18.917: *Topics in algebraic topology*

In this course, I plan to explain and prove the semi-stable conjecture of Fontaine-Jannsen following [**1**] and [**2**].

To briefly explain the statement, let V be a complete discrete valuation ring with fraction field K and perfect residue field k of mixed characteristic $(p, 0)$, and let X be a scheme over $\operatorname{Spec} V$ with generic fiber X_K and special fiber X_k,

$$\begin{array}{ccccc} X_k & \xrightarrow{\ i\ } & X & \xleftarrow{\ j\ } & X_K \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{\ i\ } & \operatorname{Spec} V & \xleftarrow{\ j\ } & \operatorname{Spec} K. \end{array}$$

Then X is said to have semi-stable reduction if, in the étale topology, X can be covered by schemes of the form

$$U = \operatorname{Spec} V[x_1, \dots, x_d]/(x_1 \dots x_r - \pi).$$

Here $1 \leq r \leq d$ and π is a generator of the maximal ideal in V. (It is expected that for any regular scheme X over $\operatorname{Spec} V$, one can find a finite extension V'/V such that $X_{V'} \to \operatorname{Spec} V'$ has semi-stable reduction.) One has the following cohomology theories

(i) the étale cohomology $H^*(X_{\bar{K}}, \mathbb{Z}_p)$, where $X_{\bar{K}} = X \times_{\operatorname{Spec} K} \operatorname{Spec} \bar{K}$; it is a $\mathbb{Z}_p$-module with a continuous $\operatorname{Gal}(\bar{K}/K)$-action;

(ii) the log-crystalline cohomology $H^*((X_k, M_k)/W)$; it is a $W(k)$-module with two operators, Frobenius and monodromy, and with a natural filtration after extending scalars to K.

The conjecture relates these objects: for $X \to \operatorname{Spec} V$ proper with semi-stable reduction, there is a natural isomorphism

$$\alpha_K \colon B_{\mathrm{st}} \otimes_{W(k)} H^q((X_k, M_k)/W) \xrightarrow{\sim} B_{\mathrm{st}} \otimes_{\mathbb{Z}_p} H^q(X_{\bar{K}}, \mathbb{Z}_p),$$

where B_{st} is a ring of "p-adic periodes" defined by Fontaine. In fact, up to torsion, one can retrieve (i) and (ii) from this common object. Time permitting, I also will explain this.

After an introduction, here is a tentative route:

(i) log schemes;

(ii) crystalline cohomology;

(iii) the Fontaine ring B_{st} and its crystalline interpretation;

(iv) syntomic cohomology and p-adic nearby cycles;

(v) proof of the conjecture.

References

[1] K. Kato, *Semi-stable reduction and p-adic étale cohomology*, Périodes p-adiques (Séminaire de Bures, 1988), Asterisque, vol. 223, 1994, pp. 269–293.

[2] T. Tsuji, *p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case*, Invent. Math. **137** (1999), 233–411.