 11/6/00

$$\begin{array}{ccccc} Y & \xrightarrow{i} & X & \xleftarrow{j} & X_K \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} k & \hookrightarrow & \operatorname{Spec} V & \xleftarrow{j} & \operatorname{Spec} K \\ & & \| & & \\ & & S & & \end{array}$$

semi-stable reduction

$$\begin{array}{ccc} (Y, M_Y) & \xrightarrow{i} & (X, M) \\ \downarrow & \text{exact cl. imm.} & \downarrow \text{smooth} \\ (\operatorname{Spec} k, L) & \xrightarrow{i} & (S, N) \end{array}$$

M, N can. log str.

$$\bar{S} = \operatorname{Spec} \bar{V}, \quad \bar{S}_n = \operatorname{Spec} \bar{V}/p^n\bar{V}$$

$$(\bar{X}_n, \bar{M}_n) = (X, M) \times_{(S,N)} (\bar{S}_n, \bar{N}_n).$$

Goal this month: If $X \to S$ is proper, there is a canonical isomorphism

$$\mathbb{Q} \otimes \lim_n H^q((\bar{X}_n, \bar{M}_n)/W_n)$$

$$\cong \left(B^+_{st,\pi} \otimes_W \lim_n H^q((Y, M_Y)/(W_n, W_n(L))) \right)^{N=0}.$$

First step: define $B^+_{st,\pi}$

$R_{\bar V} = \varprojlim_{\varphi} \bar V/p\bar V$ perfect $\mathbb{F}_p$-algebra.

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \bar V^\wedge & \xrightarrow{(-)^p} & \bar V^\wedge & \xrightarrow{(-)^p} & \bar V^\wedge & & \varprojlim_{(-)^p} \bar V^\wedge \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \wr \\ \cdots \longrightarrow & \bar V/p\bar V & \xrightarrow{\varphi} & \bar V/p\bar V & \longrightarrow & \bar V/p\bar V & & \varprojlim_{\varphi} \bar V/p\bar V \end{array}$$

Inverse: If $x = \{x^{(m)}\} \in R_{\bar V}$, choose $\tilde x^{(m)} \mapsto x^{(m)}$, let

$$\hat x^{(m)} = \lim_{k\to\infty} (\tilde x^{(m+k)})^{p^k}.$$

Then $\hat x = \{\hat x^{(m)}\} \longmapsto x = \{x^{(m)}\}$.

Valuation $v_R(x) := v_{\bar V^\wedge}(\hat x^{(0)})$.

Facts (1) $R_{\bar V}$ is a CDVR (easy);

(2) the fraction field of $R_{\bar V}$ is alg. closed. (harder). //

Define ring homomorphism

$$W(R_{\bar V}) \xrightarrow{\Theta} \bar V^\wedge$$

$$x = \{x^{(m)}\} \longmapsto \lim_{k\to\infty} (\tilde x_0^{(k)})^{p^k} + p(\tilde x_1^{(k)})^{p^{k-1}} + \cdots + p^k \tilde x_k^{(k)}$$

where $x^{(m)} \in W(\bar V/p\bar V)$, $\tilde x^{(m)} \in W(\bar V^\wedge)$, $\tilde x^{(m)} \mapsto x^{(m)}$.

(Note $W(R_{\bar V}) \xrightarrow{\sim} \varprojlim_F W(\bar V/p\bar V)$.)

⌈ Explanation: Θ is the composite

$$W(R_{\bar V}) \xrightarrow{\sim} \lim_F W(\bar V/p\bar V) \xleftarrow{\sim} \lim_F W(\bar V^\wedge)$$
$$\xrightarrow{pr} W(\bar V^\wedge) \xrightarrow{w_0} \bar V^\wedge .$$ ⌋

<u>Lemma</u> $W(R_{\bar V}) \xrightarrow{\Theta} \bar V^\wedge$ is surjective.

<u>pf</u> Write $a = \sum_{i\geq 0} a_i p^i$, $a_i \in \bar V$; let $\bar a_i = a_i + p\bar V \in \bar V/p\bar V$; choose $x_i \in R_{\bar V}$ s.t. $x_i^{(0)} = \bar a_i$; let $\underline{x_i} \in W(R_{\bar V})$ be the Teichmüller rep. $\underline{x_i} = (x_i, 0, \ldots)$. Then

$$W(R_{\bar V}) \xrightarrow{\Theta} \bar V^\wedge \longrightarrow \bar V/p\bar V$$

$$\underline{x_i} \longmapsto \lim_{k\to\infty} (x_i^{(k)})^{p^k} = x_i^{(0)} = \bar a_i$$

$$\Rightarrow \quad p^i \underline{x_i} \longmapsto p^i a_i + p^{i+1}\bar V \in \bar V/p^{i+1}\bar V$$

$$\Rightarrow \quad \sum p^i \underline{x_i} \longmapsto \sum p^i a_i \in \bar V^\wedge . \qquad //$$

$$0 \longrightarrow J \longrightarrow W(R_{\bar V}) \xrightarrow{\Theta} \bar V^\wedge \longrightarrow 0$$

<u>Def</u> $B_n = D_{J\cdot W_n(R_{\bar V})}(W_n(R_{\bar V}))$

$$\cong D_J(W(R_{\bar V})) / p^n D_J(W(R_{\bar V}))$$

<u>Lemma</u> Let R be a perfect $\mathbb{F}_p$-algebra and let A be a p-complete ring. Then the following are eq.:

(i) a ring hom. $W(R) \xrightarrow{f} A$;

(ii) a mult. map $R \xrightarrow{\rho} A$ s.t. the composite

$$R \xrightarrow{\rho} A \to A/pA$$

is a ring hom.

Pf (i) ⇒ (ii) $\rho(x) = f(\underline{x})$; $W(R)/pW(R) \xrightarrow{\sim} R$ perfect.

(ii) ⇒ (i) uniqueness.

$$x = \sum_{i=0}^{\infty} V^i(\underline{x_i}) = \sum_{i=0}^{\infty} V^i F^i(\underline{x_i}) = \sum_{i=0}^{\infty} p^i \underline{x_i}^{p^{-i}}, \text{ so}$$

$$f(x) = \sum_{i=0}^{\infty} p^i f(\underline{x_i}^{p^{-i}}) = \sum_{i=0}^{\infty} p^i \rho(x_i^{p^{-i}}).$$

Define f by this formula; show f additive, i.e.

$$\sum_{i=0}^{\infty} p^i \rho(a_i^{p^{-i}}) + \sum_{i=0}^{\infty} p^i \rho(b_i^{p^{-i}}) = \sum_{i=0}^{\infty} p^i \rho(s_i(a,b)^{p^{-i}})$$

Suffices to prove this mod p^{n+1}, for all $n \geq 0$. Let

$$x_i = a_i^{p^{-n}}, \quad y_i = b_i^{p^{-n}};$$

$$s_i(x,y) = s_i(a,b)^{p^{-n}} \quad \text{since } R \text{ perfect } \mathbb{F}_p\text{-alg.}$$

Wish to show:

$$\sum_{i=0}^{n} p^i \rho(x_i)^{p^{n-i}} + \sum_{i=0}^{n} p^i \rho(y_i)^{p^{n-i}} \equiv \sum_{i=0}^{n} p^i \rho(s_i(x,y))^{p^{n-i}} \quad (p^{n+1})$$

or

$$\underbrace{w_n(\rho(x)) + w_n(\rho(y))}_{\begin{array}{c} \| \\ w_n(S(\rho(x),\rho(y))) \end{array}} \equiv w_n(\rho(s(x,y))) \quad (p^{n+1})$$

Binomial formula $\Rightarrow$ enough to show

$$s(\rho(x),\rho(y)) \equiv \rho(s(x,y)) \quad (\mathrm{mod}\ p)$$

and this holds since, mod p, ρ is a ring hom. Proof that f is multiplicative is similar. //

<u>Prop</u> $B_n = \Gamma((\bar{S}_n/W_n)^{\sim}_{crys}, \mathcal{O}_{\bar{S}_n/W_n})$

<u>Pf</u> Since $\theta : W_n(R_{\bar{V}}) \to \bar{V}_n$ is surj. and since γ extends to $\bar{V}_n$ (since pW_n principal),

$$\begin{tikzcd} \bar{S}_n \arrow[r, "id"] \arrow[d, hook] & \bar{S}_n \arrow[d] \\ \mathrm{Spec}\, B_n \arrow[r] & \mathrm{Spec}\, W_n \end{tikzcd}$$

is an object of $(\bar{S}_n/W_n)_{crys}$. We show it is universal, i.e. for any (U, T, δ, i), there exists a unique arrow making the diagr. commute

$$\begin{tikzcd} U \arrow[r] \arrow[d, hook, "i"] & \bar{S}_n \arrow[d] \arrow[ddr, bend left] & \\ T \arrow[r, dashed] \arrow[drr, bend right] & \mathrm{Spec}\, W_n(R_{\bar{V}}) \arrow[dr] & \\ & & \mathrm{Spec}\, W_n \end{tikzcd}$$

or, equivalently,

$$\begin{tikzcd}
\Gamma(U, \mathcal{O}_U) & & \bar{V}_n \arrow[ll] & \\
\Gamma(T, \mathcal{O}_T) \arrow[u, two heads, "i^*"] & & W_n(R_{\bar{V}}) \arrow[ll, dashed] \arrow[u, two heads] & \\
& & & W_n \arrow[ul] \arrow[uul, bend right] \arrow[ulll, two heads, bend left]
\end{tikzcd}$$

Use lemma:

$$\begin{tikzcd}
\Gamma(U, \mathcal{O}_U) & \\
& R_{\bar{V}} \arrow[ul, "\rho_U"'] \arrow[dl, dashed, "\rho_T"] \\
\Gamma(T, \mathcal{O}_T) \arrow[uu, two heads] &
\end{tikzcd}$$

Let $\tilde{\rho}_U(x)$ be a lift of $\rho_U(x)$. Since the ideal I of U in T has a PD-str and since $p^n \mathcal{O}_T = 0$,

$$\rho_T(x) := \tilde{\rho}_U(x^{p^{-n}})^{p^n}$$

is well-defined and multiplicative. (The latter is also seen as a uniqueness statement.) Show

$$\rho_T(x) + \rho_T(y) \equiv \rho_T(x+y) \pmod{p}.$$

Since $R_{\bar{V}}$ perfect $\mathbb{F}_p$-alg, $(x+y)^{p^{-n}} = x^{p^{-n}} + y^{p^{-n}}$, so

$$\rho_U(x^{p^{-n}}) + \rho_U(y^{p^{-n}}) \equiv \rho_U((x+y)^{p^{-n}}) \pmod{p}.$$

Hence

$$\tilde{\rho}_U(x^{p^{-n}}) + \tilde{\rho}_U(y^{p^{-n}}) \equiv \tilde{\rho}_U((x+y)^{p^{-n}}) \pmod{p\mathcal{O}_T + I}$$

But since δ and γ are compatible, $p\mathcal{O}_T + \mathcal{J}$ has a PD-str., so their p^n th powers agree, ie

$$(\tilde{\rho}_U(x^{p^{-n}}) + \tilde{\rho}_U(y^{p^{-n}}))^{p^n} = \tilde{\rho}_U((x+y)^{p^{-n}})^{p^n}$$

$$\begin{array}{cc} \equiv (p) & \| \\ \tilde{\rho}_U(x^{p^{-n}})^{p^n} + \tilde{\rho}_U(y^{p^{-n}})^{p^n} & \rho_T(x+y) \\ \| & \\ \rho_T(x) + \rho_T(y). & \end{array}$$

//

Pick $\hat{v}^{(m)} \in \bar{V}^\wedge$ s.t. $\hat{v}^{(0)} = -p$, $(\hat{v}^{(m+1)})^p = \hat{v}^{(m)}$; let $v \in R_{\bar{V}}$ be the corresponing element, and set

$$\xi = \underline{v} + p \in W(R_{\bar{V}}).$$

<u>Lemma</u> $\mathcal{J} = (\xi)$.

<u>pf</u> Wish to show

$$0 \to W(R_{\bar{V}}) \xrightarrow{\xi} W(R_{\bar{V}}) \xrightarrow{\theta} \bar{V}^\wedge \to 0 \, ;$$

enough to show

$$0 \to W_n(R_{\bar{V}}) \xrightarrow{\xi} W_n(R_{\bar{V}}) \xrightarrow{\theta} \bar{V}_n \to 0 .$$

Indeed, derived limits vanish. Since $W(R_{\bar{V}})$ and $\bar{V}^\wedge$ torsion free, enough to show

$$0 \to R_{\bar{V}} \xrightarrow{\xi = v} R_{\bar{V}} \xrightarrow{\theta} \bar{V}/p\bar{V} \to 0$$

$$(x^{(m)}) \longmapsto x^{(0)}$$

exact.

$v \cdot y = 0 \Rightarrow y = 0$) $\quad v \cdot y = 0 \iff \hat{v}^{(m)} \cdot \hat{y}^{(m)} \in p\bar{V}^\wedge$, i.e. $\hat{v}^{(m)} \cdot \hat{y}^{(m)} = -p \cdot \hat{z}^{(m)} = (\hat{v}^{(m)})^{p^m} \cdot \hat{z}^{(m)} \Rightarrow$
$\hat{y}^{(m)} = (\hat{v}^{(m)})^{p^m - 1} \hat{z}^{(m)}$, for all $m \geq 0$, so

$$\hat{y}^{(m)} = (\hat{y}^{(m+1)})^p = (\hat{v}^{(m+1)})^{p^{m+2} - p} (\hat{z}^{(m+1)})^p$$

and $p^{m+2} - p \geq p^{m+1}$, so $(\hat{v}^{(m+1)})^{p^{m+2} - p} \in p\bar{V}^\wedge$. I.e. $y = 0$.

$\Theta(x) = 0 \Rightarrow x = v \cdot y$) $\quad (\hat{x}^{(m)})^{p^m} = \hat{x}^{(0)}$ divisible by $-p = (\hat{v}^{(m)})^{p^m}$, so $\hat{x}^{(m)}$ div. by $\hat{v}^{(m)}$; $\hat{y}^{(m)} = \hat{x}^{(m)} / \hat{v}^{(m)}$. //

Cor $D_J(W(R_{\bar{V}})) \subset W(R_{\bar{V}}) \otimes \mathbb{Q}$ subring gen. by $\xi^{[n]} = \xi^n / n!$, $n \geq 0$. //

Fontaine defines:

$$A_{cris} = \lim_n B_n = D_J(W(R_{\bar{V}}))^\wedge,$$

$$\mathrm{Fil}^r A_{cris} = (\bar{J}^{[r]})^\wedge;$$

$$B^+_{cris} = \mathbb{Q} \otimes A_{cris}$$

$$\mathrm{Fil}^r B^+_{cris} = \mathbb{Q} \otimes \mathrm{Fil}^r A_{cris};$$

$$B^+_{dR} = \lim_s B^+_{cris} / \mathrm{Fil}^s B^+_{cris}$$

$$\mathrm{Fil}^r B^+_{dR} = \lim_{s \geq r} \mathrm{Fil}^r B^+_{cris} / \mathrm{Fil}^s B^+_{cris}.$$

Note: the filtrations of A_{cris} and B^+_{cris} are not complete.

