p^r}$$

$$\xrightarrow{\text{"}\cdot\text{"}} H^q(\bar{X}_n, j_n^{\log}(0))^{\varphi = p^r} = H^q((\bar{X}_n, \bar{M}_n)/W_n)^{\varphi = p^r} \Longrightarrow$$

$$\mathbb{Q} \otimes \varprojlim_n H^q(\bar{X}_n, s_n^{\log}(r)) \longrightarrow (B_{st}^+ \otimes_{K_0} D^q)^{N=0,\ \varphi = p^r}$$

—

$$V^q(r) = \mathbb{Q} \otimes \varprojlim_n H^q(X_{\bar{K}}, \mathbb{Z}/p^n\mathbb{Z}(r))$$

$$H^q(X_{\bar{K}}, \mathbb{Z}/p^n\mathbb{Z}(r)) = H^q(\bar{X}, R\bar{j}_* \mathbb{Z}/p^n\mathbb{Z}(r))$$

$$= H^q(X_{\bar{k}}, \bar{i}^* R\bar{j}_* \mathbb{Z}/p^n\mathbb{Z}(r)) \qquad \text{(proper b.c.)}$$

$$= H^q(\bar{X}, \underbrace{\bar{i}_* \bar{i}^* R\bar{j}_* \mathbb{Z}/p^n\mathbb{Z}(r)}_{\text{p-adic nearby cycles}})$$

Thm $\quad s_n^{\log}(r) \xrightarrow{\sim} \tau_{\le r}\, \bar{i}_* \bar{i}^* R\bar{j}_* \mathbb{Z}/p^n\mathbb{Z}(r)$, $\quad 0 \le r < p-1$.

$$\begin{tikzcd}
\underline{H}^r(s_n^{\log}(r)_{\bar X})_{\bar x} \arrow[rr] & & (\bar i_* \bar i^* R^r \bar j_* \mathbb{Z}/p^n\mathbb{Z}(r))_{\bar x} \\
& K_r^M(\mathcal{O}_{\bar X,\bar x}[\tfrac{1}{p}])/p^n \arrow[ul, twoheadrightarrow] \arrow[ur, twoheadrightarrow] &
\end{tikzcd}$$

For $q \le r < p-1$, $\dim X_K \le r < p-1$, get

$$H^q(\bar X, s_n^{\log}(r)) \xrightarrow{\sim} H^q(X_{\bar K}, \mathbb{Z}/p^n\mathbb{Z}(r)) \Longrightarrow$$

$$\begin{tikzcd}
\mathbb{Q} \otimes \varprojlim_n H^q(\bar X, s_n^{\log}(r)) \arrow[r, "\sim"] \arrow[d] & V^q(r) \\
(B_{st}^+ \otimes_{K_0} D^q)^{N=0,\ \varphi = p^r} &
\end{tikzcd}$$

using $B_{st}^+ \otimes \mathbb{Q}_p(-r) \to B_{st}$,

$$V^q \longrightarrow (B_{st} \otimes_{K_0} D^q)^{N=0,\ \varphi=1}$$

9/8/00

$$\mathrm{Spec}(A) = \{ \mathfrak{p} \subset A \text{ prime ideal}, \ \mathfrak{p} \neq A \}$$

$$\begin{array}{ccc} \uparrow \wr & & \cup \\ \mathrm{Spec}(A)_f & = & \{ \mathfrak{p} \mid f \in \mathfrak{p} \} \quad \text{basis of open} \\ \uparrow \sim & & \\ \mathrm{Spec}(A_f) & & \end{array}$$

$$; \quad A_f = A[\tfrac{1}{f}] \ ;$$

$$\mathrm{Spec}(A_f) \underset{\mathrm{Spec}\,A}{\times} \mathrm{Spec}(A_g) = \mathrm{Spec}(A_f \underset{A}{\otimes} A_g) = \mathrm{Spec}(A_{fg})$$

$X = \mathrm{Spec}\, A$; $\mathcal{O}_X$ structure sheaf.

$$\Gamma(\mathrm{Spec}\, A_f, \mathcal{O}_X) = A_f$$

Stalk at $x = \mathfrak{p}$

$$\begin{array}{ccccl} \mathcal{O}_{X,x} & = & \underset{f \notin \mathfrak{p}}{\mathrm{colim}}\, A_f & = & A_{\mathfrak{p}} \quad \text{local ring} \\ \cup & & & & \cup \\ m_x & & \overline{} & & \mathfrak{p} \cdot A_{\mathfrak{p}} \quad \text{max'l ideal} \end{array}$$

$$k(x) = \mathcal{O}_{X,x} / m_x = ff(A/\mathfrak{p})$$

$$\mathrm{Spec}\, k(x) \longrightarrow \mathrm{Spec}\, \mathcal{O}_{X,x} \longrightarrow X$$

$$f(x) \in k(x) \longleftarrow\!\!\shortmid \ f \in \Gamma(X, \mathcal{O}_X)$$

$$
\begin{tikzcd}
\operatorname{Spec} k(y) \otimes_B A = f^{-1}(y) \arrow[r] \arrow[d] & X = \operatorname{Spec} A \arrow[d, "f"] \\
\operatorname{Spec} k(y) \arrow[r] & Y = \operatorname{Spec} B
\end{tikzcd}
$$

$$f^{-1}(y) = \{ k(y) \otimes_B \mathfrak{p}_x \mid \mathfrak{p}_x \subset A,\ f^{-1}(\mathfrak{p}_x) = \mathfrak{p}_y \}$$

$\subseteq)$

$$
\begin{tikzcd}
k(y) \otimes_B A & A \arrow[l] \\
k(y) \arrow[u] & B \arrow[l, "\pi"] \arrow[u, "f"]
\end{tikzcd}
\qquad \pi^{-1}(0) = \mathfrak{p}_y
$$

$\supseteq)$

$$k(y) \otimes_B \mathfrak{p}_x \to k(y) \otimes_B A \to k(y) \otimes_B A/\mathfrak{p}_x$$

prime ideal $\longleftrightarrow$ domain

$$f^{-1}(\mathfrak{p}_x) = \mathfrak{p}_y \Rightarrow \underset{\text{domain}}{A/\mathfrak{p}_x} = B/\mathfrak{p}_y \otimes_B A/\mathfrak{p}_x \quad \text{domain}$$

$$\downarrow \quad \text{localization}$$

$$ff(B/\mathfrak{p}_y) \otimes_B A/\mathfrak{p}_x \quad \text{domain}$$

$f : X \longrightarrow Y$

$\Delta : X \xrightarrow{N} X \times_Y X$;

$\Omega^1_{X/Y} := N/N^2$ q. coh. $\mathcal{O}_Y$ - module

$d : \mathcal{O}_X \longrightarrow \Omega^1_{X/Y}$, $df = f\otimes 1 - 1 \otimes f$, is the universal $\mathcal{O}_Y$ - linear derivation of $\mathcal{O}_X$ to a q. coh. $\mathcal{O}_X$ - module. I. e.

$\Omega^1_{X/Y}$ = $\mathcal{O}_X$ - module gen. by symbols df, f local section of $\mathcal{O}_X$, modulo relations

$$d(f+g) = df + dg$$

$$d(fg) = f\,dg + g\,df$$

$$d\varphi = 0 \quad , \quad \varphi \in \operatorname{im} f.$$

=

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ S \arrow[u] \arrow[ur] & \end{tikzcd}$$

$$0 \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$$

if f sm.

$$0 \to B \otimes_A \Omega^1_{A/k} \longrightarrow \Omega^1_{B/k} \longrightarrow \Omega^1_{B/A} \to 0$$

if B/A sm.

<u>Unramified</u> $f: Y \to X$ locally fin. presented and one of the following equivalent:

(i) for all $y \in Y$, $m_x \cdot \mathcal{O}_{Y,y} = m_y$ and $k(y)/k(x)$ is a fin. sep. ext. Here $x = f(y)$;

(ii) $\Omega^1_{Y/X} = 0$;

(iii) for all $x \in X$, the fiber

$$\begin{array}{ccc} Y_x & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ \operatorname{Spec} k(x) & \longrightarrow & X \end{array}$$

is a disjoint union $\coprod \operatorname{Spec} k_i$, $k_i / k(x)$ f. sep;

(iv) in every diagram

$$\begin{array}{ccc} T' & \longrightarrow & Y \\ I \downarrow i & \nearrow & \downarrow f \\ T & \longrightarrow & X \end{array}$$

T' defined in T by nilp. ideal I

there exists locally on T at most one lift.

Étale $f : Y \to X$ locally fin. presented and one of the following equivalent:

(i) flat and unramified;

(ii) in every diagram

$$\begin{tikzcd} T' \arrow[r] \arrow[d, hook, "i"'] & Y \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

$T' \overset{i}{\hookrightarrow} T$ closed imm. defined by nilp. ideal

there exists locally on T a unique lift;

(iii) for all $y \in Y$, there exist open neighborhoods $V = \operatorname{Spec} B \ni y$ and $U = \operatorname{Spec} A \ni x = f(y)$ s.t.

$$B = A[x_1, \dots, x_n] / (f_1, \dots, f_n)$$

and $\det(\partial f_i / \partial x_j) \in B^\times$.

ex (i) $A = k[t]$, $B = A[x]/(x-t^n)$; $\frac{df}{dx} = 1 \in B^\times$

(ii) $A = k[t^{\pm 1}]$, $B = k[x^{\pm 1}] = A[x]/(x^n - t)$

$$t \longmapsto x^n$$

$$\frac{df}{dx} = n x^{n-1} \in B^\times \iff n \in k^\times$$

(iii) K complete discrete valuation field, $\mathcal{O}_K \subset K$ valuation ring (i.e. $\mathcal{O}_K = \{x \in K \mid v_K(x) \geq 0\}$), L/K fin. sep. ext., $\mathcal{O}_L \subset L$ integral closure of $\mathcal{O}_K$ in L.

$$\begin{tikzcd}
\operatorname{Spec} k_L \arrow[r, hook, "i"] \arrow[d] & \operatorname{Spec} \mathcal{O}_L \arrow[d] & \operatorname{Spec} L \arrow[l, hook', "j"'] \arrow[d] \\
\operatorname{Spec} k_K \arrow[r, hook, "i"] & \operatorname{Spec} \mathcal{O}_K & \operatorname{Spec} K \arrow[l, hook', "j"']
\end{tikzcd}$$

$$m_K \mathcal{O}_L = m_L^{e_{L/K}}$$

$$e_{L/K} = \frac{|L : K|}{|k_L : k_K|}$$

<u>Smooth</u> A map $f: X \to Y$ is smooth if it is locally f. presented and one of:

(i) in every diagram

$$\begin{tikzcd} T' \arrow[r] & Y \arrow[d, "f"] \\ T \arrow[u, hook, "i"', "N"] \arrow[r] \arrow[ur, dashed, "\exists"] & X \end{tikzcd} \qquad N \text{ nilp.}$$

there exists locally on T' a lift.

(ii) flat and $\Omega^1_{Y/X}$ is locally free of rank equal to the relative dim. of Y/X;

(iii) for all $y \in Y$, there exists open affine nbhds $V \ni y$ and $U \ni x = f(y)$ s.t. $f|_V$ factors

$$V \xrightarrow{\text{étale}} \mathbb{A}^n_U \xrightarrow{pr} U \hookrightarrow X.$$

$$0 \to B \otimes_A \Omega^1_{A/k} \to \Omega^1_{B/k} \to \Omega^1_{B/A} \to 0$$

if B/A sm.

enough to show: for all B-mod. M,

$$0 \leftarrow \mathrm{Hom}_B(B \otimes_A \Omega^1_{A/k}, M) \leftarrow \mathrm{Hom}_B(\Omega^1_{B/k}, M) \leftarrow \mathrm{Hom}_B(\Omega^1_{A/k}, M) \leftarrow 0$$

B/A sm

$$\begin{tikzcd}
\mathrm{Hom}_A(\Omega^1_{A/k}, f_*M) \arrow[d, equal] & & \\
0 \leftarrow \mathrm{Der}_k(A, f_*M) & \mathrm{Der}_k(B, M) \arrow[l] & \mathrm{Der}_A(B, M) \arrow[l] \leftarrow 0
\end{tikzcd}$$

sm.

$$D \longleftarrow\!\shortmid \tilde{D}, \quad D \in \mathrm{Der}_k(A, f_*M), \quad \tilde{D} \in \mathrm{Der}_k(B, M)$$

$$\begin{tikzcd}
B \arrow[r, "\mathrm{id}"] \arrow[dr, dashed, "\exists h"] & B \\
A \arrow[u, "f"] \arrow[r] & B \oplus M \arrow[u, "pr_1"']
\end{tikzcd}
\qquad h(b) = b \oplus \tilde{D}(b) \qquad M^2 = 0$$

$$a \longmapsto f(a) \oplus D(a)$$

$=$

$$0 \to f^*\Omega^1_{X/S} \to \Omega^1_{Y/S} \to \Omega^1_{Y/X} \to 0$$

f sm.

$$\begin{tikzcd}
Y \arrow[rr, "f"] \arrow[dr] & & X \arrow[dl] \\
& S &
\end{tikzcd}$$

$$A \otimes_k A \xrightarrow{\varphi} A \oplus M$$

$$a \otimes b \longmapsto ab \oplus aDb$$

$$(a\otimes b).(a'\otimes b') \longmapsto (ab \oplus aDb).(a'b' \oplus a'Db')$$

$$= aba'b' \oplus (aba'Db' + a'b'aDb)$$

$$= aa'bb' \oplus aa'D(bb')$$

$$\begin{tikzcd} I \arrow[r] \arrow[d] & M \arrow[d] \\ A\otimes_k A \arrow[r] & A\oplus M \end{tikzcd} \qquad \overset{M^2=0}{\Longrightarrow} \qquad I/I^2 \xrightarrow{f} M$$

- $f(da) = f(1\otimes a - a\otimes 1) = 1.Da + a.D(1)$

$$= Da$$

- $f : I/I^2 \to M$ A - linear

$$\omega = \sum x_i \otimes y_i + I^2$$

$$a\omega = \sum ax_i \otimes y_i \text{ , so}$$

$$f(a\omega) = \sum ax_i Dy_i = a f(\omega)$$

- f unique: $a\otimes a' = (a\otimes 1)(1\otimes a' - a'\otimes 1) + aa'\otimes 1$

$$\sum x_i \otimes y_i + I^2 = \sum x_i dy_i + I^2$$

Def A (<u>Grothendieck</u>) <u>topology</u> on a cat. T consists of a set $\mathrm{Cov}\,T$ of families $\{U_i \xrightarrow{\varphi_i} U\}$ of maps in T called <u>coverings</u> s.t.

(i) $U' \underset{\sim}{\xrightarrow{\varphi}} U$ iso $\Rightarrow$ $\{U' \xrightarrow{\varphi} U\} \in \mathrm{Cov}\,T$;

(ii) if $\{U_i \to U\} \in \mathrm{Cov}\,T$ and $\{V_{ij} \xrightarrow{\psi_{ij}} U_i\} \in \mathrm{Cov}\,T$ then $\{V_{ij} \xrightarrow{\varphi_i \psi_{ij}} U\} \in \mathrm{Cov}\,T$;

(iii) if $\{U_i \to U\} \in T$ and $V \to U$ is a morphism in T, then $\{U_i \times_U V \to V\} \in \mathrm{Cov}\,T$.

$T^{op} \xrightarrow{F} C$ presheaf is a <u>sheaf</u> if

$M \xrightarrow{\alpha} \mathcal{O}_X$ pre-log str. (X, M)

sheaf of monoids on X_{et}

$(X,M) \to (S,N)$: $X \xrightarrow{f} S$;

$$\begin{tikzcd} f^{-1}(N) \arrow[r, "h"] \arrow[d] & M \arrow[d] \\ f^{-1}(\mathcal{O}_S) \arrow[r] & \mathcal{O}_X \end{tikzcd}$$

<u>derivation</u> of $(X,M)/(S,N)$ to $\mathcal{O}_X$-module E

$(D, D\log) : (X, M) \to E$

$D : \mathcal{O}_X \to E$ $f^{-1}(\mathcal{O}_S)$-linear derivation

$D\log : M \to E$ map of monoids.

s.t.

$D\alpha(a) = \alpha(a) D\log a$, $a \in M$,

$D\log x = 0$, $x \in \mathrm{im}(f^{-1}(N) \to M)$.

for local sections a of M and x of $\mathrm{im}(f^{-1}(N) \to M)$, respectively.

$\ulcorner$ $M^{gp} = \{ab^{-1} \mid a,b \in M\}/\sim$

$ab^{-1} \sim cd^{-1} \iff \exists s \in M: \quad sad = sbc$

$M \longrightarrow M^{gp}$ injective $\iff$ M <u>integral</u>, i.e.

$ab = ac \implies b = c$ $\lrcorner$

$$\mathcal{O}_X \xrightarrow{D} E \quad \longleftrightarrow \quad \Omega^1_{X/S} \longrightarrow E$$

$$\begin{tikzcd} M \arrow[r, "D\log"] \arrow[d] & E \\ M^{gp} \arrow[ur, dashed] & \end{tikzcd} \quad \longleftrightarrow \quad \mathcal{O}_X \otimes_{\mathbb{Z}} M^{gp} \longrightarrow E$$

$$\omega^1_{(X,M)/(S,N)} = \Omega^1_{X/S} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} M^{gp}) / \sim$$

$$d\alpha(a) \sim \alpha(a) \otimes a, \quad a \in M$$

$$1 \otimes x \sim 0, \quad x \in \operatorname{im}(f^{-1}(N) \to M).$$

Then

$$(D, D\log) : (X,M) \to E \quad \longleftrightarrow \quad \omega^1_{(X,M)/(S,N)} \longrightarrow E$$

<u>Note</u> $\omega^1_{(X,M)/(S,N)}$ need not be q-coh.

$$\begin{tikzcd} M \arrow[r, "\alpha"] & G_X \\ \alpha^{-1}(G_X^*) \arrow[u, hook] \arrow[r, "\alpha"] & G_X^* \arrow[u, hook] \end{tikzcd}$$

log str. if iso.

$$\begin{tikzcd} M \arrow[r] \arrow[rr, bend left, "\alpha"] & M^a \arrow[r] & G_X \\ \alpha^{-1}(G_X^*) \arrow[u, hook] \arrow[r] \arrow[ur, phantom, "\text{p.o.}"] & G_X^* \arrow[u] \arrow[ur, bend right] & \end{tikzcd}$$

associated log str.

$$(X, M) \rightsquigarrow (X, M^a)$$

$\ulcorner$

$$\begin{tikzcd} M \arrow[r] \arrow[rr, bend left, "v"] & M \oplus G/\sim & N \\ H \arrow[u, "t"] \arrow[r, "s"] \arrow[ur, phantom, "\text{p.o.}"] & G \arrow[u] \arrow[ur, bend right, "u"'] & \end{tikzcd}$$

group

$(m, g) \sim (m', g') \overset{\text{def}}{\Longleftrightarrow}$

$\exists\, h_1, h_2 \in H$ s.t.

$m\, t(h_1) = m'\, t(h_2)$

$g\, s(h_2) = g'\, s(h_1)$

check: $(m, g) \sim (m', g') \Rightarrow v(m)\, u(g) = v(m')\, u(g')$

$\Longleftrightarrow v(m)\, vt(h_1)\, u(g)\, us(h_2) = v(m')\, vt(h_2)\, u(g')\, us(h_1)$

since image of H in N is a group

$v(m\, t(h_1))\, u(g\, s(h_2)) = v(m' t(h_2))\, u(g' s(h_1))$ ✓

ex $G = \{1\}$: quotient $M/t(H) = M/\sim$

$m \sim m'$ if $\exists\, h_1, h_2 \in H$ s.t. $m\, t(h_1) = m'\, t(h_2)$;

generalizes orbit by group. $\lrcorner$

<u>Lemma</u> $\omega^1_{(X,M)/(S,N)} \xrightarrow{\sim} \omega^1_{(X,M^a)/(S,N)} \xrightarrow{\sim} \omega^1_{(X,M^a)/(S,N^a)}$

Pf First iso equivalent to "every derivation $(D, D\log) : (X,M) \to E$ extends uniquely to a derivation $(X, M^a) \to E$":

$$\begin{tikzcd}
 & & E \\
M \arrow[r] \arrow[urr, bend left, "D\log"] & M^a \arrow[ur, dashed] & \\
\alpha^{-1}(\mathcal{O}_X^*) \arrow[u, hook] \arrow[r] \arrow[ur, phantom, "p.o."] & \mathcal{O}_X^* \arrow[u, hook] \arrow[uur, bend right] &
\end{tikzcd} \qquad f \mapsto \frac{Df}{f}$$

Second iso is "if $D\log : M^a \to E$ vanishes on $f^{-1}(N)$, it vanishes on $f^{-1}(N^a)$"

$$\begin{tikzcd}
 & & E \\
f^{-1}(N) \arrow[r] \arrow[urr, bend left, "0"] & f^{-1}(N^a) \arrow[ur, dashed] & \\
f^{-1}\beta^{-1}(\mathcal{O}_S^*) \arrow[u] \arrow[r] \arrow[ur, phantom, "p.o."] & f^{-1}(\mathcal{O}_S^*) \arrow[u] \arrow[uur, bend right, "0"'] &
\end{tikzcd}$$

$f^{-1} \dashv f_*$ so f^{-1} preserves colimits.

