18.099: Problem Set 8

Due: Not necessarily.

This problem set is concerned with the Riemann zeta function which for real numbers $s > 1$ is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Show that the series converges absolutely for $s > 1$. We will first derive Euler's product formula that for $s > 1$,

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}},$$

where the product on the right ranges over all primes numbers p. In effect, we shall prove the following more general formula.

Theorem (Euler). *Let $f\colon \mathbb{N} \to \mathbb{R}$ be a function that satisfies $f(mn) = f(m)f(n)$, for all $m, n \in \mathbb{N}$, and suppose that the series $\sum_{n=1}^{\infty} f(n)$ converges absolutely. Then*

$$\sum_{n=1}^{\infty} f(n) = \prod_p \frac{1}{1 - f(p)},$$

where the product on the right ranges over all prime numbers p.

Here is the idea of the proof. First show that

$$\sum_{s=0}^{\infty} f(p^s) = \frac{1}{1 - f(p)}.$$

Then consider the partial product

$$P(x) = \prod_{p \leq x} \frac{1}{1 - f(p)} = \prod_{p \leq x} \Big(\sum_{s=0}^{\infty} f(p^s) \Big).$$

Conclude from the unique prime factor decomposition of positive integers that

$$P(x) = \sum_{n \in A(x)} f(n),$$

where $A(x) \subset \mathbb{N}$ is the set of natural numbers n such that if p divides n then $p \leq x$. Let $B(x) = \mathbb{N} \smallsetminus A(x)$ and use that

$$\Big| \sum_{n=1}^{\infty} f(n) - P(x) \Big| \leq \sum_{n \in B(x)} |f(n)| \leq \sum_{n > x} |f(n)|$$

to finish the proof of the lemma.

Next, we shall prove the following result.

Theorem. *The following limit formula holds.*

$$\lim_{s \to 1+} (s-1)\zeta(s) = 1.$$

To this end, first show that the series

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$$

converges for all positive real numbers s. Then show that the series converges *uniformly* on every interval of the form $[\epsilon, \infty)$ with $\epsilon > 0$. Conclude that the function $\varphi(s)$ is continuous on the open interval $(0, \infty)$. Note that for all $s > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} + 2^{-s} \sum_{n=1}^{\infty} \frac{1}{n^s},$$

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} - 2^{-s} \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and conclude that

$$\varphi(s) = (1 - 2^{1-s})\zeta(s).$$

Multiply by $(s-1)$ on both sides and apply l'Hospital's rule to show that

$$\lim_{s \to 1+} (s-1)\zeta(s) = \frac{1}{\log 2} \varphi(1).$$

The proposition follows since $\varphi(1) = \log 2$.

Use Euler's product formula, the limit formula, and the fact that the harmonic series diverges to conclude, following Euler, that there are infinitely many prime numbers. (Of course, Euklid had already given a much simpler proof of this fact.)

We next consider the gamma function which is given for real numbers $s > 0$ by

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx.$$

To show that the integral converges, treat the two summands

$$\int_0^{\infty} x^{s-1} e^{-x} dx = \int_0^{1} x^{s-1} e^{-x} dx + \int_1^{\infty} x^{s-1} e^{-x} dx$$

separately. Show that $\Gamma(1) = 1$ and that for $s > 0$,

$$\Gamma(s+1) = s\Gamma(s).$$

Hence, we can extend the definition of the gamma function to all real numbers s except the non-positive integers. The formula also shows that the function

$$s! = \Gamma(s+1)$$

extends the faculty function to all real numbers except the negative integers. We shall prove the following integral formula for the zeta function.

Theorem (Abel)**.** *For all real numbers $s > 1$,*

$$\Gamma(s)\zeta(s) = \int_0^{\infty} \frac{x^{s-1} e^{-x}}{1 - e^{-x}} dx.$$

The starting point of the proof is the geometric series. It shows that for $x > 0$,

$$\frac{e^{-x}}{1 - e^{-x}} = \sum_{n=1}^{\infty} e^{-nx}.$$

Conclude that this gives the following equality.

$$\int_0^\infty \frac{x^{s-1}e^{-x}}{1-e^{-x}}dx = \sum_{n=1}^\infty \int_0^\infty x^{s-1}e^{-nx}dx.$$

To finish the proof, substitute $y = nx$ and obtain Abel's formula.

Finally, we shall prove the following formula for the gamma function.

Theorem (Gauss). *For all real numbers s except the non-positive integers,*

$$\Gamma(s) = \lim_{n\to\infty} \frac{n^s n!}{s(s+1)(s+2)\dots(s+n)}.$$

The proof begins with the following formula valid for all positive real numbers x.

$$e^{-x} = \lim_{n\to\infty}(1-\frac{x}{n})^n.$$

Let n be a positive integer. We consider the function defined on the positive real numbers by the formula

$$f_n(x) = \begin{cases} x^{s-1}(1-\frac{x}{n})^n & \text{if } 0 < x \le n, \\ 0 & \text{if } n < x. \end{cases}$$

Show that for all positive integers n, the function $f_n(x)$ is continuous on the open interval $(0,\infty)$. Then show that for all x in this interval, the sequence $\{f_n(x)\}$ is increasing with limit $x^{s-1}e^{-x}$. Conclude that

$$\int_0^\infty x^{s-1}e^{-x}dx = \lim_{n\to\infty}\int_0^n x^{s-1}(1-\frac{x}{n})^n dx.$$

Substitute $x = ny$ to obtain the following formula.

$$\int_0^\infty x^{s-1}e^{-x}dx = \lim_{n\to\infty} n^s \int_0^1 y^{s-1}(1-y)^n dy.$$

Use the binomial formula to show that

$$\int_0^1 y^{s-1}(1-y)^n dy = \sum_{i=0}^n (-1)^i \binom{n}{i} \frac{1}{s+i}.$$

The function obtained by multiplying the right-hand side by $s(s+1)\dots(s+n)$,

$$f(s) = s(s+1)\dots(s+n)\sum_{i=0}^n (-1)^i \binom{n}{i} \frac{1}{s+i},$$

is a polynomial of degree at most n. Moreover, $f(s)$ takes the value $n!$ for each of the $(n+1)$ integers $s = 0, -1, -2, \dots, -n$. In other words, $f(s) - n!$ is a polynomial of degree at most n and has $n+1$ distinct roots. But a non-constant polynomial of degree n can have at most n distinct roots (this is a simple fact from algebra), so $f(s) - n!$ must be constant equal to zero. It is now easy to finish the proof.