Natural additive map:

$$\mathbb{Z}_p(1) \xrightarrow{\beta} \mathrm{Fil}^1 A_{cris}$$

defined as composite of

$$\begin{tikzcd}
 & & (1+J)^\times \arrow[d] \\
\mathbb{Z}_p(1) \arrow[r] \arrow[urr, bend left] \arrow[d, equal] & R_{\bar V}^\times \arrow[r, "-"] \arrow[d, equal] & W(R_{\bar V})^\times \arrow[d, "\theta"] \\
\lim\limits_{m\geq 0} \mu_{p^m} \arrow[r, hook] \arrow[rr, bend right, "0"'] & \lim\limits_{m\geq 0} (\bar V^\wedge)^\times \arrow[r, "pr_0"] & (\bar V^\wedge)^\times
\end{tikzcd}$$

and

$$(1+\bar J)^\times \xrightarrow{\log} \mathrm{Fil}^1 A_{cris}$$

$$1-x \longmapsto -\sum_{s\geq 1} \frac{x^s}{s} = \sum_{s \geq 1} (s-1)!\, x^{[s]}$$

Let $\varepsilon \in \mathbb{Z}_p(1)$ be a generator; define

$$t = \beta(\varepsilon)$$

$$B_{cris} = B_{cris}^+[t^{-1}], \qquad B_{dR} = B_{dR}^+[t^{-1}].$$

Then B_{dR}^+ is a CDVR with residue field $\bar K^\wedge$ and fraction field B_{dR}; t is a uniformizer so

$$gr^i B_{dR} \cong \bar K^\wedge(i).$$

Since B_{dR}^+ is a CDVR, it contains a maximal subfield, and $\theta : B_{dR}^+ \to \bar{K}^\wedge$ maps this subfield isomorphically onto $\hat{K}^\wedge$. Hence we have

$$\hat{K}^\wedge [[t]] \xrightarrow{\sim} B_{dR}^+ .$$

Let $\pi \in V$ be a uniformizer; choose $s \in R_{\bar{V}}$ s.t. $\hat{s}^{(0)} = \pi$; then $\theta(\underline{s}) = \pi$, so $\underline{s} \cdot \pi^{-1}$ lies in the kernel

$$\begin{array}{c} (1+\mathrm{Fil}^1 B_{dR}^+)^\times \longrightarrow (B_{dR}^+)^\times \xrightarrow{\theta} (\bar{V}^\wedge)^\times \\ \big\downarrow \log \\ \mathrm{Fil}^1 B_{dR}^+ \end{array}$$

Define

$$u_s := \log(\underline{s} \cdot \pi^{-1}) \in \mathrm{Fil}^1 B_{dR}^+ .$$

$$B_{st,\pi}^+ = B_{cryp}^+ [u_s] \subset B_{dR}^+$$

$$B_{st,\pi} = B_{cryp}[u_s] = B_{st,\pi}^+[t^{-1}] \subset B_{dR} .$$

Note: if $s, s' \in R_{\bar{V}}$ both lift π, then $u_s - u_{s'} \in \mathbb{Z}_p \cdot t \in B_{cryp}^+$. So $B_{st,\pi}^+$ only depends on π.

$K \subset L \subset \bar{K}$ fin. ext.

$(S_{L,n}, N_{L,n}) = (\operatorname{Spec} \mathcal{O}_{L,n}, \text{ can. log str.})$

$(S_{L,n}/W_n)^{\log}_{\mathrm{crys}} = ((S_{L,n}, N_{L,n})/(W_n, \mathcal{O}^*_{W_n}, pW_n, \gamma))_{\mathrm{crys}}$.

<u>Lemma</u> The natural map is an isomorphism

$$\operatorname*{colim}_L H^q((S_{L,n}/W_n)_{\mathrm{crys}}, \mathcal{O}_{S_{L,n}/W_n}) \xrightarrow{\sim} \operatorname*{colim}_L H^q((S_{L,n}/W_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{S_{L,n}/W_n})$$

and the common group is zero if $q > 0$.

Pf Let $k_L = \mathcal{O}_L/\mathfrak{m}_L$ and choose $\pi_L \in \mathcal{O}_L$ uniformizer; let $(V_L, M_{V_L}) = (\operatorname{Spec} W(k_L)[T_L], \text{"} T_L = 0 \text{"})$; exact closed immersions

$$\begin{tikzcd}
(S_L, N_L) \arrow[r, hook, "i_L"] & (V_L, M_{V_L}) & \pi_L \mapsfrom T_L \\
(S_{L,n}, M_{L,n}) \arrow[u, hook] \arrow[r, hook, "i_{L,n}"] \arrow[d, hook] & (V_{L,n}, M_{V_{L,n}}) \arrow[u] & \\
(E_{L,n}, M_{E_{L,n}}) \arrow[ur, "\lambda"'] & &
\end{tikzcd}$$

$(E_{L,n}, M_{E_{L,n}})$ divided power envelope.

usual divided power envelope of $S_{L,n}$ in $V_{L,n}$

$\lambda^* M_{V_{L,n}}$

Since $V_{L,n}/W_n$ and $(V_{L,n}, N_{L,n})/(W_n, \mathcal{O}^*_{W_n})$ are both smooth, we have

$$\begin{tikzcd}
R u_{S_{L,n}/W_n *}(\mathcal{O}_{S_{L,n}/W_n}) \arrow[r, "\sim"] \arrow[d] & \mathcal{O}_{E_{L,n}} \otimes_{\mathcal{O}_{V_{L,n}}} \Omega^{\bullet}_{V_{L,n}/W_n} \arrow[d] \\
R u^{\log}_{S_{L,n}/W_n *}(\mathcal{O}_{S_{L,n}/W_n}) \arrow[r, "\sim"] & \mathcal{O}_{E_{L,n}} \otimes_{\mathcal{O}_{V_{L,n}}} \omega^{\bullet}_{(V_{L,n}, M_{V_{L,n}})/W_n}
\end{tikzcd}$$

Here $\Omega^1_{V_{L,n}/W_n}$ and $\omega^1_{(V_{L,n}, M_{V_{L,n}})/W_n}$ are generated by $d\pi_L$ and $d\log \pi_L$, respectively. Pick $\pi_{L'} \in \bar{K}$ s.t. $\pi_{L'}^{p^n} = \pi_L$, let $L' = L(\pi_{L'})$. Then

$$\Omega^1_{V_{L,n}} \to \Omega^1_{V_{L',n}}, \quad \omega^1_{(V_{L,n}, M_{V_{L,n}})} \to \omega^1_{(V_{L',n}, M_{V_{L',n}})}$$

are zero. //

Fix a uniformizer $\pi \in K$, consider again

$$\begin{tikzcd}
(S_n, N_n) \arrow[r, hook] \arrow[dr, hook, "J_{E_n}"'] & (V_n, M_{V_n}), & \pi \leftarrow\!\shortmid T \\
& (E_n, M_{E_n}) \arrow[u] \arrow[loop right, "F_{E_n}"] & \text{PD-map w.r.t. } p\mathcal{O}_{E_n} + J_{E_n}
\end{tikzcd}$$

$$\| $$

$$(D_{(S_n,N_n),\gamma}(V_n, M_{V_n}), M_D)$$

$$\| $$

$$(D_{(S_1,N_1),\gamma}(V_n, M_{V_n}), M_D)$$

$$\circlearrowleft \varphi \qquad \circlearrowleft F_{V_n}: \begin{array}{l} T \longmapsto T^p \\ W \xrightarrow{\sigma} W \end{array}$$

$$(X/E_n)^{\log}_{\mathrm{crys}} := ((X,M)/(E_n, M_{E_n}, p\mathcal{O}_{E_n} + J_{E_n}, [-]))_{\mathrm{crys}}$$

$$\mathcal{G}\ \begin{cases} B_n = \operatorname{colim}_L H^0((S_{L,n}/W_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{S_{L,n}/W_n}) \\ \downarrow \qquad\qquad \downarrow \\ P_n := \operatorname{colim}_L H^0((S_{L,n}/E_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{S_{L,n}/E_n}) \\ \downarrow J_{P_n} \qquad\qquad \downarrow \\ \mathcal{O}_{E,n} = \operatorname{colim}_L \Gamma(S_{L,n}, \mathcal{O}_{S_{L,n}/E_n}) \end{cases}$$

since θ epi

$$0 \to J_{S_{L,n}/E_n} \to \mathcal{O}_{S_{L,n}/E_n} \to i_{S_{L,n}/E_n *}\mathcal{O}_{S_{L,n}} \to 0 \quad \Rightarrow$$

$$J_{P_n} = \operatorname{colim}_L H^0((S_{L,n}/E_n)^{\log}_{\mathrm{crys}}, J_{S_{L,n}/E_n})$$

The PD-str. on $J_{S_{L,n}/E_n}$ induces one on J_{P_n}.

Prop For every p^nth root β of $\pi \in \bar{K}$, there is a canonical element $v_\beta \in 1 + J_{P_n}$ s.t. there is a PD-isomorphism

$$B_n\langle v\rangle \xrightarrow{\sim} P_n, \quad v^{[i]} \mapsto (v_\beta - 1)^{[i]}$$

Pf Let $L \subset \bar{K}$, $\beta \in L$; write $(S', N') = (S_L, N_L)$, let $(S', N') \hookrightarrow (U', M_{U'})$ be a cl. imm. with

$(V', M_{V'}) \to W$ sm. e.g. $(\operatorname{Spec} W(k')[T'], \text{"} T' = 0 \text{"})$, $\pi' \leftarrow T'$. let

$$(V'', M_{V''}) = (V', M_{V'}) \underset{W}{\times} (V, M)$$

s.t. we have

$$(S', N') \hookrightarrow (S', N') \times (S, N) \hookrightarrow (V'', M_{V''}).$$

For example, $(V'', M_{V''}) = (\operatorname{Spec} W(k')[T', T], \text{"}T'=0\text{"}, \text{"}T=0\text{"})$, $\pi' \leftarrow T'$, $\pi \leftarrow T$. Consider

$$\begin{tikzcd}
(S'_n, N'_n) \arrow[r, hook] \arrow[d, hook, "J_{E'_n}"'] & (V'_n, M_{V'_n}) \\
(E'_n, M_{E'_n}) \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(S'_n, N'_n) \arrow[r, hook, "\text{not exact}"] \arrow[d, hook, "J_{E''_n}"'] & (V''_n, M_{V''_n}) \\
(E''_n, M_{E''_n}) \arrow[ur] &
\end{tikzcd}$$

Choose local section $\tilde{\beta}$ of $M_{E'_n}$ which maps to $\beta \in \Gamma(S'_n, N'_n)$. Then $\tilde{\beta}^{p^n}$ is indept. of choice of $\tilde{\beta}$, so they glue to give a global section $\alpha \in \Gamma(E'_n, M_{E'_n})$ which maps to $\pi \in \Gamma(S'_n, N'_n)$.

$$\begin{tikzcd}
(E''_n, M_{E''_n}) \arrow[r, "g"] \arrow[d] & (E'_n, M_{E'_n}) \arrow[d] & \text{PD-mor} \\
(V''_n, M_{V''_n}) \arrow[r, "pr_1"] & (V'_n, M_{V'_n}) &
\end{tikzcd}$$

Then

$$\begin{array}{ccc}
\Gamma(E''_n, M_{E''_n}) & \longrightarrow & \Gamma(S'_n, N'_n) \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
g^*(\alpha), T & \longmapsto & \pi
\end{array}$$

By def^n of log-PD-envelope,

$$(S_n', N_n') \xrightarrow{i} (E_n'', M_{E_n''})$$

is an <u>exact</u> cl. imm. So exact seq. of monoids (i.e. the orbits by the action of the group on the left on the middle term is iso., via the given map, to the right hand term)

$$1 + J_{E_n''} \to M_{E_n''} \to i_* N_{S_n'}$$

Let $v_{\beta,L} \in \Gamma(E_n'', 1 + J_{E_n''})$ be the unique element s.t.

$$v_{\beta,L} \cdot g^*(\alpha) = T. \ /$$

Since $(V_n', M_{V_n'}) \to W$ is sm., so is

$$(V_n', M_{V_n'}) \times_W (E_n, M_{E_n}) \to (E_n, M_{E_n})$$

Moreover

$$(E_n'', M_{E_n''}) \cong D_{(S_n', N_n'), p\mathcal{O}_{E_n} + J_{E_n}}((V_n', M_{V_n'}) \times_W (E_n, M_{E_n}))$$

So

$$Ru^{\log}_{S_n'/E_n *}(\mathcal{O}_{S_n'/E_n}) \xrightarrow{\sim} \mathcal{O}_{E_n''} \otimes_{\mathcal{O}_{V_n'}} \omega^{\bullet}_{(V_n', M_{V_n'})/W_n}$$

The prop. is thus reduced to:

<u>Lemma</u> $\Gamma(E_n', \mathcal{O}_{E_n'})\langle V\rangle \xrightarrow{\sim} \Gamma(E_n'', \mathcal{O}_{E_n''})$, $V \mapsto V_{\beta,L} - 1$.

pf Tsuji, J. reine angew. Math. 472 (1996), 69-138; lemma 2.2.2. //

<u>Lemma</u> Let (S, L, I, γ) be a PD-scheme with fine log str. and consider the diagram of fine log schemes over (S, L)

$$\begin{tikzcd}
(X,M) \arrow[rr, hook, "i"] \arrow[dr, hook] & & (Y,N) \\
& (D, M_D) \arrow[ur] & \\
& & \\
(X,M) \arrow[uuu, equal] \arrow[rr, hook, "i'"] \arrow[dr, hook] & & (Y', N') \arrow[uuu, "f"'] \\
& (D', M_{D'}) \arrow[ur] \arrow[uuu, "f_D"] &
\end{tikzcd}$$

where i, i' are closed imm., f is smooth, (Y,N) and (Y',N') are smooth over (S,L), and (D, M_D) and $(D', M_{D'})$ are the PD-envelopes. Let $x \in X$ and pick $t_1, \ldots, t_r \in N'_{\bar{x}}$ s.t.

$$\mathcal{O}_{Y', \bar{x}}\{d\log t_1, \ldots, d\log t_r\} \xrightarrow{\sim} \omega^1_{(Y',N')/(Y,N), \bar{x}}$$

and $s_1, \ldots, s_r \in N_{\bar{x}}$ s.t. $i^*_{\bar{x}}(s_i) = i'^*_{\bar{x}}(t_i)$. Let $u_1, \ldots, u_r \in 1 + J_{D',x}$ be the unique elements s.t. $t_i = f^*_{D,\bar{x}}(s_i)$. Then there is a PD-iso:

$$\mathcal{O}_{D,\bar{x}}\langle T_1, \ldots, T_r\rangle \xrightarrow{\sim} \mathcal{O}_{D',\bar{x}}$$

$$T_i \longmapsto u_i - 1 .$$

//

$$\begin{tikzcd}
(S_n,N_n) \arrow[r, hook] \arrow[d, hook, "J_{E_n}"'] \arrow[dd, equal] & (V_n,M_{V_n}) = (\mathrm{Spec}\, W_n[T], \text{"}T=0\text{"}) \\
(E_n,M_{E_n}) \arrow[ur] & \\
(S_n,N_n) \arrow[r, hook] \arrow[d, hook, "J_{E_n(1)}"'] & (V_n,M_{V_n}) \times_{W_n} (V_n,M_{V_n}) \arrow[uu, "p_i\ (sm)"'] \\
(E_n(1),M_{E_n(1)}) \arrow[ur] &
\end{tikzcd}$$

— not exact

$$p_1^*(T) = u \cdot p_2^*(T), \qquad u \in \Gamma(E_n(1), 1+J_{E_n(1)})$$

$$p_2^{-1}\mathcal{O}_{E_n}\langle u-1\rangle \xrightarrow{\sim} \mathcal{O}_{E_n(1)} \xleftarrow{\sim} p_1^{-1}\mathcal{O}_{E_n}\langle u^{-1}-1\rangle$$

For $(X,M) \to (S,N)$ syntomic (explain later) $\Rightarrow$

$$p_i^* H^q((X_n/E_n)^{\log}_{crys}, \mathcal{O}_{X_n/E_n}) \xrightarrow{\sim} H^q((X_n/E_n(1))^{\log}_{crys}, \mathcal{O}_{X_n/E_n(1)})$$

get

$$\begin{tikzcd}
H^q(X_n/E_n) \arrow[r, "\varepsilon"] \arrow[ddr, "\nabla"'] & p_{2*}p_1^* H^q(X_n/E_n) \arrow[d, equal] & \\
& H^m(X_n/E_n) \otimes_{\Gamma(E_n,\mathcal{O}_{E_n})} \Gamma(E_n(1), \mathcal{O}_{E_n(1)}) \arrow[d] & u^{-1}-1 \arrow[dd, mapsto] \\
& H^m(X_n/E_n) \otimes_{\Gamma(E_n,\mathcal{O}_{E_n})} \Gamma(E_n(1), \mathcal{O}_{E_n(1)}/J_\Delta^2) \arrow[d, equal] & \\
& H^m(X_n/E_n) \cdot d\log T & d\log T
\end{tikzcd}$$

$$\nabla(x) =: N(x) \cdot d\log T$$

Consider $(X, M) = (S_L, N_L) \longrightarrow (S, N)$, $\beta \in L$ p^n th root of $\pi \in K$. Will recall

$$H^0((S_{L,n}/W_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{S_{L,n}/W_n})\langle v\rangle$$

$$\xrightarrow{\sim} H^0((S_{L,n}/E_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{S_{L,n}/E_n}), \quad v \longmapsto v_{\beta,L} - 1,$$

and show

$$N((v_{\beta,L}-1)^{[i]}) = (v_{\beta,L}-1)^{[i-1]} \cdot v_{\beta,L}$$

Write $(S'_n, N'_n) = (S_{L,n}, N_{L,n})$.

$$\begin{tikzcd}
(S'_n, N'_n) \arrow[r, hook] \arrow[d, equal] \arrow[dr, hook] & (V'_n, M_{V'_n}) = (\mathrm{Spec}\, W'_n[T'], \text{"}T'=0\text{"}) & T' \\
 & (E'_n, M_{E'_n}) \arrow[u] & T' \arrow[u, mapsto] \\
(S'_n, N'_n) \arrow[r, hook] \arrow[dr, hook] & (V''_n, M_{V''_n}) = (\mathrm{Spec}\, W'_n[T', T], \text{"}T'T=0\text{"}) \arrow[uu, bend right, "\mathrm{proj}"'] & \\
 & (E''_n, M_{E''_n}) \arrow[u] \arrow[uu, "g"] &
\end{tikzcd}$$

$\pi', \pi \longleftarrow T', T$ — not exact — factor:

$$(S'_n, N'_n) \overset{\text{exact}}{\hookrightarrow} (Z, M_Z) \xrightarrow{\text{ét}} (V''_n, M_{V''_n})$$

$$\parallel$$

$$(\mathrm{Spec}\, W'_n[T', x, x^{-1}], \text{"}T'=0\text{"})$$

$$\pi', \frac{\pi'^e}{\pi} \longleftarrow T', x \,;\; T', T'^e x^{-1} \longleftarrow T', T$$

(Note $\pi'^{e} + \pi\theta(\pi') = 0$, $\frac{\pi'^{e}}{\pi} = -\theta(\pi') \in \mathcal{O}_L^{\times}$.)