//

ex (i) $P \to \Gamma(X, \mathcal{O}_X) \Longleftrightarrow P_X \to \mathcal{O}_X \Rightarrow$

$M = P_X^a \to \mathcal{O}_X$

(ii) $U \overset{j}{\hookrightarrow} X$ open immersion

$$\begin{tikzcd} M \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & \mathcal{O}_X \arrow[d] \\ j_* \mathcal{O}_U^* \arrow[r] & j_* \mathcal{O}_U \end{tikzcd}$$

" reg. functions on X which are invertible on U. "

Pathological unless $Z = X \setminus U$ is a "reduced divisor w. normal crossings"

(iii) $X = \operatorname{Spec} V_K \overset{j}{\hookleftarrow} U = \operatorname{Spec} K$; if $V_K \xrightarrow{\text{étale}} V_L$

$$\begin{tikzcd} \Gamma(V_L, M) \arrow[r] \arrow[d] & \Gamma(V_L, \mathcal{O}_X) = V_L \arrow[d] \\ \Gamma(V_L, j_* \mathcal{O}_U^*) \arrow[r] \arrow[d, equal] & \Gamma(V_L, j_* \mathcal{O}_U) \arrow[ddd, equal] \\ \Gamma(V_L \otimes_{V_K} K, \mathcal{O}_U^*) \arrow[d, equal] & \\ (V_L \otimes_{V_K} K)^* \arrow[d, equal] & \\ L^* \arrow[r, hook] & L \end{tikzcd}$$

i.e. $\Gamma(V_L, M) = L^* \cap V_L \hookrightarrow V_L = \Gamma(V_L, \mathcal{O}_X)$

$$\mathbb{N}_0 \xrightarrow{\sim} V_K \quad ; \quad (\pi_K) = m_K$$
$$i \longmapsto \pi_K^i$$

$$\mathbb{N}_{0,X}^a \xrightarrow{\sim} M$$

$$\begin{tikzcd}
\mathbb{N}_0 = \Gamma(V_L, \mathbb{N}_{0,X}) \arrow[r] \arrow[rr, bend left] & \Gamma(V_L, M) & = V_L \cap L^* \\
\{1\} = \Gamma(V_L, \alpha^{-1}(\mathcal{O}_X^*)) \arrow[u] \arrow[r] \arrow[ur, phantom, "\text{p.o.}"] & \Gamma(V_L, \mathcal{O}_X^*) \arrow[u] & = V_L^* \arrow[u]
\end{tikzcd}$$

$=$

$$X = \operatorname{Spec} V[x_1, \dots, x_n]/(x_1 \cdots x_r - \pi)$$

$$\mathbb{N}_0^r \longrightarrow \Gamma(X, \mathcal{O}_X)$$
$$(i_1, \dots, i_r) \longmapsto x_1^{i_1} \cdots x_r^{i_r}$$

$$U = \operatorname{Spec} K[x_1, \dots, x_n]/(x_1 \cdots x_r - \pi) \overset{j}{\hookrightarrow} X$$

<u>Lemma</u> $(\mathbb{N}_0^r)^a_X \xrightarrow{\sim} M$.

$$\begin{tikzcd}
M \arrow[r, hook] \arrow[d] & \mathcal{O}_X \arrow[d] \\
j_*(\mathcal{O}_U^*) \arrow[r, hook] & j_* \mathcal{O}_U
\end{tikzcd}$$

$\mathcal{O}_{X,x}$ <u>regular</u> if $S_{k(x)}(m_x/m_x^2) \xrightarrow{\sim} gr^{\cdot}_{m_x}(\mathcal{O}_{X,x})$

$$\mathcal{O}^h_{X,x} = \operatorname*{colim}_{\substack{V \xrightarrow{\text{étale}} X \\ \nwarrow \ \nearrow \\ x}} \Gamma(V, \mathcal{O}_X) = \text{henselization of } \mathcal{O}_{X,x}$$

$$\bar{x} = \operatorname{Spec} \overline{k(x)} \longrightarrow \operatorname{Spec} k(x) \xrightarrow{x} X$$

$$\mathcal{O}_{X,\bar{x}} = \operatorname*{colim}_{\substack{V \xrightarrow{\text{étale}} X \\ \nwarrow \ \nearrow \\ \bar{x}}} \Gamma(V, \mathcal{O}_X) = \text{strict henselization of } \mathcal{O}_{X,x} = \mathcal{O}^{sh}_{X,x}$$

<u>exercise</u> Show: $\mathcal{O}_{X,x}$ regular $\Rightarrow$ $\mathcal{O}^h_{X,x}$, $\mathcal{O}_{X,\bar{x}}$ regular

D divisor on X;

$U = X \setminus \mathrm{supp}(D) \xhookrightarrow{j} X$;

$$0 \to \mathcal{O}^*_{X,\bar{x}} \to (j_* \mathcal{O}^*_U)_{\bar{x}} \xrightarrow{\mathrm{res}} \mathbb{Z}^s \to 0$$

s = # irr. comp. of supp(D) which passes through x

res = residue of $\frac{f}{g}$ along these comp.

=

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}^*_{X,\bar{x}} & \to & (j_* \mathcal{O}^*_U)_{\bar{x}} & \to & \mathbb{Z}^s & \to & 0 \\ & & \| & & \wr & & \wr & & \\ 0 & \to & \mathcal{O}^*_{X,\bar{x}} & \to & M_{\bar{x}} & \to & \mathbb{N}_0^s & \to & 0 \end{array}$$

$(\mathcal{C}, J)$ site (= cat. $\mathcal{C}$ w. topology J)

$\mathcal{C}^\wedge = Sets^{\mathcal{C}^{op}}$ = presheaves on $\mathcal{C}$

$\cup$

$\mathcal{C}^\sim$ = (full) subcat. of sheaves (wrt. J)

(topos) $\mathcal{C}^\sim \underset{U}{\overset{(-)^\sim}{\rightleftarrows}} \mathcal{C}^\wedge$ $\qquad (-)^\sim \dashv U$

$F^\sim \longleftarrow F$

associated sheaf

$\varepsilon \quad (UG)^\sim \xrightarrow{\sim} G$

$(-)^\sim$ preserves finite limits (and all colimits)

---

a point in $\mathcal{C}^\sim$ is a map of topoi

$x : Sets \longrightarrow \mathcal{C}^\sim$

i.e. an adjoint pair of functors

$Sets \underset{x^*}{\overset{x_*}{\rightleftarrows}} \mathcal{C}^\sim \quad ; \quad x^* \dashv x_*$

s.t. x^* preserves finite limits. ($x^* F =: F_x$ stalk at x)

$\mathcal{C}^\sim$ has enough points if there is a collection of points x_i s.t.

$F \xrightarrow{\sim} G \iff F_{x_i} \xrightarrow{\sim} G_{x_i}$, for all i.

ob $X_{\acute{e}t} = \{ U \xrightarrow{\text{étale}} X \}$; mor $X_{\acute{e}t} = \left\{ \begin{tikzcd} U \arrow[rr, "\text{étale}"] \arrow[dr] & & V \arrow[dl] \\ & X & \end{tikzcd} \right\}$

topology: $\left\{ \begin{tikzcd} U_i \arrow[rr, "\varphi_i"] \arrow[dr] & & U \arrow[dl] \\ & X & \end{tikzcd} \right\}$ cover $\underset{\text{def}}{\Longleftrightarrow} \bigcup_i \varphi_i(U_i) = U$

For $x \in X$, choose sep. alg. closure $k(x) \hookrightarrow \overline{k(x)}$; get

$$\bar{x} = \operatorname{Spec} \overline{k(x)} \to \operatorname{Spec} k(x) \to X .$$

If $F \in X_{\acute{e}t}^{\sim}$, define

$$\bar{x}^* F = \operatorname*{colim}_{\begin{tikzcd}[row sep=small, column sep=small] U \arrow[rr, "\acute{e}t"] & & X \\ & \bar{x} \arrow[ul] \arrow[ur] & \end{tikzcd}} F(U) . \qquad (F_{\bar{x}})$$

$$(\bar{x}_* F)(U \to X) = \begin{cases} F & , \text{ if } \exists \begin{tikzcd}[row sep=small, column sep=small] U \arrow[rr] & & X \\ & \bar{x} \arrow[ul] \arrow[ur] & \end{tikzcd} \\ \varnothing & , \text{ else} . \end{cases}$$

<u>exercise</u> (in cat. th.) note that the definition of $(\bar{x}^*, \bar{x}_*)$ makes sense for $F \in X_{\acute{e}t}^{\wedge}$; call this pair $(\bar{x}^!, \bar{x}_!)$; then $\bar{x}^* = \bar{x}^! \circ U$, $\bar{x}_* = (-)^{\sim} \circ \bar{x}_!$

Show that for $F \in X_{\acute{e}t}^{\wedge}$, $\bar{x}^* F^{\sim} \xrightarrow{\sim} \bar{x}^! F$.

(Hint: use $\bar{x}_! E \xrightarrow{\sim} U \bar{x}_* E$ and $(UG)^{\sim} \xrightarrow{\sim} G$.)

$$\mathcal{O}_{X,\bar{x}} = \varinjlim_{\substack{V \xrightarrow{\text{étale}} X \\ \nwarrow \ \nearrow \\ \bar{x}}} \Gamma(V, \mathcal{O}_X) = \mathcal{O}^{sh}_{X,x}$$ strict henselization of $\mathcal{O}_{X,x}$

Th $\mathcal{O}_{X,\bar{x}}$ is noetherian (resp. regular, resp. a valuation ring, resp. CM, resp. krull) iff $\mathcal{O}_{X,x}$ is, //

Recall: $\mathcal{O}_{X,x}$ regular $\underset{\text{def}}{\Longleftrightarrow}$ noetherian and

$$S_{k(x)}(m_x/m_x^2) \xrightarrow{\sim} gr_{m_x}(\mathcal{O}_{X,x})$$

$X \to \operatorname{Spec} K$, K field, loc. fin. presented:

X reg. $\Longleftrightarrow$ $X \to \operatorname{Spec} K$ sm $\Longleftrightarrow$ $\forall x$:

$$\begin{tikzcd}
x \arrow[d, phantom, "\cap" rotate=90] & x \arrow[d, phantom, "\cap" rotate=90] \\
U \arrow[r, hook] \arrow[d, "\text{étale}"] & X \arrow[d] \\
\mathbb{A}^d_K \arrow[r] & \operatorname{Spec} K
\end{tikzcd}$$

$X \to \operatorname{Spec} V$, V CDVR, loc. fin. presented.

expect X reg $\Longleftrightarrow$ $X \to \operatorname{Spec} V$ potentially of semi-stable reduction

$X \to \operatorname{Spec} V$ semi-stable reduction $\Longleftrightarrow$ $\forall\, \bar{x}$

$$\begin{tikzcd}
 & \bar{x} \arrow[dl] \arrow[dr] & \\
U \arrow[rr, "\text{étale}"] \arrow[d, "\text{étale}"] & & X \arrow[d] \\
\operatorname{Spec} V[x_1, \dots, x_d]/(x_1 \cdots x_r - \pi) \arrow[rr] & & \operatorname{Spec} V
\end{tikzcd}$$

irreducible regular
noetherian, dim $= d$

potentially : $\exists\, L/K$ fin. s.t. $X_L \to \operatorname{Spec} V_L$ semi-stable

$X = \operatorname{Spec} V[x_1, \ldots, x_d]/(x_1 \cdots x_r - \pi)$ irreducible regular noetherian scheme of dimension d.

$$\begin{tikzcd}
X_k \arrow[r, hook, "i"] \arrow[d] & X \arrow[d] & X_K \arrow[l, hook', "j"'] \arrow[d, "\text{smooth}"] \\
\operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} V & \operatorname{Spec} K \arrow[l, hook']
\end{tikzcd}$$

log str. defined by X_k:

$$\begin{tikzcd}
M \arrow[r, hook] \arrow[d, hook] & \mathcal{O}_X \arrow[d, hook] \\
j_* \mathcal{O}^*_{X_K} \arrow[r] & j_* \mathcal{O}_{X_K}
\end{tikzcd} ;$$

also

$$\begin{aligned} P = \mathbb{N}_0^r &\longrightarrow \Gamma(X, \mathcal{O}_X) \\ (i_1, \ldots, i_r) &\longmapsto x_1^{i_1} \cdots x_r^{i_r} \end{aligned} \qquad \Longrightarrow$$

$$\begin{tikzcd}
P_X \arrow[r] \arrow[rr, bend left, "\alpha"] & P_X^a \arrow[r] & \mathcal{O}_X \\
\alpha^{-1}(\mathcal{O}_X^*) \arrow[u, hook] \arrow[r] & \mathcal{O}_X^* \arrow[u, hook] \arrow[ur, hook] &
\end{tikzcd}$$

canonical map

$$P_X^a \longrightarrow M \,;$$

the comp. $P \to \Gamma(X, \mathcal{O}_X) \to \Gamma(X, j_* \mathcal{O}_{X_K})$ lands in $\Gamma(X, j_* \mathcal{O}_{X_K}^*) \hookrightarrow \Gamma(X, j_* \mathcal{O}_{X_K})$, so get $P \to \Gamma(X, M)$, hence $P_X \to M$; also $\mathcal{O}_X^* \hookrightarrow M$ since log-str.; agree on $\alpha^{-1}(\mathcal{O}_X^*)$ so have map

<u>Lemma</u> This is an isomorphism.

<u>Pf</u> Enough to show that map induces iso

$$P^a_{X,\bar{x}} \xrightarrow{\sim} M_{\bar{x}} .$$

$$\begin{array}{ccccc} M_{\bar{x}} & \longrightarrow & \mathcal{O}_{X,\bar{x}} & = & \mathcal{O}^{sh}_{X,x} \\ \cap & & \cap & & \\ (j_* \mathcal{O}^*_{X_K})_{\bar{x}} & \hookrightarrow & (j_* \mathcal{O}_X)_{\bar{x}} & & \\ \| & & \| & & \\ \mathcal{O}_{X,\bar{x}}[\tfrac{1}{\pi}]^* & \hookrightarrow & \mathcal{O}_{X,\bar{x}}[\tfrac{1}{\pi}] & & \end{array}$$

$\mathcal{O}_{X,x}$ (noeth.) reg. local ring $\Longrightarrow$ $\mathcal{O}_{X,\bar{x}}$ ditto

$\xRightarrow{\text{thm.}}$ $\mathcal{O}_{X,\bar{x}}$ is a UFD, i.e.

can write $f \in \mathcal{O}_{X,\bar{x}}$ as

$$f = u \, p_1^{i_1} \cdots p_m^{i_m},$$

with $u \in \mathcal{O}_{X,\bar{x}}^*$ and $p_1, \dots, p_m \in \mathcal{O}_{X,\bar{x}}$ prime elements (i.e. $p_s \mathcal{O}_{X,\bar{x}} \subset \mathcal{O}_{X,\bar{x}}$ prime ideal). Note

$$x_i \in \mathcal{O}_{X,\bar{x}} \text{ prime} \iff x_i \in m_{\bar{x}}$$

so $\qquad\qquad$ (x_i passes through x)

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{X,\bar{x}}^* \arrow[r] & \mathcal{O}_{X,\bar{x}}[\frac{1}{\pi}]^* \arrow[r] & \mathbb{Z}^s \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_{X,\bar{x}}^* \arrow[r] \arrow[u, equal] & M_{\bar{x}} \arrow[r] \arrow[u, hook] & \mathbb{N}_0^s \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

where $s = \#\{i \mid x_i \in m_{\bar{x}}\}$. Clearly

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{X,\bar{x}}^* \arrow[r] & M_{\bar{x}} \arrow[r] & \mathbb{N}_0^s \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_{X,\bar{x}}^* \arrow[r] \arrow[u, equal] & P^a_{X,\bar{x}} \arrow[r] \arrow[u] & \mathbb{N}_0^s \arrow[r] \arrow[u, "\sim"'] & 0 \\
0 \arrow[r] & \alpha^{-1}(\mathcal{O}_X^*)_{\bar{x}} \arrow[r] \arrow[u, hook, "\text{p-c.}"'] \arrow[d, equal] & P_{X,\bar{x}} \arrow[r] \arrow[u, hook] \arrow[d, equal] & \mathbb{N}_0^s \arrow[r] \arrow[u, "\sim\ \text{since p-c.}"'] & 0 \\
& \mathbb{N}_0^{r-s} & \mathbb{N}_0^r & &
\end{tikzcd}$$

//

<u>Lemma</u> $\Gamma(U, \Omega^1_{X/S} \oplus (\mathcal{O}_X \underset{\mathbb{Z}}{\otimes} P_X^{gp})) = \Gamma(U, \Omega^1_{X/S}) \oplus (\Gamma(U, \mathcal{O}_X) \underset{\mathbb{Z}}{\otimes} P^{gp})$

pf (1) $(P_X)^{gp} = (P^{gp})_X$: indeed $(-)^{gp} \dashv U$, $(-)_X \dashv \Gamma(X, -)$, and U and $\Gamma(X, -)$ commute.

(2) $\Gamma(U, \mathcal{O}_X \underset{\mathbb{Z}}{\otimes} A_X) = \Gamma(U, \mathcal{O}_X) \underset{\mathbb{Z}}{\otimes} A$; A ab. gp.

By def$^{\underline{n}}$, $\mathcal{O}_X \underset{\mathbb{Z}}{\otimes} A_X$ is the sheaf associated w. the presheaf

$$U \longmapsto \Gamma(U, \mathcal{O}_X) \underset{\mathbb{Z}}{\otimes} A .$$

But this is already a sheaf, in fact, a q. coh. $\mathcal{O}_X$-module:

$$\Gamma(U, \mathcal{O}_X) \underset{\mathbb{Z}}{\otimes} A = \Gamma(U, \mathcal{O}_X) \underset{\Gamma(X, \mathcal{O}_X)}{\otimes} (\Gamma(X, \mathcal{O}_X) \underset{\mathbb{Z}}{\otimes} A) .$$

Hence

$$\Gamma(U, \Omega^1_{X/S} \oplus (\mathcal{O}_X \underset{\mathbb{Z}}{\otimes} P_X^{gp}))$$

$$= \Gamma(U, \Omega^1_{X/S}) \oplus \Gamma(U, \mathcal{O}_X \underset{\mathbb{Z}}{\otimes} P_X^{gp})$$

$$= \Gamma(U, \Omega^1_{X/S}) \oplus \Gamma(U, \mathcal{O}_X) \underset{\mathbb{Z}}{\otimes} P^{gp} .$$

//

<u>Lemma</u> If in a sequence of $\mathcal{O}_X$-modules

$$0 \to M' \to M \to M'' \to 0,$$

M' and M (resp. M' and M'') are q-coh. then so is M'' (resp. M).

<u>Pf</u> Consider $U \xrightarrow{\text{étale}} X$; we may assume that U and X are both affine.

$$\begin{tikzcd}
0 \arrow[d, "{\Gamma(X,\mathcal{O}_X) \to \Gamma(U,\mathcal{O}_X) \text{ flat}}"] & 0 \arrow[d] \\
\Gamma(U,\mathcal{O}_X) \otimes_{\Gamma(X,\mathcal{O}_X)} \Gamma(X,M') \arrow[r] \arrow[d] & \Gamma(U,M') \arrow[d] \\
\Gamma(U,\mathcal{O}_X) \otimes_{\Gamma(X,\mathcal{O}_X)} \Gamma(X,M) \arrow[r] \arrow[d] & \Gamma(U,M) \arrow[d] \\
\Gamma(U,\mathcal{O}_X) \otimes_{\Gamma(X,\mathcal{O}_X)} \Gamma(X,M'') \arrow[r] \arrow[d, "{M' \text{ q-coh.} + X \text{ affine}}"] & \Gamma(U,M'') \arrow[d, "{\text{since } M' \text{ q-coh. and } U \text{ affine}}"] \\
0 & 0
\end{tikzcd}$$

//

9/20/00

Def A map $(X, M) \xrightarrow{f} (Y, N)$ of schemes w. fine log. str. is smooth (resp. étale) if $X \to Y$ is locally of f. presentation and if in every diagram

$$\begin{tikzcd}
(T', L') \arrow[r, "s"] \arrow[d, "i"'] & (X, M) \arrow[d, "f"] \\
(T, L) \arrow[r, "t"] \arrow[ur, dashed, "g"] & (Y, N)
\end{tikzcd},$$

where i is an exact closed immersion defined by a nilpotent ideal I, there exist étale locally (resp. there exists a unique) lift g.

---

chart : $P_X \to M$ s.t. $P_X^a \xrightarrow{\sim} M$.

q. coh. : chart exists étale locally.

coh. : —— " —— w. P f.g.

integral : —— " —— w. P integral

fine : coh. + integral.

$X \xrightarrow{f} Y$, $N \to \mathcal{O}_Y \rightsquigarrow f^*N \to \mathcal{O}_X$:

$$\begin{tikzcd}
f^{-1}N \arrow[r] \arrow[dr] & f^{-1}\mathcal{O}_Y \arrow[r] & \mathcal{O}_X \\
f^*N := & (f^{-1}N)^a \arrow[ur] &
\end{tikzcd}$$

inverse image

ex $S = \operatorname{Spec} \mathbb{Z}[P]$ can. log. str.:

$$P \to \mathbb{Z}[P] \implies P_S^a \to S.$$

$$P_X \to M \quad = \quad X \xrightarrow{f} \operatorname{Spec} \mathbb{Z}[P]$$

chart.

$$+$$

$$f^*(P_S^a) \xrightarrow{\sim} M.$$

$(T', L') \xrightarrow{i} (T, L)$ is a closed imm. (resp. an exact closed imm.) if $T' \hookrightarrow T$ is a cl. imm. and $i^*L \to L'$ is surj. (resp. an iso.)

