18.099: Problem Set 7

Due: Tuesday, November 25.

The problem set this time shows, in particular, that a finite set of contractions on euklidean space determines a unique fractal.

Let (X, d) be a metric space. A map $f\colon X \to X$ is called a *contraction* if there exists a constant $0 \leq \lambda < 1$ such that for all $x, y \in X$,

$$d(f(x), f(y)) \leq \lambda \cdot d(x, y).$$

Show that a contraction $f\colon X \to X$ on a *complete* metric space has a unique *fixed point*, that is, a point $x \in X$ such that $f(x) = x$.

We shall apply the result above to the set $\mathcal{K}$ of non-empty compact subsets of $\mathbb{R}^n$ ($n \geq 1$ is fixed) equipped with the Hausdorff metric which we proceed to define. We define the distance from a point $x \in \mathbb{R}^n$ to a non-empty compact subset $K \subset \mathbb{R}^n$ to be

$$d(x, K) = \inf\{d(x, y) \mid y \in K\} = \min\{d(x, y) \mid y \in K\}.$$

And if $\epsilon \geq 0$, we define the ϵ-enlargement of a non-empty compact subset $K \subset \mathbb{R}^n$ by

$$K_\epsilon = \{x \in \mathbb{R}^n \mid d(x, K) \leq \epsilon\}.$$

Then, the Hausdorff distance between two non-empty compact subsets $K, K' \subset \mathbb{R}^n$ is defined to be

$$d(K, K') = \inf\{\epsilon \geq 0 \mid K \subset K'_\epsilon \text{ and } K' \subset K_\epsilon\}.$$

Show first that $(\mathcal{K}, d)$ is a metric space. Then show that this metric space is *complete*.

Prove the following result.

Theorem. *Let $f_1, \dots, f_N\colon \mathbb{R}^n \to \mathbb{R}^n$ be a finite family of contractions. Then exists a unique non-empty compact subset $K \subset \mathbb{R}^n$ such that*

$$K = f_1(K) \cup \dots \cup f_N(K).$$