Show $(Z, M_Z) \longrightarrow (V_n'', M_{V_n''})$ étale: chart:

$$\begin{tikzcd}
W_n[T', x, x^{-1}] & W_n'[T', T] \arrow[l] \\
\mathbb{N}_0 \times \mathbb{N}_0 \arrow[u] & \mathbb{N}_0 \times \mathbb{N}_0 \arrow[l] \arrow[u]
\end{tikzcd}$$

$T'^{i} x^{-j} \leftarrow\!\shortmid (i,j)$ $\qquad$ $T'^{i} T^{j} \leftarrow\!\shortmid (i,j)$

$P = \mathbb{N}_0 \times \mathbb{N}_0$, $\quad (i, ei+j) \leftarrow\!\shortmid (i,j)$, $\quad \mathbb{N}_0 \times \mathbb{N}_0 = Q$

$$\begin{tikzcd}
W_n'[T', T] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P] \arrow[rr] \arrow[dr, "\sim"'] & & W_n'[T', x^{\pm 1}] \\
& W_n'[T', x^{-1}] \arrow[ur, "\text{étale}"'] &
\end{tikzcd}$$

so OK

Suppose $\pi' = \beta$, $e = p^n$; $L = K(\beta)$.

$$\tilde{\beta} = T' \in \Gamma(E_n', \mathcal{O}_{E_n'}) \longmapsto \beta \in \Gamma(S_n', \mathcal{O}_{S_n'})$$

$$\alpha = \tilde{\beta}^{p^n} = T'^{p^n}$$

$$v \cdot g^*(\alpha) = T \iff v \cdot T'^{p^n} = T'^{p^n} \cdot x^{-1}$$

so

$$v_{\beta, K(\beta)} = x^{-1}.$$

Now calculate $N(V_\beta)$.

$$\begin{tikzcd}
(S'_n, N'_n) \arrow[rr, hook] \arrow[dr] \arrow[dd, equal] & & (V''_n, M_{V''_n}) = (\operatorname{Spec} W'_n[T', T], \text{"}T'T = 0\text{"}) \\
& (E''_n, M_{E''_n}) \arrow[ur] & \\
(S'_n, N'_n) \arrow[rr, hook] \arrow[dr] & & (V''_n(1), M_{V''_n(1)}) = (\operatorname{Spec} W'_n[T', T_1, T_2], \text{"}T'T_1T_2 = 0\text{"}) \arrow[uu, "p_i'"'] \\
& (E''_n(1), M_{E''_n(1)}) \arrow[ur] \arrow[uu, "p_i'"] &
\end{tikzcd}$$

$$p_i': \quad T' \longmapsto T', \qquad T \longmapsto T_i'$$

Factor:

$$\begin{tikzcd}
(S'_n, N'_n) \arrow[r, hook] \arrow[d, equal] & (\operatorname{Spec} W'_n[T', x^{\pm 1}], \text{"}T' = 0\text{"}) \arrow[r] & (\operatorname{Spec} W'_n[T', T], \text{"}T'T = 0\text{"}) \\
(S'_n, N'_n) \arrow[r, hook] & (\operatorname{Spec} W'_n[T', x_1^{\pm 1}, x_2^{\pm 1}], \text{"}T' = 0\text{"}) \arrow[r] \arrow[u, "p_i'"] & (\operatorname{Spec} W'_n[T', T_1, T_2], \text{"}T'T_1T_2 = 0\text{"}) \arrow[u]
\end{tikzcd}$$

$$T' \longmapsto T', \qquad x \longmapsto x_i'$$

$\underline{i=2}|$ $\quad t = T_1 \;:\; \mathcal{O}_{V''_n(1)} \cdot d\log T_1 \xrightarrow{\sim} \omega^1_{(V''_n(1), M_{V''_n(1)})/(V''_n, M_{V''_n})}$

$\quad s = T \;:\; \iota^*(T) = \iota'^*(T_1)$

$$T_1 = p_2^*(T) \cdot u = T_2 \cdot u$$

$$T' p^n x_1^{-1} = T' p^n x_2^{-1} u \longrightarrow u = x_1^{-1} x_2$$

so

$$p_1^{-1} \mathcal{O}_{E''_n} \langle x_1 x_2^{-1} - 1 \rangle \xrightarrow{\sim} \mathcal{O}_{E''_n(1)} \xleftarrow{\sim} p_2^{-1} \mathcal{O}_{E''_n} \langle x_1^{-1} x_2 - 1 \rangle$$

$$\left.\begin{array}{lllll} p_1^*(v_\beta) & = & p_1^*(x^{-1}) & = & x_1^{-1} \\ p_2^*(v_\beta) & = & p_2^*(x^{-1}) & = & x_2^{-1} \end{array}\right\} \Rightarrow$$

$$\mathbb{1} \otimes v_\beta \xmapsto{p_2^*} x_2^{-1} = x_1^{-1} \cdot x_1 x_2^{-1} \xleftarrow{p_1^*} v_\beta \otimes x_1 x_2^{-1}$$

$$\| $$

$$v_\beta \otimes \mathbb{1} + \underbrace{v_\beta \otimes (x_1 x_2^{-1} - \mathbb{1})}$$

$$\nabla(v_\beta) = v_\beta \, d\log T$$

$$\nabla((v_\beta - 1)^{[i]}) = (v_\beta - 1)^{[i-1]} \, \nabla(v_\beta - 1)$$

$$= (v_\beta - 1)^{[i]} \, \nabla(v_\beta) = (v_\beta - 1)^{[i-1]} \, v_\beta \cdot d\log T$$

So

$$N((v_\beta - 1)^{[i]}) = (v_\beta - 1)^{[i-1]} \cdot v_\beta$$

We also have

$$\begin{tikzcd}
(S_n', N_n') \arrow[r, hook] \arrow[d, equal] & (V_n', M_{V_n'}) \underset{W_n}{\times} (E_n, M_{E_n}) \arrow[r, "sm"] & (E_n, M_{E_n}) \\
(E_n', M_{E_n'}) \arrow[u, hook] \arrow[ur] & & \\
(S_n', N_n') \arrow[u, "pi'"] \arrow[r, hook] \arrow[d, hook] & (V_n', M_{V_n'}) \underset{W_n}{\times} (E_n(1), M_{E_n(1)}) \arrow[uu, "pi'"] \arrow[r, "sm"] & (E_n(1), M_{E_n(1)}) \arrow[uu, "pi'"] \\
(E_n'', M_{E_n''}) \arrow[ur] & &
\end{tikzcd}$$

so

$$\begin{array}{ccc}
Ru^{\log}_{S_n'/E_n} & \xrightarrow{\sim} & \mathcal{O}_{E_n'} \underset{\mathcal{O}_{V_n'}}{\otimes} \omega^{1}_{(V_n', M_{V_n'})/W_n} \\
\downarrow pi'^{*} & & \downarrow pi'^{*} \\
Ru^{\log}_{S_n'/E_n(1)} & \xrightarrow{\sim} & \mathcal{O}_{E_n''} \underset{\mathcal{O}_{V_n'}}{\otimes} \omega_{(V_n', M_{V_n'})/W_n}
\end{array}$$

Thus the monodromy operator we evaluated <u>IS</u>

$$H^0((S_n'/E_n)^{\log}_{\text{crys}}, \mathcal{O}_{S_n'/E_n}) \circlearrowleft N.$$

Let $(X,M) \xrightarrow{f} (Y,N)$ be a map of log schemes with fine log str. Then f is <u>integral</u> if for every $(Y',N') \to (Y,N)$, the log str. on

$$(Y',N') \underset{(Y,N)}{\times} (X,M)$$

is fine, or equivalently, if there exists a chart $(P_X \to M,\ Q_Y \to N,\ Q \to P)$ of f s.t.

$$\mathbb{Z}[Q] \to \mathbb{Z}[P]$$

is flat.

The map $(X,M) \xrightarrow{f} (Y,N)$ is <u>syntomic</u> if it is integral, if $X \to Y$ is flat and locally fin. presented, and if étale locally on X, there exists a factorization

$$\begin{tikzcd}(X,M) \arrow[r, hook, "i"] \arrow[rr, bend right, "f"'] & (Z,M_Z) \arrow[r, "h"] & (Y,N)\end{tikzcd}$$

with i an exact closed immersion, $X \hookrightarrow Z$ a regular closed immersion, and h smooth.

<u>ex</u> $(\operatorname{Spec} \mathcal{O}_L, N_L) \xrightarrow{f} (\operatorname{Spec} \mathcal{O}_K, N_K)$ is syntomic

$$\begin{tikzcd}(\operatorname{Spec} \mathcal{O}_L, N_L) \arrow[r, "f"] \arrow[d, hook] & (\operatorname{Spec} \mathcal{O}_K, N_K) \\ (\operatorname{Spec} \mathcal{O}_K[T_L], \text{“} T_L = 0 \text{”}) \arrow[ur] & \end{tikzcd}$$

$\pi_L \leftarrow\!\!\dashv T_L$

Chart: choose unif. $\pi_K \in K$, $\pi_L \in L$.

$$\begin{array}{cccccc} i & Q = \mathbb{N}_0 & \longrightarrow & \mathcal{O}_K & & i \longmapsto \pi_K^i \\ \downarrow & \downarrow & & \downarrow f^* & & \\ (e_{L/K}\, i, i) & P = \mathbb{N}_0 \times \mathbb{N}_0 & \longrightarrow & \mathcal{O}_L & & (i,j) \longmapsto \pi_L^i \left(\frac{\pi_K}{\pi_L^{e_{L/K}}} \right)^j \end{array}$$

then

$$\begin{array}{ccccc} \mathbb{Z}[Q] & = & \mathbb{Z}[x] & & x \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z}[P] & = & \mathbb{Z}[y,z] & & y^{e_{L/K}} z \end{array} \quad \text{flat.} \quad //$$

Thm (Syntomic base change) Suppose given a diagr. of fine log schemes

$$\begin{array}{ccc} (X', M') & \xrightarrow{g'} & (X, M) \\ \downarrow f' & & \downarrow f \\ (Y', N') & \xrightarrow{g} & (Y, N) \\ \downarrow & & \downarrow \\ (S', L', I', \gamma') & \xrightarrow{h} & (S, L, I, \gamma) \end{array} \qquad m\mathcal{O}_S = 0$$

with f syntomic, $X \to Y$ quasi-compact and quasi-separated. Then for every q-coh. flat crystal of $\mathcal{O}_{X/S}$-modules E, there is a natural iso

$$Lg^*_{crys}(Rf_{crys*}(E)) \xrightarrow{\sim} Rf'_{crys*}(g'^*_{crys}(E)). \quad //$$

<u>Note</u> $Rf_{crys*}(E)$ coh. bd.

<u>pf</u> (first step) must show for all $\begin{array}{ccc}(U',M_{U'}) & \to & (Y',N') \\ \int & & \downarrow \\ (T',M_{T'}) & \to & (S',L')\end{array}$ in $(Y'/S')^{log}_{crys}$ that

$$Lg^{*}_{crys}(Rf_{crys*}(E))\begin{pmatrix}(U',M_{U'}) & \to & (Y',N') \\ \int & & \downarrow \\ (T',M_{T'}) & \to & (S',L')\end{pmatrix}$$

$$\xrightarrow{\sim} Rf'_{crys*}(g'^{*}_{crys}(E))\begin{pmatrix}(U',M_{U'}) & \to & (Y',N') \\ \int & & \downarrow \\ (T',M_{T'}) & \to & (S',L')\end{pmatrix}$$

is an iso. in derived cat. of q. coh. $\mathcal{O}_{T'}$-mod. on T'_{et}. Given a comm. diagr.

$$\begin{tikzcd}
(U',M_{U'}) \arrow[rr] \arrow[dd, hook] \arrow[dr] & & (Y',N') \arrow[dr, "g"] \arrow[dd] & \\
& (U,M_U) \arrow[rr] \arrow[dd, hook] & & (Y,N) \arrow[dd] \\
(T',M_{T'}) \arrow[rr] \arrow[dr, "u"'] & & (S',L') \arrow[dr, "h"] & \\
& (T,M_T) \arrow[rr] & & (S,L)
\end{tikzcd}$$

with the front face in $(Y/S)^{log}_{crys}$, there is a natural iso.

$$L u^* \left(Rf_{crys*}(E)_{\begin{pmatrix} (U, M_U) & \to & (Y, N) \\ \cap & & \downarrow \\ (T, M_T) & \to & (S, L) \end{pmatrix}} \right)$$

$$\xrightarrow{\sim} L g^*_{crys} \left(Rf_{crys*}(E) \right)_{\begin{pmatrix} (U', M_{U'}) & \to & (Y', N') \\ \cap & & \downarrow \\ (T', M_{T'}) & \to & (S', L') \end{pmatrix}}$$

This follows from def^n of g^*_{crys}. Next, give a new description of

$$Rf_{crys*}(E)_{\begin{pmatrix} (U, M_U) & \to & (Y, N) \\ \cap & & \downarrow \\ (T, M_T) & \to & (S, L) \end{pmatrix}} .$$

Will not discuss remaining part of proof. But we will give this new description in a special case:

Prop Let

$$\begin{array}{ccc} (X', M') & \xrightarrow{g'} & (X, M) \\ \downarrow f' & \text{cart} & \downarrow f \\ (Y', N') & \xrightarrow{g} & (Y, N) \\ \cap i' & & \cap i \\ (S', L', I', \gamma') & \xrightarrow{h} & (S, L, I, \gamma) \end{array}$$

be as before and assume, in addition, that i and i' are exact closed immersions given by PD-subideals of I and I'. Let

$$f_{X/S}^{\log} : (X/S)_{crys}^{\log} \xrightarrow{u_{X/S}^{\log}} X_{et}^{\sim} \longrightarrow S_{et}^{\sim}$$

$$f'^{\log}_{X'/S'} : (X'/S')_{crys}^{\log} \xrightarrow{u_{X'/S'}^{\log}} X'^{\sim}_{et} \longrightarrow S'^{\sim}_{et}$$

Then for every q.coh. flat crystal E of $\mathcal{O}_{X/S}$-modules,

$$Lh^*(Rf^{\log}_{X/S*}(E)) \xrightarrow{\sim} Rf'^{\log}_{X'/S'*}(g'^*_{crys}(E)).$$

Pf

$$\begin{array}{ccc} (Y,N) & \xrightarrow{id} & (Y,N) \\ \int i & & \downarrow \\ (S,L) & \xrightarrow{id} & (S,L) \end{array} \in (Y/S)^{\log}_{crys}$$

claim: iso. in $S_{et}^{\sim}$

$$Rf_{crys*}(E)\begin{pmatrix} (Y,N) \longrightarrow (Y,N) \\ \int i \qquad \downarrow \\ (S,L) \longrightarrow (S,L) \end{pmatrix} = Rf_{X/S*}(E).$$