Thm The following are equivalent:

(i) $(X, M) \xrightarrow{f} (Y, N)$ is smooth (resp. étale);

(ii) étale locally on X and Y, there exists a chart $(P_X \to M, Q_Y \to N, Q \to P)$ of f s.t. the kernel and the torsion part of the cokernel (resp. the kernel and the cokernel) of $P^{gp} \to Q^{gp}$ are finite groups whose orders are invertible on X, and s.t.

$$X \to Y \times_{\operatorname{Spec} \mathbb{Z}[Q]} \operatorname{Spec} \mathbb{Z}[P]$$

is smooth (resp. étale). //

ex $(\operatorname{Spec} V[x_1, \dots, x_d]/(x_1 \cdots x_r - \pi), M) \to (\operatorname{Spec} V, N)$

is smooth: $((\mathbb{N}_0^r)_X \to M, (\mathbb{N}_0)_S \to N, \mathbb{N}_0 \xrightarrow{\Delta} \mathbb{N}_0^r)$

is a chart; $\mathbb{Z} \xrightarrow{\Delta} \mathbb{Z}^r$ has no kernel and torsion-free cokernel. Finally

$$\begin{array}{ccc} X & = & \operatorname{Spec} V[t_1, \dots, t_d]/(t_1 \cdots t_r - \pi) \\ \downarrow & & \downarrow \text{ smooth} \\ Y \times_{\operatorname{Spec} V[t]} \operatorname{Spec} V[t_1, \dots, t_r] & = & \operatorname{Spec} V \otimes_{V[t]} V[t_1, \dots, t_r] \\ & & \pi \mapsfrom t \qquad t \mapsto t_1 \cdots t_r \\ & & \| \\ & & \operatorname{Spec} V[t_1, \dots, t_r]/(t_1 \cdots t_r - \pi) \end{array}$$

Lemma $(X, f^*N) \xrightarrow{f} (Y, N)$ is sm. (resp. étale) iff $X \xrightarrow{f} Y$ is sm. (resp. étale). //

Lemma Let P and Q be f.g. integral monoids, $Q \to P$ a map of monoids, R a ring, and suppose the kernel and the torsion part of the cokernel (resp. the kernel and the cokernel) are finite grps whose orders inv. in R. Then

$$(\operatorname{Spec} R[P], P^a) \longrightarrow (\operatorname{Spec} R[Q], Q^a)$$

is sm. (resp. étale).

Given these, prove (ii) ⟹ (i):

$$\begin{tikzcd}
(X,M) \arrow[dr, "\text{sm. by lemma 1}"] \arrow[drr, bend left] \arrow[ddr, bend right, "f"'] & & \\
& (Y \times_{\operatorname{Spec} \mathbb{Z}[Q]} \operatorname{Spec} \mathbb{Z}[P], P^a) \arrow[r] \arrow[d, "\text{sm.}"] & (\operatorname{Spec} \mathbb{R}[P], P^a) \arrow[d, "\text{sm. by lemma 2}"] \arrow[dl, phantom, "\xLeftarrow{\text{p.b.}}"] \\
& (Y,N) \arrow[r] & (\operatorname{Spec} \mathbb{R}[Q], Q^a)
\end{tikzcd}$$

//

pf of lemma 2: find (unique) lifting

$$\begin{tikzcd}
(T', L') \arrow[r, "s"] \arrow[d, "i"', "I"] & (\operatorname{Spec} \mathbb{R}[P], P^a) \arrow[d, "f"] \\
(T, L) \arrow[r, "t"] \arrow[ur, dashed] & (\operatorname{Spec} \mathbb{R}[Q], Q^a)
\end{tikzcd}$$

$I^2 = 0$

or equivalently, a map of sheaves on $T_{\acute{e}t}$:

$$\begin{tikzcd}
i_* L' & P_T \arrow[l] \arrow[dl, dashed] \\
L \arrow[u] & Q_T \arrow[l] \arrow[u]
\end{tikzcd}$$

<u>Claim</u> $L/(1+I)^* \xrightarrow{\sim} i_* L'$

$$\begin{tikzcd}
P_T \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right, "\text{define now}"'] & & \\
& L \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & i_*L' \arrow[d] \\
& L^{gp} \arrow[r] & (i_*L')^{gp}
\end{tikzcd}$$

since $(1+I)^* \cong I$ and T affine

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(T,(1+I)^*) \arrow[r] \arrow[d, "\wr", "(n\in R^*)"'] & \Gamma(T,L^{gp}) \arrow[r] \arrow[d] & \Gamma(T,(i_*L')^{gp}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(T,(1+I)^*)[\frac{1}{n}] \arrow[r] \arrow[ur] & \Gamma(T,L^{gp})[\frac{1}{n}] \arrow[r] \arrow[ur] & \Gamma(T,(i_*L')^{gp})[\frac{1}{n}] \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[r] & Q^{gp} \arrow[r] \arrow[d] \arrow[ul] & P^{gp} \arrow[r] \arrow[d] \arrow[ul] & C \arrow[r] \arrow[d] \arrow[ul] & 0 \\
& 0 \arrow[r] & Q^{gp}[\frac{1}{n}] \arrow[r] & P^{gp}[\frac{1}{n}] \arrow[r] \arrow[l, bend right] & C[\frac{1}{n}] \arrow[r] \arrow[l, bend right] & 0
\end{tikzcd}$$

$n = \# K \cdot \# \operatorname{tors}(C)$

f.g. free

$$\Longrightarrow \quad P \longrightarrow P^{gp} \longrightarrow \Gamma(T, L^{gp}) \qquad \text{unique if } C[\tfrac{1}{n}] = 0.$$

Prove claim: $L = H^a_T$, $L' = H^a_{T'}$

$$\begin{tikzcd}
(1+I)^* \arrow[r] & \mathcal{O}_T^* \arrow[r] \arrow[d] & i_*\mathcal{O}_{T'}^* \arrow[d] \\
(1+I)^* \arrow[r] & L \arrow[r] \arrow[d] & i_*L' \arrow[d] \\
& L/\mathcal{O}_T^* \arrow[r, "\sim", "\text{show}"'] & i_*L'/i_*\mathcal{O}_{T'}^*
\end{tikzcd}$$

$(I$ nilp$)$

$T' \overset{i}{\hookrightarrow} T$ is a surjective closed imm.

$$
\begin{tikzcd}
H \arrow[r, "\sim"] \arrow[dd, equal] & L_{\bar{x}} / G^*_{T,\bar{x}} \arrow[d] \\
& (i_* L')_{\bar{x}} / (i_* G^*_{T'})_{\bar{x}} \arrow[d, equal] \\
H \arrow[r, "\sim"] & L'_{\bar{z}} / G^*_{T',\bar{z}}
\end{tikzcd}
\qquad i(\bar{z}) = \bar{x}
$$

Remember: if $Z \overset{i}{\hookrightarrow} X$ is a cl. imm. then

$$
(i_* F)_{\bar{x}} = \begin{cases} F_{\bar{z}} & , \text{ if } \bar{x} = i(\bar{z}) \\ \emptyset & , \text{ else.} \end{cases}
$$

ex $\quad (X,M) = (\mathrm{Spec}\, V[x_1,\dots,x_d]/(x_1 \cdots x_r - \pi), (\mathbb{N}_0^r)^a_X)$

$$\begin{array}{ccc} \downarrow & & \downarrow \text{ smooth} \\ (S,N) & = & (\mathrm{Spec}\, V, (\mathbb{N}_0)^a_S) \end{array}$$

The $\mathcal{O}_X$-module $\omega^1_{(X,M)/(S,N)}$ is q-coh.,

$$\Gamma(X, \omega^1_{(X,M)/(S,N)}) = \left(\Gamma(X, \Omega^1_{X/S}) \oplus \Gamma(X, \mathcal{O}_X) \underset{\mathbb{Z}}{\otimes} P^{gp}\right) \Big/ R$$

where R is the $\Gamma(X, \mathcal{O}_X)$-submodule gen. by:

$$(d\alpha(a), 0) - (0, \alpha(a) \otimes a), \quad a \in P$$

$$(0, 1 \otimes x), \qquad x \in \mathrm{im}\,(Q \to P).$$

Get:

$$\Gamma(X, \omega^1_{(X,M)/(S,N)}) = \frac{\Gamma(X, \mathcal{O}_X)\{d\log x_1, \dots, d\log x_r, dx_{r+1}, \dots, dx_d\}}{\Gamma(X, \mathcal{O}_X)\{d\log x_1 + \dots + d\log x_r\}}$$

i.e. $\omega^1_{(X,M)/(S,N)}$ is a free $\mathcal{O}_X$-module of rk. $d-1$.

Compare:

$$\Gamma(X, \Omega^1_{X/S}) = \frac{\Gamma(X, \mathcal{O}_X)\{dx_1, \dots, dx_d\}}{\Gamma(X, \mathcal{O}_X)\{d(x_1 \cdots x_r)\}},$$

not free

Prop If $(X,M) \xrightarrow{f} (Y,N)$ is smooth, then $\omega^1_{(X,M)/(Y,N)}$ is a projective q-coh. $\mathcal{O}_X$-module.

pf Show : $F \to F'' \to 0$ exact seq. q-coh. $\mathcal{O}_X$-mod.
$\Rightarrow \mathrm{Hom}_{\mathcal{O}_X}(\omega^1_{X/Y}, F) \to \mathrm{Hom}_{\mathcal{O}_X}(\omega^1_{X/Y}, F'') \to 0$ exact.

$$\mathrm{Hom}_{\mathcal{O}_X}(\omega^1_{X/Y}, F) = \mathrm{Der}_{(Y,N)}((X,M), F)$$

and a derivation $(D, D\log) : (X,M) \to F$ defines a lifting

$$\begin{tikzcd}
(X,M) \arrow[r, equal] \arrow[d, hook, "i"] & (X,M) \arrow[d] \\
(\underline{\mathrm{Spec}}\, \mathcal{O}_X \oplus F, M \oplus F) \arrow[r] \arrow[ur, dashed] & (Y,N)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r, equal] & \mathcal{O}_X \arrow[dl, "\mathrm{id}\oplus D"'] \\
\mathcal{O}_X \oplus F \arrow[u, "pr_1"] & f^{-1}\mathcal{O}_Y \arrow[l, "\varphi"'] \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
M \arrow[r, equal] & M \arrow[dl, "\mathrm{id}\oplus D\log"'] \\
M \oplus F \arrow[u, "pr_1"] & f^{-1}N \arrow[l, "\varphi"'] \arrow[u]
\end{tikzcd}$$

Conversely, a lift defines a derivation.

⌜

$$\left\{ \begin{array}{c} Y \\ \downarrow \\ X \end{array} \text{ affine} \right\} \rightleftarrows \left\{ \text{q. coh. } \mathcal{O}_X\text{-algebras} \right\}$$

$$(Y \xrightarrow{f} X) \longmapsto \mathcal{O}_X \to f_* \mathcal{O}_Y$$

$$\underline{\mathrm{Spec}}\, \mathcal{A} \longleftarrow\!\shortmid \mathcal{O}_X \to \mathcal{A}$$

adjoint equivalence. ⌟

$$(X, M) \xhookrightarrow{i} (\underline{\mathrm{Spec}}\, \mathcal{O}_X \oplus F, M \oplus F)$$

$\underline{\text{exact}}$ closed imm.; i.e. cl. imm. ✓ and

$$i^*(M \oplus F) \xrightarrow{\sim} M \quad ; \quad ✓$$

$$\left(i^*(M\oplus F)/\mathcal{O}_X^* \right)_{\bar{x}} \xrightarrow{\sim} \left(M/\mathcal{O}_X^* \right)_{\bar{x}}$$

$$\| \ (*)$$

$$\left(i^{-1}\left(M\oplus F/(\mathcal{O}_X \oplus F)^* \right) \right)_{\bar{x}}$$

$$\| \ (+)$$

$$\left(M \oplus F/(\mathcal{O}_X \oplus F)^* \right)_{i(\bar{x})}$$

$$\|$$

$$M_{\bar{x}} \oplus F_{\bar{x}} / \mathcal{O}^*_{X,\bar{x}} \oplus F_{\bar{x}} \xrightarrow{\sim} M_{\bar{x}}/\mathcal{O}^*_{X,\bar{x}}$$

(+) $\bar{x}^* \dashv \bar{x}_*$, $i^{-1} \dashv i_*$ and $i_*(\bar{x}_* E) = i(\bar{x})_* E$.

(*) see p. 5

$$\begin{tikzcd}
(X,M) \arrow[rr, equal] \arrow[dd, hook] \arrow[dr, equal] & & (X,M) \arrow[dr, equal] & \\
& (X,M) \arrow[ur, dashed, "?", "\tilde{g}"'] \arrow[rr, equal, "1"] \arrow[dl, dash] \arrow[dd, hook] & & (X,M) \arrow[dd, "f"] \\
(\underline{\mathrm{Spec}}\, \mathcal{O}_X \oplus F, M \oplus F) \arrow[rr, hook] \arrow[dr, leftrightarrow] & & (Y,N) \arrow[dr, equal] & \\
& (\underline{\mathrm{Spec}}\, \mathcal{O}_X \oplus F'', M \oplus F'') \arrow[rr] \arrow[uurr, "g"'] & & (Y,N)
\end{tikzcd}$$

i.e. must find lift

$$\begin{tikzcd}
(\underline{\mathrm{Spec}}\, \mathcal{O}_X \oplus F'', M \oplus F'') \arrow[r, "s"] \arrow[d, hook, "j"', "0 \oplus F'"] & (X,M) \arrow[d, "f"] \\
(\underline{\mathrm{Spec}}\, \mathcal{O}_X \oplus F, M \oplus F) \arrow[r] \arrow[ur, dashed, "\tilde{g}"] & (Y;N)
\end{tikzcd}$$

This exist since j is an exact cl. imm. (w. ideal $0 \oplus F' \subset \mathcal{O}_X \oplus F$) and since f is smooth. //

Cor Suppose $(X,M) \xrightarrow{f} (Y,N)$ is smooth, Then X has a (Zariski) covering $\{U_i \to X\}$ s.t. for all i, $\omega^1_{(X,M)/(Y;N)}|_{U_i}$ is a f.g. free $\mathcal{O}_{U_i}$-module.

pf Since $X \to Y$ is loc. finitely presented and since M and N are fine, $\omega^1_{(X,M)/(Y,N)}$ is a locally fin. presented $\mathcal{O}_X$-module. Use :

<u>Lemma</u> Let A be a ring, $\mathfrak{p} \subset A$ a prime ideal, and M a f. presented A-module. Suppose $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module of rk r. Then there exist $f \in A \setminus \mathfrak{p}$ s.t. M_f is a free A_f-module of rk. r.

(Matsumura, p. 28). //

—

If $f: X \to Y$ is a map of schemes and $N \to \mathcal{O}_Y$ a log str, then

$$f^{-1}(N/\mathcal{O}_Y^*) \xrightarrow{\sim} f^*(N)/\mathcal{O}_X^*$$

Indeed this follows from the cartesian sq:

$$\begin{tikzcd} f^{-1}(\mathcal{O}_Y^*) \arrow[r] \arrow[d, hook] & \mathcal{O}_X^* \arrow[d, hook] \\ f^{-1}(\mathcal{O}_Y) \arrow[r] & \mathcal{O}_X \end{tikzcd}$$

On stalks this is

$$\begin{tikzcd} \mathcal{O}_{Y,\bar{y}}^* \arrow[r] \arrow[d, hook] & \mathcal{O}_{X,\bar{x}}^* \arrow[d, hook] \\ \mathcal{O}_{Y,\bar{y}} \arrow[r] & \mathcal{O}_{X,\bar{x}} \end{tikzcd}, \qquad \bar{y} = f(\bar{x})$$

which is cartesian since $\mathcal{O}_{Y,\bar{y}} \to \mathcal{O}_{X,\bar{x}}$ is a <u>local</u> homomorphism.

$W(A) = A^{\mathbb{N}_0}$ as a set

$w : W(A) \longrightarrow A^{\mathbb{N}_0}$, $w(a_0, a_1, \dots) = (w_0, w_1, \dots)$

ghost map

$$w_0 = a_0$$
$$w_1 = a_0^p + p a_1$$
$$w_2 = a_0^{p^2} + p a_1^p + p^2 a_2$$
$$w_n = a_0^{p^n} + p a_1^{p^{n-1}} + \dots + p^{n-1} a_{n-1}^p + p^n a_n$$
$$\vdots$$

<u>Lemma</u> (Dwork) Suppose $f : A \to A$ is a ring hom. s.t. $f(x) \equiv x^p \pmod{pA}$, $\forall x \in A$. Then $(x_0, x_1, \dots)$ is in the image of the ghost map if and only if for all $n \geq 0$,

$$x_n \equiv f(x_{n-1}) \pmod{p^n A};$$

Pf The binomial formula shows:

$$x \equiv y \pmod{pA} \Rightarrow x^{p^{n-1}} \equiv y^{p^{n-1}} \pmod{p^n A}$$

Show $w_n \equiv f(w_{n-1}) \pmod{p^n A}$:

$$f(w_{n-1}) = f(a_0^{p^{n-1}} + p a_1^{p^{n-2}} + \dots + p^{n-1} a_{n-1})$$
$$= f(a_0)^{p^{n-1}} + p f(a_1)^{p^{n-2}} + \dots + p^{n-1} f(a_{n-1})$$
$$\equiv a_0^{p^n} + p a_1^{p^{n-1}} + \dots + p^{n-1} a_{n-1}^p \pmod{p^n A}$$

$$\equiv a_0^{p^n} + p a_1^{p^{n-1}} + \cdots + p^{n-1} a_{n-1}^p + p^n a_n = w_n \pmod{p^n A}$$

Conversely, inductively choose $a_n \in A$ s.t.

$$p^n a_n = w_n - f(w_{n-1}). \quad //$$

Note (i) If A has a lifting of the frob, on A/pA, then $w(W(A)) \subset A^{\mathbb{N}_0}$ is a <u>subring</u>;

(ii) A p-tors. free $\Rightarrow$ $w : W(A) \hookrightarrow A^{\mathbb{N}_0}$ injective.

Define ring-str. : $U = \mathbb{Z}[a_0, a_1, a_2, \ldots ; b_0, b_1, \ldots]$, $f : U \to U$, $f(a_i) = a_i^p$, $f(b_i) = b_i^p$. Let $a = (a_0, a_1, \ldots)$, $b = (b_0, b_1, \ldots) \in U$ and define $s = (s_0, s_1, \ldots) \in U$ (resp. $p = (p_0, p_1, \ldots) \in U$) by

$$w(s(a,b)) = w(a) + w(b)$$

$$w(p(a,b)) = w(a) \cdot w(b).$$

If A is a ring and $a, b \in W(A)$, there is a unique ring hom $U \to A$ s.t. $W(U) \to W(A)$ maps a to a and b to b. Define

$$a + b := s(a,b)$$

$$a \cdot b := p(a,b).$$

---- show this is a ring.

<u>Note</u> $s_n = s_n(a_0, \dots, a_n;\ b_0, \dots, b_n)$

$p_n = p_n(a_0, \dots, a_n;\ b_0, \dots, b_n)$

So get ring str. on

$$W_n(A) = A^n.$$

Define <u>restriction</u>, <u>Frobenius</u>, <u>Verschiebung</u>, and <u>Teichmüller</u>:

$R : W_n(A) \to W_{n-1}(A)$, $R(a_0, \dots, a_{n-1}) = (a_0, \dots, a_{n-2})$;

$F : W_n(A) \to W_{n-1}(A)$, $F(w_0, \dots, w_{n-1}) = (w_1, \dots, w_{n-2})$;

$V : W_{n-1}(A) \to W_n(A)$, $V(a_0, \dots, a_{n-2}) = (0, a_0, \dots, a_{n-2})$;

$\underline{\ \ }_n : A \to W_n(A)$, $\underline{x}_n = (x, 0, \dots, 0)$.

$\ulcorner$ To define F, consider $U = \mathbb{Z}[a_0, a_1, \dots]$, let $(f_0, f_1, \dots) \in W(U)$ be the unique vector s.t.

$$w(f_0, f_1, \dots) = (w_1, w_2, \dots);$$

for any ring A define

$$F : W_n(A) \to W_{n-1}(A)$$

$$F(a_0, \dots, a_{n-1}) = (f_0(a_0, a_1), \dots, f_{n-2}(a_0, \dots, a_{n-1})). \lrcorner$$

<u>exercise</u> Show :

R, F are ring hom;

$V : F_* W_{n-1}(A) \to W_n(A)$ map of $W_n(A)$-mod.;

(i.e. V additive and $x \cdot Vy = V(Fx \cdot y)$)

$\underline{\ \ }_n : A \to W_n(A)$ multiplicative;

$FV = p$; $F \underline{x}_n = \underline{x}^p_{n-1}$

<u>Lemma</u> If $a = (a_0, a_1, \dots, a_{n-1}) \in W_n(A)$ then

$$a = \sum_{i=0}^{n-1} V^i(\underline{a_i}_{n-i}).$$

Pf The two sides have same image under the ghost map. So the formula holds if A is p-tors. free. But for any ring A, we can find a p-tors. free ring $\tilde{A}$ and a surj. $\tilde{A} \to A$. Hence the formula holds in general. //

<u>exercise</u> Show that

$$0 \to W_m(A) \xrightarrow{V^n} W_{m+n}(A) \xrightarrow{R^n} W_m(A) \to 0$$

is exact.

