

R ring $\rightsquigarrow (X, \mathcal{O}_X)$ ringed space

Point set: $X = \text{Spec } R = \{ \mathfrak{p} \subseteq R \text{ prime ideal} \}$

$$\begin{array}{ccc} \psi & & \psi \\ x & \longleftrightarrow & \mathfrak{p}_x \end{array}$$

$\mathcal{O}_{X,x} := R_{\mathfrak{p}_x}$ local ring at $x \in X$

$\mathfrak{m}_x := \mathfrak{p}_x R_{\mathfrak{p}_x}$ maximal ideal

$k(x) := \mathcal{O}_{X,x} / \mathfrak{m}_x$ residue field at $x \in X$.

$R \rightarrow R / \mathfrak{p}_x \hookrightarrow k(x)$ value of f

ψ $f \mapsto f(x)$ at $x \in X$

$$f(x) = 0 \iff f \in \mathfrak{p}_x$$

$$f(x) = 0, \forall x \in X \iff f \in \bigcap_{x \in X} \mathfrak{p}_x \iff$$

f is nilpotent.

Topology: Zariski topology on X :

closed subsets: $I \subseteq R$ ideal

$$\begin{aligned} V(I) &= \{x \in X \mid \forall f \in I : f(x) = 0\} \\ &= \{x \in X \mid \beta_x \supset I\}. \end{aligned}$$

$$\bigcap_{\alpha \in A} V(I_\alpha) = V(\sum_{\alpha \in A} I_\alpha)$$

$$V(I) \cup V(J) = V(I \cap J) = V(IJ)$$

$$\Gamma \quad V(I) \cup V(J) \subset V(I \cap J) \subset V(IJ) \quad \text{trivial}$$

$$V(I) \cup V(J) \supset V(IJ) \Leftrightarrow$$

$$X \setminus (V(I) \cup V(J)) \subset X \setminus V(IJ)$$

$$x \notin V(I) \cup V(J) \Leftrightarrow \exists f \in I, g \in J : f(x) \neq 0, g(x) \neq 0$$

$$\Rightarrow \exists f \in I, g \in J : (fg)(x) \neq 0 \Rightarrow x \notin V(IJ). \quad \perp$$

Distinguished open : $f \in R$

$$D(f) = \{x \in X \mid f(x) \neq 0\}$$

$$= \{x \in X \mid f \notin \beta_x\}$$

Basis of open sets :

$$X \setminus V(I) = \{x \in X \mid \exists f \in I : f(x) \neq 0\} = \bigcup_{f \in I} D(f)$$

$$\overline{\{x\}} = V(\sqrt{I_x}) \text{ irreducible}$$

$$\begin{aligned} \overline{\{x\}} &= \bigcap_{V(I) \ni x} V(I) = \bigcap_{\sqrt{I_x} \supset I} V(I) \\ &= V\left(\sum_{\sqrt{I_x} \supset I} I\right) = V(\sqrt{I_x}). \end{aligned}$$

$$\begin{aligned} \overline{\{x\}} = W \cup W' &\Rightarrow x \in W \text{ or } x \in W' \Rightarrow \\ \overline{\{x\}} = W \text{ or } \overline{\{x\}} = W'. \end{aligned}$$

$$\text{Spec } R/I \xrightarrow{\sim} V(I) \subset \text{Spec } R$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \overline{I} & \longmapsto & \pi^{-1}(\overline{I}) \end{array}$$

Lemma $\text{Spec } R = \bigcup_{\alpha \in A} D(f_\alpha) \iff (f_\alpha |_{\alpha \in A}) = R.$

Pf $\text{Spec } R = \bigcup_{\alpha \in A} D(f_\alpha) \iff \forall x \in X, \exists \alpha \in A :$

$f_\alpha(x) \neq 0 \iff \forall x \in X : (f_\alpha |_{\alpha \in A}) \neq \mathcal{P}_x$

$\iff (f_\alpha |_{\alpha \in A}) = R$ since every ideal

is contained in a max'l ideal. //

Cor $\text{Spec } R$ is quasi-compact. //

Structure sheaf $\mathcal{O}_X$:

$$R_f = R[\frac{1}{f}].$$

$$D(f) \supset D(g) \Leftrightarrow g \in \sqrt{(f)}$$

$$\nexists D(f) \supset D(g) \Leftrightarrow$$

$$\exists x \in X: x \notin D(f) \text{ and } x \in D(g) \Leftrightarrow$$

$$\exists x \in X: f \notin \mathcal{O}_x \text{ and } g \in \mathcal{O}_x \Leftrightarrow$$

$$g \notin \bigcap_{f \in \mathcal{I}} \mathcal{O}_x = \sqrt{(f)}. \quad "$$

$D(f) \supset D(g) \rightsquigarrow$ can. ring homom.

