18.099: Problem Set 5

Due: Tuesday, October 22.

The purpose of this problem set is to write an exposé about the p-adic integers. You are free to organize the material as you see fit.

Let p be a prime number. We define a p-adic integer a to be a sequence $a = (a_0, a_1, a_2, \dots)$ of integers $0 \le a_i < p$. It is common to display the sequence a as an infinite "sum"

$$a = a_0 + a_1 p + a_2 p^2 + a_3 p^3 + \cdots .$$

Every non-negative integer n can be written uniquely as a sum

$$n = n_0 + n_1 p + n_2 p^2 + n_3 p^3 + \cdots ,$$

with $0 \le n_i < p$ and with all but finitely many n_i equal to zero. This way we obtain an injective map

$$\iota\colon \mathbb{N}_0 \hookrightarrow \mathbb{Z}_p$$

from the set of non-negative integers to the set of p-adic integers. We define sum and product of p-adic integers by the usual formulas such that $\iota(m+n) = \iota(m) + \iota(n)$ and $\iota(m \cdot n) = \iota(m) \cdot \iota(n)$. Show that

$$-1 = (p-1) + (p-1) \cdot p + (p-1) \cdot p^2 + (p-1) \cdot p^3 + \cdots$$

such that we can extend the map ι to an injective map

$$\iota\colon \mathbb{Z} \hookrightarrow \mathbb{Z}_p$$

by sending a negative integer $-n$ to $-1 \cdot \iota(n)$. In the following, if n is an integer, we will sometimes abuse notation and also write n for the p-adic integer $\iota(n)$.

Let a be a p-adic integer. We define $0 \le v_p(a) \le \infty$ to be the smallest i for which $a_i \ne 0$. Show that $v_p(a)$ is equal to the number of times that $p = \iota(p)$ divides a. We define the p-adic metric on the set $\mathbb{Z}_p$ by

$$d_p(a,b) = \begin{cases} p^{-v_p(a-b)} & \text{if } a \ne b, \\ 0 & \text{if } a = b. \end{cases}$$

Show that d_p satisfies the stronger triangle inequality

$$d_p(a,c) \le \max\{d_p(a,b), d_p(b,c)\},$$

and conclude that d_p is a metric. A metric that satisfies the stronger triangle inequality is called an ultra-metric. Show that all triangles in an ultra-metric space are isosceles. Show that the metric space $(\mathbb{Z}_p, d_p)$ is complete.

Let $B_r(a) = \{b \in \mathbb{Z}_p \mid d_p(a,b) < r\}$ be the open ball of radius $r > 0$ around a. Prove the following statements.

(i) If $b \in B_r(a)$, then $B_r(a) = B_r(b)$.

(ii) The subset $B_r(a) \subset \mathbb{Z}_p$ is both open and closed.

(iii) The intersection $B_r(a) \cap B_s(b)$ is non-empty if and only if one of the two balls contains the other.

A metric space (X, d) is called *totally bounded* if for every $\epsilon > 0$, there exists a finite cover of X be open balls of radius ϵ. Show that $(\mathbb{Z}_p, d_p)$ is totally bounded. Show that a metric space (X, d) is compact if and only if it is complete and totally bounded. (The "if" part is the more difficult. Show that every sequence $\{x_n\}$ in X has a subsequence that is Cauchy.) Conclude that $(\mathbb{Z}_p, d_p)$ is compact.