<u>Lemma</u> If A is an $\mathbb{F}_p$-algebra then

$$F = W(\varphi) : W(A) \to W(A).$$

<u>Pf</u> Show $f_n \equiv a_n^p \pmod{pA}$. This follows inductively from:

$$f_0^{p^n} + p f_1^{p^{n-1}} + \cdots + p^n f_n = a_0^{p^{n+1}} + p a_1^{p^n} + \cdots + p a_n^p + p^{n+1} a_{n+1}. \quad //$$

<u>Cor</u> If A is an $\mathbb{F}_p$-alg, $VF = FV = p$. //

$$\Rightarrow \quad V^m W(A) \cdot V^n W(A) \subset V^{m+n} W(A) :$$

$$\begin{aligned} V^m x \cdot V^n y &= V^m(x \cdot F^m V^n y) \qquad (m \geq n) \\ &= V^m(x \cdot V^n F^m y) \\ &= V^{m+n}(F^n x \cdot F^m y). \end{aligned}$$

$$\Rightarrow \quad gr_V^* W(A) \quad \text{graded ring}$$

$$\begin{aligned} \varphi_*^n A &\xrightarrow{\sim} gr_V^n W(A) \qquad \text{~~bijection~~} \\ a &\longmapsto V^n \underline{a} \qquad A\text{-mod-iso.} \end{aligned}$$

$$\Rightarrow \quad \bigoplus_{n \geq 0} \varphi_*^n A \xrightarrow{\sim} gr_V^* W(A), \quad A \ \mathbb{F}_p\text{-alg.}$$

$$\varphi^m A \otimes \varphi^n A \to \varphi^{m+n} A, \quad x \otimes y \mapsto \varphi^n x \cdot \varphi^m y.$$

Suppose A p-tors. free, $f: A \to A$ lift of frob of A/pA

$$\begin{tikzcd} & W(A) \arrow[dr, hook, "w"] & \\ A \arrow[ur, dashed, "s_f"] \arrow[rr] & & A^{\mathbb{N}_0} \end{tikzcd}$$

$$a \longmapsto (a, f(a), f^2(a), \ldots) ;$$

[exercise Show that for any ring A,

$$F : W(A) \longrightarrow W(A)$$

lifts the frobenius of $W(A)/pW(A)$.]

$$t_f : A \xrightarrow{s_f} W(A) \longrightarrow W(A/pA).$$

$$t_f(p^n A) \subset p^n W(A/pA) = V^n F^n W(A/pA) \subset V^n W(A/pA)$$

A/pA $\mathbb{F}_p$-alg.

$$\Rightarrow \quad t_f : A/p^n A \longrightarrow W_n(A)$$

Prop If A/pA is a perfect $\mathbb{F}_p$-algebra then

$$t_f : A/p^n A \xrightarrow{\sim} W_n(A).$$

Pf $t_f(p^n x) = p^n t_f(x) = V^n F^n t_f(x) = V^n t_f(F^n x) \Big\} \Rightarrow$

$$t_f(f^n x) \equiv \underline{\varphi^n x} \pmod{VW(A)}$$

$$t_f(p^n x) \equiv V^n(\underline{\varphi^n \bar{x}}) \quad (\text{mod } V^{n+1}W(A))$$

Hence we have a commutative diagram

$$\begin{tikzcd}
A/pA \arrow[r, "\varphi^n", "\sim"'] \arrow[d, "p^n", "\sim"'] & \varphi^n_* A/pA \arrow[d, "\sim"] & a \arrow[d, mapsto] \\
p^n A/p^{n+1}A \arrow[r, "t_f"] & gr^n_V W(A/pA) & V^n \underline{a} .
\end{tikzcd}$$

//

<u>Cor</u> Suppose that A is p-tors. free and that the p-adic topology on A is complete and separated. If there exists a lift $f: A \to A$ of the frob. on A/pA, then

$$A \xrightarrow[\sim]{t_f} W(A/pA).$$

<u>ex</u> $W(\mathbb{F}_p) \simeq \mathbb{Z}_p$. //

Lemma For any ring A, $F : W(A) \to W(A)$ lifts the frob. of $W(A)/pW(A)$, i.e. for all $a \in W(A)$,

$$F(a) \equiv a^p \pmod{pW(A)}.$$

Pf By naturality, it suffices to consider the universal case $U = \mathbb{Z}[a_0, a_1, a_2, \ldots]$, $a = (a_0, a_1, \ldots)$, Recall that U has a lift of the frob. of U/pU given by $\varphi : U \to U$, $\varphi(a_i) = a_i^p$. So the image of the ghost map

$$w : W(U) \longrightarrow U^{\mathbb{N}_0}$$

is given by the lemma of Dwork. Write

$$w(a) = (w_0, w_1, w_2, \ldots).$$

Then

$$w(F(a) - a^p) = (w_1 - w_0^p, w_2 - w_1^p, w_3 - w_2^p, \ldots)$$

and we must show

(i) $w_n - w_{n-1}^p$ is divisible by p;

(ii) $\left(\frac{w_1 - w_0^p}{p}, \frac{w_2 - w_1^p}{p}, \ldots\right)$ is in the image of the ghost map.

To show (i) note:

$$w_n - w_{n-1}^p \equiv w_n - \varphi(w_{n-1}) \quad (\text{mod } pA)$$

$$= p^n a_n \quad (\text{since universal case})$$

Prove (ii) use lemma of Dwork: must show

$$\frac{w_n - w_{n-1}^p}{p} \equiv \varphi\left(\frac{w_{n-1} - w_{n-2}^p}{p}\right) \quad (p^{n-1}U) \iff$$

$$w_n - w_{n-1}^p \equiv \varphi(w_{n-1} - w_{n-2}^p) \quad (p^n U) \iff$$

$$w_n - \varphi(w_{n-1}) \equiv w_{n-1}^p - \varphi(w_{n-2}^p) \quad (p^n U)$$

But both sides are $\equiv 0$ $(p^n U)$ by Dwork's lemma. //

10/6/00

Def A ring, $I \subset A$ ideal. A PD-str. on I consists of maps $\gamma_i : I \to A$, $i \geq 0$, s.t.

(i) $\gamma_0(x) = 1$, $\gamma_1(x) = x$, $\gamma_i(x) \in I$, $i \geq 1$, $x \in I$;

(ii) $\gamma_k(x+y) = \sum_{i+j=k} \gamma_i(x)\gamma_j(y)$, $x, y \in I$;

(iii) $\gamma_k(\lambda x) = \lambda^k \gamma_k(x)$, $\lambda \in A$, $x \in I$;

(iv) $\gamma_i(x)\gamma_j(x) = \dfrac{(i+j)!}{i!\,j!}\gamma_{i+j}(x)$, $x \in I$;

(v) $\gamma_i(\gamma_j(x)) = \dfrac{(ij)!}{i!\,(j!)^i}\gamma_{ij}(x)$, $x \in I$.

Ex (1) If A is a $\mathbb{Q}$-algebra, then every ideal has a unique PD-str:

$$\gamma_n(x) = \frac{x^n}{n!}.$$

(2) If V is a DVR of mixed char. $(0, p)$ then $m = (\pi)$ has a (unique) PD-str. iff $e \leq p-1$; $m^e = (p)$. Indeed, we must have

$$\gamma_n(x) = \frac{x^n}{n!} \in K$$

and this defines a PD-str. on m iff.

$$\frac{x^n}{n!} \in m \quad , \quad \forall x \in m, \ n \geq 1 \iff$$

$$\frac{\pi^n}{n!} \in m \quad , \quad \forall n \geq 1 \iff$$

$$v_\pi\left(\frac{\pi^n}{n!}\right) \geq 1 \quad , \quad \forall n \geq 1 \iff$$

$$n - e\, v_p(n!) \geq 1$$

<u>Lemma</u> If $n = \sum_{i=0}^{m} a_i p^i$, $0 \leq a_i < p$, then

$$v_p(n!) = \frac{1}{p-1} \sum_{i=0}^{m} a_i (p^i - 1) \quad //$$

$$n - e\, v_p(n!) = \sum_{i=0}^{m} a_i p^i - \frac{e}{p-1} \sum_{i=0}^{m} a_i (p^i - 1) \geq 1$$

$$\iff \quad e \leq p - 1 .$$

(3) If k is a perfect field of char. $p > 0$, then $W(k)$ is a CDVR with $e = 1$, so $pW(k)$ has a unique PD-str.

<u>exercise</u> Assume $mA = 0$, some $m \in \mathbb{N}$, and $I \subset A$ f.g. Show that if I has a PD-str. then I is nilpotent.

<u>Lemma</u> The forgetful functor has a left adjoint.

$$\{A\text{-PD-algebras}\} \rightleftarrows \{A\text{-modules}\}$$

$$(B, J, \delta) \longmapsto J$$

$$(\Gamma_A(M), \Gamma_A^+(M), [\ ]) \longleftarrow M. \quad //$$

<u>Def</u> Let (A, I, γ) be a PD-ring and B an A-algebra. Say γ <u>extends</u> to B if $(B, IB, \bar{\gamma})$,

$$\bar{\gamma}_n(xb) := \gamma_n(x)\, b^n,$$

is well-defined.

<u>exercise</u> Suppose $I \subset A$ is principal. Show γ extends to any B.

Prop Let (A, I, γ) be a PD-ring. Then the forgetful functor has a left adjoint:

$$\begin{array}{ccc} \{(A,I,\gamma) \to (A', I', \gamma') \text{ PD-morphism}\} & & D_{B,\gamma}(J) \\ \uparrow \perp & & \text{I} \\ \{(A,I) \to (B,J)\ ;\ IB \subset J\} & & (B,J) \end{array}$$

Pf Let $K \subset \Gamma_B(J)$ be the ideal gen. by

$$x^{[1]} - x \quad , \quad x \in J ,$$

$$(f(y))^{[n]} - f(\gamma_n(y)) \quad , \quad y \in I .$$

Claim $K \cap \Gamma_B^+(J) \subset \Gamma_B^+(J)$ sub-PD-ideal.

$$\Rightarrow (D, \bar{J}, [\;]) = \left(\Gamma_B(J)/K, \; \Gamma_B^+(J)+K/K, \; [\;]\right)$$

PD-ring.

//

(A, I, γ), (B, J, δ) PD-rings, B A-algebra.
Say γ and δ are <u>compatible</u> if γ extends to $\bar{\gamma}$ on IB and $\bar{\gamma} = \delta$ on $IB \cap J$.

Equivalently: $K = IB + J$ has a PD-str. $\bar{\delta}$ and

$$(A, I, \gamma) \to (B, K, \bar{\delta}) \leftarrow (B, J, \delta)$$

are PD-morphisms.

<u>Prop</u> Let (A, I, γ) be a PD-ring. Then the forgetful functor has a left-adjoint:

$$\begin{array}{ccc} \left\{ \begin{array}{l} (A', I', \gamma') \ A\text{-PD-alg}, \\ \gamma \text{ and } \gamma' \text{ compatible} \end{array} \right\} & & (D_{B,\gamma}(J), \bar{J}, [\ ]) \\ \uparrow \downarrow & & \uparrow \\ \{ (B, J),\ B\ A\text{-algebra} \} & & (B, J) \end{array}$$

<u>pf</u> Let $K = IB + J$, s.t. $IB \subset K$; let

$$D_{B,\gamma}(J) := D_{B,\gamma}(K)$$

$$\bar{J} \subset \bar{K} \text{ sub-PD ideal gen. by } J.$$

//

ex (i) B $\mathbb{Q}$-alg, $D_{B,\gamma}(J) \xleftarrow{\sim} B$.

(ii) $D_{S_A(M)}(S_A^+(M)) \xrightarrow{\sim} \Gamma_A(M)$. //

Define for (A, I, γ)

$$I^{[n]} = (\gamma_{i_1}(x_1) \cdots \gamma_{i_s}(x_s) \mid i_1 + \cdots + i_s \geq n)$$

Thm $(\mathcal{O}, m)$ reg. local ring. Then

$$\Gamma_k(m/m^2) \xrightarrow{\sim} gr^{\cdot} D_{\mathcal{O}}(m) \;//$$

Note (i) By constr. γ extends to $D_{B,\gamma}(J)/\bar{J}$;
if γ extends to B/J, then

$$B/J \xrightarrow{\sim} D_{B,\gamma}(J)/\bar{J}.$$

(ii) if $mA = 0$, some $m \in \mathbb{N}$, then $\bar{J}$ is a nil-ideal

(A, I, γ), (B, J, δ) PD-rings, B A-algebra; say γ and δ are <u>compatible</u> if γ extends to $\bar{\gamma}$ on IB and $\bar{\gamma} = \delta$ on $IB \cap J$.

Equivalently: $K = IB + J$ has PD-str. $\bar{\delta}$,

$$(A, I, \gamma) \longrightarrow (B, K, \bar{\delta}) \longleftarrow (B, J, \delta)$$

are PD-morphisms.

<u>Prop</u> Let (A, I, γ) be a PD-ring. The forgetful functor has a left adjoint:

$$\left\{ \begin{matrix} (B, J, \delta) \ A\text{-PD alg} \\ \delta \text{ and } \gamma \text{ compatible} \end{matrix} \right\} \begin{matrix} \longleftarrow \\ \longrightarrow \end{matrix} \left\{ \begin{matrix} (B, J) \\ B \ A\text{-alg.} \end{matrix} \right\}$$

$$(B, J, \delta) \longmapsto (B, J)$$

$$(D_{B,\gamma}(J), \bar{J}, [\,]) \longleftarrow\!\shortmid (B, J)$$

<u>pf</u> Let $K = IB + J$; define

$$D_{B,\gamma}(J) = D_{B,\gamma}(K) = \Gamma_B(K) / (x^{[1]} - x,\ y^{[n]} - \gamma_n(y) \mid x \in K, y \in I)$$

$\bar{J} \subset \bar{K}$ sub PD-ideal gen. by J. //

<u>Rem</u> (i) $B/J \xrightarrow{\sim} D_{B,\gamma}(J)/\bar{J} \iff \gamma$ extends to B/J;

(ii) If $A \to B$ is flat, then γ extends to B.

<u>Prop'</u> Let (S, I, γ) be a PD-scheme. The forgetful functor has a right adjoint:

$$\left\{ \begin{array}{c} X \overset{J}{\hookrightarrow} Y \\ \searrow \ \swarrow \\ S \end{array} \quad \begin{array}{l} \text{PD-str. } \delta \\ \text{on } J \text{ comp.} \\ \text{w/ } \gamma \end{array} \right\} \underset{U}{\overset{\longleftarrow}{\longrightarrow}} \left\{ \begin{array}{c} X \overset{J}{\hookrightarrow} Y \\ \searrow \ \swarrow \\ S \end{array} \right\}$$

$$(X, Y, \delta) \longmapsto (X, Y)$$

$$(\bar{X}, \bar{Y}, [\,J\,]) \longleftarrow\!\!\shortmid \ (X, Y) \qquad //$$

$$\mathcal{O}_{\bar{Y}} = \mathcal{D}_{\mathcal{O}_Y, \gamma}(J) \quad ; \quad \bar{Y} = \mathrm{Spec}_Y\, \mathcal{O}_{\bar{Y}}$$
$$\supset$$
$$\mathcal{O}_{\bar{X}} = \mathcal{O}_{\bar{Y}} / \bar{J} \quad ; \quad \bar{X} = \mathrm{Spec}_Y\, \mathcal{O}_{\bar{X}}$$

iso. if γ extends to X

$$\begin{tikzcd} \bar{X} \arrow[r, hook] \arrow[d] & \bar{Y} \arrow[d] \arrow[dr, dashed] & \\ X \arrow[r, hook] \arrow[rr, dashed, bend right] & Y \arrow[r, dashed] & X' \arrow[r, hook] & Y' \end{tikzcd}$$

<u>Rem</u> If $m\,\mathcal{O}_S = 0$, some $m \in \mathbb{Z}$, then $\tilde{J}$ is a nil-ideal; so $\bar{X}^{\sim} \xrightarrow{\sim} \bar{Y}^{\sim}$.

(S, I, γ) PD-scheme, $m\mathcal{O}_S = 0$, some $m \in \mathbb{N}$;

X scheme over S, γ extends to X.

Define crystalline site $\mathrm{Crys}(X/S)$:

objects: (U, T, δ)

$$\begin{tikzcd} T \arrow[d] & U \arrow[l, hook', "J"'] \arrow[r, "\text{étale}"] \arrow[d] & X \arrow[d] \\ S \arrow[r, equal] & S \arrow[r, equal] & S \end{tikzcd}$$

δ PD-str. on J comp. w. γ;

morphisms:

$$\begin{tikzcd} T \arrow[d, "h"] & U \arrow[l, hook'] \arrow[r] \arrow[d, "g"] & X \arrow[d, equal] \\ T' \arrow[d] & U' \arrow[l, hook'] \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r, equal] & S \arrow[r, equal] & S \end{tikzcd}$$

h PD-morphism

(note: g automatically étale)

coverings: $\{(U_\alpha, T_\alpha, \delta_\alpha) \to (U, T, \delta)\}$

s.t. $\{T_\alpha \to T\}$ étale cover.

<u>Lemma</u> $\mathrm{Crys}(X/S)$ has finite limits.

Pf Show $(U_1, T_1, \delta_1) \times (U_2, T_2, \delta_2)$ exists:

$$\begin{tikzcd}
U_1 \times_X U_2 \arrow[rr] & & T_1 \times_S T_2 & \text{locally closed} \\
& \overline{U_1 \times_X U_2} \arrow[ul] \arrow[rr, "{[-]}"'] & & \overline{T_1 \times_S T_2} \arrow[ul] \\
U \arrow[uu] \arrow[ur, dashed] \arrow[rr, "\delta"'] & & T \arrow[uu] \arrow[ur, dashed] &
\end{tikzcd}$$

$\left.\begin{array}{l} \gamma \text{ extends to } X \\ U_1 \times_X U_2 \to X \text{ flat} \end{array}\right\} \Rightarrow U_1 \times_X U_2 \xrightarrow{\sim} \overline{U_1 \times_X U_2}$

so $(\overline{U_1 \times_X U_2}, \overline{T_1 \times_S T_2}, [\,]) = (U_1, T_1, \delta_1) \times (U_2, T_2, \delta_2)$.

Show $\mathrm{Crys}(X/S)$ has equalizers : exercise. //

$(X/S)_{crys} := \mathrm{Crys}(X/S)^{\sim}$ crystalline topos.

<u>Lemma</u> The following are equivalent:

(1) $F \in (X/S)_{crys}$;

(2) for every obj. (U, T, δ) in $\mathrm{Crys}(X/S)$,

F_T obj. of $\tilde{T_{et}}$;

for every mor. $u : (U_1, T_1, \delta_1) \to (U, T, \delta)$ in $\mathrm{Crys}(X/S)$,

$\rho_u : u^{-1} F_T \to F_{T_1}$ mor. of $\tilde{T}_{1,et}$

such that :

(i) for all $(U_2, T_2, \delta_2) \xrightarrow{v} (U_1, T_1, \delta_1) \xrightarrow{u} (U, T, \delta)$,

$$\begin{tikzcd}
v^{-1}u^{-1}F_T \arrow[r, "v^{-1}\rho_u"] \arrow[d, phantom, "\parallel"] & v^{-1}F_{T_1} \arrow[r, "\rho_v"] & F_{T_2} \arrow[d, phantom, "\parallel"] \\
(uv)^{-1}F_T \arrow[rr, "\rho_{uv}"] & & F_{T_2}
\end{tikzcd}$$

(ii) If $T_1 \longrightarrow T$ is étale, then

$$\rho_u : u^{-1}F_T \xrightarrow{\sim} F_{T_1}.$$

//

$\Longrightarrow$ $(X/S)_{crys}$ has enough points:

$$F_{(U,T,\delta,\bar{x})} = (F_T)_{\bar{x}}$$

$$= \operatorname{colim} F(U', T', \delta')$$

$$\begin{tikzcd}
 & \bar{x} \arrow[dl] \arrow[dr] & \\
T' \arrow[rr, "\text{ét}"] & & T \\
T' \times_T U = U' \arrow[u, hook] \arrow[rr] \arrow[d] & & U \arrow[u, hook] \arrow[d] \\
X \arrow[rr, phantom, "="] & & X
\end{tikzcd}$$

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & S \end{array} \Rightarrow g_{crys} : (X'/S')_{crys} \to (X/S)_{crys}.$$

Problem: would like to define

$$(g_* F')(T) = F'(g^{-1}(T)) = \mathrm{Crys}(X'/S')^{\wedge}(g^* \tilde{T}, F')$$

can define this, it's just not representable..

$$\begin{array}{ccc} ? & & T \\ \uparrow & & \uparrow \\ X' \times_X U & \to & U \\ \downarrow & & \downarrow \\ X' & \xrightarrow{g} & X \end{array}$$

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$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd} \quad \Longrightarrow \quad g_{crys} : (X'/S')_{crys} \longrightarrow (X/S)_{crys}$$

PD-mor

$$g^*_{crys} F = \operatorname{ass}\left((U', T', \delta') \longmapsto \overbrace{\operatorname*{colim}_{\begin{tikzcd}[row sep=small] T' \arrow[r] & T \\ U' \arrow[u, hook] \arrow[r] \arrow[d] & U \arrow[u, hook] \arrow[d] \\ X' \arrow[r, "g"] & X \end{tikzcd}} F(U, T, \delta)}^{g^{\#} F(U', T', \delta')} \right)$$

$T' \to T$ PD-mor

$$g_{crys *} F'(U, T, \delta) = (X'/S')_{crys}\big(\underbrace{g^*_{crys}(U, T, \delta)^{\sim}}_{\text{calc. this}}, F'\big)$$

$$\operatorname*{colim}_{\begin{tikzcd}[row sep=small] T' \arrow[r] & T_1 \\ U' \arrow[u, hook] \arrow[r] \arrow[d] & U_1 \arrow[u, hook] \arrow[d] \\ X' \arrow[r, "g"] & X \end{tikzcd}} \overbrace{(U, T, \delta)^{\sim}(U_1, T_1, \delta_1)}^{\text{Crys}(X/S)((U_1, T_1, \delta_1), (U, T, \delta))}$$

$$\xrightarrow{\sim} \left\{ \begin{tikzcd} T' \arrow[r, "h"] & T \\ U' \arrow[u, hook] \arrow[r] \arrow[d] & U \arrow[u, hook] \arrow[d] \\ X' \arrow[r, "g"] & X \end{tikzcd} \; ; \; h \text{ PD-mor} \right\}$$