$$R_f \longrightarrow R_g \quad g^n = b f^c.$$

$$\frac{a}{f^c} \longmapsto \frac{a b^c}{g^{nc}}$$

If $D(f) \supset D(g) \supset D(h)$, then

$$\begin{array}{ccc} R_f & \longrightarrow & R_g \\ & \searrow & \swarrow \\ & R_h & \end{array}$$

commutes.

lemma let $g_\alpha \in R_{f_\alpha}$, $\alpha \in A$, and suppose that for all $\alpha, \beta \in A$, g_α and g_β have the same image in $R_{f_\alpha f_\beta}$. Then there exists $g \in R_f$ with image $g_\alpha \in R_\alpha$, for all $\alpha \in A$.

Pf Consider only the case $f=1$. Choose finite subcover, i.e. $f_1, \dots, f_k \in \{f_\alpha \mid \alpha \in A\}$ s.t. $(f_1, \dots, f_k) = R$. Suppose we can find $g_i \in R$ with image $g_i' \in R_{f_i}$, $1 \leq i \leq k$. Then the image g_i' of g in $R_{f_\alpha f_i}$ is equal to g_α since g_i' and g_α have the same image in $R_{f_\alpha f_i}$, $1 \leq i \leq k$: This is the conclusion of the previous lemma.

To find $g \in R$, write $g_i = \frac{b_i}{f_i}$, $1 \leq i \leq k$.

$$\left. \begin{aligned} g_i &= \frac{b_i}{f_i^n} \in R_{f_i} \\ g_j &= \frac{b_j}{f_j^n} \in R_{f_j} \end{aligned} \right\} \Rightarrow$$

$$\frac{b_i f_j^n}{(f_i f_j)^n} = \frac{b_j f_i^n}{(f_i f_j)^n} \in R_{f_i f_j}$$

i.e. $\exists m_{ij} \geq 0$ s.t.

$$(f_i f_j)^{m_{ij}} \cdot (b_i f_j^n - b_j f_i^n) = 0 \in R.$$

Choose $M \geq m_{ij}$, $1 \leq i, j \leq k$; let $N = M + n$, $b'_i = b_i f_i^M$. Then

$$g_i = \frac{b'_i}{f_i^N} \in R_{f_i}$$

$$b'_i f_j^N = b'_j f_i^N \in R$$

Since $X = \bigcup_{1 \leq i \leq k} D(f_i) = \bigcup_{1 \leq i \leq k} D(f_i^N)$,

$1 \in (f_1^N, \dots, f_k^N)$, say

$1 = h_1 f_1^N + \dots + h_k f_k^N$; let

$$g = h_1 b'_1 + \dots + h_k b'_k \in R.$$

Then

$$\begin{aligned} f_j^N g &= \sum_{1 \leq i \leq k} f_j^N h_i b'_i \\ &= \sum_{1 \leq i \leq k} f_i^N h_i b_j = b_j. \end{aligned}$$

so

$$g \mapsto \frac{b'_j}{f_j^N} = g_j \in R_{f_j}. \quad //$$

For $U \subset X$ open, define

$$\Gamma(U, \mathcal{O}_X) \subset \prod_{x \in U} \mathcal{O}_{X,x}$$

to be the subset of tuples $\{s_x\}_{x \in U}$ s.t. there exists a covering $U = \bigcup D(f_\alpha)$ and elements $s_\alpha \in R_{f_\alpha}$ s.t. for all $x \in U$ and $x \in D(f_\alpha)$, the can. map $R_{f_\alpha} \rightarrow R_{\beta_x} = \mathcal{O}_{X,x}$ takes s_α to s_x .

$$\Gamma(U, \mathcal{O}_X) \subset \prod_{x \in U} \mathcal{O}_{X,x}$$

$\downarrow$

$\downarrow \text{pr}$

$$\Gamma(U, \mathcal{O}_X) \subset \prod_{x \in U} \mathcal{O}_{X,x}$$

so have a presheaf on X .