Show the global sections are the same:

$$f_{crys*}(E)(S) = (Y/S)^{\log}_{crys}(S^{\sim}, f_{crys*}(E))$$

$$= (X/S)^{\log}_{crys}(f^*_{crys}(S^{\sim}), E)$$

$$f_{X/S*}(E)(S) = \Gamma(X, u_{X/S*}(E)) = \Gamma((X/S)^{\log}_{crys}, E)$$

$$= (X/S)^{\log}_{crys}(e, E).$$

Need $f^*_{crys}(S^\sim) = e \in (X/S)^{\log}_{crys}$: by def$^{\underline{n}}$

$$f^*_{crys}(S^\sim)\left(\begin{array}{ccc}(U,M_U) & \to & (X,M)\\ \downarrow & & \downarrow \\ (T,M_T) & \to & (S,L)\end{array}\right)$$

$$= \left\{ \begin{tikzcd}
(U,M_U) \arrow[rr] \arrow[dr] \arrow[dd] & & (X,M) \arrow[dr] \arrow[dd] & \\
& (Y,N) \arrow[rr, "id"] \arrow[dd] & & (Y,N) \arrow[dd] \\
(T,M_T) \arrow[rr] \arrow[dr] & & (S,L) \arrow[dr] & \\
& (S,L) \arrow[rr, "id"] & & (S,L)
\end{tikzcd} \right\}$$

= one-pt set. //

<u>Cor</u> Suppose in addition that S, S' are affine. Then there is natural iso.

$$\Gamma(S', \mathcal{O}_{S'}) \otimes^{L}_{\Gamma(S,\mathcal{O}_S)} R\Gamma((X/S)^{\log}_{crys}, E)$$

$$\xrightarrow{\sim} R\Gamma((X'/S')^{\log}_{crys}, g'^{*}_{crys}(E)).$$

pf This is the composite

$$\Gamma(S', \mathcal{O}_{S'}) \otimes^{L}_{\Gamma(S,\mathcal{O}_S)} R\Gamma((X/S)^{\log}_{crys}, E)$$

$$\xleftarrow{\sim} \Gamma(S', \mathcal{O}_{S'}) \otimes^{L}_{\Gamma(S,\mathcal{O}_S)} R\Gamma(S, Rf^{\log}_{X/S*}(E))$$

$$\xrightarrow{?} R\Gamma(S', Lh^*(Rf^{\log}_{X/S*}(E)))$$

$$\xrightarrow[\text{(prop)}]{\sim} R\Gamma(S', Rf'^{\log}_{X'/S'*}(g'^*_{crys}(E)))$$

$$\xrightarrow{\sim} R\Gamma((X'/S')^{\log}_{crys}, g'^*_{crys}(E)) .$$

Must show (?) iso. Let $M^\cdot$ represent $Rf^{\log}_{X/S*}(E)$. Then $M^\cdot$ is a cx. of q. coh. $\mathcal{O}_S$-modules s.t. $M^\cdot$ and $Lh^*(M^\cdot)$ are coh. bd. Using

$$\tau_{\leq s-1}M^\cdot \to \tau_{\leq s}M^\cdot \to H^s(M^\cdot)[s] \to \tau_{\leq s-1}M^\cdot[1]$$

we, inductively, are left to show

$$\Gamma(S', \mathcal{O}_{S'}) \otimes^{L}_{\Gamma(S,\mathcal{O}_S)} R\Gamma(S, H^s(M^\cdot)[s])$$

$$\xrightarrow{\sim} R\Gamma(S', Lh^*(H^s(M^\cdot)[s])) .$$

But S and S' are aff. and $H^s(M^\cdot)[s]$ q. coh., so $R\Gamma = \Gamma$ and this is clear. //

Use this on

$$\begin{tikzcd}
(X_n, M_n) \arrow[r, "id"] \arrow[d, "f"] & (X_n, M_n) \arrow[d, "f"] \\
(S_n, N_n) \arrow[r, "id"] \arrow[d, hook] & (S_n, N_n) \arrow[d, hook] \\
(E_n(1), M_{E_n(1)}) \arrow[r, "P_i'"] & (E_n, M_{E_n})
\end{tikzcd}$$

to get

$$\Gamma(E_n(1), \mathcal{O}_{E_n(1)}) \underset{P_i^{*} \uparrow \; \Gamma(E_n, \mathcal{O}_{E_n})}{\otimes} H^m((X_n/E_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{X_n/E_n})$$

$$\xrightarrow{\sim} H^m((X_n/E_n(1))^{\log}_{\mathrm{crys}}, \mathcal{O}_{X_n/E_n(1)}).$$

This gives the monodromy operator

$$H^m((X_n/E_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{X_n/E_n}) \circlearrowleft N$$

11/27/00

Lemma Given a diagr. of fine log schemes

$$\begin{tikzcd} (X,M) \arrow[r, "\text{syn}"] \arrow[d, hook, "\text{cl. imm.}"'] & (S,N) \arrow[d, hook, "\text{exact cl. imm. def. by PD-subideal of } I"] \\ (Z,M_Z) \arrow[r, "\text{sm}"] & (T,M_T,I,\gamma) \end{tikzcd}$$

let $(D,M_D) = D_{(X,M),\gamma}(Z,M_Z)$. Then $D \to T$ is flat.

Pf see Tsuji 4.5.5. //

Recall:

$$\begin{tikzcd} (S_n,N_n) \arrow[r, hook, "T \leftarrow T"] \arrow[dr, hook] & (V_n, M_{V_n}) = (\mathrm{Spec}\, W_n[T], \text{"}T=0\text{"}) \\ & (E_n, M_{E_n}) \arrow[u] \quad \text{PD-envelope} \end{tikzcd}$$

$$P_n := \operatorname{colim}_L H^0((S_{L,n}/E_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{S_{L,n}/E_n})$$

Prop P_n is flat over $\Gamma(E_n, \mathcal{O}_{E_n})$.

Pf let $R_{E_n} = \Gamma(E_n, \mathcal{O}_{E_n})$ and let A be an R_{E_n}-module. Must show $\mathrm{Tor}_i^{R_{E_n}}(A, P_n) = 0$, $i>0$.

If F q.coh. crystal of $\mathcal{O}_{S_n/E_n}$-modules, let

$$F_L = f^*F = \mathcal{O}_{S_{L,n}/E_n} \otimes_{f^{-1}\mathcal{O}_{S_n/E_n}} f^{-1}F, \quad S_{L,n} \xrightarrow{f} S_n;$$

then

$$\operatorname*{colim}_{L} H^m((S_{L,n}/E_n)^{\log}_{\mathrm{crys}}, F_L) = 0, \quad m > 0. \;/$$

A R_{E_n}-module $\rightsquigarrow$ q. coh. $\mathcal{O}_{E_n}$ - module $\tilde{A}$.

$\rightsquigarrow$ q. coh. crystal $\mathcal{O}_{S_n/E_n}$-modules F_A :

$$(F_A)\begin{pmatrix} \begin{tikzcd} (U,M_U) \arrow[r] \arrow[d, hook] & (S_n,N_n) \arrow[d, hook] \\ (T,M_T) \arrow[r, "g"] & (E_n,M_{E_n}) \end{tikzcd} \end{pmatrix} := g^*\tilde{A}$$

Choose

$$\begin{tikzcd} & (S_{L,n},N_{L,n}) \arrow[r, "\mathrm{sqn}"] \arrow[dl] \arrow[d, hook, "\text{exact}"] & (S_n,N_n) \arrow[d, hook, "J_{E_n}"] \\ (D,M_D) \arrow[dr] & {} & {} \\ & (Z,M_Z) \arrow[r, "\mathrm{sm}"] & (E_n,M_{E_n}, p\mathcal{O}_{E_n}+J_{E_n}, [-]) \end{tikzcd}$$

with Z affine. Then

$$R\Gamma((S_{L,n}/E_n)^{\log}_{\mathrm{crys}}, F_{A,L}) \cong \Gamma(D, (F_{A,L})_D \otimes_{\mathcal{O}_Z} \omega^{\bullet}_{(Z,M_Z)/(E_n,M_{E_n})})$$

$$= A \otimes_{R_{E_n}} \Gamma(D, \mathcal{O}_D \otimes_{\mathcal{O}_Z} \omega^{\bullet}_{(Z,M_Z)/(E_n,M_{E_n})})$$

$$= A \otimes^{L}_{R_{E_n}} \underbrace{R\Gamma((S_{L,n}/E_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{S_{L,n}/E_n})}_{\text{coh. bd.}}$$

since $D \to E_n$ flat.

Sp. seq.

$$E_2^{s,t} = \mathrm{Tor}_{-s}^{R_{E_n}}(A, H^t((S_{L,n}/E_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{S_{L,n}/E_n}))$$

$$\Rightarrow H^{s+t}((S_{L,n}/E_n)^{\log}_{\mathrm{crys}}, F_{A,L})$$

taking colimit over L:

$$E_2^{s,t} = \begin{cases} \mathrm{Tor}_{-s}^{R_{E_n}}(A, P_n) & , t=0 \\ 0 & , t>0 \end{cases}$$

$$\Rightarrow \begin{cases} \operatorname{colim}_L H^0((S_{L,n}/E_n)^{\log}_{\mathrm{crys}}, F_{A,L}) & , s=t=0 \\ 0 & , \text{else} \end{cases}$$

so $\mathrm{Tor}_i^{R_{E_n}}(A, P_n) = 0$, $i>0$. //

Define

$$(X_L, M_L) = (X, M) \times_{(S,N)} (S_L, N_L)$$

$$H^m((\bar{X}_n/E_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{\bar{X}_n/E_n})$$

$$:= \operatorname{colim}_L H^m((X_{L,n}/E_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{X_{L,n}/E_n}).$$

Prop Suppose $(X,M) \to (S,N)$ sm., $X \to S$ q. cpt. and q. sep. Then

$$P_n \otimes_{R_{E_n}} H^m((X_n/E_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{X_n/E_n}) \xrightarrow{\sim} H^m((\bar{X}_n/E_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{\bar{X}_n/E_n}).$$

Pf Will show:

$$R\Gamma((S_{L,n}/E_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{S_{L,n}/E_n}) \otimes^{L}_{R_{E_n}} R\Gamma((X_n/E_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{X_n/E_n}) \qquad (+)$$

$$\xrightarrow{\sim} R\Gamma((X_{L,n}/E_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{X_{L,n}/E_n})$$

This gives a sp. seq. (since coh. bd.)

$$E_2^{s,t} = \bigoplus_{i+j=t} \mathrm{Tor}_{-s}^{R_{E_n}} \big(H^i((S_{L,n}/E_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{S_{L,n}/E_n}),$$
$$H^j((X_n/E_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{X_n/E_n}) \big)$$

$$\Rightarrow H^{s+t}((X_{L,n}/E_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{X_{L,n}/E_n})$$

and hence by taking colimit over $L \subset \bar{K}$:

$$E_2^{s,t} = \begin{cases} P_n \otimes_{R_{E_n}} H^t((X_n/E_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{X_n/E_n}) & , s=0 \\ 0 & , s>0 \end{cases}$$

$$\Rightarrow H^{s+t}((\bar{X}_n/E_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{\bar{X}_n/E_n}).$$

Assume first that there exists a cartesian sq. with Z (and hence X_n) affine

$$\begin{tikzcd} & (X_n, M_n) \arrow[r, "\mathrm{sm.}"] \arrow[d, "\mathrm{exact}"] & (S_n, N_n) \arrow[d, "\supset E_n"] \\ (D, M_D) \arrow[ur, hook] \arrow[r, "\mathrm{id}"'] & (Z, M_Z) \arrow[r, "\mathrm{sm.}"] & (E_n, M_{E_n}) \end{tikzcd}$$

Write $(S_n', N_n') = (S_{L,n}, N_{L,n})$ and choose

$$\begin{tikzcd}
(S_n', N_n') \arrow[r, hook, "\text{exact}"] \arrow[dr, hook] & (V', M_{V'}) \arrow[r, "\text{sm}"] & (E_n, M_{E_n}) \\
& (E', M_{E'}) \arrow[u] &
\end{tikzcd}$$

PD-envelope comp. w. $p\mathcal{O}_{E_n} + J_{E_n}$

Let

$$\begin{tikzcd}
(X_n', M_n') = (X_n, M_n) \times_{(S_n, N_n)} (S_n', N_n') \arrow[r, hook] \arrow[dr, hook] & (Z, M_Z) \times_{(E_n, M_{E_n})} (E', M_{E'}) =: (Z', M_{Z'}) \\
& (D', M_{D'}) \arrow[u]
\end{tikzcd}$$

PD-envelope comp. w. $p\mathcal{O}_{E_n} + J_{E_n}$

Then

$$Ru^{\log}_{S_n'/E_n *}(\mathcal{O}_{S_n'/E_n}) \xrightarrow{\sim} \mathcal{O}_{E'} \otimes_{\mathcal{O}_{V'}} \omega^{\cdot}_{(V', M_{V'})/(E_n, M_{E_n})}$$

$$Ru^{\log}_{X_n/E_n *}(\mathcal{O}_{X_n/E_n}) \xrightarrow{\sim} \mathcal{O}_D \otimes_{\mathcal{O}_Z} \omega^{\cdot}_{(Z, M_Z)/(E_n, M_{E_n})}$$

$$\cong \omega^{\cdot}_{(Z, M_Z)/(E_n, M_{E_n})}$$

$$Ru^{\log}_{X_n'/E_n *}(\mathcal{O}_{X_n'/E_n}) \xrightarrow{\sim} \mathcal{O}_{D'} \otimes_{\mathcal{O}_{Z'}} \omega^{\cdot}_{(Z', M_{Z'})/(E_n, M_{E_n})}$$

$$\cong \mathcal{O}_{E'} \otimes_{\mathcal{O}_{V'}} \omega_{(Z', M_{Z'})/(E_n, M_{E_n})}$$

since $D = Z$, $D' = Z' \times_{V'} E' = Z \times_{E_n} E'$.

Since all schemes are affine and all modules q.coh., the map $(+)$ is repr. by

$$\Gamma(E', \mathcal{O}_{E'} \underset{\mathcal{O}_{V'}}{\otimes} \omega^{\cdot}_{(V', M_{V'})/(E_n, M_{E_n})}) \overset{L}{\underset{R_{E_n}}{\otimes}} \Gamma(Z, \omega^{\cdot}_{(Z, M_Z)/(E_n, M_n)})$$

$$\xrightarrow{\sim} \Gamma(D', \mathcal{O}_{D'} \underset{\mathcal{O}_{Z'}}{\otimes} \omega^{\cdot}_{(Z', M_{Z'})/(E_n, M_{E_n})})$$

and since $E' \to E_n$ is flat, $\otimes^L = \otimes$.
But locally this is

$$\Lambda^{\cdot}_{\Gamma(E', \mathcal{O}_{E'})}\{x_1, \dots, x_r\} \underset{\Gamma(E_n, \mathcal{O}_{E_n})}{\otimes} \Lambda^{\cdot}_{\Gamma(Z, \mathcal{O}_Z)}\{y_1, \dots, y_s\}$$

$$\xrightarrow{\sim} \Lambda^{\cdot}_{\underbrace{\Gamma(E', \mathcal{O}_{E'}) \underset{\Gamma(E_n, \mathcal{O}_{E_n})}{\otimes} \Gamma(Z, \mathcal{O}_Z)}_{\Gamma(D', \mathcal{O}_{D'})}}\{x_1, \dots, x_r, y_1, \dots, y_s\}$$

and hence an iso. of cx.

In general, we can — by top'l inv. of the étale site — find (Z, M_Z) étale locally on (X_n, M_n). One considers check complexes for such a covering. //

Last time:

$$P_n \otimes_{P_{E_n}} H^m((X_n, M_n)/(E_n, M_{E_n})) \xrightarrow{\sim} H^m((\bar{X}_n, \bar{M}_n)/(E_n, M_{E_n}))$$

Today: relate

$$P_{E_n} \otimes_{W_n} H^m((X_n, M_n)/(W_n, N_n^{\circ})) \rightsquigarrow H^m((X_n, M_n)/(E_n, M_{E_n})).$$

can. log. str.