As (U', T', δ') varies this IS a sheaf on Crys(X'/S'), i.e. this is $g^*_{crys}(U, T, \delta)^{\sim}$

Points: given $(U, T, \delta) \in \mathrm{Crys}(X/S)$, $\bar{x} \longrightarrow T$, let

$$\bar{x}^* F \;=\; \operatorname*{colim} F(U_1, T_1, \delta_1)$$

$$\begin{tikzcd}
& \bar{x} \arrow[dl] \arrow[dr] & \\
T_1 \arrow[rr, "\text{et}"'] & & T \\
U_1 \arrow[u, hook] \arrow[rr] \arrow[d, hook] & & U \arrow[u, hook] \arrow[d, hook] \\
X \arrow[rr, equal] & & X
\end{tikzcd}$$

$$\bar{x}_* E \;=\; \mathrm{ass}\left((U_1, T_1, \delta_1) \longmapsto \begin{cases} E, & \text{if } \exists \\ \emptyset, & \text{else} \end{cases} \right)$$

$$\begin{tikzcd}
& \bar{x} \arrow[dl] \arrow[dr] & \\
T_1 \arrow[rr] & & T \\
U_1 \arrow[u, hook] \arrow[rr] \arrow[d, hook] & & U \arrow[u, hook] \arrow[d, hook] \\
X \arrow[rr, equal] & & X
\end{tikzcd}$$

This gives enough points. $\lrcorner$

Let $J \xrightarrow{F} (X/S)^\sim$ be a __finite__ diagram.
Must show

$$g^*_{crys}\left(\lim_J F_j\right) \xrightarrow{\sim} \lim_J g^*_{crys} F_j$$

Check on stalks.

$$(g^*_{crys} F)_{(U', T', \delta', \bar{x}')} \;=\; (g^{\#} F)_{(U', T', \delta', \bar{x}')}$$

$$= \operatorname*{colim} g^{\#} F(U'', T'', \delta'')$$

$$\begin{tikzcd}
& \bar{x}' \arrow[dl] \arrow[dr] & \\
T'' \arrow[rr, "\text{et}"'] & & T' \\
U'' \arrow[u, hook] \arrow[rr] \arrow[d, hook] & & U' \arrow[u, hook] \arrow[d, hook] \\
X' \arrow[rr, equal] & & X'
\end{tikzcd}$$

$$
= \operatorname*{colim}_{
\begin{tikzcd}[ampersand replacement=\&]
 \& \& \bar{x} \arrow[dl] \arrow[dr] \& \\
T \& T'' \arrow[l] \arrow[rr, "et"'] \& \& T' \\
U \arrow[u, "\iota"] \arrow[d] \& U'' \arrow[l] \arrow[rr] \arrow[u, "\iota"] \arrow[d] \& \& U' \arrow[u, "\iota"] \arrow[d] \\
X \& X' \arrow[l, "g"'] \arrow[rr, equal] \& \& X'
\end{tikzcd}} F(U, T, \delta)
$$

Enough to show indexing category $\mathcal{E}$ is filtered:
(finite limits and filtered colimits commute)

(i) $\mathcal{E}$ is non-empty

(ii) given $a, b \in \operatorname{ob} \mathcal{E}$, exists

$$
\begin{tikzcd}
a \arrow[dr] & \\
 & c \\
b \arrow[ur] &
\end{tikzcd}
$$

(iii) given $a \overset{u}{\underset{v}{\rightrightarrows}} b$ in $\mathcal{E}$, exists

$$
a \overset{u}{\underset{v}{\rightrightarrows}} b \xrightarrow{w} c, \quad wu = wv. \quad \checkmark
$$

Show (ii): remember $\mathrm{Crys}(X/S)$ has finite limits.
given two objects of $\mathcal{E}$:

$$
\begin{tikzcd}
 & & \bar{x}' \arrow[dl] \arrow[d] \arrow[dr] & & \\
T_1 & T_1'' \arrow[l] \arrow[r, "et"'] & T' & T_2'' \arrow[l, "et"] \arrow[r] & T_2 \\
U_1 \arrow[u, "\iota"] \arrow[d] & U_1'' \arrow[l] \arrow[r] \arrow[u, "\iota"] \arrow[d] & U' \arrow[u, "\iota"] \arrow[d] & U_2'' \arrow[l] \arrow[r] \arrow[u, "\iota"] \arrow[d] & U_2 \arrow[u, "\iota"] \arrow[d] \\
X & X' \arrow[l, "g"'] \arrow[r, equal] & X' \arrow[r, equal] & X' \arrow[r, "g"] & X
\end{tikzcd}
$$

to get $(\bar{u})$ we take

$$\begin{array}{ccc}
(U,T,\delta) := & (U_1,T_1,\delta_1) & \times\ (U_2,T_2,\delta_2) \\
\uparrow & & \uparrow \\
(U'',T'',\delta'') := & (U_1'',T_1'',\delta_1'') & \times\ (U_2'',T_2'',\delta_2'') \\
\downarrow & & \downarrow \\
(U',T',\delta') & = & (U',T',\delta')
\end{array}$$

Only need to check: $T'' \longrightarrow T_i''$ étale --- //

<u>Lemma</u> A sheaf $F \in (X/S)_{crys}$ is equivalent to:

(i) for every object $(U,T,\delta) \in Crys(X/S)$, a sheaf $F_T \in T_{et}^{\sim}$;

(ii) for every mor. $(U,T,\delta) \xrightarrow{u} (U',T',\delta')$ in $Crys(X/S)$, a mor. in $T_{et}^{\sim}$

$$\rho_u : u^{-1}F_{T'} \longrightarrow F_T$$

s.t. for $(U,T,\delta) \xrightarrow{u} (U',T',\delta') \xrightarrow{v} (U'',T'',\delta'')$,

$$\begin{array}{ccccc}
v^{-1}u^{-1}F_{T''} & \xrightarrow{v^{-1}\rho_u} & v^{-1}F_{T'} & \xrightarrow{\rho_v} & F_T \\
\| & & & & \| \\
(uv)^{-1}F_{T''} & & \xrightarrow{\rho_{uv}} & & F_T
\end{array}$$

and s.t. if $T \xrightarrow{u} T'$ is étale,

$$\rho_u : u^{-1}F_{T'} \xrightarrow{\sim} F_T . \qquad //$$

example Structure sheaf $\mathcal{O}_{X/S}$, $(\mathcal{O}_{X/S})_T = \mathcal{O}_T$, i.e.

$$\mathcal{O}_{X/S}(U,T,\delta) := \Gamma(T,\mathcal{O}_T). \quad //$$

Terminal object e of $(X/S)_{crys}$:

$$e = \mathrm{ass}\,((U,T,\delta) \longmapsto \{0\})$$

global sections of $F \in (X/S)_{crys}$

$$\Gamma(X/S, F) := (X/S)_{crys}(e, F)$$

$$= \lim_{(U,T,\delta) \in \mathrm{Crys}(X/S)} F(U,T,\delta).$$

Cohomology:

$$H^*(X/S, \mathcal{O}_{X/S}) = R^*\Gamma(X/S, \mathcal{O}_{X/S})$$

$X \xrightarrow{f} S$ fixed

$(\mathcal{P}_{X/S} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} \mathcal{O}_X \ , \mathcal{O}_X)$ Hopf algebroid / $f^{-1}\mathcal{O}_S$:

$$\mathcal{O}_X \overset{\eta_L}{\underset{\eta_R}{\rightrightarrows}} \mathcal{P}_{X/S} \qquad \begin{array}{l} a \longmapsto a\otimes 1 \\ a \longmapsto 1\otimes a \end{array}$$

$$\mathcal{P}_{X/S} \xrightarrow{\delta} \mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S} \qquad a\otimes b \longmapsto a\otimes 1 \otimes_{\mathcal{O}_X} 1 \otimes b$$

$$\mathcal{P}_{X/S} \xrightarrow{\varepsilon} \mathcal{O}_X \qquad a\otimes b \longmapsto ab$$

$$\mathcal{P}_{X/S} \xrightarrow{\tau} \mathcal{P}_{X/S} \qquad a\otimes b \longmapsto b\otimes a.$$

represents groupoid.

F, G $\mathcal{O}_X$-modules

$F \xrightarrow{h} G$ $f^{-1}\mathcal{O}_S$-linear $\Longleftrightarrow$

$$\mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} F = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} F \xrightarrow{\bar{h}} G \quad \mathcal{O}_X\text{-linear}$$

$$\bar{h}(a\otimes b \otimes_{\mathcal{O}_X} x) = a h(bx).$$

h $\mathcal{O}_X$-linear $\Longleftrightarrow$ $\bar{h}$ annihilates $I \otimes_{\mathcal{O}_X} F$.

Def $\mathrm{Diff}^n_{X/S}(F,G) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} F, G)$.

$$\mathcal{P}^n_{X/S} = \mathcal{P}_{X/S} / I^{n+1} \quad ; \quad I = \ker(\varepsilon).$$

Note:

$$\begin{tikzcd} \mathcal{P}_{X/S} \arrow[r, "\delta"] \arrow[d] & \mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S} \arrow[d] \\ \mathcal{P}^{m+n}_{X/S} \arrow[r, dashed, "\delta^{m,n}"] & \mathcal{P}^m_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S} \end{tikzcd}$$

Composition:

$$\begin{array}{ccccc} \mathrm{Diff}^m_{X/S}(E,F) & \otimes & \mathrm{Diff}^n_{X/S}(F,G) & \longrightarrow & \mathrm{Diff}^{m+n}_{X/S}(E,G) \\ \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\ f & \otimes & g & \longmapsto & f \circ g \end{array}$$

$$\mathcal{P}^{m+n}_{X/S} \otimes_{\mathcal{O}_X} E \xrightarrow{\delta \otimes E} \mathcal{P}^m_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} E \xrightarrow{\mathcal{P} \otimes g} \mathcal{P}^m_{X/S} \otimes F \xrightarrow{f} G.$$

Prop Suppose $X \to S$ is smooth, let $x_1, \dots, x_d$ be local sections which defines $X \to \mathbb{A}^d_S$ étale. Let $\xi_i = 1 \otimes x_i - x_i \otimes 1$. Then

(i) $\mathcal{P}^n_{X/S}$ is a free $\mathcal{O}_X$-module with basis $\xi^\alpha := \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d}$, $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_1 + \dots + \alpha_d \leq n$;

(ii) if D_α, $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_1 + \dots + \alpha_d \leq n$, is the dual basis of $\mathrm{Diff}^n_{X/S}(\mathcal{O}_X, \mathcal{O}_X)$,

$$D_\alpha \circ D_{\alpha'} = \binom{\alpha + \alpha'}{\alpha} D_{\alpha+\alpha'},$$

$$\alpha + \alpha' = (\alpha_1 + \alpha_1', \dots, \alpha_d + \alpha_d') ; \quad \binom{\beta}{\alpha} = \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_d}{\alpha_d}.$$

Pf (sketch) $X \xrightarrow{\Delta} X \times_S X$ is a locally closed imm. of sm. S-schemes, hence a regular imm. $\Rightarrow$

$$\begin{array}{ccc} S_{\mathcal{O}_X}(I/I^2) & \xrightarrow{\sim} & gr_I^{\cdot}(\mathcal{P}_{X/S}) \\ \| & & \\ \Omega^1_{X/S} & & \end{array} \qquad //$$

—

Def A ***stratification*** on an $\mathcal{O}_X$-module E consists of compatible $\mathcal{P}^n_{X/S}$-linear isomorphisms:

$$\mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} E \underset{\sim}{\xrightarrow{\varepsilon_n}} E \otimes \mathcal{P}^n_{X/S}$$

s.t. ε_0 is the canonical iso. and s.t. $\forall m, n \geq 0$:

$$\begin{tikzcd}[column sep=small]
\mathcal{P}^m_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} E \arrow[rr, "\delta^{m,n*}(\varepsilon_{m+n})"] \arrow[dr, "\mathcal{P} \otimes \varepsilon_n"'] & & E \otimes_{\mathcal{O}_X} \mathcal{P}^m_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S} \\
& \mathcal{P}^m_{X/S} \otimes_{\mathcal{O}_X} E \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S} \arrow[ur, "\varepsilon_m \otimes \mathcal{P}"'] &
\end{tikzcd}$$

commutes.

Lemma The following are equivalent:

(i) a $\mathcal{P}^1_{X/S}$-linear iso. $\mathcal{P}^1_{X/S} \otimes_{\mathcal{O}_X} E \underset{\sim}{\xrightarrow{\varepsilon_1}} E \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X/S}$ which reduces to the identity mod $\Omega^1_{X/S}$;

(ii) a connection $\nabla : E \to E \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$.

pf $\quad \mathbb{1} \otimes x \xmapsto{\varepsilon_1} x \otimes \mathbb{1} + \nabla(x); \qquad \mathbb{1} = 1 \otimes 1$

$$\nabla(ax) = \varepsilon_1(\mathbb{1} \otimes ax) - ax \otimes \mathbb{1} = \varepsilon_1(1 \otimes a \otimes 1 \otimes x) - x \otimes a \otimes 1.$$

$$= (1 \otimes a)\varepsilon_1(\mathbb{1} \otimes x) - (a \otimes 1)(x \otimes \mathbb{1})$$

$$= (1 \otimes a)\varepsilon_1(\mathbb{1} \otimes x) - (1 \otimes a)(x \otimes \mathbb{1}) + (1 \otimes a)(x \otimes \mathbb{1}) - (a \otimes 1)(x \otimes \mathbb{1})$$

$$= (1 \otimes a)\nabla(x) + da . (x \otimes \mathbb{1})$$

$$= a\nabla(x) + x \otimes da. \qquad //$$

Prop Suppose $X \to S$ is smooth and E an $\mathcal{O}_X$-module. The following are equivalent:

(i) a stratification on E;

(ii) a map of pro-$\mathcal{O}_X$-algebras

$$\mathrm{Diff}^{\cdot}_{X/S}(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \mathrm{Diff}^{\cdot}_{X/S}(E, E);$$

(iii) a crystal on $\mathrm{Inf}(X/S)$, i.e. for every nilpotent thickening $X \xleftarrow{\text{ét}} U \hookrightarrow T$, an $\mathcal{O}_T$-module E_T and every $(U', T') \xrightarrow{u} (U, T)$ an isomorphism

$$\rho_u : u^* E_T = \mathcal{O}_{T'} \otimes_{u^{-1}\mathcal{O}_T} u^{-1} E_T \xrightarrow{\sim} E_{T'}$$

s.t. for every two maps

$$(U'', T'') \xrightarrow{v} (U', T') \xrightarrow{u} (U, T)$$

the following diagram commutes:

$$\begin{tikzcd} v^* u^* E_T \arrow[r, "v^*\rho_u"] \arrow[d, "\cong"] & v^* E_{T'} \arrow[d, "\rho_v"] \\ (uv)^* E_T \arrow[r, "\rho_{uv}"] & E_{T''} \end{tikzcd}.$$

Pf (i) => (iii) for (U,T) small enough, we can choose a lifting

$$\begin{tikzcd} U \arrow[r] \arrow[d, "I"'] & X \arrow[d, "sm."] \\ T \arrow[r] \arrow[ur, "g"] & S \end{tikzcd};$$

$$E_T := g^* E$$

Given $(U',T') \xrightarrow{u} (U,T)$ and lifts g', g, ug will be another lift of $U' \to X$.

$$\begin{tikzcd} T' \arrow[r, "(g', ug)"] \arrow[dr, "\bar{g}"'] & X \times_S X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \\ & P^n_{X/S} \arrow[u] & \end{tikzcd}$$

(if $(I')^n = 0$)

$$\begin{array}{ccc} E_{T'} & = & \bar{g}^* p_1^* E \\ \uparrow \sim & & \uparrow \bar{g}^*(\varepsilon_n) \\ u^* E_T & = & \bar{g}^* p_2^* E \end{array}$$

The cocycle condition allows us to glue the E_{T_α} in a covering $\{T_\alpha \to T\}$ and shows the diagram in (iii) commutes.

(iii) $\Rightarrow$ (i) $E = E_X$; to get ε_n use:

$$\begin{tikzcd} P^n_{X/S} \arrow[r, "\tau"] \arrow[rr, bend left, "p_1"] & P^n_{X/S} \arrow[r, "p_2"] & X \\ X \arrow[u, hook] \arrow[r, equal] & X \arrow[u, hook] \arrow[r, equal] & X \arrow[u, equal] \end{tikzcd}$$

$$\varepsilon_n := \rho_\tau : p_1^* E_X \xrightarrow{\sim} p_2^* E_X$$

$$\begin{array}{ccc} p_1^* E_X & \xrightarrow{\sim} & p_2^* E_X \\ \| & & \| \\ \mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} E & & E \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S} \end{array}$$

To get cocycle condition look at the three projections $p_{12}, p_{13}, p_{23} : P^n_{X/S}(2) \to P^n_{X/S}(1)$.

(i) $\Rightarrow$ (ii) $\mathcal{P}^n_{X/S} \to \mathcal{O}_X$ $\mathcal{O}_X$ - linear $\Rightarrow$

$$\begin{tikzcd} E \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S} \arrow[r] & E \quad \mathcal{O}_X\text{- linear} \\ \mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} E \arrow[u, "\varepsilon_n"] \arrow[ur] & \end{tikzcd}$$

(ii) $\Rightarrow$ (i) see handout.

Prop If X/S is smooth, the following are eq:

(i) a stratified $\mathcal{O}_X$-module E;

(ii) a crystal of $\mathcal{O}_{X/S}$-modules on $\mathrm{Inf}(X/S)$;

(iii) an $\mathcal{O}_X$-module E and a ring hom:

$$\mathrm{Diff}_{X/S}(\mathcal{O}_X, \mathcal{O}_X) \xrightarrow{\nabla} \mathrm{Diff}_{X/S}(E, E).$$

pf (sketch) (i) $\Leftrightarrow$ (ii): proved last time.

(i) $\Rightarrow$ (iii): given $D: \mathcal{P}^n_{X/S} \to \mathcal{O}_X$, $\mathcal{O}_X$-linear, define $\nabla(D)$ by

$$\mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} E \underset{\sim}{\xrightarrow{\varepsilon_n}} E \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S} \xrightarrow{E \otimes D} E \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong E.$$

(iii) $\Rightarrow$ (i): given ∇ define ε_n by:

$$\begin{array}{ccc} \mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} E & \xrightarrow{\widetilde{ev}} & \mathrm{Hom}_{\mathcal{O}_X}(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} E, E), E) \\ \downarrow \varepsilon_n & & \downarrow \mathrm{Hom}(\nabla, E) \\ E \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S} & \xrightarrow{\widetilde{E \otimes ev}} & \mathrm{Hom}_{\mathcal{O}_X}(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}^n_{X/S}, \mathcal{O}_X), E) \end{array},$$

where the lower horizontal map is an iso. since $\mathcal{P}^n_{X/S}$ is locally free. Check: ε_n $\mathcal{P}^n_{X/S}$-lin. //

$$E \xrightarrow{\nabla} E \otimes_{\mathcal{O}_X} \Omega^1_{X/S} \rightsquigarrow E \otimes_{\mathcal{O}_X} \Omega^k_{X/S} \xrightarrow{\nabla^k} E \otimes_{\mathcal{O}_X} \Omega^{k+1}_{X/S}$$

$$x \otimes \omega \longmapsto \nabla(x) \wedge \omega + x \otimes d\omega$$

<u>Lemma</u> $K = \nabla^1 \circ \nabla : E \to E \otimes_{\mathcal{O}_X} \Omega^2_{X/S}$ $\mathcal{O}_X$-linear.

<u>Pf</u> $\nabla^1(a \cdot x \otimes \omega) = \nabla^1(ax \otimes \omega) := \nabla(ax) \wedge \omega + ax \otimes d\omega$

$= a\nabla(x) \wedge \omega + (x \otimes da) \wedge \omega + ax \otimes d\omega$

$= a\nabla^1(x \otimes \omega) - (x \otimes \omega) \wedge da$

$\nabla^1 \circ \nabla(ax) = \nabla^1(a\nabla(x) + x \otimes da)$

$= a\nabla^1(\nabla(x)) - \nabla(x) \wedge da + \nabla(x) \wedge da + x \otimes d^2 a$

$= a\nabla^1(\nabla(x))$. //

<u>exercise</u> $\nabla^{k+1} \circ \nabla^k (x \otimes \omega) = K(x) \wedge \omega$. //

<u>Def</u> ∇ is <u>integrable</u> (or <u>flat</u>) if $K = 0$.

<u>Lemma</u> Suppose X/S smooth and let $x_1, \dots, x_d$ be local sections of $\mathcal{O}_X$ which define $X \to \mathbb{A}^d_S$ étale. Let E be an $\mathcal{O}_X$-module with conn. ∇ and define θ_i by

$$\mathrm{Diff}^1_{X/S}(\mathcal{O}_X, \mathcal{O}_X) \xrightarrow{\nabla} \mathrm{Diff}^1_{X/S}(E, E)$$

$$\partial/\partial x_i \longmapsto \theta_i .$$

Then $K = \sum_{1 \le i < j \le d} [\theta_i, \theta_j] \otimes (dx_i \wedge dx_j)$.