Lemma $\mathcal{O}_X$ is a sheaf. //

Prop The canonical map

$$R_f \rightarrow \Gamma(D(f), \mathcal{O}_X)$$

is an isomorphism.

Pf Then map $R_f \rightarrow \prod_{x \in D(f)} \mathcal{O}_{X,x}$ is injective by (the proof of) Lemma 1. It lands in $\Gamma(D(f), \mathcal{O}_X)$ and is surj. by Lemma 2. //

$$\dim R/\mathfrak{p}_x + \dim R_{\mathfrak{p}_x} \leq \dim R$$

$$\begin{array}{ccc} | & & | \\ \dim. \text{ of } \overline{\{x\}} & & \text{codim. of } \overline{\{x\}} \end{array}$$

Thm If R is both an integral domain and a finite generated algebra over a field, then

$$\dim R/\mathfrak{p} + \dim R_{\mathfrak{p}} = \dim R.$$

Pt Eisenbud p. 290. "

Lemma Let A be a noetherian ring, and let $I \subset A$ be an ideal.

Then the partially ordered set of prime ideals $\mathfrak{p} \supset I$ has finitely many minimal elements.

Pf Suppose this is not true. Since A is noetherian, there exists an ideal $I \subset A$ that is maximal among ideals in A for which the statement fails. This ideal I cannot be a prime ideal. So there exist $f, g \in A$ s.t. $fg \in I$ but $f \notin I$ and $g \notin I$. If $\mathfrak{p} \supset I$ then $fg \in \mathfrak{p}$ so either $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. Hence, if $\mathfrak{p} \supset I$ then either $\mathfrak{p} \supset (I, f)$ or $\mathfrak{p} \supset (I, g)$. It follows that set of primes ideals that are minimal among

prime ideals that contain I is contained in the union of the sets of prime ideals that are minimal among prime ideals that contain (I, f) or (I, g) . Since $(I, f) \not\supseteq I$ and $(I, g) \not\supseteq I$, these sets are finite by the maximality of I . This proves the lemma. //

$$V(I) = V(\sqrt{I})$$

$$\sqrt{I} = \bigcap_{\substack{\mathfrak{P} \supset I}} \mathfrak{P} = \bigcap_{\substack{\mathfrak{P} \supset I \\ \text{minimal}}} \mathfrak{P}$$

$$V(I) = \bigcup_{\substack{\mathfrak{P} \supset I \\ \text{minimal}}} V(\mathfrak{P})$$

ex $I = (x, y) = (x) \cap (y)$ //

Thm (Krull's principal ideal thm.)

Let A be a noetherian ring,

let $I = (a_1, \dots, a_r)$ be an ideal generated by r elements, and

let $\mathfrak{p}$ be minimal among prime ideals that contain I . Then

$$\text{ht } \mathfrak{p} = \dim A_{\mathfrak{p}} \leq r.$$

Pf See Eisenbud, p. 235. "

$$X = \text{Spec } A$$

$$\cup \quad \cup$$

$$Z = \text{Spec } A/I$$

$$\text{codim}_X Z := \min_{Z \in Z} \dim \mathcal{O}_{X, Z}$$

$$= \min_{\mathfrak{p} \supset I} \dim A_{\mathfrak{p}}$$

$$= \min_{\substack{\mathfrak{p} \supset I \\ \text{minimal}}} \dim A_{\mathfrak{p}}$$

$$\mathfrak{p} \supset I$$

minimal

so the theorem says that

$$\text{codim}_X Z \leq r.$$

Ex If $p \in A$ is not a unit and not a zero-divisor, then

$$\text{codim}_A A/pA = 1. \quad //$$

Thm Let k be a field, and let A be a finitely generated k -algebra. Suppose that A is an integral domain, and let K be the quotient field. Then

$$\dim A = \text{tr. deg}_k K.$$

Moreover, all maximal chains of prime ideals in A have the same length.

Pf Eisenbud, p. 290. //

Ex let $m_x \subset A$ be a maximal ideal. Then $k(x) = A/m_x$ is again a finitely generated k -algebra. Hence

$$\text{tr. deg}_k k(x) = \dim k(x) = 0.$$

So $k(x)$ is a finite algebraic extension of k . //

Cor For every prime ideal $\mathfrak{p} \subset A$,

$$\dim A_{\mathfrak{p}} + \dim A/\mathfrak{p} = \dim A.$$

Pf The left-hand side is the maximal length of all chains of prime ideals that include $\mathfrak{p}$.

