18.099: Problem Set 6

Due: Tuesday, November 4.

The purpose this time is to write a small paper about Cantor's construction of the real numbers. In short, this is the construction of the real numbers as the *completion* of the rational numbers with respect to the (usual) euclidean metric.

We consider sequences $\{a_n\}$ of rational numbers. We say that the sequence $\{a_n\}$ converges to $a \in \mathbb{Q}$ if for all $\epsilon > 0$ ($\epsilon \in \mathbb{Q}$), there exists a positive integer N such that for all $n \geq N$,

$$|a - a_n| < \epsilon.$$

Similarly, we say that $\{a_n\}$ is a Cauchy sequence if for all $\epsilon > 0$ ($\epsilon \in \mathbb{Q}$), there exists a positive integer N such that for all $m, n \geq N$,

$$|a_m - a_n| < \epsilon.$$

These are the definitions except that we only consider $\epsilon > 0$ that are rational numbers. (We are constructing the real numbers.)

We say that two Cauchy sequences of rational numbers $\{a_n\}$ and $\{b_n\}$ are *equivalent* and write $\{a_n\} \sim \{b_n\}$ if the sequence $\{a_n - b_n\}$ converges to zero. Show that this relation is an *equivalence relation*, that is, the following three conditions are satisfied.

Symmetry: $\{a_n\} \sim \{a_n\}$.

Reflexivity: $\{a_n\} \sim \{b_n\}$ implies $\{b_n\} \sim \{a_n\}$.

Transitivity: $\{a_n\} \sim \{b_n\}$ and $\{b_n\} \sim \{c_n\}$ implies $\{a_n\} \sim \{c_n\}$.

In general, if we are given an equivalence relation on a set X, then a subset $A \subset X$ is called an *equivalence class* if the following two conditions are satisfied.

(i) $a, a' \in A$ implies $a \sim a'$.

(ii) $a \in A$, $x \in X$ and $a \sim x$ implies $x \in A$.

Show that every element $x \in X$ belongs to a unique equivalence class.

Consider again the set of Cauchy sequences of rational numbers with the equivalence relation defined earlier. Following Cantor, we define a *real number* to an equivalence class of Cauchy sequences of rational numbers. So the set $\mathbb{R}$ is the set of equivalence classes of Cauchy sequences of rational numbers. The addition and multiplication on the set of real numbers is given by term-wise adding and multiplying Cauchy sequences of rational numbers. Show that the addition and multiplication is well-defined. Show that $\mathbb{R}$ is a field and identify $\mathbb{Q}$ with a subfield of $\mathbb{R}$.

We next define the order on the set or real numbers. A Cauchy sequence of rational numbers $\{a_n\}$ is called *positive* if there exists a rational number $\epsilon > 0$ and a positive integer N such that for all $n \geq N$, $a_n > \epsilon$. Show that if $\{a_n\} \sim \{b_n\}$ and if $\{a_n\}$ is positive, then $\{b_n\}$ is positive. We then define $x \in \mathbb{R}$ to be positive if $\{a_n\} \in x$ is positive. Show that for every $x \in \mathbb{R}$ exactly one of the following three statements is true.

(i) $x = 0$.

(ii) x is positive.

(iii) $-x$ is positive.

Show that with this ordering $\mathbb{R}$ is an ordered field.

Let $\{a_n\} \in x \in \mathbb{R}$. Show that if we consider $\{a_n\}$ as a Cauchy sequence of real numbers, then $\{a_n\}$ converges to x. Show that $\mathbb{R}$ is complete. Show that $\mathbb{R}$ has the least upper bound property.