Pf Evaluate $\theta_i = \nabla(\partial/\partial x_i)$. As an $\mathcal{O}_X$-module

$$\mathcal{P}^1_{X/S} = \mathcal{O}_X\{1\} \oplus \mathcal{O}_X\{dx_1, \dots, dx_d\}$$

and $\partial/\partial x_i$ is the element the dual basis of $\mathrm{Diff}^1_{X/S}(\mathcal{O}_X, \mathcal{O}_X)$ which corresponds to dx_i, i.e.

$$\partial/\partial x_i (dx_j) = \delta_{ij} .$$

By definition, θ_i is the composite

$$\mathcal{P}^1_{X/S} \underset{\mathcal{O}_X}{\otimes} E \underset{\sim}{\xrightarrow{\varepsilon}} E \underset{\mathcal{O}_X}{\otimes} \mathcal{P}^1_{X/S} \xrightarrow{1 \otimes \partial/\partial x_i} E \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X \cong E$$

$$1 \otimes x \longmapsto x \otimes 1 + \nabla(x)$$

i.e. $\nabla(x) = \sum_{i=1}^{d} \theta_i(x) \otimes dx_i$.

$$K(x) = \nabla^1(\nabla(x)) = \sum_{i=1}^{d} \nabla^1(\theta_i(x) \otimes dx_i)$$

$$= \sum_{i=1}^{d} \nabla(\theta_i(x)) \wedge dx_i$$

$$= \sum_{i=1}^{d} \sum_{j=1}^{d} (\theta_j(\theta_i(x)) \otimes dx_j) \wedge dx_i$$

$$= \sum_{1 \le i < j \le d} [\theta_i, \theta_j](x) \otimes (dx_i \wedge dx_j). \quad //$$

Cor $K = 0 \iff [\theta_i, \theta_j] = 0, \ 1 \le i < j \le d.$ //

Prop Let S be defined over $\mathbb{Q}$ and suppose X/S is smooth. Then the following are equivalent for an $\mathcal{O}_X$-module E:

(i) a stratification on E;

(ii) an integrable connection on E.

Pf Let $x_1, \ldots, x_d$ be local sections which defines $X \to \mathbb{A}^d_S$ étale, let D_α, $\alpha = (\alpha_1, \ldots, \alpha_d)$, be the corresponding basis of $\mathrm{Diff}_{X/S}(\mathcal{O}_X, \mathcal{O}_X)$. Let $D_i = \partial/\partial x_i$, recall

$$D^\alpha := D_1^{\alpha_1} \cdots D_d^{\alpha_d} = (\alpha_1!) D_{\alpha_1} \cdots (\alpha_d)! D_{\alpha_d}$$

$$= \alpha! D_\alpha \ ; \quad \alpha! := (\alpha_1)! \cdots (\alpha_d)!.$$

So $D_\alpha = \frac{1}{\alpha!} D^\alpha$, i.e. $\mathrm{Diff}_{X/S}(\mathcal{O}_X,\mathcal{O}_X)$ is gen. as an $\mathcal{O}_X$-algebra by $D_1, \dots, D_\ell$. We wish to extend

$$\begin{tikzcd}
\mathrm{Diff}^1_{X/S}(\mathcal{O}_X,\mathcal{O}_X) \arrow[r] \arrow[d, hook] & \mathrm{Diff}^1_{X/S}(E,E) \arrow[d, hook] \\
\mathrm{Diff}_{X/S}(\mathcal{O}_X,\mathcal{O}_X) \arrow[r, dashed] & \mathrm{Diff}_{X/S}(E,E).
\end{tikzcd}$$

$$\partial/\partial x_i \longmapsto \theta_i$$

We can do this if and only if the θ_i's commute in $\mathrm{Diff}_{X/S}(E,E)$, i.e. iff $K=0$. //

(S, I, γ), $m\mathcal{O}_S = 0$, some $m \in \mathbb{N}$.

X/S, γ extends to X.

$$D_{X/S}(\nu) := D_X(\underbrace{X \times_S \cdots \times_S X}_{\nu}) \; ; \quad \mathcal{D}_{X/S}(\nu) = \mathcal{O}_{D_{X/S}(\nu)}$$

$$D^n_{X/S}(\nu) := D^n_{X/S}(X \times_S \cdots \times_S X) \; ; \quad \mathcal{D}^n_{X/S}(\nu) = \mathcal{D}_{X/S}(\nu) / \bar{J}^{[n]} .$$

$(\mathcal{D}_{X/S}(1), \mathcal{O}_X)$ Hopf algebroid / $f^{-1}\mathcal{O}_S$.

$$\mathcal{O}_X \underset{\eta_L}{\overset{\eta_R}{\rightrightarrows}} P_{X/S} \to \mathcal{D}_{X/S}(1)$$

$$\begin{tikzcd}
\mathcal{D}_{X/S}(1) \arrow[r, "\delta"] & \mathcal{D}_{X/S}(1) \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}(1) \\
P_{X/S} \arrow[u] \arrow[r, "\delta"] & P_{X/S} \otimes_{\mathcal{O}_X} P_{X/S} \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{D}_{X/S}(1) \arrow[rr, "\varepsilon"] & & \mathcal{O}_X \\
& P_{X/S} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{D}_{X/S}(1) \arrow[r, "\tau"] & \mathcal{D}_{X/S}(1) \\
P_{X/S} \arrow[u] \arrow[r] & P_{X/S} \arrow[u]
\end{tikzcd}$$

Def $\text{PD-Diff}^n_{X/S}(E,F) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{D}^n_{X/S}(1) \otimes_{\mathcal{O}_X} E, F)$;

$\text{HPD-Diff}_{X/S}(E,F) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{D}_{X/S}(1) \otimes_{\mathcal{O}_X} E, F)$. //

Prop Suppose X/S is smooth, let $x_1, \dots, x_d$ be local sections which defines $X \to \mathbb{A}^d_S$ étale. Let $\xi_i :=$ $1 \otimes x_i - x_i \otimes 1$. Then

(i) $\mathcal{D}_{X/S}(1) = \mathcal{O}_X \langle \xi_1, \dots, \xi_d \rangle$;

(ii) if D_α is the dual $\xi^{[\alpha]} := \xi^{[\alpha_1]} \cdots \xi^{[\alpha_d]}$, then $D_\alpha \circ D_{\alpha'} = D_{\alpha+\alpha'}$.

Pf We have

$$\Gamma_{\mathcal{O}_X}(I/I^2) \xrightarrow{\sim} gr^{\cdot}_{\mathfrak{J}}(\mathcal{D}_{X/S}(1)) ,$$

and since $\mathfrak{J} \subset \mathcal{D}_{X/S}(1)$ is a nil-ideal, we get (i). By def^{n}, $(D_\alpha \circ D_{\alpha'})(\xi^{[k]})$ is the image of $\xi^{[k]}$ under the composition

$$\begin{array}{ccccccc} \mathcal{D}_{X/S}(1) & \xrightarrow{\delta} & \mathcal{D}_{X/S}(1) \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}(1) & \xrightarrow{\mathcal{D} \otimes D_{\alpha'}} & \mathcal{D}_{X/S}(1) & \xrightarrow{D_\alpha} & \mathcal{O}_X \\ \xi^{[k]} & \longmapsto & \sum_{i+j=k} \xi^{[i]} \otimes \xi^{[j]} & \longmapsto & \xi^{[k-\alpha']} & \longmapsto & \delta_{\alpha,\alpha'} . \end{array}$$

//

<u>Lemma</u> Let A be a ring and (B, I, γ), (C, J, δ) two A-PD-algebras s.t. the projections $B \to B/I$ and $C \to C/J$ have a section. Then the ideal $I \otimes_A C + B \otimes_A J \subset B \otimes_A C$ has a unique PD-str. ε s.t. the maps

$$(B, I, \delta) \to (B \otimes_A C, I \otimes_A C + B \otimes_A J, \varepsilon) \leftarrow (C, J, \delta)$$

are PD-morphisms. //

<u>Cor</u> If $m\mathcal{O}_S = 0$, then the can. map is an iso:

$$\mathcal{D}_{X/S}(\nu) \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}(\nu') \xrightarrow{\sim} \mathcal{D}_{X/S}(\nu+\nu').$$

<u>Pf</u> We need $m\mathcal{O}_S = 0$ in order to define $\mathcal{D}_{X/S}(\nu)$. The various projections define

$$\begin{array}{ccc} \mathcal{D}_{X/S}(\nu) \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}(\nu') & \leftarrow & \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} \cdots \otimes_{f^{-1}\mathcal{O}_S} \mathcal{O}_X \\ & \searrow \quad \swarrow & \\ & \mathcal{O}_X & \end{array}$$

and since $\bar{J} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}(\nu') + \mathcal{D}_{X/S}(\nu) \otimes \bar{J}$ has a PD-str., we get

$$\mathcal{D}_{X/S}(\nu) \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}(\nu') \leftarrow \mathcal{D}_{X/S}(\nu+\nu').$$

//

Def A Hopf algebroid (in a topos T) consists of two rings A, D (in T) and five ring homomorphisms

$$p_1, p_2 : A \longrightarrow D$$

$$\Delta : D \longrightarrow A$$

$$p_{13} : D \longrightarrow D \underset{p_2 \nwarrow A \nearrow p_1}{\otimes} D$$

$$\tau : D \longrightarrow D$$

s.t.

(1) $\Delta \circ p_1 = id_A = \Delta \circ p_2$

(2) $p_{13} \circ p_1 = p_{12} \circ p_1$, $p_{13} \circ p_2 = p_{23} \circ p_2$

$\left(p_{12} = D \otimes p_1 : A \otimes_A D \to D \otimes_A D ;\ p_{23} = p_2 \otimes D : D \otimes_A A \longrightarrow D \otimes_A D \right)$

(3) $(\Delta \otimes D) \circ p_{13} = (D \otimes \Delta) \circ p_{13} = id_B$

(4) $(p_{13} \otimes D) \circ p_{13} = (D \otimes p_{13}) \circ p_{13}$

(5) $\tau \circ \tau = id_D$

(6) $\tau \circ p_1 = p_2$, $\tau \circ p_2 = p_1$

(7) $\Delta \circ \tau = \tau$

(8) the following diagrams commute:

$$\begin{array}{ccc} D\otimes_A D & \xrightarrow{D\otimes\tau} & D \\ \uparrow p_{13} & & \uparrow p_1 \\ D & \xrightarrow{\Delta} & A \end{array} \qquad \begin{array}{ccc} D\otimes_A D & \xrightarrow{\tau\otimes D} & D \\ \uparrow p_{13} & & \uparrow p_2 \\ D & \xrightarrow{\Delta} & A \end{array} \quad . \quad //$$

ex (i) $A = \mathcal{O}_X$, $D = \mathcal{P}_{X/S} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} \mathcal{O}_X$,

$p_1, p_2 : \mathcal{O}_X \to \mathcal{P}_{X/S}$, $\quad a \mapsto a\otimes 1,\ 1\otimes a$;

$\Delta \ : \mathcal{P}_{X/S} \to \mathcal{O}_X$, $\quad a\otimes b \mapsto ab$;

$p_{13} \ : \mathcal{P}_{X/S} \longrightarrow \mathcal{P}_{X/S} \otimes_{\mathcal{O}} \mathcal{P}_{X/S}$, $\quad a\otimes b \mapsto a\otimes 1 \otimes_{\mathcal{O}_X} 1\otimes b$;

$\tau \ : \mathcal{P}_{X/S} \to \mathcal{P}_{X/S}$, $\quad a\otimes b \mapsto b\otimes a$.

(ii) (S, I, γ), $m\mathcal{O}_S = 0$, X/S, γ extends to X:

$A = \mathcal{O}_X$, $D = \mathcal{D}_{X/S} = \mathcal{D}_{\mathcal{P}_{X/S},\gamma}(J)$; the maps above trivially extend, except

$$\begin{array}{ccc} \mathcal{D}_{X/S} & \overset{p_{13}}{\dashrightarrow} & \mathcal{D}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S} \\ \uparrow & & \uparrow \\ \mathcal{P}_{X/S} & \xrightarrow{p_{13}} & \mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S} \end{array}$$

To verify axioms use iso

$$\mathcal{D}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S} \xrightarrow{\sim} \mathcal{D}_{X/S}(2).$$

For example, axiom (2):

$$\begin{tikzcd}
\mathcal{D}_{X/S} \arrow[r, "p_{13}"] & \mathcal{D}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S} \arrow[r, "\sim"] & \mathcal{D}_{X/S}^{(2)} \\
\mathcal{O}_X \arrow[u, "p_1"] \arrow[r, "p_1"] & \mathcal{D}_{X/S} \arrow[u, "p_{12}"] &
\end{tikzcd}$$

follows from the commutativity of

$$\begin{tikzcd}
X \times_S X \arrow[d, "p_1"] & X \times_S X \times_S X \arrow[l, "p_{13}"'] \arrow[d, "p_{12}"] \\
X & X \times_S X \arrow[l, "p_1"'] .
\end{tikzcd}$$

//

Def A comodule over a Hopf algebroid (A, D) is an A-module E and an additive map

$$E \xrightarrow{\theta} E \otimes_{A, p_1} D = p_1^* E$$

s.t.

(1) θ is p_2-linear, (2)
$$\begin{tikzcd}
E \arrow[r, "\theta"] \arrow[dr, "\sim"'] & E \otimes_A D \arrow[d, "E \otimes \Delta"] \\
& E \otimes_A A
\end{tikzcd}$$

(3) the following diagram commutes

$$\begin{tikzcd}
E \arrow[r, "\theta"] \arrow[d, "\theta"] & E \otimes_A D \arrow[d, "\theta \otimes D"] \\
E \otimes_A D \arrow[r, "E \otimes p_{13}"] & E \otimes_A D \otimes_A D .
\end{tikzcd}$$

Def A stratification on an A-module E relative to the Hopf algebroid is a map of D-modules

$$p_2^* E \xrightarrow{\varepsilon} p_1^* E$$

s.t. the following diagrams commute:

(1)
$$\begin{tikzcd} \Delta^* p_2^* E \arrow[rr, "\Delta^*\varepsilon"] & & \Delta^* p_1^* E \\ & E \arrow[ul, "\sim"] \arrow[ur, "\sim"'] & \end{tikzcd}$$

(2)
$$\begin{tikzcd} p_{13}^* p_2^* E \arrow[rr, "p_{13}^*\varepsilon"] \arrow[d, equal] & & p_{13}^* p_1^* E \arrow[d, equal] \\ p_{23}^* p_2^* E \arrow[dr, "p_{23}^*\varepsilon"] & & p_{12}^* p_1^* E \\ & p_{23}^* p_1^* E = p_{12}^* p_2^* E \arrow[ur, "p_{12}^*\varepsilon"'] & \end{tikzcd}$$

An A-linear map $f: E \to F$ between two stratified A-modules is horizontal if the diagram

$$\begin{tikzcd} p_2^* E \arrow[r, "\varepsilon"] \arrow[d, "p_2^* f"] & p_1^* E \arrow[d, "p_1^* f"] \\ p_2^* F \arrow[r, "\varepsilon"] & p_1^* F \end{tikzcd}$$

commutes. //

Lemma ε is an isomorphism.

pf The diagram (2) in the definition of a stratification gives the following diagram:

$$\begin{tikzcd}[column sep=small]
 & p_2^* E \arrow[rr] \arrow[d, equal] & & p_2^* E \arrow[d, equal] & \\
 & (\tau\otimes D)^* p_{13}^* p_2^* E \arrow[rr] \arrow[d, equal] & & (\tau\otimes D)^* p_{13}^* p_1^* E \arrow[d, equal] & \\
p_2^* E = (\tau\otimes D)^* p_{23}^* p_2^* E \arrow[dr, "\varepsilon"'] & & & (\tau\otimes D)^* p_{12}^* p_1^* E = p_2^* E & \\
 & p_1^* E = (\tau\otimes D)^* p_{23}^* p_1^* E \arrow[rr, equal] & & (\tau\otimes D)^* p_{12}^* p_2^* E = p_1^* E \arrow[u, "\tau^*\varepsilon"'] &
\end{tikzcd}$$

The top arrow is

$$(\tau\otimes D)^* p_{13}^* \varepsilon = p_2^* \Delta^* \varepsilon = p_2^* id_E = id_{p_2^* E}$$

so $\tau^*\varepsilon$ is a left inverse of ε. Using $D\otimes\tau$ one sees $\tau^*\varepsilon$ is a right inverse. //

$$
\begin{tikzcd}
A \arrow[r, "p_1"] \arrow[d, "p_2"] & D \arrow[d, "p_{23}"] \\
D \arrow[r, "p_{12}"] & D \otimes_A D
\end{tikzcd}
$$

cocartesian $\Rightarrow$

$$p_{23*}\, p_{12}^* = p_1^*\, p_{2*}$$

<u>Def</u> A <u>comodule</u> over (A, D) is an A-module E together —

$$E \xrightarrow{G} p_{2*} p_1^* E$$

s.t. the following diagr. of A-modules commute:

(1)
$$
\begin{tikzcd}
E \arrow[r, "G"] \arrow[dr, "\sim"'] & p_{2*} p_1^* E \arrow[d] \\
& p_{2*} \Delta_* \Delta^* p_1^* E
\end{tikzcd}
$$

(2)
$$
\begin{tikzcd}
E \arrow[r, "G"] \arrow[d, "G"] & p_{2*} p_1^* E \arrow[d, "p_{2*} p_1^* G"] \\
p_{2*} p_1^* E \arrow[d, "\mathrm{can}"] & p_{2*} p_1^* p_{2*} p_1^* E \arrow[d, equal] \\
p_{2*} p_{13*} p_{13}^* p_1^* E \arrow[r, equal] & p_{2*} p_{23*} p_{12}^* p_1^* E
\end{tikzcd}
$$

$$p_2^* E \xrightarrow{\varepsilon} p_1^* E \quad \rightsquigarrow \quad E \xrightarrow{\theta} p_{2*} p_1^* E$$

cocycle cond. $\rightsquigarrow$ coassociativity :

$$\begin{tikzcd}[column sep=small]
 & & E \arrow[dll] \arrow[drr] \arrow[ddll, bend right=60, "\theta"'] \arrow[ddrr, bend left=60, "\theta"] & & \\
p_{2*}p_2^*E \arrow[r] \arrow[d] & p_{2*}p_{23*}p_{23}^*p_2^*E \arrow[r, phantom, "="] \arrow[d] & & p_{2*}p_{13*}p_{13}^*p_2^*E \arrow[d] & p_{2*}p_2^*E \arrow[l] \arrow[d] \\
p_{2*}p_1^*E \arrow[r] \arrow[ddrrr, bend right, "p_{2*}p_1^*\theta"'] & p_{2*}p_{23*}p_{23}^*p_1^*E \arrow[d, phantom, "\shortparallel"] & & p_{2*}p_{13*}p_{13}^*p_1^*E \arrow[d, phantom, "\shortparallel"] & p_{2*}p_1^*E \arrow[l] \\
 & p_{2*}p_{23*}p_{12}^*p_2^*E \arrow[rr] & & p_{2*}p_{23*}p_{12}^*p_1^* \arrow[d, phantom, "\shortparallel"] & \\
 & & & p_{2*}p_1^*p_{2*}p_1^*E &
\end{tikzcd}$$

<u>Lemma</u> Let X/S be smooth, $m\mathcal{O}_S = 0$, let $x_1, \dots, x_d$ be local coordinates, and let $\xi_i = 1 \otimes x_i - x_i \otimes 1 \in \mathcal{D}_{X/S}$. Then, locally,

$$\mathcal{O}_X \langle \xi_1, \dots, \xi_d \rangle \xrightarrow{\sim} \mathcal{D}_{X/S}.$$

Pf Let N be s.t. $J^{N+1}\mathcal{D}_{X/S} = 0$. Then, locally,

$$\begin{aligned}
\mathcal{D}_{X/S} := \mathcal{D}_{\mathcal{O}_{X/S},\gamma}(J) &\xrightarrow{\sim} \mathcal{D}_{\mathcal{O}_{X/S}/J^{N+1},\gamma}(J/J^{N+1}) \\
&\xleftarrow{\sim} \mathcal{D}_{\mathcal{O}_X[\xi_1,\dots,\xi_d]/(\xi_1,\dots,\xi_d)^{N+1},\gamma}((\xi_1,\dots,\xi_d)/(\xi_1,\dots,\xi_d)^{N+1}) \\
&\xleftarrow{\sim} \mathcal{D}_{\mathcal{O}_X[\xi_1,\dots,\xi_d],\gamma}((\xi_1,\dots,\xi_d)) \\
&= \mathcal{O}_X\langle \xi_1, \dots, \xi_d \rangle .
\end{aligned}$$

//

<u>exercise</u> Let (A, I, γ) be a PD-ring, let B be an A-algebra, let $J, K \subset B$ be ideals and suppose $K D_{B,\gamma}(J) = 0$. Show $D_{B,\gamma}(J) \xrightarrow{\sim} D_{B/K,\gamma}(J/J\cap K)$. //

Recall curvature formula: let ∇ be a conn. on $\mathcal{O}_X$-module E, let X/S be smooth and let $x_1, \dots, x_d$ be local coord.

$$\begin{tikzcd}
E \arrow[r, "\nabla"] \arrow[rr, bend right, "\nabla_{x_i}"'] & E \otimes_{\mathcal{O}_X} \Omega^1_{X/S} \arrow[r, "E \otimes \partial/\partial x_i"] & E
\end{tikzcd}$$

$$K^{\nabla} = \sum_{1 \le i < j \le d} [\nabla_{x_i}, \nabla_{x_j}] \otimes dx_i \wedge dx_j .$$

Prop Let X/S be smooth and let E be an $\mathcal{O}_X$-module. The following are equivalent:

(i) an $(\mathcal{O}_X, \mathcal{D}_{X/S})$-comodule str. on E;

(ii) an integrable quasi-nilpotent connection on E.

pf Choose local coordinates $x_1, .., x_d$ s.t.

$$\mathcal{D}_{X/S} = \mathcal{O}_X\langle \xi_1, \dots, \xi_d\rangle, \qquad \xi_i = 1\otimes x_i - x_i \otimes 1.$$

The data of (i) and (ii) correspond as follows:

$$E \xrightarrow{\theta} E \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}$$

$$m \longmapsto \sum_q \nabla_q^b(m) \otimes \xi^{[q]}$$

$$\nabla_q^b = \prod_{i=1}^d (\nabla_{x_i})^{q_i}$$

$$\begin{tikzcd} E \arrow[r, "\nabla"] \arrow[rr, bend right, "\nabla_{x_i}"'] & E \otimes_{\mathcal{O}_X} \Omega^1_{X/S} \arrow[r, "E\otimes \partial/\partial x_i"] & E \end{tikzcd}$$

(i) $\Rightarrow$ (ii)

$$\begin{tikzcd} E \arrow[r, "\theta"] \arrow[rr, bend right, "\nabla_q^b"'] & E \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S} \arrow[r, "E\otimes D_q"] & E \end{tikzcd} ; \quad D_q(\xi^{[q']}) = \delta_{q,q'}$$

We show $\nabla_q^b \circ \nabla_{q'}^b = \nabla_{q+q'}^b$;

$$
\begin{tikzcd}
E \arrow[r, "\Theta"] \arrow[d, "\Theta"] \arrow[rr, bend left, "\nabla^b_{q'}"] \arrow[ddrr, bend right, "\nabla^b_{q+q'}"'] & E \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S} \arrow[r, "E \otimes D_{q'}"] \arrow[d, "\Theta \otimes \mathcal{D}"] & E \arrow[d, "\Theta"] \arrow[dd, bend left=60, "\nabla_{q'}"] \\
E \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S} \arrow[r, "E \otimes \delta"] \arrow[drr, "E \otimes D_{q+q'}"] & E \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S} \otimes \mathcal{D}_{X/S} \arrow[r, "E \otimes \mathcal{D} \otimes D_{q'}"] & E \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S} \arrow[d, "E \otimes D_q"] \\
 & & E
\end{tikzcd}
$$

Hence $\nabla^b_q = \prod_{i=1}^{d} (\nabla_{x_i})^{q_i}$. It follows that

(1) $[\nabla_{x_i}, \nabla_{x_j}] = 0 \Rightarrow K^\nabla = 0$

(2) given $U \to X$ open, $m \in \Gamma(U, E)$, there exists $\{U_\alpha \to U\}$ covering, $e_{\alpha,i} \in \mathbb{Z}$ s.t.

$$(\nabla_{x_i})^{e_{\alpha,i}}(m|_{U_\alpha}) = 0,$$

i.e. ∇ is quasi-nilpotent.