The right-hand side is the maximal length of all chains of prime ideals. //

$$X = \operatorname{Spec} A$$

$$V = \operatorname{Spec} A/I$$

$$W = \operatorname{Spec} A/J$$

$$V \times_X W = \operatorname{Spec} A/I \otimes_A A/J = \operatorname{Spec} A/I+J$$

- scheme theoretic intersection.

We say that V and W intersect properly if

$$\dim(V \times_X W) \leq \dim V + \dim W - \dim X.$$

If A is a f.g. k -algebra and an integral domain, and if V and W are irreducible, then this is equivalent to

$$\operatorname{codim}_X(V \times_X W) \geq \operatorname{codim}_X V + \operatorname{codim}_X W.$$

Let A be a f.g. k -algebra and an integral domain. Let $f \in A$ be an element that is not a unit and not a zero-divisor, and let

$$D = \operatorname{Spec} A/fA \subset X = \operatorname{Spec} A.$$

Then

$$\operatorname{codim}_X D = 1.$$

Let $\mathfrak{p} \subset A$ be a prime ideal and

$$V = \operatorname{Spec} A/\mathfrak{p} \subset X.$$

Then V and D intersect properly if and only if $f \notin \mathfrak{p}$. In this case either $V \times_X D = \emptyset$ or

$$\dim(V \times_X D) = \dim V - 1.$$

We define the intersection

cycle to be the formal linear combination

$$[A/\mathfrak{p}] \cdot [A/\mathfrak{f}A]$$

$$= \sum_{\mathfrak{p}'} \text{length}_{A_{\mathfrak{p}'}} (A/\mathfrak{p} + \mathfrak{f}A)_{\mathfrak{p}'} \cdot [A/\mathfrak{p}']$$

where the sum ranges over prime ideals $\mathfrak{p}' \subset A$ s.t.

$$\dim A/\mathfrak{p}' = \dim A/\mathfrak{p} - 1.$$

ex $A = k[x, y]$

$$\mathfrak{p} = (y - x^n)$$

$$\mathfrak{f} = y$$

$$\bigvee_x D = \text{Spec } k[x, y]/(y, y - x^n)$$

$$= \text{Spec } k[x]/(x^n).$$

$$[Y] \cdot [D] = n \cdot [A/(x, y)]$$

//

A sequence $a_1, \dots, a_n$ of elements of A is a regular sequence if $(a_1, \dots, a_n) \subsetneq A$ and if for all $1 \leq i \leq n$, a_i is a non-zero-divisor in $A/(a_1, \dots, a_{i-1})$. If A is a f.g. k -algebra and an integral domain, then

$$\begin{aligned} \dim A/(a_1, \dots, a_i) \\ = \dim A - i. \end{aligned}$$

Suppose $V = \text{Spec } A/\mathfrak{p}$ intersects the subschemes $\text{Spec } A/(a_1, \dots, a_j)$ properly. This happens if and only if every irreducible comp. of $\text{Spec } A/\mathfrak{p} + (a_1, \dots, a_j)$ has $\dim$. $\dim A/\mathfrak{p} - j$, if $r \geq j$, and is empty otherwise.

Prop Let $a_1, \dots, a_n$ be a regular sequence in A , let $V = \text{Spec } A/\mathfrak{p}$ be a closed, integral subscheme, and suppose that V intersects all the subschemes $\text{Spec } A/(a_{i_1}, \dots, a_{i_j})$ properly. Then the intersection cycle

$$[V] \cdot (a_1) \cdot \dots \cdot (a_n)$$

is independent of the ordering of the a_i .

Pf It suffices to show

$$[V] \cdot (a) \cdot (b) = [V] \cdot (b) \cdot (a).$$

Let m be minimal among prime ideals that contain $\mathfrak{p} + aA + bA$.

To calculate the multiplicity of m in the two intersection cycles, we may localize at m

and divide out $\mathfrak{p}$. Hence, we can assume that A is a local domain of dimension 2, and that $\mathfrak{m}$ is the only prime ideal of A that contains (a, b) .

