18.099: Problem Set 3

Due: Tuesday, September 23.

Two sets A and B are said to have the *same cardinality* if there exists a bijection between them.

1. Do the sets $[0,1]$ and $(0,1]$ have the same cardinality?

2. Suppose that $B \subset A$ and that there exists an injective map $f\colon A \to B$. Show that A and B have the same cardinality.

(*Hint*: First, define a sequence of subsets

$$A_1 \supset B_1 \supset A_2 \subset B_2 \supset A_3 \supset B_3 \supset \dots$$

as follows. Let $A_1 = A$, $B_1 = B$, and for $n > 1$, let $A_n = f(A_{n-1})$ and $B_n = f(B_{n-1})$. Then, show that the map $h\colon A \to B$ given by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_n \smallsetminus B_n \text{ for some } n, \\ x & \text{otherwise,} \end{cases}$$

is bijective.)

3. Prove the following theorem.

Theorem (Schroeder-Bernstein)**.** *Let A and B be two sets and suppose that there exists injective maps $f\colon A \to B$ and $g\colon B \to A$. Then A and B have the same cardinality.*