(ii) $\Rightarrow$ (i) we define $\nabla^b_q = \prod_{i=1}^{d} (\nabla_{x_i})^{q_i}$, and since ∇ is quasi-nilpotent, we can define Θ by the formula above. Finally

$[\nabla_{x_i}, \nabla_{x_j}] = 0 \Rightarrow \nabla^b_q \circ \nabla^b_{q'} = \nabla^b_{q+q'}$ and the above diagram hence shows that $\mathfrak{G}$ is coassociative. //

Prop Let X/S be smooth. The following categories are equivalent:

(i) stratified $\mathcal{O}_X$-modules and horizontal maps;

(ii) crystals of $\mathcal{O}_{X/S}$-modules.

Pf (i) ⇒ (ii) given (U, T, δ), if a lift exists

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] \arrow[ur, "g"] & S \end{tikzcd}$$

we choose one and define $E_T := g^*E$. Then, given $(U', T', \delta) \xrightarrow{u} (U, T, \delta)$, $g': T' \to X$, $g: T \to X$, get

$$\begin{tikzcd} T' \arrow[r, "(g', gu)"] \arrow[dr, "h"'] & X \times_S X \arrow[r, shift left] \arrow[r, shift right] & X \\ & D_{X/S} \arrow[u] \arrow[ur, shift left, "p_1"] \arrow[ur, shift right, "p_2"'] & \end{tikzcd}$$

$$\begin{array}{ccc} u^* E_T & \xrightarrow[\sim]{\rho_u} & E_{T'} \\ \| & & \| \\ h^* p_2^* E & \xrightarrow[\sim]{h^* \varepsilon} & h^* p_1^* E \end{array}$$

[exercise Show

$$\begin{tikzcd} & v^* u^* E_T \arrow[dl, "v^* \rho_u"'] \arrow[dr, "\rho_{uv}"] & \\ v^* E_{T'} \arrow[rr, "\rho_v"'] & & E_{T''} \end{tikzcd}$$

commutes.]

Given (U, T, δ) s.t. a lift $g : T \to X$ does not exist, choose a covering $\{T_\alpha \xrightarrow{u_\alpha} T\}$ s.t. lifts $g_\alpha : T_\alpha \to X$ exist, for all α. Define

$$E_T \longrightarrow \prod_\alpha u_{\alpha *} E_{T_\alpha} \rightrightarrows \prod_{\alpha, \beta} u_{\alpha\beta} E_{T_{\alpha\beta}} .$$

See also SGA 4, III 4.1.

(ii) $\Rightarrow$ (i) $E := E_X$.

$$\begin{tikzcd}
D_{X/S} \arrow[r, "\tau"] \arrow[rr, bend left, "p_2"] & D_{X/S} \arrow[r, "p_1"] & X \\
X \arrow[u, hook] \arrow[r, equal] & X \arrow[u, hook] \arrow[r, equal] & X \arrow[u, equal]
\end{tikzcd}
\qquad
\begin{tikzcd}
p_2^* E \arrow[r, "\varepsilon"] \arrow[d, equal] & p_1^* E \arrow[d, equal] \\
\tau^* E_{D_{X/S}} \arrow[r, "\rho_\tau"] & E_{D_{X/S}}
\end{tikzcd}$$

exercise Verify the cocycle condition.

$$L(E)_Y = p_{1*} p_2^* E$$

$$L(E)_Y \xrightarrow{\theta} p_{2*} p_1^* L(E)_Y$$ defined as

$$p_{1*} p_2^* E \longrightarrow p_{1*} p_{13*} p_{13}^* p_2^* E$$

$$\parallel$$

$$p_{1*} p_{12*} p_{23}^* p_2^* E$$

$$\parallel$$

$$p_{1*} p_2^* p_{1*} p_2^* E \to p_{1*} \tau_* \tau^* p_2^* p_{1*} p_2^* E$$

$$\parallel$$

$$p_{2*} p_1^* p_{1*} p_2^* E$$

$u \in HPD\text{-}\mathrm{Diff}_{Y/S}(E, F)$, i.e. $p_{1*} p_2^* E \xrightarrow{u} F$

$$L(E)_Y \xrightarrow{L(u)_Y} L(F)_Y$$ symbol:

$$p_{1*} p_2^* E \longrightarrow p_{1*} p_{13*} p_{13}^* p_2^* E$$

$$\parallel$$

$$p_{1*} p_{12*} p_{23}^* p_2^* E$$

$$\parallel$$

$$p_{1*} p_2^* p_{1*} p_2^* E \xrightarrow{p_{1*} p_2^* u} p_{1*} p_2^* F$$

<u>exercise</u> Show the following:

(i) $L(-)_Y$ is a functor;

(ii) $L(u)_Y$ is horizontal. //

 10/27/00

<u>Lemma</u> There is a natural map of complexes of $\mathcal{O}_Y$-modules with an $(\mathcal{O}_Y, \mathcal{D}_{Y/S})$-stratification:

$$\mathcal{O}_Y \longrightarrow L(\Omega^{\cdot}_{Y/S}).$$

($\mathcal{O}_Y$ is given the stratification of the canonical isomorphism $p_2^* \mathcal{O}_Y \xrightarrow{\sim} p_1^* \mathcal{O}_Y$.)

<u>pf</u> The adjoint of the can. iso.

$$p_1^* \mathcal{O}_Y \xrightarrow{\sim} \mathcal{D}_{Y/S} \xleftarrow{\sim} p_2^* \mathcal{O}_Y$$

is a natural map of $\mathcal{O}_Y$-modules

$$\mathcal{O}_Y \longrightarrow p_{1*} p_2^* \mathcal{O}_Y =: L(\mathcal{O}_Y)_Y.$$

[<u>exercise</u> Show that this is horizontal.]

We must show that the composite is zero:

$$\begin{array}{ccccc}
\mathcal{O}_Y & \longrightarrow & L(\mathcal{O}_Y)_Y & \xrightarrow{L(d)_Y} & L(\Omega^1_{Y/S})_Y \\
\| & & \| & & \| \\
\mathcal{O}_Y & \longrightarrow & \mathcal{D}_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y & \longrightarrow & \mathcal{D}_{Y/S} \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S}
\end{array}$$

$$\mathcal{D}_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y \xrightarrow{p_{13} \otimes \mathcal{O}_Y} \mathcal{D}_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y \xrightarrow{\mathcal{D} \otimes \bar{d}} \mathcal{D}_{Y/S} \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S}$$

$$a \longmapsto a \otimes 1 \otimes_{\mathcal{O}_Y} 1 \qquad a \otimes 1 \otimes d(1) = 0$$

$$a \otimes 1 \otimes_{\mathcal{O}_Y} 1 \mapsto a \otimes 1 \otimes_{\mathcal{O}_Y} 1 \otimes_{\mathcal{O}_Y} 1 \mapsto a \otimes 1 \otimes d(1)$$

$$\begin{tikzcd}
\Omega^k_{Y/S} \arrow[rr, "d"] \arrow[dr, "p_2\otimes\Omega"'] & & \Omega^{k+1}_{Y/S} \\
& \mathcal{D}^1_{Y/S}\otimes_{\mathcal{O}_Y}\Omega^k_{Y/S} \arrow[ur, "\bar{d}"'] &
\end{tikzcd}$$

<u>Claim</u> $\bar{d}(a\otimes b\otimes\omega) = a\,db\wedge\omega + ab\,d\omega$

pf $\mathcal{P}_{Y/S}\otimes_{\mathcal{O}_Y}\Omega^k_{Y/S} \xrightarrow{\sim} \mathcal{O}_Y\otimes_{f^{-1}\mathcal{O}_S}\Omega^k_{Y/S} \xrightarrow{6\otimes d} \mathcal{O}_Y\otimes_{f^{-1}\mathcal{O}_S}\Omega^{k+1}_{Y/S} \longrightarrow \Omega^{k+1}_{Y/S}$

$a\otimes b\otimes\omega \longmapsto a\otimes b\omega \longmapsto a\otimes d(b\omega) \longmapsto a\,d(b\omega)$;

this factors through the projection

$$\mathcal{P}_{Y/S}\otimes_{\mathcal{O}_Y}\Omega^k_{Y/S} \longrightarrow \mathcal{P}^1_{Y/S}\otimes_{\mathcal{O}_Y}\Omega^k_{Y/S} \xrightarrow{\sim} \mathcal{D}^1_{Y/S}\otimes_{\mathcal{O}_Y}\Omega^k_{Y/S} \qquad /$$

It remains to show that $L(\Omega^\bullet_{Y/S})_Y$ is a complex, i.e. $L(d)_Y^2 = 0$. Since $L(-)_Y$ is a functor, it suffices to show that $d^2=0$ as an HPD-diff. op!

<u>Claim</u> $\bar{d}\circ\bar{d} = 0$

pf By def^{n}, $\bar{d}\circ\bar{d}$ is the composite (we use that d has order ≤ 1):

$$\mathcal{D}^2_{Y/S}\otimes_{\mathcal{O}_Y}\Omega^k_{Y/S} \xrightarrow{p_{13}\otimes\Omega} \mathcal{D}^1_{Y/S}\otimes_{\mathcal{O}_Y}\mathcal{D}^1_{Y/S}\otimes_{\mathcal{O}_Y}\Omega^k_{Y/S} \xrightarrow{\mathcal{D}\otimes\bar{d}} \mathcal{D}^1_{Y/S}\otimes_{\mathcal{O}_Y}\Omega^{k+1}_{Y/S} \xrightarrow{\bar{d}} \Omega^{k+2}_{Y/S}$$

$$\xi^{[2]}\otimes\omega \longmapsto \xi\otimes\xi\otimes\omega \longmapsto \xi\otimes dx_i\wedge\omega \longmapsto dx_i\wedge dx_i\wedge\omega = 0$$

$$a\otimes b\otimes\omega \longmapsto a\otimes 1\otimes 1\otimes b\otimes\omega \longmapsto a\otimes 1\otimes d(b\omega) \longmapsto a\,d^2(b\omega) = 0.$$

The proof is complete. //

<u>Lemma</u> Let Y/S be smooth, let $x_1, \dots, x_d$ be local coord. on Y, let $\xi_i = 1 \otimes x_i - x_i \otimes 1$, let $\omega \in \Omega^k_{Y/S}$ and $a \in \mathcal{O}_Y$ be local sections. Then

$$L(\Omega^k_{Y/S})_Y \xrightarrow{L(d)_Y} L(\Omega^{k+1}_{Y/S})_Y$$

$$a\,\xi_1^{[k_1]} \cdots \xi_d^{[k_d]} \otimes \omega \longmapsto a \sum_{i=1}^{d} \xi_1^{[k_1]} \cdots \xi_i^{[k_i-1]} \cdots \xi_d^{[k_d]} \otimes dx_i \wedge \omega$$

$$+\, a\,\xi_1^{[k_1]} \cdots \xi_d^{[k_d]} \otimes d\omega .$$

<u>pf</u> By def^{n}, $L(d)_Y$ is the composite

$$\begin{tikzcd}
\mathcal{D}_{Y/S} \otimes_{\mathcal{O}_Y} \Omega^k_{Y/S} \arrow[r, "p_{13} \otimes \Omega"] & \mathcal{D}_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y/S} \otimes_{\mathcal{O}_Y} \Omega^k_{Y/S} \arrow[r, "\mathcal{D} \otimes \bar{d}"] \arrow[d, "\mathcal{D} \otimes pr \otimes \Omega"'] & \mathcal{D}_{Y/S} \otimes_{\mathcal{O}_Y} \Omega^{k+1}_{Y/S} \\
& \mathcal{D}_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{D}^1_{Y/S} \otimes_{\mathcal{O}_Y} \Omega^k_{Y/S} \arrow[ur, "\mathcal{D} \otimes \bar{d}"'] &
\end{tikzcd}$$

Suppose $d = 1$.

$$a\,\xi^{[k]} \otimes \omega \longmapsto a\,(1 \otimes \xi + \xi \otimes 1)^{[k]} \otimes \omega$$

$$= a \sum_{i+j=k} \xi^{[i]} \otimes \xi^{[j]} \otimes \omega$$

$$\xmapsto{\mathcal{D} \otimes pr \otimes \Omega} a\,\xi^{[k-1]} \otimes \xi \otimes \omega + a\,\xi^{[k]} \otimes 1 \otimes \omega$$

$$\xmapsto{\mathcal{D} \otimes \bar{d}} a\,\xi^{[k-1]} \otimes dx \wedge \omega + a\,\xi^{[k]} \otimes d\omega .$$

The case $d > 1$ is similar. //

Let A be a ring (in a topos T), let

$$B = A[\xi_1, \ldots, \xi_d]$$

$$C = A\langle \xi_1, \ldots, \xi_d \rangle .$$

There is an A-linear derivation ($\Rightarrow$ int. conn.)

$$C \xrightarrow{\nabla} C \otimes_B \Omega^1_{B/A}$$

$$\xi_i^{[k]} \longmapsto \xi_i^{[k-1]} \otimes d\xi_i$$

<u>Lemma</u> $A \xrightarrow{\sim} C \otimes_B \Omega^{\bullet}_{B/A}$.

<u>Pf</u> Write $K^{\bullet}(\xi_1, \ldots, \xi_d) = C \otimes_B \Omega^{\bullet}_{B/A}$, then

$$K^{\bullet}(\xi_1, \ldots, \xi_d) \otimes_A K^{\bullet}(\eta_1, \ldots, \eta_e) \xleftarrow{\sim} K^{\bullet}(\xi_1, \ldots, \xi_d, \eta_1, \ldots, \eta_e).$$

Since these are complexes of <u>flat</u> A-modules, we are reduced, inductively, to consider the case $d = 1$:

$$0 \to A \to A\langle \xi \rangle \to A\langle \xi \rangle \, d\xi \to 0$$

$$\xi^{[k]} \longmapsto \xi^{[k-1]} \cdot d\xi \; . \qquad //$$

<u>Thm</u> (Poincaré lemma) Suppose Y/S is smooth, $m\mathcal{O}_S = 0$; then

$$\mathcal{O}_{Y/S} \xrightarrow{\sim} L(\Omega^{\bullet}_{Y/S})_Y$$

is a resolution of the crystals.

Pf The calc. on p. 3 gives an iso. of cx.

$$L(\Omega^{\bullet}_{Y/S})_Y \xrightarrow{\sim} \mathcal{O}_Y\langle \xi_1, \dots, \xi_d \rangle \otimes_{\mathcal{O}_Y[\xi_1, \dots, \xi_d]} \Omega^{\bullet}_{\mathcal{O}_Y[\xi_1, \dots, \xi_d]/\mathcal{O}_Y}.$$

The map is induced from $dx_i \mapsto d\xi_i$. Hence the theorem follows from the lemma on p. 4. //

$$(X/S)_{crys} \underset{i_{X/S}}{\overset{u_{X/S}}{\rightleftarrows}} X_{et} \qquad u_{X/S} \circ i_{X/S} = \mathrm{Id}$$

$$\begin{array}{ccccc} u_{X/S\,!} & \dashv & u^*_{X/S} & \dashv & u_{X/S\,*} \\ \shortparallel & & \shortparallel & & \shortparallel \\ i^*_{X/S} & \dashv & i_{X/S\,*} & \dashv & i^!_{X/S} \end{array} \qquad \begin{array}{l} u_{X/S\,*} \circ i_{X/S\,*} = \mathrm{Id} \iff \\ i^*_{X/S} \circ u^*_{X/S} = \mathrm{Id} \end{array}$$

$$u_{X/S\,!}(E)(U \to X) = E(U \overset{\mathrm{Id}}{\hookleftarrow} U \to X)$$

i.e. $u_{X/S\,!}(E) = E_X$

$$u^*_{X/S}(F)(T \overset{\delta}{\hookleftarrow} U \to X) = F(U)$$

$$u_{X/S\,*}(E)(U \to X) = X^{\sim}_{et}((U \to X)^{\sim}, u_{X/S\,*}(E))$$

$$\overset{\text{must}}{=} (X/S)_{crys}(u^*_{X/S}(U \to X)^{\sim}, E).$$

Since, clearly, $u^*_{X/S}$ commutes with colimits, we get $u^*_{X/S} \dashv u_{X/S\,*}$. Indeed, every sheaf is a colimit of representable sheaves. Also, $u_{X/S\,!}$ clearly preserves (all) limits, so $i_{X/S}$ is a map of topoi.

<u>Lemma</u>

$$\begin{tikzcd} (X/S)_{crys} \arrow[rr, "u_{X/S}"] \arrow[dr, "\Gamma(X/S,-)"'] & & X^{\sim}_{et} \arrow[dl, "\Gamma(X,-)"] \\ & \text{Sets} & \end{tikzcd}$$

commutes.

Pf $\Gamma(X, u_{X/S*}(E)) = \tilde{X_{et}}((X \xrightarrow{id} X)^\sim, u_{X/S*}(E))$

$= (X/S)_{crys}(u^*_{X/S}(X \xrightarrow{id} X)^\sim, E) \overset{?}{=} (X/S)_{crys}(e, E)$

$u^*_{X/S}(X \xrightarrow{id} X)^\sim (T \overset{\delta}{\hookleftarrow} U \to X)$

$$:= (X \xrightarrow{id} X)^\sim (U \xrightarrow{f} X) = \left\{ \begin{tikzcd} U \arrow[r] \arrow[dr, "f"'] & X \arrow[d, "id"] \\ & X \end{tikzcd} \right\};$$

this is a one-pt. set, i.e. $u^*_{X/S}(X \xrightarrow{id} X)^\sim = e$. //

We now consider abelian groups in $(X/S)_{crys}$. Since $u_{X/S*}$ has an exact left adjoint, it preserves injectives, so we have a Grothendieck sp. seq:

$$\begin{array}{ccc} E_2^{s,t} = R^s\Gamma(X, R^t u_{X/S*}(E)) & \Rightarrow & R^{s+t}\Gamma((X/S), E) \\ \| & & \| \\ H^s_{et}(X, R^t u_{X/S*}(E)) & & H^{s+t}(X/S, E) \end{array}$$

<u>Lemma</u>

$$\begin{tikzcd} (Y/S)_{crys} \arrow[rr, "u_{Y/S}"] \arrow[dr, "\Gamma(Y/S,-)"'] & & Y^{\sim}_{et} \arrow[dl, "\Gamma(Y,-)"] \\ & Sets & \end{tikzcd}$$

commutes.

pf $\Gamma(Y, u_{Y/S*}(E)) = Y^{\sim}_{et}((Y \xrightarrow{id} Y)^{\sim}, u_{Y/S*}(E))$

$= (Y/S)_{crys}(u^*_{Y/S}(Y \xrightarrow{id} Y)^{\sim}, E) \overset{?}{=} (Y/S)_{crys}(e, E).$

$u^*_{Y/S}(Y \xrightarrow{id} Y)^{\sim}(T \xleftarrow{s} U \xrightarrow{f} Y)$

$:= (Y \xrightarrow{id} Y)^{\sim}(U \xrightarrow{f} Y) = \left\{ \begin{array}{ccc} U & \longrightarrow & Y \\ \downarrow f & & \downarrow id \\ Y & = & Y \end{array} \right\}$

This is a one-pt. set, so $u^*_{Y/S}(Y \xrightarrow{id} Y)^{\sim} = e$. //

Recall $u_{Y/S!} \dashv u^*_{Y/S} \dashv u_{Y/S*}$, so $u^*_{Y/S}$ exact, so $u_{Y/S*}$ preserves injectives. Hence if E is an ab. grp. in $(Y/S)_{crys}$, have sp. seq.

$$\begin{array}{ccc} E_2^{s,t} = R^s\Gamma(Y, R^t u_{Y/S*}(E)) & \Rightarrow & R^{s+t}\Gamma(Y/S, E) \\ \| & & \| \\ H^s_{et}(Y, R^t u_{Y/S*}(E)) & & H^{s+t}(Y/S, E) \end{array}$$

or

$$H^*_{et}(Y, Ru_{Y/S*}(E)) \cong H^*(Y/S, E)$$

Lemma $E \xrightarrow{\sim} Ru_{Y/S*} L(E)_Y$, Y/S smooth.