The intersection cycle $[V] \cdot (a)$ is given by

$$\sum_{\mathfrak{p}} \text{length}_{A_{\mathfrak{p}}} (A_{\mathfrak{p}} / aA_{\mathfrak{p}}) \cdot [A/\mathfrak{p}]$$

where the sum runs over the prime ideals that are minimal among prime ideals that contain a . So $[V] \cdot (a) \cdot (b)$ is

$$\begin{aligned} & \sum_{\mathfrak{p}} \text{length}_{A_{\mathfrak{p}}} (A_{\mathfrak{p}} / aA_{\mathfrak{p}}) \\ & \quad \cdot \text{length}_{A/\mathfrak{p}A} (A/\mathfrak{p} + bA) \end{aligned}$$

We apply the following lemma to A/aA :

lemma let (B, m) be a 1-dim'l noeth. local ring with minimal primes $\mathfrak{p}_1, \dots, \mathfrak{p}_r$, and let b be an element of B that is not contained in any of the $\mathfrak{p}_i$. Then for every f.g. B -module M ,

$$\begin{aligned} \text{length}_B (M/bM) &= \text{length}_B ({}_b M) \\ &= \sum_{1 \leq i \leq r} \text{length}_{B_{\mathfrak{p}_i}} (M_{\mathfrak{p}_i}) \\ &\quad \cdot \text{length}_B (B/\mathfrak{p}_i + bB). \end{aligned}$$

Pf Since length is an Euler characteristic, it suffices to consider $M = B/m$ and $M = B/\mathfrak{p}_i$. If $M = B/m$ then

$$\text{length}_B (M/bM) = \begin{cases} 1 & b \in m \\ 0 & b \notin m \end{cases}$$

$$\text{length}_B(bM) = \begin{cases} 1 & b \in m \\ 0 & b \notin m \end{cases}$$

and $M_{\mathfrak{p}_i} = 0$, for all $1 \leq i \leq r$.

If $M = B/\mathfrak{p}_i$ then

$$\text{length}_{B_{\mathfrak{p}_j}}(M_{\mathfrak{p}_j}) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

and since $b \notin \mathfrak{p}_i$, $bM = 0$, so

$$\begin{aligned} & \text{length}_B(M/bM) - \text{length}_B(bM) \\ &= \text{length}_B(M/bM) \\ &= \text{length}_B(B/\mathfrak{p}_i + bB). \quad // \end{aligned}$$

The lemma shows that the intersection mult. in question is

$$\text{length}_A(A/(a,b)) - \text{length}_A(b(A/aA)).$$

But there is an A -linear
isomorphism

$$b(A/aA) \xrightarrow{\sim} a(A/bA)$$

that to the b -torsion elem.

$x + aA$ of A/aA assigns the

a -torsion elem. $y + bA$ of

A/bA where y is the unique

elem. of A s.t. $bx = ay$. //

A : f.g. k -algebra, k field.

$\mathfrak{p}_1, \dots, \mathfrak{p}_n \subset A$: minimal prime ideals

$\dim A_{\mathfrak{p}_i} = 0$: $A_{\mathfrak{p}_i}$ artinian ring

Define the fundamental cycle

$$[A] \in \mathbb{Z}_*(A) = \bigoplus_{r \geq 0} \mathbb{Z}_r(A)$$

to be

$$[A] = \sum_{1 \leq i \leq n} \text{length}_{A_{\mathfrak{p}_i}}(A_{\mathfrak{p}_i}) \cdot [A/\mathfrak{p}_i].$$

Note that $\text{Spec}(A/\mathfrak{p}_i)$ is an irred. comp. of $\text{Spec}(A)$ with the reduced scheme structure. The way this comp. sits in $\text{Spec}(A)$ is as the closed subscheme $\text{Spec}(A/\mathfrak{I}_i)$ where

$$\mathfrak{I}_i = \ker(A \rightarrow A_{\mathfrak{p}_i}).$$

Let D and D' be two divisors on X that intersect properly: no closed integral subscheme $V \subset X$ of codim. 1 is contained in both $\text{supp}(D)$ and $\text{supp}(D')$.

Then both $D \cdot [D']$ and $D' \cdot [D]$ are defined.