Recall syntomic base change:

$$\begin{tikzcd}
(X', M') \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & (X, M) \arrow[d, "\text{syntomic}"] \\
(S', N') \arrow[r] \arrow[d, hook] & (S, N) \arrow[d, hook, "\text{- PD-subideal}"] \\
(T', M_{T'}, I', \gamma') \arrow[r] & (T, M_T, I, \gamma)
\end{tikzcd}$$

affine

Then

$$\Gamma(T', \mathcal{O}_{T'}) \otimes^L_{\Gamma(T, \mathcal{O}_T)} R\Gamma((X/T)^{\log}_{\mathrm{crys}}, \mathcal{O}_{X/T})$$

$$\xrightarrow{\sim} R\Gamma((X'/T')^{\log}_{\mathrm{crys}}, \mathcal{O}_{X'/T'})$$

Suppose first $V = W$, $\pi = p$. Then

$$\begin{tikzcd}
(Y, M_Y) \arrow[r, "id"] \arrow[d] & (Y, M_Y) \arrow[d] \\
(W_1, N_1^\circ) \arrow[r, "id"] \arrow[d, hook] & (W_1, N_1^\circ) \arrow[d, hook] \\
(E_n^\circ, M_{E_n^\circ}) \arrow[r] & (W_n, N_n^\circ)
\end{tikzcd} \Rightarrow$$

$$R_{E_n^\circ} \otimes^L_{W_n} R\Gamma((Y/W_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{Y/W_n})$$

$$\xrightarrow{\sim} R\Gamma((Y/E_n^\circ)^{\log}_{\mathrm{crys}}, \mathcal{O}_{Y/E_n^\circ}) \quad \text{— done.}$$

In general, choose r s.t. $\pi^{p^r} \in p\mathcal{O}_K$; consider

$$\begin{tikzcd}
(X_1, M_1) \arrow[r] \arrow[d] & (Y, M_Y) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\text{cart.}"] & (X_1, M_1) \arrow[d] \\
(S_1, N_1) \arrow[r] \arrow[d, hook] & (W_1, N_1^\circ) \arrow[r, hook] \arrow[d, hook] & (S_1, N_1) \arrow[d, hook] \\
(E_n, M_{E_n}) \arrow[r, "\partial"] & (E_n^\circ, M_{E_n^\circ}) \arrow[r, "\mathrm{can}"] & (E_n, M_{E_n})
\end{tikzcd}$$

$$\begin{array}{ccccc}
T & \leftarrow & T & T^{p^r} \longleftarrow & T
\end{array}$$

$$\begin{tikzcd}
(S_1, N_1) \arrow[r] \arrow[d, hook] & (W_1, N_1) \arrow[r, hook] \arrow[d, hook] & (S_1, N_1) \arrow[r, hook] \arrow[d, hook] & (\operatorname{Spec} W_n[T], \text{"}T=0\text{"}) \\
(E_n, M_{E_n}) \arrow[r, "\partial"] & (E_n^\circ, M_{E_n^\circ}) \arrow[r] & (E_n, M_{E_n})
\end{tikzcd}$$

PD-envelopes comp. w. pW_n

$$\begin{tikzcd} R_{E_n} & R_{E_n^\circ} \arrow[l, "g"'] & R_{E_n} \arrow[l, "\mathrm{can}"'] \arrow[ll, bend right, "\varphi^r"'] \\ T^{p^r} & T \arrow[l, mapsto] & T \arrow[l, mapsto] \\ \varphi^r(a) & a \arrow[l, mapsto] & \end{tikzcd}$$

Right hand square + syn. b/c :

$$R_{E_n^\circ} \underset{\mathrm{can}\nwarrow R_{E_n}}{\otimes} R\Gamma((X_1/E_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{X_1/E_n}) \xrightarrow{\sim} R\Gamma((Y/E_n^\circ)^{\log}_{\mathrm{crys}}, \mathcal{O}_{Y/E_n^\circ})$$

$$\xRightarrow{R_{E_n} \underset{g \nwarrow R_{E_n^\circ}}{\otimes} -}$$

$$R_{E_n} \underset{\varphi^r \nwarrow R_{E_n}}{\otimes} R\Gamma((X_1/E_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{X_1/E_n})$$

$$\xrightarrow{\sim} R_{E_n} \underset{g \nwarrow R_{E_n^\circ}}{\otimes} R\Gamma((Y/E_n^\circ)^{\log}_{\mathrm{crys}}, \mathcal{O}_{Y/E_n^\circ})$$

$$\xleftarrow{\sim} R_{E_n} \underset{g \nwarrow R_{E_n^\circ}}{\otimes} R_{E_n^\circ} \underset{W_n}{\otimes} R\Gamma((Y/W_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{Y/W_n})$$

$$\xleftarrow{\sim} R_{E_n} \underset{\varphi^r \nwarrow W_n}{\otimes} R\Gamma((Y/W_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{Y/W_n});$$

So can compare (limit systems) by

$$
\begin{array}{ll}
R\Gamma((X_1/E_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{X_1/E_n}) & \\
\exists \psi \big\downarrow \quad \big\uparrow 1\otimes \varphi^r & \psi\varphi^r = p^r,\ \varphi^r\psi = p^r \\
 & \text{(Hyodo - Kato)} \\
R_{E_n} \underset{\substack{R_{E_n} \\ \nwarrow \\ \varphi^r}}{\overset{L}{\otimes}} R\Gamma((X_1/E_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{X_1/E_n}) & \\
\big\downarrow \wr & \\
R_{E_n} \underset{\substack{R_{E_n^0} \\ \nwarrow \\ s}}{\overset{L}{\otimes}} R\Gamma((Y/E_n^0)^{\log}_{\mathrm{crys}}, \mathcal{O}_{Y/E_n^0}) & \\
\big\uparrow \wr & \\
R_{E_n} \underset{\substack{W_n \\ \nwarrow \\ \varphi^r}}{\overset{L}{\otimes}} R\Gamma((Y/W_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{Y/W_n}) & \\
\exists \psi \big\uparrow \quad \big\downarrow 1\otimes \varphi^r & \psi\varphi^r = p^r,\ \varphi^r\psi = p^r \\
 & \text{(Hyodo - Kato)} \\
R_{E_n} \underset{W_n}{\overset{L}{\otimes}} R\Gamma((Y/W_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{Y/W_n}) & \\
\big\downarrow \wr & R_{E_n}/W_n \text{ flat} \\
R_{E_n} \underset{W_n}{\otimes} R\Gamma((Y/W_n)^{\log}_{\mathrm{crys}}, \mathcal{O}_{Y/W_n}) &
\end{array}
$$

Hyodo - Kato uses $(X, M) \to (S, N)$ smooth with $(Y, M_Y) \to (W, N^0)$ of Cartier type.

Suppose $(Y, M_Y) \to (\operatorname{Spec} k, N_1^0)$ is smooth with $Y \to \operatorname{Spec} k$ proper. It can be shown that

$$H^m((Y,M_Y)/(W,N^0)) := \varprojlim_n H^m((Y,M_Y)/(W_n,N_n^0))$$

is a finite W-module and that the seq. is exact

$$0 \to H^m((Y,M_Y)/(W,N^0))/p^n \to H^m((Y,M_Y)/(W_n,N_n^0))$$

$$\to {}_{p^n}H^m((Y,M_Y)/(W,N^0)) \to 0$$

Since $W_n \to P_n$ is flat, it follows that

$$(\varprojlim_n P_n) \otimes_W H^m((Y,M_Y)/(W,N^0))$$

$$\xrightarrow{\sim} \varprojlim_n (P_n \otimes_{W_n} H^n((Y,M_Y)/(W_n,N_n^0))).$$

From what we have proved earlier, we get

$$(\mathbb{Q}\otimes \varprojlim_n P_n) \otimes_W H^m((Y,M_Y)/(W,N^0))$$

$$\xrightarrow{\sim} \mathbb{Q}\otimes \underbrace{H^m((\bar{X},\bar{M})/(E,M_E))}_{\begin{array}{c}\shortparallel \\ \varprojlim_n H^m((\bar{X}_n,\bar{M}_n)/(E_n,M_{E_n}))\end{array}}$$

It is further known that the monodromy op. on $\mathbb{Q}\otimes H^m((Y,M_Y)/(W,N^0))$ is <u>nilpotent</u>. Hence

$$(\mathbb{Q}\otimes \varprojlim_n P_n)^{N\text{-nilp.}} \underset{W}{\otimes} H^m((Y,M_Y)/(W,N^0))$$

$$\xrightarrow{\sim} (\mathbb{Q}\underset{W}{\otimes} H^m((\bar{X},\bar{M})/(E,M_E)))^{N\text{-nilp.}}$$

One defines

$$\begin{array}{ccc} B^+_{st} & = & (\mathbb{Q}\otimes \varprojlim_n P_n)^{N\text{-nilp.}} \\ \cup & & \cup \\ B^+_{cris} & = & \mathbb{Q}\otimes \varprojlim_n B_n \end{array}$$

There is a natural map (in fact iso)

$$\mathbb{Q}\otimes H^m((\bar{X},\bar{M})/W) \xrightarrow{\sim} (\mathbb{Q}\otimes H^m((\bar{X},\bar{M})/(E,M_E))^{N=0},$$

and hence

$$\mathbb{Q}\otimes H^m((\bar{X},\bar{M})/W) \xrightarrow{\sim} (B^+_{st,\pi}\underset{W}{\otimes} H^m((Y,M_Y)/(W,N^0)))^{N=0}.$$

This is Kato's crystalline interpretation of RHS.
We evaluate $B^+_{st,\pi}$.

Let $\pi \in \mathcal{O}_K$ be a unif. and $\beta \in \mathcal{O}_{\bar{K}}$ a p^nth root of π. Then

$$P_n \xleftarrow{\sim} B_n\langle v_\beta - 1\rangle ,$$

$$N((v_\beta - 1)^{[i]}) = (v_\beta - 1)^{[i-1]} v_\beta$$

<u>Lemma</u> The seq. is exact:

$$0 \to \bigoplus_{0 \le j < i} B_n \cdot u^{[j]} \longrightarrow P_n \xrightarrow{N^i} P_n \longrightarrow 0$$

$$u^{[j]} \longmapsto (\log v_\beta)^{[j]}$$

Pf Since $p^n B_n = 0$,

$$\log v_\beta = \log(1-(1-v_\beta)) = -\sum_{s \ge 0} (-1)^s (s-1)! (v_\beta - 1)^{[s]}$$

is well-defined.

<u>Claim</u> $N((\log v)^{[j]}) = (\log v)^{[j-1]}$.

We may consider the equation in the subring $\mathbb{Z}/p^n\langle v_\beta - 1\rangle \subset B_n\langle v_\beta - 1\rangle$. Since N is a derivation

$$N(\log v_\beta) = \frac{N(v_\beta)}{v_\beta} = 1 ,$$

so it suffices to show that if $f \in \mathbb{Z}/p^n\{(v_\beta - 1)^{[j]} \mid j > 0\}$, then

$$N(f^{[j]}) = f^{[j-1]} N(f).$$

But $\mathbb{Z}/p^n\langle v_\beta - 1\rangle$ is a subquotient of $\mathbb{Q}\langle v_\beta - 1\rangle$ and the formula for N makes perfect sense there. And in $\mathbb{Q}\langle v_\beta - 1\rangle$, we have

$$N(f^{[j]}) = \frac{1}{j!} N(f^j) = \frac{j}{j!} f^{j-1} N(f) = f^{[j-1]} N(f).$$

The claim follows. /

We show that the map is injective:

$$B_n\langle u\rangle \longrightarrow P_n \quad , \quad u \longmapsto \log v_\beta .$$

If not, there exists an equation in P_n

$$\sum_{j=0}^{m} a_j (\log v_\beta)^{[j]} = 0$$

with $a_j \in B_n$ not all zero. Assume m is minimal with this property. Applying N we get

$$\sum_{j=0}^{m-1} a_{j+1} (\log v_\beta)^{[j-1]} = 0 \quad ⚡.$$

Next, we show inductively that

$$\ker(N^i) = \bigoplus_{0 \le j < i} B_n \cdot (\log v_\beta)^{[j]}$$

If $i = 0$, we have

$$N\Big(\sum_{j=0}^{m} a_j (v_\beta - 1)^{[j]}\Big) = \sum_{j=1}^{m} a_j \cdot (v_\beta - 1)^{[j-1]} v_\beta$$

$$= \sum_{j=1}^{m} a_j \cdot \big(j (v_\beta - 1)^{[j]} + (v_\beta - 1)^{[j-1]} \big)$$

$$= \sum_{j=0}^{m} (a_{j+1} + j a_j)(v_\beta - 1)^{[j]} = 0 \iff a_j = 0, \; j > 0 .$$

Induction step:

$$f \in \ker(N^i) \Rightarrow N(f) \in \ker(N^{i-1}) \Rightarrow N(f) = \sum_{j=0}^{i-2} a_j (\log v_\beta)^{[j]};$$

let $g = \sum_{j=0}^{i-2} a_j (\log v_\beta)^{[j+1]}$. Then

$$N(f-g) = 0 \Rightarrow f-g \in B_n .$$

Finally N is surjective: if $g = \sum_{j\geq 0} b_j (v_\beta - 1)^{[j]}$, let

$$f = \sum_{j\geq 1} a_j (v_\beta - 1)^{[j]}, \qquad a_j = b_j - (j-1) a_{j-1} .$$

The sum is finite and $N(f) = g$. //

Choose $s = \{s_n\}_{n\geq 0}$, $s_0 = \pi$, $s_{n+1}^p = s_n$, $n \geq 0$. Get

$$0 \to \bigoplus_{0\leq j<i} \lim_n B_n \cdot u_s \to \lim_n P_n \xrightarrow{N^i} \lim_n P_n \to 0$$

$$u_s \longmapsto \{\log v_{s_n}\}_{n\geq 1}$$

Indeed, $B_n \twoheadrightarrow B_{n-1}$, so no derived limits. It follows that

$$(\lim_n B_n)\langle u_s \rangle \xrightarrow{\sim} (\lim_n P_n)^{N\text{-nilp.}}$$

$$(\mathbb{Q} \otimes \lim_n B_n)[u_s] \xrightarrow{\sim} (\mathbb{Q} \otimes \lim_n P_n)^{N\text{-nilp.}}$$

$$\parallel \qquad\qquad\qquad \parallel$$

$$B^+_{cris}[u_s] \xrightarrow{\sim} B^+_{st,\pi}$$

12/4/00

$(X,M) \to W$ syntomic, M fine; suppose $\exists$

$$\begin{tikzcd} (X,M) \arrow[r, hook] & (Z,M_Z) \arrow[r, "sm"] \arrow[loop below, "f"] & W \end{tikzcd}$$

lift of frob. on (Z_1, M_{Z_1})

$$\begin{tikzcd} (X_n,M_n) \arrow[r, hook] \arrow[dr, hook, "J_{D_n}"'] & (Z_n,M_{Z_n}) \\ & (D_n, M_{D_n}) \arrow[u] \end{tikzcd}$$

PD - envelope comp. w. can. PD-str. on pW_n

then

$$Ru^{\log}_{X_n/W_n *}(\mathcal{O}_{X_n/W_n}) \xrightarrow{\sim} \mathcal{O}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \omega^{\bullet}_{(Z_n,M_{Z_n})/W_n} =: j^{\log}_{n,X}(0)$$

$$\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow \text{ "1"}$$

$$Ru^{\log}_{X_n/W_n *}(\mathcal{J}^{[r]}_{X_n/W_n}) \xrightarrow{\sim} \mathcal{J}^{[r-\bullet]}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \omega^{\bullet}_{(Z_n,M_{Z_n})/W_n} =: j^{\log}_{n,X}(r)$$

where

$$0 \to \mathcal{J}_{X_n/W_n} \to \mathcal{O}_{X_n/W_n} \to i_{X_n/W_n *}\mathcal{O}_{X_n} \to 0$$

⌈For $\begin{matrix} U & \to & X_n \\ \downarrow & & \downarrow \\ T & \to & W_n \end{matrix}$ in $(X_n/W_n)^{\log}_{crys}$, the ass. sheaves on T_{et} are:

$$0 \to \mathcal{J}_T \to \mathcal{O}_T \to i_* \mathcal{O}_U \to 0$$ ⌋

Since (D_n, M_{D_n}) is also the PD-envelope

$$\begin{array}{ccc} (X_1, M_1) & \hookrightarrow & (Z_n, M_{Z_n}) \\ p\mathcal{O}_{D_n} + J_{D_n} \subset & & \nearrow \\ & & (D_n, M_{D_n}) \end{array}$$

we have a lift of frob., $f : \mathcal{O}_{D_n} \to \mathcal{O}_{D_n}$. This is a PD-morphism w.r.t. $p\mathcal{O}_{D_n} + J_{D_n}$, but we will not consider this ideal.