Thm If Y/S is smooth, then

$$Ru_{Y/S*}\mathcal{O}_{Y/S} \xrightarrow{\sim} Ru_{Y/S*}L(\Omega^{\cdot}_{Y/S})_Y \xleftarrow{\sim} \Omega^{\cdot}_{Y/S}.$$

Pf The equivalence on the left was proved last time (uses sm.); the right hand eq. is lemma. //

General theory: C site, T ob. of C.

$$C/T \xrightarrow{j_T} C, \quad \begin{matrix} X \\ \downarrow \\ T \end{matrix} \longmapsto X$$

$$\begin{tikzcd} (C/T)^{\wedge} \arrow[r, shift left=2, "j_{T!}"] \arrow[r, shift right=2, "j_{T*}"'] & C^{\wedge} \arrow[l, "j_T^*"'] \end{tikzcd} \qquad j_{T!} \dashv j_T^* \dashv j_{T*}$$

Kan extensions

$$j_T^*(F)(X \to T) = F(X)$$

$$j_{T!}(F)(X) = \operatorname{colim}\left((j_T^{op}/X) \xrightarrow{pr} (C/T)^{op} \xrightarrow{F} \text{Sets}\right)$$

$$= \coprod_{X \to T} F(X \to T)$$

$$j_{T*}(F)(X) = \lim\left(\underset{\substack{\| \\ (j_T/X)^{op}}}{(X/j_T^{op})} \xrightarrow{pr} (C/T)^{op} \xrightarrow{F} \text{Sets}\right)$$

In application, (j_T/X) has terminal object.

$C^\sim \xrightarrow{i} C^\wedge$ map of topoi.

<u>Lemma</u> (i) j_T^* and j_{T*} preserve sheaves;

(ii) $j_{T!} := i^* j_{T!} i_* \dashv j_T^*$

(iii) $(C/T)^\sim \xrightarrow[\sim]{j_{T!}} C^\sim/T^\sim$.

Pf (ii) $C^\sim(i^* j_{T!} i_* E, F) = C^\wedge(j_{T!} i_* E, i_* F)$

$= (C/T)^\wedge(i_* E, j_T^* i_* F) \overset{(i)}{=} (C/T)^\wedge(i_* E, i_* j_T^* F)$

$= (C/T)^\sim(E, j_T^* F)$. //

$\Longrightarrow$ map of topoi $(C/T)^\sim \xrightarrow{j_T} C^\sim$

$C = \mathrm{Crys}(Y/S)$, $T = (Y \xleftarrow{id} Y \xrightarrow{id} Y)$

C/T ob

$$\begin{tikzcd}
U \arrow[r, "\text{ét}"] \arrow[d, hook, "\delta"'] & Y \arrow[d] \\
T \arrow[r] \arrow[ur, "g"'] & S
\end{tikzcd}$$

mor:

$$\begin{tikzcd}
U \arrow[rr] \arrow[dr] \arrow[dd, hook, "\delta"'] & & Y \arrow[dr, phantom, "="] & \\
& U' \arrow[rr] \arrow[dd, hook, "\delta'"] & & Y \arrow[dd] \\
T \arrow[rr] \arrow[dr, "\text{PD-mor.}"'] & & S \arrow[dr, phantom, "="] & \\
& T' \arrow[rr] & & S
\end{tikzcd}$$

$$j_Y^*(E)\left(\begin{tikzcd} U \arrow[r] \arrow[d, hook, "\delta"] & Y \arrow[d] \\ T \arrow[r] \arrow[ur] & S \end{tikzcd}\right) = E\left(\begin{tikzcd} U \arrow[r] \arrow[d, hook] & Y \arrow[d] \\ T \arrow[r] & S \end{tikzcd}\right).$$

$$\left(j_Y /\ \begin{tikzcd} U \arrow[r] \arrow[d, hook, "\delta"'] & Y \arrow[d] \\ T \arrow[r] & S \end{tikzcd}\right)$$ over cat — terminal ob:

$$\begin{tikzcd} U \arrow[r] \arrow[d, hook] & Y \arrow[d] \\ D_{U,\delta,0}(T \times_S Y) \arrow[r] \arrow[ur, "p_2"'] & S \end{tikzcd}, \qquad \begin{tikzcd}[column sep=small] U \arrow[rr] \arrow[dd, hook] \arrow[dr, equal] & & Y \arrow[dd] \arrow[dr, equal] & \\ & U \arrow[rr] & & Y \arrow[dd] \\ D_{U,\delta,0}(T \times_S Y) \arrow[rr, hook] \arrow[dr, "p_1"'] & & S \arrow[dr, equal] & \\ & T \arrow[rr] & & S \end{tikzcd}$$

so

$$j_{Y*}(E)\left(\begin{tikzcd} U \arrow[r] \arrow[d, hook, "\delta"] & Y \arrow[d] \\ T \arrow[r] & S \end{tikzcd}\right) = E\left(\begin{tikzcd} U \arrow[r] \arrow[d, hook] & Y \arrow[d] \\ D_{U,\delta,0}(T \times_S Y) \arrow[r] \arrow[ur, "p_2"] & S \end{tikzcd}\right)$$

—

Consider diagram of topoi:

$$\begin{tikzcd} & (Y/S)_{crys}/Y \arrow[dl, "j_Y"'] \arrow[dr, "\varphi_Y"] & \\ (Y/S)_{crys} \arrow[rr, "u_{Y/S}"] & & \tilde{Y}_{et} \end{tikzcd}$$

$$\varphi_Y^*(F)\left(\begin{tikzcd} U \arrow[r] \arrow[d, hook] & Y \arrow[d] \\ T \arrow[r] \arrow[ur] & S \end{tikzcd}\right) = F(U \to Y)$$

$$\varphi_{Y*}(F)(U \to Y) = F\left(\begin{tikzcd} U \arrow[r] \arrow[d, "\mathrm{id}"] & Y \arrow[d] \\ U \arrow[r] \arrow[ur] & S \end{tikzcd}\right)$$

$$j_Y^* u_{Y/S}^*(F)\left(\begin{tikzcd} U \arrow[r] \arrow[d, hook] & Y \arrow[d] \\ T \arrow[r] \arrow[ur] & S \end{tikzcd}\right) = u_{Y/S*}(F)\left(\begin{tikzcd} U \arrow[r] \arrow[d, hook] & Y \arrow[d] \\ T \arrow[r] & S \end{tikzcd}\right)$$

$$= F(U \to Y) = \varphi_Y^*(F)\left(\begin{tikzcd} U \arrow[r] \arrow[d, hook] & Y \arrow[d] \\ T \arrow[r] & S \end{tikzcd}\right) \quad \checkmark$$

<u>Lemma</u> If Y/S is sm, $j_{Y*}\varphi_Y^*(E) = L(E)$.

Pf The sheaf on Y corresponding to LHS is

$$j_{Y*}\varphi_Y^*(E)_Y = p_{1*}p_2^*(E) = L(E)_Y$$

$$Y \xleftarrow{p_1} D_{Y,0,0}(Y \times_S Y) \xrightarrow{p_2} Y$$

$$\| \qquad \text{—— } \gamma \text{ ext. to } Y + Y \xrightarrow{\Delta} Y \times_S Y \text{ has section}$$

$$D_{Y,\gamma}(Y \times_S Y)$$

Must show that $j_{Y*}\varphi_Y^*(E)$ is a crystal, or equivalently:

$$\begin{tikzcd} T \arrow[d, "g"] & D_{U,\delta,0}(T \times_S Y) \arrow[l] \arrow[d] \\ Y & D_{Y,0,0}(Y \times_S Y) \arrow[l] \arrow[ul, phantom, "\text{cart.}"] \end{tikzcd} \quad //$$

<u>proof</u> of lemma on p. 2: sp. seq.

$$E_2^{s,t} = R^s u_{Y/S*}(R^t j_{Y*}(E)) \Rightarrow R^{s+t} \varphi_{Y*}(E)$$

j_{Y*} exact $\Rightarrow$ $R^s u_{Y/S*}(j_{Y*}(E)) \xrightarrow{\sim} R^s \varphi_{Y*}(E)$

φ_{Y*} exact $\Rightarrow$ $R^s \varphi_{Y*}(E) = 0, \quad s > 0.$ //

Given a cartesian diagram

$$\begin{array}{ccc} X_0 & \overset{i}{\hookrightarrow} & X \\ \downarrow & & \downarrow \\ S_0 & \hookrightarrow & S \end{array}$$

$(S, \mathcal{J}, \gamma)$ PD-scheme

$S_0 \hookrightarrow S$ closed immersion defined by a sub PD-ideal $K \subset \mathcal{J}$.

Theorem:

$$H^i_{cris}(X_0/S, \mathcal{O}_{X_0/S}) = H^i_{cris}(X/S, \mathcal{O}_{X/S})$$

Applications:

(a) If $X \to S$ is smooth then by a previous result $H^i_{cris}(X/S, \mathcal{O}_{X/S}) = H^i_{dR}(X/S)$.

Hence:

$$\begin{pmatrix} \text{cristalline} \\ \text{coh. of } X_0/S \end{pmatrix} = \begin{matrix} \text{deRham cohomology} \\ \text{of } \underline{\text{any}} \text{ lift.} \end{matrix}$$

(b) Assume S_0 in characteristic p.
Then we have

$$F : X_0 \longrightarrow X_0$$

frobenius map of X_0. This gives

$$F : (X_0/S)_{cris} \longrightarrow (X_0/S)_{cris} \quad \text{(ringed topoi)}$$

and hence

$$\begin{tikzcd}
F^* : H^i_{cris}(X_0/S, \mathcal{O}) \arrow[r] \arrow[d, equal] & H^i_{cris}(X_0/S, \mathcal{O}) \arrow[d, equal] \\
H^i_{cris}(X/S, \mathcal{O}) \arrow[r] \arrow[d, equal] & H^i_{cris}(X/S, \mathcal{O}) \arrow[d, equal] \\
H^i_{dR}(X/S) \arrow[r] & H^i_{dR}(X/S)
\end{tikzcd}$$

$X \longrightarrow S$ smooth

So Frobenius acts on the de Rham coh of any lift.

General Case of closed immersion

$$\begin{tikzcd} Y \arrow[r, hook, "i"] \arrow[dr] & X \arrow[d] \\ & (S, \mathcal{J}, \delta) \end{tikzcd}$$

$\mathcal{F}$ sheaf on $\mathrm{Cris}(Y/S)$
$(U, T, \delta) \in \mathrm{Cris}(X/S)$.

Set

$$\begin{tikzcd} & & & D_V(T) \arrow[d, "p"] \\ V := Y \times_X U \arrow[urrr, hook] \arrow[r, hook] \arrow[d] & U \arrow[rr, hook] \arrow[d] & & T \\ Y \arrow[r, hook] & X \end{tikzcd}$$

$T \xleftarrow{p} D_V(T)$ = spec of the divided power hull of $\mathcal{O}_T$ in $\mathcal{J}(V \hookrightarrow T)$, divided powers taken compatible with those on $\mathcal{J}(U \subset T)$ and on $\mathcal{J}$

structure morphism

$(V, D_V(T), [\,]) \in \mathrm{Cris}(X/S)$

Lemma

$\mathrm{Hom}_{\mathrm{Cris}(Y/S)}((U', T', \delta'), (V, D_V(T), [\,])) \overset{bij}{=}$

~~Hom~~ $\{\alpha : T' \to T$ s.t. $\alpha(U') \subset U$, α compatible w. δ', δ and $\begin{tikzcd} U' \arrow[r] \arrow[d] \arrow[dr, phantom, "\sigma"] & U \arrow[d] \\ Y \arrow[r] & {} \end{tikzcd}$

Corollary

(a) $i_{cris*}(\mathcal{F})_{(U,T,\delta)} = p_*\left(\mathcal{F}_{(V, D_V(T), [\,])}\right)$

(b) i_{cris*} is an exact functor on sheaves of abelian groups and on $\mathcal{O}$-modules

Proof of (b): p on sheaves of abelian groups is exact as it is a closed immersion on the underlying topological spaces. Result for $\mathcal{O}$-modules follows from result of abelian sheaves. $\square$

REMARK ALSO TRUE BUT HARDER: $i_{cris*}(\mathcal{O}_{Y/S})$ is a crystal of $\mathcal{O}_{X/S}$-modules

Back to our case

$$\begin{tikzcd} X_0 = V(\mathcal{K}\mathcal{O}_X) \arrow[r, hook] \arrow[dr] & X \arrow[d] \\ & (S, \mathcal{J}, \gamma) \quad \mathcal{K} \subset \mathcal{J}. \end{tikzcd}$$

Then

$$V = U_0 = V(\mathcal{K}\cdot\mathcal{O}_U) \subset U \subset T$$

Furthermore

$$\underbrace{\mathcal{K}\cdot\mathcal{O}_T + \mathcal{J}(U \subset T)}_{\text{pd ideal}} \subset \underbrace{\mathcal{J}\cdot\mathcal{O}_T + \mathcal{J}(U \hookrightarrow T)}_{\text{pd-ideal}}$$

pd ideal $\Longleftarrow$ pd-ideal

Hence $\boxed{D_{U_0}(T) = T}$

<u>Corollary</u> $i_{cris\,*}(\mathcal{O}_{X_0/S}) = \mathcal{O}_{X/S}$

<u>Proof.</u> This follows from (a) above;

$$i_{cris\,*}(\mathcal{F})_{(U,T,\delta)} = p_*(\mathcal{F}_{(U_0,T,[\,])})$$

so $p = id$ and hence:

$$i_{cris}(\mathcal{O}_T)_{(U,T,\delta)} = p_* \mathcal{O}_T = \mathcal{O}_T.$$

Finally, use the spectral sequence

$$H^p_{cris}((X_0/S), R^q i_{cris\,*}(\mathcal{F})) \Rightarrow H^{p+q}_{cris}(X_0/S, \mathcal{F})$$

Since $i_{cris\,*}$ is exact all higher direct images are zero.

Example

$$H^i\left(\mathrm{Spec}(\mathbb{F}_p[t]) / \mathbb{Z}/p^n\mathbb{Z},\ \mathcal{O}\right)$$

$$= H^i_{dR}\left(\mathrm{Spec}(\mathbb{Z}/p^n\mathbb{Z}[t]) / \mathbb{Z}/p^n\mathbb{Z}\right)$$

$$= H^i\left(\begin{array}{c}\text{the complex} \\ \mathbb{Z}/p^n\mathbb{Z}[t] \xrightarrow{d} \mathbb{Z}/p^n\mathbb{Z}[t]\,dt\end{array}\right)$$

$$= \begin{cases} 0 & i \neq 0, 1 \\ \bigoplus_{k \geq 0} \mathbb{Z}/p^{\mathrm{ord}_p(k)}\mathbb{Z} \cdot p^{n-\mathrm{ord}_p(k)} \cdot t^k & i = 0 \\ \bigoplus_{k \geq 0} \mathbb{Z}/p^{\min(n,\, \mathrm{ord}_p(k+1))}\, t^k\, dt & i = 1 \end{cases}$$

$$\frac{t^{k+1}}{k+1}$$

Hence

$$H^i_{cris}(\mathbb{A}^1_{\mathbb{F}_p}/\mathbb{Z}_p, \mathcal{O}) \stackrel{def}{=}$$

$$= \varprojlim_n H^i(\mathbb{A}^1/_{\mathbb{Z}_p/p^n\mathbb{Z}_p}, \mathcal{O})$$

$$\cong \begin{cases} \mathbb{Z}_p & \text{for } i=0 \\ \bigoplus_{h\geq 0} \mathbb{Z}/p^{\mathrm{ord}_p(h+1)}\mathbb{Z} \cdot t^a dt & i=1 \\ 0 & i\geq 2 \end{cases}$$

<u>Generally speaking:</u>

cristalline Coh does NOT work unless $X_0 \to S$ is proper (and smooth over $S_0 \hookrightarrow S$).

Prop Let $X \overset{i}{\hookrightarrow} Y$ be a closed imm. with Y/S sm. The following are equivalent:

(i) crystals E on $(X/S)_{crys}$;

(ii) $\mathcal{D}_{X,\gamma}(Y)$-modules E with a stratification relative to $(\mathcal{D}_{X,\gamma}(Y), \mathcal{D}_{X,\gamma}(Y \times_S Y))$.

pf (i) $\Rightarrow$ (ii) $\quad E = E_{D_{X,\gamma}(Y)}$

$$\begin{tikzcd}
D_X(Y \times_S Y) \arrow[r, "\tau"] \arrow[rr, bend left, "p_2"] & D_X(Y \times_S Y) \arrow[r, "p_1"] & D_X(Y) \\
X \arrow[u, "\tilde{i}"] \arrow[r, equal] & X \arrow[u, "\tilde{i}"] \arrow[r, equal] & X \arrow[u, "\tilde{i}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
p_2^* E \arrow[r, "\varepsilon"] \arrow[d, equal] & p_1^* E \arrow[d, equal] \\
\tau^* E_{D_X(Y \times_S Y)} \arrow[r, "\rho_\tau"] & E_{D_X(Y \times_S Y)}
\end{tikzcd}$$

(ii) $\Rightarrow$ (i) for (T, U, δ) small enough, have lift:

$$\begin{tikzcd}
 & D_{U,\gamma}(Y) \arrow[r] & D_{X,\gamma}(Y) \arrow[dr] & \\
U \arrow[ur] \arrow[rr] \arrow[d, hook, "\delta"'] & & X \arrow[r, hook] \arrow[d] & Y \arrow[d] \\
T \arrow[uur, dashed] \arrow[uurr, dashed, "h"] \arrow[urrr, dashed, "g"] \arrow[rr] & & S \arrow[r] & S
\end{tikzcd}
\qquad E_T := h^* E.$$

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Lemma Let
$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr] & Y \arrow[d] \\ & S \end{tikzcd}$$
be a closed imm. Then

$$\mathcal{D}_{X,\gamma}(Y) \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y/S}(1) \xrightarrow{\sim} \mathcal{D}_{X,\gamma}(Y \times_S Y) \longleftarrow \mathcal{D}_{Y/S}(1) \otimes_{\mathcal{O}_Y} \mathcal{D}_{X,\gamma}(Y).$$

pf Will need (and prove) this only if $Y \to S$ is smooth. Then, locally,

$$\begin{tikzcd} \mathcal{O}_Y = \mathcal{O}_S[x_1, \dots, x_d] \arrow[rr, "J"] & & \mathcal{O}_X \\ \mathcal{O}_{Y \times_S Y} = \mathcal{O}_S[x_1, \dots, x_d, y_1, \dots, y_d] \arrow[u, "{(\xi_1, \dots, \xi_d)}"'] \arrow[urr, "J'"] \arrow[rr, "K"'] & & \mathcal{O}_{X \times_S Y} \arrow[u, "{(\xi_1, \dots, \xi_d)}"'] \end{tikzcd}$$

$$\begin{aligned}
\mathcal{D}_{X,\gamma}(Y \times_S Y) &:= \mathcal{D}_{J',\gamma}(\mathcal{O}_{Y \times_S Y}) \\
&\cong \mathcal{D}_{(\xi_1, \dots, \xi_d), [-]}(\mathcal{D}_{K,\gamma}(\mathcal{O}_{Y \times_S Y})) \\
&= \mathcal{D}_{(\xi_1, \dots, \xi_d), [-]}(\mathcal{D}_{J,\gamma}(\mathcal{O}_Y)[y_1, \dots, y_d]) \\
&= \mathcal{D}_{(\xi_1, \dots, \xi_d), [-]}(\mathcal{D}_{J,\gamma}(\mathcal{O}_Y)[\xi_1, \dots, \xi_d]) \\
&= \mathcal{D}_{J,\gamma}(\mathcal{O}_Y)\langle \xi_1, \dots, \xi_d \rangle \\
&= \mathcal{D}_{X,\gamma}(Y) \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y,\gamma}(Y \times_S Y).
\end{aligned}$$

//

Let $X \xrightarrow{i} Y$ be a closed imm., Y/S sm., and let $J \subset \mathcal{O}_Y$ be the ideal of X. Then

$$\mathcal{D}_{X,\gamma}(Y) \xrightarrow{\nabla} \mathcal{D}_{X,\gamma}(Y) \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S}$$

$$x^{[k]} \longmapsto x^{[k-1]} \otimes dx \quad , \quad x \in J.$$

Prop The following are equivalent:

(i) $\mathcal{D}_{X,\gamma}(Y)$-modules E with a stratification rel to $(\mathcal{D}_{X,\gamma}(Y), \mathcal{D}_{X,\gamma}(Y \times_S Y))$;

(ii) $\mathcal{D}_{X,\gamma}(Y)$-modules E with a stratification rel. to $(\mathcal{O}_Y, \mathcal{D}_{Y/S}(1))$ which is compatible with the canonical stratification on $\mathcal{D}_{X,\gamma}(Y)$ rel. to $(\mathcal{O}_Y, \mathcal{D}_{Y/S}(1))$;

(iii) $\mathcal{D}_{X,\gamma}(Y)$-modules E with a quasi-nilpotent integrable connection (as an $\mathcal{O}_Y$-module) which is compatible with the canonical connection on $\mathcal{D}_{X,\gamma}(Y)$.