Prop If D and D' intersect properly, then

$$D \cdot [D'] = D' \cdot [D]$$

in $\mathbb{Z}^2(X)$.

and let

$$Z'_*(X) \subset Z_*(X)$$

be the subgroup of cycles in good position:

$$\alpha = \sum_V n_V \cdot [V]$$

$$V \subset \text{supp}(D) \Rightarrow n_V = 0.$$

Then we define

$$Z'_r(X) \longrightarrow Z_{r-1}(X)$$

$$[V] \longmapsto D \cdot [V]$$

as follows. By restriction along $j: V \hookrightarrow X$, we get a divisor j^*D on V ; we let $D \cdot [V] = [j^*D]$ be the associated Weil divisor.

$$\text{ord}_A : K^* \longrightarrow \mathbb{Z}.$$

Now let D be a Cartier divisor on X with local equations f_α .

The f_α are determined by D up to multiplication by sections of $\mathcal{O}_X^*$. If x is a codimension 1 point in U_α , we define

$$\text{ord}_x(D) = \text{ord}_{\mathcal{O}_{X,x}}(f_\alpha).$$

Then the Weil divisor (= codim. 1 cycle) associated with D is defined by

$$[D] = \sum_{\substack{x \in X \\ \text{codim } 1}} \text{ord}_x(D) \cdot [\overline{\{x\}}].$$

Let D be a divisor on X ,

If $Z \subset X$ is a closed integral subscheme, and if D is a divisor on X s.t. $Z \not\subset \text{supp}(D)$, then local equations for D define local equations for a divisor j^*D on Z .

Let A be a one-dimensional domain with quotient field K .

For a non-zero element $a \in A$, we define

$$\text{ord}_A(a) = \text{length}_A(A/aA),$$

and for $r = a/b \in K$, we define

$$\text{ord}_A(r) = \text{ord}_A(a) - \text{ord}_A(b).$$

Since $\text{length}_A(-)$ is an Euler characteristic, $\text{ord}_A(-)$ is a well-defined homomorphism

m_x : sheaf of meromorphic fct.

$$\Gamma(U, m_x) := (\text{non-zero-div.})^{-1} \Gamma(U, \mathcal{O}_x).$$

A Cartier divisor D on X is a section of $m_x^* / \mathcal{O}_x^*$. It is determined by an open covering $\{U_\alpha\}$, sections $f_\alpha \in \Gamma(U_\alpha, m_x^*)$ s.t. on $U_\alpha \cap U_\beta$, f_α / f_β is a section of $\mathcal{O}_x^*$. Call f_α local equations for D .

Define

$$\text{supp}(D) \subset X$$

to be the set of $x \in X$ s.t.

local equations for D are not

in $\mathcal{O}_{x,x}^* \subset m_{x,x}^*$. A divisor D

is effective if local equations

are sections of $\mathcal{O}_x$.

$$f^*[V] = \sum_{\substack{w \in V \times (\mathbb{Z} \times X) \\ \mathbb{Z} \quad Y}} \text{length}_{\mathcal{O}_{V \times (\mathbb{Z} \times X), w}^{\mathbb{Z} \times (\mathbb{Z} \times X), w}} (\mathcal{O}_{V \times (\mathbb{Z} \times X), w}^{\mathbb{Z} \times (\mathbb{Z} \times X), w}) \cdot [w]$$

$$= \sum_{\substack{w \in V \times (\mathbb{Z} \times X) \\ \mathbb{Z} \quad Y}} \text{length}_{\mathcal{O}_{\mathbb{Z} \times X, w}^{\mathbb{Z} \times X, w}} (\mathcal{O}_{\mathbb{Z} \times X, w}^{\mathbb{Z} \times X, w} / m_v \mathcal{O}_{\mathbb{Z} \times X, w}^{\mathbb{Z} \times X, w}) \cdot [w]$$

By lemma (i), if w is an irred. comp. of $\mathbb{Z} \times X$, then

$$V = \overline{f(w)}$$

is an irred. comp. of $\mathbb{Z}$. By lemma (ii),

$$\cdot \text{length}_{\mathcal{O}_{\mathbb{Z} \times X, w}^{\mathbb{Z} \times X, w}} (\mathcal{O}_{\mathbb{Z} \times X, w}^{\mathbb{Z} \times X, w})$$

$$= \text{length}_{\mathcal{O}_{\mathbb{Z}, v}^{\mathbb{Z}, v}} (\mathcal{O}_{\mathbb{Z}, v}^{\mathbb{Z}, v})$$

$$\cdot \text{length}_{\mathcal{O}_{\mathbb{Z} \times X, w}^{\mathbb{Z} \times X, w}} (\mathcal{O}_{\mathbb{Z} \times X, w}^{\mathbb{Z} \times X, w} / m_v \mathcal{O}_{\mathbb{Z} \times X, w}^{\mathbb{Z} \times X, w})$$

$$\text{So } f^*([Z]) = [\mathbb{Z} \times X].$$

//

Lemma If $f: X \rightarrow Y$ is flat, then for every closed subscheme $Z \subset Y$,

$$f^*[Z] = [Z \times_Y X].$$

Prf Consider

$$\begin{array}{ccc} Z \times_Y X & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Z & \longrightarrow & Y \end{array}$$