<u>Lemma</u> $f(J_{D_n}^{[r]}) \subset p^r \mathcal{O}_{D_n}$, $0 \leq r < p$.

<u>pf</u> Let $x \in J_{D_n}$, write $f(x) = x^p + py$, $y \in \mathcal{O}_{D_n}$; must show $f(x^{[r']}) \in p^r \mathcal{O}_{D_n}$, $r' \geq r$.

$$f(x^{[r']}) = f(x')^{[r']} = (x^p + py)^{[r']}$$

$$= (p!\, x^{[p]} + py)^{[r']} = p^{[r']}((p-1)!\, x^{[p]} + y)^{[r']}$$

and $p^{[r']} \in p^r \mathcal{O}_{D_n}$, provided $0 \leq r < p$. //

<u>Lemma</u> For all r, m and n, the seq. is exact:

$$0 \to J_{D_m}^{[r]} \xrightarrow{p^n} J_{D_{m+n}}^{[r]} \longrightarrow J_{D_n}^{[r]} \to 0$$

<u>pf</u> Étale locally, $(X_n, M_n) \hookrightarrow (Z_n, M_{Z_n})$ is an <u>exact</u> closed immersion with X_n defined by a reg. seq. $(f_1, \dots, f_s)$ in Z_n. In particular, D_n is the PD-envelope of X_n in Z_n. Consider

$$\begin{tikzcd}
X_n \arrow[r, hook] \arrow[d, "f"] & Z_n \arrow[d, "f'"] & & f_s \\
W_n \arrow[r, hook] & \mathbb{A}^s_{W_n} = \operatorname{Spec} W_n[t_1, \dots, t_s] & & t_s \arrow[u, mapsto]
\end{tikzcd}$$

By EGA 4, 0, 15.1.21, f' is flat on a nbhd of X_n in Z_n, so by flat base change for PD-envelope [Ber, 3.21],

$$D_n \xrightarrow{\sim} Z_n \times_{\mathbb{A}^s_{W_n}} D_{W_n}(\mathbb{A}^s_{W_n})$$

Enough to consider $W_n \hookrightarrow \mathbb{A}^s_{W_n}$, where

$$\begin{aligned}
\mathcal{O}_{D_n} &= \mathcal{O}_{W_n}\langle t_1, \dots, t_s \rangle \\
&\cup \\
J^{[r]}_{D_n} &= \mathcal{O}_{W_n}\{ t_1^{[i_1]} \cdots t_s^{[i_s]} \mid i_1 + \dots + i_s \geq r \}.
\end{aligned}$$

//

The two lemmas and $f(\omega^q_{(Z_n, M_n)/W_n}) \subset p^q \omega_{(Z_n, M_{Z_n})/W_n}$ give:

$$\begin{tikzcd}
0 \arrow[d] & & 0 \\
j^{\log}_{r,X}(r) \arrow[d, "p^n"] & & j^{\log}_{r,X}(r) \arrow[u] \\
j^{\log}_{n+r,X}(r) \arrow[rr, "f"] \arrow[d] & & j^{\log}_{n+r,X}(r) \arrow[u] \\
j^{\log}_{n,X}(r) \arrow[rr, dashed, "\text{''}\frac{f}{p^r}\text{''}"] \arrow[d] & & j^{\log}_{n,X}(r) \arrow[u, "p^r"] \\
0 & & 0 \arrow[u]
\end{tikzcd}$$

For $0 \le r < p$, define $s^{\log}_{n,X}(r)$ as mapping fiber

$$
\begin{tikzcd}
s^{\log}_{n,X}(r) \arrow[r] & j^{\log}_{n,X}(r) \arrow[r, "1-\frac{f}{p^r}"] & j'^{\log}_{n,X}(r) \arrow[r] & s^{\log}_{n,X}(r)[1] \\
& Ru^{\log}_{X_n/W_n *}(J^{[r]}_{X_n/W_n}) \arrow[u, "\sim"'] & Ru^{\log}_{X_n/W_n *}(\mathcal{O}_{X_n/W_n}) \arrow[u, "\sim"'] &
\end{tikzcd}
$$

so $s^{\log}_{n,X}(r)$ well-defined up to can qis:

$$
\begin{tikzcd}
(Z, M_Z) & (Z, M_Z) \times_W (Z', M_{Z'}) \arrow[l] \arrow[r] & (Z', M_{Z'}) \\
& (X, M) \arrow[ul, hook] \arrow[u, hook] \arrow[ur, hook] &
\end{tikzcd}
$$

The composite

$$s^{\log}_{n,X}(r) \longrightarrow j^{\log}_{n,X}(r) \xrightarrow{\text{``}1\text{''}} j^{\log}_{n,X}(0)$$

induces

$$H^m(X_n, s^{\log}_{n,X}(r)) \longrightarrow H^m((X_n, M_n)/W_n) \, ;$$

this is far from an iso. Goal:

Thm (Kato) If $0 \le r < p-1$, there is a qis:

$$s^{\log}_{n,\bar{X}}(r) \xrightarrow{\sim} \tau_{\le r} \bar{i}^* R \bar{j}_* \mathbb{Z}/p^n(r) .$$

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Syntomic cx. $s^{\log}_{n,X}(r)$, $0 \le r < p$:

$$\begin{array}{ccc} (X,M) & \hookrightarrow & (Z,M_Z) \\ \downarrow \text{sm} & & \downarrow \text{sm} \\ (S,N) & \xrightarrow[\text{syn}]{} & W \end{array} \qquad \begin{array}{ccc} (X_n,M_n) & \hookrightarrow & (Z_n,M_{Z_n}) \\ & \text{exact} \nwarrow & \nearrow \\ & & (D_n,M_{D_n}) \end{array}$$

$$\Downarrow$$

$$M_{D_n}/1+J_{D_n} \xrightarrow{\sim} M_n$$

$$j^{\log}_{n,X}(r) := J^{[r-\cdot]}_{D_n} \underset{\mathcal{O}_{Z_n}}{\otimes} \omega^{\cdot}_{(Z_n,M_{Z_n})/W_n} \quad \left(\xleftarrow{\sim} Ru_{X_n/W_n *}(J^{[r]}_{X_n/W_n})\right)$$

$$s^{\log}_{n,X}(r) \longrightarrow j^{\log}_{n,X}(r) \xrightarrow{1-\frac{\varphi}{p^r}} j^{\log}_{n,X}(0) \longrightarrow s^{\log}_{n,X}(r)[1]$$

Define symbol map :

$$(M^{gp}_{n+1})^{\otimes r} \longrightarrow \underline{H}^r(s^{\log}_{n,X}(r))$$

$$\begin{array}{cccc} & \deg 0 & & \deg 1 \\ C_{n+1} : & 1+J_{D_{n+1}} & \longrightarrow & M^{gp}_{D_{n+1}} \\ \downarrow \text{qis} & \downarrow & & \downarrow \\ M^{gp}_{n+1}[1] & 0 & \longrightarrow & M^{gp}_{n+1} \end{array}$$

define chain map $C_{n+1} \xrightarrow{6} s^{\log}_{n,X}(1)$, then

$$M^{gp}_{n+1} \xleftarrow{\sim} \underline{H}^1(C_{n+1}) \longrightarrow \underline{H}^1(s^{\log}_{n,X}(1)) ;$$

use product structure.

$$\begin{tikzcd}
1+\mathcal{J}_{D_{n+1}} \arrow[r, "\sigma^0"] \arrow[d] & \mathcal{J}_{D_n} \arrow[d] \arrow[dr, "1-\frac{\varphi}{p}"] & \\
M^{gp}_{D_{n+1}} \arrow[r, "\sigma^1"] \arrow[d] & \mathcal{O}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \omega^1_{(Z_n, M_{Z_n})}/W_n \arrow[d, "d"] \arrow[dr, "1-\varphi\wedge\frac{d\varphi}{p}"] \;\oplus & \mathcal{O}_{D_n} \arrow[d, "-d"] \\
0 \arrow[r, "\sigma^2"] \arrow[d] & \mathcal{O}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \omega^2_{(Z_n, M_{Z_n})}/W_n \arrow[d] \;\oplus & \mathcal{O}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \omega^1_{(Z_n, M_{Z_n})}/W_n \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

$$\sigma^0(a) = \log a$$

$$\sigma^1(b) = d\log b \oplus \frac{\log(b^p\varphi(b)^{-1})}{p}$$

$$b^p\varphi(b)^{-1} \in 1 + p\,\mathcal{O}_{D_{n+1}} \Rightarrow \log(b^p\varphi(b)^{-1}) \in p\,\mathcal{O}_{D_{n+1}} \xleftarrow[\sim]{p} \mathcal{O}_{D_n}$$

Let $n = 1$, write

$$(M^{gp}_{D_2})^{\otimes r} \longrightarrow \underline{H}^r(s^{\log}_{1,X}(r))$$

$$a_1 \otimes \cdots \otimes a_r \longmapsto \{\bar{a}_1, \ldots, \bar{a}_r\}$$

Use symbol map to define filtration

$$\underline{H}^r(s^{\log}_{1,X}(r)) \supset U^0 \supset V^0 \supset U^1 \supset V^1 \supset U^2 \supset V^2 \supset \cdots$$

U^0 = subsheaf gen. by $\{\bar{a}_1, \ldots, \bar{a}_r\}$, $a_i \in M_2^{gp}$

V^0 = subsheaf gen. by $\{\bar{a}_1, \ldots, \bar{a}_{r-1}, \pi\}$, $a_i \in M_2^{gp}$.

$\vdots$

U^m = subsheaf gen. by $\{1+\pi^m x, \bar{a}_1, \ldots, \bar{a}_{r-1}\}$, $a_i \in M_2^{gp}$, $x \in \mathcal{O}_{X_2}$

V^m = subsheaf gen. by $\{1+\pi^m x, \bar{a}_1, \ldots, \bar{a}_{r-2}, \pi\}$, $a_i \in M_2^{gp}$, $x \in \mathcal{O}_{X_2}$.

and $\{1+\pi^{m+1} x, \bar{a}_1, \ldots, \bar{a}_{r-1}\}$

Write

$$gr_0^m \, \underline{H}^r(s_{1,X}^{\log}(r)) = U^m / V^m ;$$

$$gr_1^m \, \underline{H}^r(s_{1,X}^{\log}(r)) = V^m / U^{m+1} ,$$

Consider i^* of sheaves above but abreviate this; note that i^* is exact. Define

$$\omega^r_{(Y,M_Y)/(W_1,N_1^0),\log} \subset \omega^r_{(Y,M_Y)/(W_1,N_1^0)}$$

to be the subsheaf gen. locally for the étale top. on Y by sections of the form

$$d\log a_1 \wedge \cdots \wedge d\log a_r \quad , \quad a_i \in M_Y .$$

$\ulcorner$ If $(Y, M_Y) \to (W_1, N_1^0)$ is smooth of Cartier type, one can show that the seq. on $\tilde{Y}_{et}$ is exact:

$$0 \to \omega^r_{(Y,M_Y)/(W_1,N_1^0),\log} \to \omega^r_{(Y,M_Y)/(W_1,N_1^0)} \xrightarrow{1-C^{-1}} \omega_{(Y,M_Y)/(W_1,N_1^0)} \to 0$$

where C^{-1} is the inverse Cartier operator. $\lrcorner$

Prop Suppose $p > 2$. Then

(1) $U^0 = \underline{H}^r(s_{1,X}^{\log}(r))$, $U^m = 0$, if $m \geq pe/(p-1)$,

(2) $gr_0^0\, \underline{H}^r(s_{1,X}^{\log}(r)) \xrightarrow{\sim} \omega^r_{(Y,M_Y)/(W_1,N_1^0)}$, log

$\{\bar{a}_1, \dots, \bar{a}_r\} \longmapsto d\log \bar{a}_1 \wedge \dots \wedge d\log \bar{a}_r$

$gr_1^0\, \underline{H}^r(s_{1,X}^{\log}(r)) \xrightarrow{\sim} \omega^{r-1}_{(Y,M_Y)/(W_1,N_1^0)}$, log

$\{\bar{a}_1, \dots, \bar{a}_{r-1}, \pi\} \longmapsto d\log \bar{a}_1 \wedge \dots\, d\log \bar{a}_{r-1}$

(3) if $0 < m < pe/(p-1)$ and $p \nmid m$:

$gr_0^m\, \underline{H}^r(s_{1,X}^{\log}(r)) \xrightarrow{\sim} \omega^{r-1}_{(Y,M_Y)/(W_1,N_1^0)} / B^{r-1}$

$\{1+\pi^m x, \bar{a}_1, \dots, \bar{a}_{r-1}\} \longmapsto \bar{x}\, d\log \bar{a}_1 \wedge \dots \wedge d\log \bar{a}_{r-1}$

$gr_1^m\, \underline{H}^r(s_{1,X}^{\log}(r)) \xrightarrow{\sim} \omega^{r-2}_{(Y,M_Y)/(W_1,N_1^0)} / Z^{r-2}$

$\{1+\pi^m x, \bar{a}_1, \dots, \bar{a}_{r-2}, \pi\} \longmapsto \bar{x}\, d\log \bar{a}_1 \wedge \dots \wedge d\log \bar{a}_{r-2}$

(4) if $0 < m < pe/(p-1)$ and $p \mid m$:

$gr_0^m\, \underline{H}^r(s_{1,X}^{\log}(r)) \xrightarrow{\sim} \omega^{r-1}_{(Y,M_Y)/(W_1,N_1^0)} / Z^{r-1}$

$\{1+\pi^m x, \bar{a}_1, \dots, \bar{a}_{r-1}\} \longmapsto \bar{x}\, d\log \bar{a}_1 \wedge \dots \wedge d\log \bar{a}_{r-1}$

$gr_1^m\, \underline{H}^r(s_{1,X}^{\log}(r)) \xrightarrow{\sim} \omega^{r-2}_{(Y,M_Y)/(W_1,N_1^0)} / Z^{r-2}$

$\{1+\pi^m x, \bar{a}_1, \dots, \bar{a}_{r-2}, \pi\} \longmapsto \bar{x}\, d\log \bar{a}_1 \wedge \dots \wedge d\log \bar{a}_{r-2}$.

We begin the proof of the structure of

$$\underline{H}^r(s_{1,X}^{\log}(r)) \xleftarrow{\sim} \underline{H}^r(\sigma_{\geq r-1} s_{1,X}^{\log}(r)) \xrightarrow{\sim} \underline{H}^r(\underbrace{\sigma_{<r+2}\,\sigma_{\geq r-1}\, s_{1,X}^{\log}(r)}_{S^\bullet})$$

The cx. $S^\bullet$ is mapping fiber of

$$\begin{tikzcd}
 & \mathcal{O}_{D_1} \otimes_{\mathcal{O}_{Z_1}} \omega^{r-2}_{(Z_1,M_{Z_1})/W_1} \arrow[d, "d"] & \deg r-2 \\
J_{D_1} \otimes_{\mathcal{O}_{Z_1}} \omega^{r-1}_{(Z_1,M_{Z_1})/W_1} \arrow[r, "1-\frac{\varphi}{p^r}"] \arrow[d, "d"] & \mathcal{O}_{D_1} \otimes_{\mathcal{O}_{Z_1}} \omega^{r-1}_{(Z_1,M_{Z_1})/W_1} \arrow[d, "d"] & \deg r-1 \\
\mathcal{O}_{D_1} \otimes_{\mathcal{O}_{Z_1}} \omega^{r}_{(Z_1,M_{Z_1})/W_1} \arrow[r, "1-\frac{\varphi}{p^r}"] \arrow[d] & \mathcal{O}_{D_1} \otimes_{\mathcal{O}_{Z_1}} \omega^{r}_{(Z_1,M_{Z_1})/W_1} & \\
\mathcal{O}_{D_1} \otimes_{\mathcal{O}_{Z_1}} \omega^{r+1}_{(Z_1,M_{Z_1})/W_1} & \text{write:} &
\end{tikzcd}$$

$$S^\bullet \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow S^\bullet[1].$$

Define filtration $\tilde{U}^m S^\bullet$, $0 \leq m \leq pe$; examine spectral sequence:

$$E_1^{s,t} = \underline{H}^{s+t}(gr^s_{\tilde{U}} S^\bullet), \quad E_\infty^{s,t} = gr^{s+t}_{\tilde{U}} \underline{H}^{s+t}(S^\bullet).$$

Will show that the sp. seq. collapses at E_1 and that $\tilde{U}^\bullet \underline{H}^r(S^\bullet) = U^\bullet \underline{H}^r(S^\bullet)$.

We assume:

$$\begin{tikzcd}
(X,M) \arrow[r, hook, "i"] \arrow[d] \arrow[dr, phantom, "\text{cart.}"] & (Z,M_Z) \arrow[d, "\text{sm}"] & \\
(S,N) \arrow[r, hook] & (V,M_V) \arrow[dr, "\text{sm.}"] & = (\operatorname{Spec} W[T], \text{"}T=0\text{"}) \\
& & W
\end{tikzcd}$$

$$\pi \longmapsfrom T$$

$$(D_n, M_{D_n}) = D_{(X_n,M_n),\gamma}(Z_n, M_{Z_n})$$

$$= (D_{X_n,\gamma}(Z_n), \lambda^{-1} M_{Z_n}) \qquad \text{since } i \text{ exact}$$

$$\mathcal{O}_{Z_n}/(T^e + p\,\theta(T)) \xrightarrow{\sim} i_* \mathcal{O}_{X_n} \qquad \text{since cart.}$$

so in $\mathcal{O}_{D_1}$:

$$T^{pe} = (T^e)^p = p!\,(T^e)^{[p]} = 0,$$

but $T^i \neq 0$, $i < pe$, and

$$\lambda^{-1}(T^i \mathcal{O}_{Z_1} / T^{i+1}\mathcal{O}_{Z_1}) \xrightarrow{\sim} T^i \mathcal{O}_{D_1} / T^{i+1}\mathcal{O}_{D_1}$$

Let m' be the smallest integer $\geq \max\{e + \frac{m}{p}, m\}$;

note $m \geq e + \frac{m}{p} \iff m \geq pe/(p-1)$ so

$m' = m \iff m \geq pe/(p-1)$. Define

$$\begin{tikzcd}
 & (T^m\mathcal{O}_{D_1}+J^{[p]}_{D_1})\otimes_{\mathcal{O}_{Z_1}}\omega^{r-2}_{(Z_1,M_{Z_1})/W_1} \arrow[d, "d"] \\
(T^{m'}\mathcal{O}_{D_1}+J^{[p]}_{D_1})\otimes_{\mathcal{O}_{Z_1}}\omega^{r-1}_{(Z_1,M_{Z_1})/W_1} \arrow[r, "1-\frac{\varphi}{p^r}"] \arrow[d, "d"] & (T^m\mathcal{O}_{D_1}+J^{[p]}_{D_1})\otimes_{\mathcal{O}_{Z_1}}\omega^{r-1}_{(Z_1,M_{Z_1})/W_1} \arrow[d, "d"] \\
(T^m\mathcal{O}_{D_1}+J^{[p]}_{D_1})\otimes_{\mathcal{O}_{Z_1}}\omega^{r}_{(Z_1,M_{Z_1})/W_1} \arrow[r, "1-\frac{\varphi}{p^r}"] \arrow[d, "d"] & (T^m\mathcal{O}_{D_1}+J^{[p]}_{D_1})\otimes_{\mathcal{O}_{Z_1}}\omega^{r}_{(Z_1,M_{Z_1})/W_1} \\
(T^m\mathcal{O}_{D_1}+J^{[p]}_{D_1})\otimes_{\mathcal{O}_{Z_1}}\omega^{r+1}_{(Z_1,M_{Z_1})/W_1} &
\end{tikzcd}$$

$$\tilde{U}^m S^{\cdot} \to \tilde{U}^m A^{\cdot} \longrightarrow \tilde{U}^m B^{\cdot} \to \tilde{U}^m S^{\cdot}[1]$$

⌈ Since $2<p$, $\frac{\varphi}{p}(J^{[2]}_{D_1}) = p\cdot\frac{\varphi}{p^2}(J^{[2]}_{D_1}) = 0$, so $\frac{\varphi}{p}$ maps $J^{[p]}_{D_1}\subset J^{[2]}_{D_1}$ to zero. Hence $1-\frac{\varphi}{p^r}$ preserves filt. ⌋

<u>Lemma</u> $U^m\underline{H}^r(S^{\cdot}) \subset \tilde{U}^m\underline{H}^r(S^{\cdot})$.

<u>pf</u> Let $a_1,\dots,a_r\in M^{gp}_{D_2}$ with images $\bar{a}_1,\dots,\bar{a}_r\in M^{gp}_2$. Then the symbol $\{\bar{a}_1,\dots,\bar{a}_r\}\in\underline{H}^r(S^{\cdot})$ is repr. by

$$d\log a_1\wedge\dots\wedge d\log a_r \oplus \sum_{i=1}^{r}(-1)^{i-1}\frac{\log(a_i^p\,\varphi_{D_2}(a_i)^{-1})}{p}$$

$$\cdot\, d\log a_1\wedge\dots\wedge d\log a_{i-1}\wedge\frac{\varphi}{p}(d\log a_{i+1})\wedge\dots\wedge\frac{\varphi}{p}(d\log a_r).$$

(exercise: prove this.) So it suffices to show

$$d\log(1+T^m x) \in (T^m \mathcal{O}_{D_1} + J_{D_1}^{[p]}) \otimes_{\mathcal{O}_{Z_1}} \omega^1_{(Z_1, M_{Z_1})/W_1},$$

$$\frac{\log((1+T^m x)^p \varphi_{D_2}(1+T^m x)^{-1})}{p} \in T^m \mathcal{O}_{D_1} + J_{D_1}^{[p]}.$$

The first of these follows from:

$$d\log(1+T^m x) = -\sum_{i\geq 1} (-1)^i (i-1)!\,(T^m x)^{[i-1]}\, d(T^m x).$$

To prove the second, write

$$\varphi_{D_2}(x) = x^p + py$$

$$z = \frac{y}{1+T^{mp}x^p}$$

Then

$$\left.\begin{aligned} \varphi_{D_2}(1+T^m x) &= (1+T^{pm}x^p)(1+pT^{pm}z) \\ (1+T^m x)^p &= 1+T^{pm}x^p + pT^m w, \quad w \in \mathcal{O}_{D_2} \end{aligned}\right\} \Rightarrow$$

$$(1+T^m x)^p \varphi_{D_2}(1+T^m x)^{-1} = \frac{1+pT^m w(1+T^{pm}x^p)^{-1}}{1+pT^{pm}z}$$

<u>Lemma</u> $\tilde{U}^m \underline{H}^r(S^\cdot) = 0$ if $m > pe/(p-1)$.

<u>Pf</u> If $m > pe/(p-1)$ then $m > e + \frac{m}{p}$ so $m' = m$.
So $\tilde{U}^m A^{r-1} = \tilde{U}^m B^{r-1}$ (and $\tilde{U}^m A^r = \tilde{U}^m B^r$, always), and since $\frac{\varphi}{p^r}$ is nilpotent,

$$\tilde{U}^m A^i \xrightarrow[\sim]{1-\frac{\varphi}{p^r}} \tilde{U}^m B^i, \quad i = r-1, r.$$

Get

$$\begin{tikzcd} & \underline{H}^{r-1}(\tilde{U}^m A^\cdot) \arrow[r, two heads] & \underline{H}^{r-1}(\tilde{U}^m B^\cdot) \arrow[dll, bend left] \\ \underline{H}^r(\tilde{U}^m S^\cdot) \arrow[r] & \underline{H}^r(\tilde{U}^m A^\cdot) \arrow[r, tail] & \underline{H}^r(\tilde{U}^m B^\cdot) \end{tikzcd}$$

//

Consider the diagram:

$$\begin{tikzcd} & & & (1+\pi^m \mathcal{O}_{X_2}) \otimes (M_2^{gp})^{\otimes r-1} \arrow[d] & \\ 0 \arrow[r] & U^{m+1}\underline{H}^r(S^\cdot) \arrow[r] \arrow[d, hook] & U^m \underline{H}^r(S^\cdot) \arrow[r] \arrow[d, hook] & gr_U^m \underline{H}^r(S^\cdot) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \tilde{U}^{m+1}\underline{H}^r(S^\cdot) \arrow[r] & \tilde{U}^m \underline{H}^r(S^\cdot) \arrow[r] & gr_{\tilde{U}}^m \underline{H}^r(S^\cdot) \arrow[r] \arrow[d, hook, "\beta"] & 0 \\ & & & \underline{H}^r(gr_{\tilde{U}}^m S^\cdot) & \end{tikzcd}$$

We will show that all differentials from $E_1^{s,r-1-s}$ are zero, s.t. $E_\infty^{s,r-s} \rightarrowtail E_1^{s,r-s}$ giving the map β.

Next, we will show that the composite of the right hand vertical maps is an epimorphism. It follow that the spectral seq. collapses at E_1, so β is an iso. An easy downward induction on m, starting with any $m \geq pe/p-1$, then shows:

$$U^m \underline{H}^r(S^\cdot) \xrightarrow{\sim} \tilde{U}^m \underline{H}^r(S^\cdot),$$

$$gr^m_U \underline{H}(S^\cdot) \xrightarrow{\sim} \underline{H}^r(gr^m_{\tilde{U}} S^\cdot).$$

We evaluate $E_1^{s,t} = \underline{H}^{s+t}(gr^s_{\tilde{U}} S^\cdot)$; we use

$$
\begin{aligned}
& \qquad\qquad\qquad\qquad 0 \longrightarrow \underline{H}^{r-2}(gr^m_{\tilde{U}} B^\cdot) \\
& \longrightarrow \underline{H}^{r-1}(gr^m_{\tilde{U}} S^\cdot) \longrightarrow \underline{H}^{r-1}(gr^m_{\tilde{U}} A^\cdot) \longrightarrow \underline{H}^{r-1}(gr^m_{\tilde{U}} B^\cdot) \\
& \longrightarrow \underline{H}^{r}(gr^m_{\tilde{U}} S^\cdot) \longrightarrow \underline{H}^{r}(gr^m_{\tilde{U}} A^\cdot) \longrightarrow \underline{H}^{r}(gr^m_{\tilde{U}} B^\cdot) \\
& \longrightarrow \underline{H}^{r+1}(gr^m_{\tilde{U}} S^\cdot) \longrightarrow 0
\end{aligned}
$$

From definitions:

$$\underline{H}^{r-2}(gr^m_{\tilde{U}} B^\cdot) = \underline{Z}^{r-2}\left((T^m G_{Z_1}/T^{m+1}G_{Z_1}) \underset{G_{Z_1}}{\otimes} \omega^\cdot_{(Z_1,M_{Z_1})/W_1}\right), \quad 0 \le m < pe,$$

$$
\begin{aligned}
\underline{H}^{r-1}(gr^m_{\tilde{U}} A^\cdot) &= (T^{e+\frac{m}{p}} G_{Z_1}/T^{e+\frac{m}{p}+1}G_{Z_1}) \underset{G_{Z_1}}{\otimes} \omega^{r-1}_{(Z_1,M_{Z_1})/W_1}, \quad 0 \le m < \frac{pe}{p-1},\ p \mid m, \\
&= 0, \quad \text{—''—},\ p \nmid m, \\
&= \underline{Z}^{r-1}\left((T^m G_{Z_1}/T^{m+1}G_{Z_1}) \underset{G_{Z_1}}{\otimes} \omega^\cdot_{(Z_1,M_{Z_1})/W_1}\right), \quad \frac{pe}{p-1} \le m < pe
\end{aligned}
$$

$$\underline{H}^{r-1}(gr^m_{\tilde{U}} B^\cdot) = \underline{H}^{r-1}\left((T^m G_{Z_1}/T^{m+1}G_{Z_1}) \underset{G_{Z_1}}{\otimes} \omega^\cdot_{(Z_1,M_{Z_1})/W_1}\right), \quad 0 \le m < pe,$$

$$
\begin{aligned}
\underline{H}^{r}(gr^m_{\tilde{U}} A^\cdot) &= \underline{Z}^{r}\left((T^m G_{Z_1}/T^{m+1}G_{Z_1}) \underset{G_{Z_1}}{\otimes} \omega^\cdot_{(Z_1,M_{Z_1})/W_1}\right), \quad 0 \le m < \frac{pe}{p-1}, \\
&= \underline{H}^{r}\left((T^m G_{Z_1}/T^{m+1}G_{Z_1}) \underset{G_{Z_1}}{\otimes} \omega^\cdot_{(Z_1,M_{Z_1})/W_1}\right), \quad \frac{pe}{p-1} \le m < pe,
\end{aligned}
$$

$$\underline{H}^{r}(gr^m_{\tilde{U}} B^\cdot) = (T^m G_{Z_1}/T^{m+1}G_{Z_1}) \underset{G_{Z_1}}{\otimes} \omega^r_{(Z_1,M_{Z_1})/W_1} / B^r, \quad 0 \le m < pe,$$

$$\underline{H}^{r+1}(gr^m_{\tilde{U}} A^\cdot) = (T^m G_{Z_1}/T^{m+1}G_{Z_1}) \underset{G_{Z_1}}{\otimes} \omega^{r+1}_{(Z_1,M_{Z_1})/W_1} / B^{r+1}, \quad 0 \le m < pe.$$

$$\begin{array}{ccccccc}
(Y,M_Y) & \hookrightarrow & (X_1,M_1) & \hookrightarrow & (Z_1,M_{Z_1}) & & \\
\downarrow & \text{cart.} & \downarrow & \text{cart.} & \downarrow & & \\
(W_1,N_1^\circ) & \hookrightarrow & (S_1,N_1) & \hookrightarrow & (V_1,M_{V_1}) & \longrightarrow & W_1
\end{array}$$

$$\begin{array}{ccccccccc}
0 \to & \mathcal{O}_Y \otimes_{\mathcal{O}_{Z_1}} \omega^{\cdot -1}_{(Z_1,M_{Z_1})/(V_1,M_{V_1})} & \xrightarrow{\wedge d\log T} & \mathcal{O}_Y \otimes_{\mathcal{O}_{Z_1}} \omega^{\cdot}_{(Z_1,M_{Z_1})/W_1} & \longrightarrow & \mathcal{O}_Y \otimes_{\mathcal{O}_{Z_1}} \omega^{\cdot}_{(Z_1,M_{Z_1})/(V_1,M_{V_1})} & \to 0 \\
& \downarrow \wr & & \cdot T^m \downarrow \wr & & \downarrow \wr & \\
0 \to & \omega^{\cdot -1}_{(Y,M_Y)/(W_1,N_1^\circ)} & \longrightarrow & (T^m\mathcal{O}_{Z_1}/T^{m+1}\mathcal{O}_{Z_1}) \otimes_{\mathcal{O}_{Z_1}} \omega^{\cdot}_{(Z_1,M_{Z_1})/W_1} & \longrightarrow & \omega^{\cdot}_{(Y,M_Y)/(W_1,N_1^\circ)} & \to 0
\end{array}$$

<u>Lemma</u> (i) If $p \mid m$, the map is an iso:

$$\mathcal{O}_Y \otimes_{\mathcal{O}_{Z_1}} \omega^{i}_{(Z_1,M_{Z_1})/W_1} \xrightarrow[\sim]{T^m\varphi \otimes (\frac{\varphi}{p})^{\wedge i}} \underline{H}^i((T^m\mathcal{O}_{Z_1}/T^{m+1}\mathcal{O}_{Z_1}) \otimes_{\mathcal{O}_{Z_1}} \omega^{\cdot}_{(Z_1,M_{Z_1})/W_1})\,;$$

(ii) if $p \nmid m$, $\underline{H}^i((T^m\mathcal{O}_{Z_1}/T^{m+1}\mathcal{O}_{Z_1}) \otimes_{\mathcal{O}_{Z_1}} \omega^{\cdot}_{(Z_1,M_{Z_1})/W_1}) = 0$.

<u>Pf</u> The statement (i) is the Cartier isomorphism. The statement (ii) follows from the fact (see Tsuji) that the connecting homomorphism in the cohomology seq. ass. w. the seq. of cx. above is given by mult. by $(-1)^i m$. //

<u>Cor</u> $\underline{H}^{r-1}(\mathrm{gr}^m_{\tilde{U}} A^\cdot) \longrightarrow \underline{H}^{r-1}(\mathrm{gr}^m_{\tilde{U}} B^\cdot)$ is iso, $0 \le m < \frac{pe}{p-1}$, and epi, $\frac{pe}{p-1} \le m < pe$. //

Cor There are natural iso:

$$\underline{H}^r(gr^m_{\tilde U} S^\cdot) \cong \ker\Big(\underline{Z}^r(G_Y \otimes_{G_{Z_1}} \omega^\cdot_{(Z_1,M_{Z_1})/W_1}) \xrightarrow{1-\frac{\Phi}{p^r}} \underline{H}^r(G_Y\otimes_{G_{Z_1}}\omega^\cdot_{(Z_1,M_{Z_1})/W_1})\Big), \quad m=0,$$

$$\cong \underline{B}^r\big((T^mG_{Z_1}/T^{m+1}G_{Z_1})\otimes_{G_{Z_1}}\omega^\cdot_{(Z_1,M_{Z_1})/W_1}\big), \quad 0<m<\frac{pe}{p-1},$$

$$= 0, \quad \frac{pe}{p-1}\le m< c.$$

Lemma All diff. leaving $E_1^{m,r-1-m}$ are zero.

pf If $pe/(p-1)\le m<m$, $E_1^{m,r-m}=0$, so can assume $0\le m<pe/(p-1)$. Wish to show

$$\begin{tikzcd}
\underline{H}^{r-1}(\tilde U^m S^\cdot) \arrow[r, two heads, "?"] & \underline{H}^{r-1}(gr^m_{\tilde U} S^\cdot) \\
\underline{Z}^{r-1}(\tilde U^m \mathcal{B}^\cdot) \arrow[u] \arrow[r, two heads, "?"] & \underline{Z}^{r-2}(gr^m_{\tilde U}\mathcal{B}^\cdot) \arrow[u, two heads, "\text{— since } m<\frac{pe}{p-1}"'] \\
\underline{Z}^{r-2}\big((T^mG_{D_1}+J_{D_1}^{[p]})\otimes_{G_{Z_1}}\omega^\cdot_{(Z_1,M_{Z_1})/W_1}\big) \arrow[u, equal] \arrow[r, two heads, "?"] & \underline{Z}^{r-2}\big((T^mG_{Z_1}/T^{m+1}G_{Z_1})\otimes_{G_{Z_1}}\omega^\cdot_{(Z_1,M_{Z_1})/W_1}\big) \arrow[u, equal] \\
\underline{Z}^{r-2}\big((T^mG_{Z_1}+J_{D_1}^{[p]})\otimes_{G_{Z_1}}\omega^\cdot_{(Z_1,M_{Z_1})/W_1}\big) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Since epi on boundaries, can instead show

$$\begin{tikzcd}
\underline{H}^{r-2}\big((T^mG_{Z_1}+J_{D_1}^{[p]})\otimes_{G_{Z_1}}\omega^\cdot_{(Z_1,M_{Z_1})/W_1}\big) \arrow[r, two heads, "?"] & \underline{H}^{r-2}\big((T^mG_{Z_1}/T^{m+1}G_{Z_1})\otimes_{G_{Z_1}}\omega^\cdot_{(Z_1,M_{Z_1})/W_1}\big) \\
\omega^{r-2}_{(Z_1,M_{Z_1})/W_1} \arrow[u] \arrow[r, two heads] & G_Y\otimes_{G_{Z_1}}\omega^{r-2}_{(Z_1,M_{Z_1})/W_1} \arrow[u, "T^m\varphi\wedge\left(\frac{\varphi}{p}\right)^{\wedge r-2}"']
\end{tikzcd}$$

If $p \nmid m$, upper right hand term zero, so OK; if $p | m$, right hand vertical map is iso, by a previous lemma. //

<u>Lemma</u> Let $a_i \in M^{gp}_{Z_1}$ with image $\bar{a}_i \in M^{gp}_Y$, let $x \in \mathcal{O}_Y$. Then if $p \nmid m$ (resp. if $p | m$), the seq. is exact:

$$0 \to \omega^{r-2}_{(Y,M_Y)/(W_1,N_1^0)}/Z^{r-2} \to B^{r-1}((T^m\mathcal{O}_{Z_1}/T^{m+1}\mathcal{O}_{Z_1})\otimes_{\mathcal{O}_{Z_1}}\omega^{\cdot}_{(Z_1,M_{Z_1})/W_1})$$

$$\to \omega^{r-1}_{(Y,M_Y)/(W_1,N_1^0)}/B^{r-1} \to 0$$

$$(\text{resp.} \quad \to \omega^{r-1}_{(Y,M_Y)/(W_1,N_1^0)}/Z^{r-1} \to 0\ ),$$

the two maps given by

$$x\, d\log\bar{a}_1\wedge\cdots\wedge d\log\bar{a}_{r-2} \mapsto d(T^m x\otimes d\log a_1\wedge\cdots\wedge d\log a_{r-2}\wedge d\log T)$$

$$d(T^m x\otimes d\log a_1\wedge\cdots\wedge d\log a_{r-1}) \mapsto x\, d\log\bar{a}_1\wedge\cdots\wedge d\log\bar{a}_{r-1}.$$

<u>Pf</u> If $p \nmid m$, $B^{r-1} = Z^{r-1}$ and hence

$$\begin{tikzcd}
0 \arrow[r] & \omega^{r-2}_{(Y,M_Y)/(W_1,N_1^0)} \arrow[r] & (T^m\mathcal{O}_{Z_1}/T^{m+1}\mathcal{O}_{Z_1})\otimes_{\mathcal{O}_{Z_1}}\omega^{r-1}_{(Z_1,M_1)/W_1} \arrow[r] & \omega^{r-1}_{(Y,M_Y)/(W_1,N_1^0)} \arrow[r] & 0 \\
0 \arrow[r] & Z^{r-2} \arrow[u, phantom, "\cup"] \arrow[r] & Z^{r-1} \arrow[u, phantom, "\cup"] & \\
& & B^{r-1} \arrow[u, phantom, "\parallel"] \arrow[r] & B^{r-1} \arrow[uu, phantom, "\cup"] \arrow[r] & 0.
\end{tikzcd}$$

If $p | m$, the map

$$(T^m\mathcal{O}_{Z_1}/T^{m+1}\mathcal{O}_{Z_1})\otimes_{\mathcal{O}_{Z_1}}\omega^{\cdot}_{(Z_1,M_{Z_1})/W_1} \to \omega^{\cdot}_{(Y,M_Y)/(W_1,N_1^0)} \to 0$$

induces epi. on cohomology, hence on cycles. //

We can now prove the following lemma. This completes the proof that the sp. seq. collapses, that $U^m \underline{H}^r(S') = \tilde{U}^m \underline{H}^r(S')$, and evaluates $gr^{\cdot}_U \underline{H}^r(S')$ in terms of diff.

<u>Lemma</u> (1) if $0 < m < pe/(p-1)$, the symbol

$$\{1+\pi^m \bar{x}, \bar{a}_1, \ldots, \bar{a}_{r-1}\} \in U^m \underline{H}^r(S') \subset \tilde{U}^m \underline{H}^r(S')$$

is represented in

$$E_1^{m,r-m} = \underline{H}^r(gr^m_{\tilde{U}} S') \xrightarrow{\sim} B^r((T^m \mathcal{O}_{Z_1}/T^{m+1}\mathcal{O}_{Z_1}) \underset{\mathcal{O}_{Z_1}}{\otimes} \omega^{\cdot}_{(Z_1, M_{Z_1})/W_1})$$

by $d(T^m x \otimes d\log a_1 \wedge \ldots \wedge d\log a_{r-1})$, $a_i \in M^{gp}_{Z_1} \longmapsto \bar{a}_i \in M^{gp}_Y$;

(2) if $m = 0$, the symbol

$$\{\bar{a}_1, \ldots, \bar{a}_r\} \in U^0 \underline{H}^r(S') \subset \underline{H}^r(S')$$

is repr. in

$$E_1^{0,r} = \underline{H}^r(gr^0_{\tilde{U}} S') \xrightarrow{\sim} \ker\left(\underline{Z}^r(\mathcal{O}_Y \underset{\mathcal{O}_{Z_1}}{\otimes} \omega^{\cdot}_{(Z_1,M_{Z_1})/W_1}) \to \underline{H}^r(\mathcal{O}_Y \underset{\mathcal{O}_{Z_1}}{\otimes} \omega^{\cdot}_{(Z_1,M_{Z_1})/W_1})\right)$$

by $1 \otimes d\log a_1 \wedge \ldots \wedge d\log a_r$.

Pf (1) follows from the above, but (2) requires more work – see Tsuji. //

 12/13/00

Define

$$\mathbb{Z}/p^n(1) \longrightarrow \underline{H}^0(s_{n,\bar{X}}^{\log}(1))$$

$$0 \to \underline{H}^0(s_{n,\bar{X}}^{\log}(1)) \to \underline{H}^0(j_{n,\bar{X}}^{\log}(1)) \xrightarrow{1-\frac{\varphi}{p}} \underline{H}^0(j_{n,X}^{\log}(0))$$

$$\begin{array}{ccccc} \mathbb{Z}/p^n(1) & \longrightarrow & J_{D_n} & \xrightarrow[(1)]{1-\frac{\varphi}{p}} & \mathcal{O}_{D_n} \\ & & (2) \downarrow d & & \\ & & \mathcal{O}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \omega(Z_n, M_{Z_n}) / W_n & & \end{array}$$

$$\xi \longmapsto \log(\tilde{\xi}^{p^n}) \qquad (\tilde{\xi}^{p^n} \in 1 + J_{D_n},$$
indept. of $\tilde{\xi} \mapsto \xi$)

(1) $\varphi(\tilde{\xi}) \equiv \tilde{\xi}$ (mod $p\,\mathcal{O}_{D_n}$) $\Longrightarrow$

$$\varphi(\tilde{\xi})^{p^n} \equiv \tilde{\xi}^{p^{n+1}} \pmod{p^{n+1}\mathcal{O}_{D_n}}$$

$\quad\shortparallel$

$\varphi(\tilde{\xi}^{p^n}) \quad \Longrightarrow \quad \varphi(\tilde{\xi}^{p^n}) = \tilde{\xi}^{p^{n+1}}$, so

$$\varphi(\log(\tilde{\xi}^{p^n})) := \log(\tilde{\xi}^{p^{n+1}}) = p\log(\tilde{\xi}^{p^n}) \quad \checkmark$$

(2) $d\log(\tilde{\xi}^{p^n}) = p^n\, d\log(\tilde{\xi}) = 0$, (note $\log(\tilde{\xi})$ is not defined.)

$$\underline{H}^0(s_{n,\bar{X}}^{\log}(1)) \otimes \underline{H}^r(s_{n,\bar{X}}^{\log}(r))$$

$$\downarrow$$

$$\underline{H}^r(s_{n,\bar{X}}^{\log}(r+1))$$

In all,

$$\underline{H}^q(s_{n,\bar{X}}^{\log}(r))(r-q) \longrightarrow \underline{H}^r(s_{n,\bar{X}}^{\log}(r))$$

<u>Thm</u> This is an iso.

—

<u>Thm</u> Suppose that $(X,M) \to (S,N)$ is smooth with $(Y,M_Y) \to (W_1,N_1^0)$ of Cartier type. Then for $0 \le r < p-1$:

(1) $\underline{H}^r(s_{n,X}^{\log}(r)) \xrightarrow{\sim} i^* R^r j_* \mathbb{Z}/p^n\mathbb{Z}(r)$

(2) $s_{n,\bar{X}}^{\log}(r) \longrightarrow \tau_{\le r}\, \bar{i}^* R \bar{j}_* \mathbb{Z}/p^n\mathbb{Z}(r)$ q is.

$$H^m(X_K, \mathbb{Z}/p^n\mathbb{Z}(r)) \xleftarrow{\sim} H^m(X, Rj_* \mathbb{Z}/p^n\mathbb{Z}(r))$$

$$\xleftarrow{\sim} H^m(\operatorname{Spec} \mathcal{O}_K, Rf_* Rj_* \mathbb{Z}/p^n\mathbb{Z}(r))$$

$$\cong H^m(\operatorname{Spec} k, i^* Rf_* Rj_* \mathbb{Z}/p^n\mathbb{Z}(r))$$

(proper b/c) $$\cong H^m(\operatorname{Spec} k, Rf'_* i^* Rj_* \mathbb{Z}/p^n\mathbb{Z}(r))$$

$$\xrightarrow{\sim} H^m(Y, i^* Rj_* \mathbb{Z}/p^n\mathbb{Z}(r))$$

For $0 \leq m < r < p-1$:

$$H^m(\bar{Y}, \bar{i}^* R\bar{j}_* \mathbb{Z}/p^n\mathbb{Z}(r)) \xleftarrow{\sim} H^m(\bar{X}_n, s^{\log}_{n,\bar{X}}(r))$$

$$\xrightarrow{\text{"}\frac{1}{p^r}\text{"}} H^m(\bar{X}_n, j^{\log}_{n,\bar{X}}(0))$$

$$\xleftarrow{\sim} H^m((\bar{X}_n, \bar{M}_n)/W_n)$$

$$\xrightarrow{\sim} H^m((\bar{X}_n, \bar{M}_n)/(E_n, M_{E_n}))^{N=0}$$

$$\xleftarrow{\sim} \left(H^0((\bar{S}_n, \bar{N}_n)/(E_n, M_{E_n})) \otimes_{\Gamma(E_n, \mathcal{O}_{E_n})} H^m((X_n, M_n)/(E_n, M_{E_n}))\right)^{N=0}$$

$$\xleftarrow{\sim} \left(\underbrace{H^0((\bar{S}_n, \bar{N}_n)/(E_n, M_{E_n}))}_{P_n} \otimes_{W_n} H^m((Y, M_Y)/(W_n, N_n^0))\right)^{N=0}$$

After $\mathbb{Q} \otimes \varprojlim_n (-)$:

$$H^m(X_{\bar{K}}, \mathbb{Q}_p(r)) \cong \left((\mathbb{Q} \otimes \varprojlim_n P_n) \otimes_W H^m((Y, M_Y)/(W, N^\circ)) \right)^{N=}$$

$$\xleftarrow{\sim} \left(B^+_{st,\pi} \otimes_W H^m((Y, M_Y)/(W, N^\circ)) \right)^{N=0}$$

So get natural map

$$H^m(X_{\bar{K}}, \mathbb{Q}_p(r)) \longrightarrow B^+_{st,\pi} \otimes_W H^m((Y, M_Y)/(W, N^\circ))$$

Let $\mathbb{Z}_p(1) \longrightarrow B^+_{st,\pi}$ be the map

$$\mathbb{Z}/p^n\mathbb{Z}(1) \longrightarrow B_n = H^0((\bar{S}_n, \bar{N}_n)/W_n)$$

$$\xi \longmapsto \log(\tilde{\xi}^{p^n})$$

Pick gen $\varepsilon = (\varepsilon_n) \in \mathbb{Z}_p(1)$, let $t \in \varprojlim_n B_n$ be the image.

$$B_{crys} = B^+_{crys}[t^{-1}]$$

$$B_{st,\pi} = B^+_{st,\pi}[t^{-1}]$$

$$\left(B^+_{st,\pi} = B^+_{crys}[u_s] \right)$$

Now, finally, we have

$$H^m(X_{\bar{K}}, \mathbb{Q}_p) \xleftarrow{\sim} \mathbb{Q}_p(-r) \underset{\mathbb{Q}_p}{\otimes} H^m(X_{\bar{K}}, \mathbb{Q}_p(r))$$

$$\longrightarrow \mathbb{Q}_p(-r) \underset{\mathbb{Q}_p}{\otimes} B^+_{st,\pi} \underset{W}{\otimes} H^m((Y, M_Y)/(W, N^0))$$

$$\longrightarrow B_{st,\pi} \underset{W}{\otimes} H^m((Y, M_Y)/(W, N^0)),$$

and, extending scalars, we get

$$\boxed{B_{st,\pi} \underset{\mathbb{Q}_p}{\otimes} H^m(X_{\bar{K}}, \mathbb{Q}_p) \longrightarrow B_{st,\pi} \underset{W}{\otimes} H^m((Y, M_Y)/(W, N^0))}$$