By definition

$$[Z] = \sum_{\substack{v \in Z \\ \text{generic}}} \text{length}_{\mathcal{O}_{Z,v}}(\mathcal{O}_{Z,v}) \cdot [v]$$

$$[Z \times_Y X] = \sum_{\substack{w \in Z \times_Y X \\ \text{generic}}} \text{length}_{\mathcal{O}_{Z \times_Y X, w}}(\mathcal{O}_{Z \times_Y X, w}) \cdot [w]$$

is exact, and since B is flat over A , so is

$$0 \rightarrow B \xrightarrow{f(a)} B \rightarrow B/f(a)B \rightarrow 0$$

So $f(a)$ is a non-zero-divisor

in B . Let $\mathfrak{p} \subset B$ be a minimal prime. Then $f(a) \notin \mathfrak{p}$ if $a \neq 0$.

$$\text{So } f^{-1}(\mathfrak{p}) = \{0\}.$$

(ii) let

$$A = I_0 > I_1 > \dots > I_r = 0$$

be a chain of ideal with

$$I_i / I_{i+1} \cong A/m. \text{ Then}$$

$$B = I_0 B > I_1 B > \dots > I_r B = 0$$

is a chain with

$$I_i B / I_{i+1} B \cong \underset{\uparrow}{I_i / I_{i+1}} \underset{A}{\otimes} B \cong B/mB$$

since flat.

//

Lemma Let A and B be rings and let $f: A \rightarrow B$ be a flat local homomorphism. Then:

(i) $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is onto.

(ii) If A and B are artinian local, and if $\mathfrak{m} \subset A$ is the maximal ideal, then

$$\text{length}_B(B) = \text{length}_A(A)$$

$$= \text{length}_B(B/\mathfrak{m}B).$$

Prf. Given $\mathfrak{p} \subset A$, we must find

$\mathfrak{p} \subset B$ s.t. $\mathfrak{p} = f^{-1}(\mathfrak{p})$. Since

$B/\mathfrak{p}B$ is flat over $A/\mathfrak{p}$, we can

assume $\mathfrak{p} = 0$. Then, if $a \in A$ is

non-zero, the sequence

$$0 \rightarrow A \xrightarrow{a} A \rightarrow A/aA \rightarrow 0$$

is irreducible, and if X has pure dimension $\dim Y + n$, then f has relative dimension n .

Ex If Z is of pure dimension n , then the projection

$$\text{pr}_1: X = Y \times Z \rightarrow Y$$

is flat of relative dimension n . //

Let $f: X \rightarrow Y$ be flat of relative dimension n . Define

$$f^*: \mathbb{Z}_r(Y) \rightarrow \mathbb{Z}_{r+n}(X)$$

by

$$f^*([Z]) = [Z \times_Y X]$$

$$\mathbb{Z}_*(X) \hookrightarrow \mathbb{Z}_*(Y).$$

In particular, if $Z \subset X$ is a closed integral subscheme, the two definitions of $[Z] \in \mathbb{Z}_*(X)$ agree.

We say that X has pure dimension r if $\dim X_i = r$, $1 \leq i \leq n$.

In this case, $[X] \in \mathbb{Z}_r(X)$.

Let $f: X \rightarrow Y$ be a flat map.

The map f is said to have relative dimension n if for

every closed integral subscheme $V \subset Y$, and every irreducible component V' of $V \times_Y X$,

$$\dim V' = \dim V + n.$$

If $f: X \rightarrow Y$ is flat, if Y

X : scheme of finite type / k

$X_1, \dots, X_n$: irred. comp. considered as closed (integral) subschemes with the reduced scheme str.

$x_i \in X_i$: generic point: $X_i = \overline{\{x_i\}}$

Define the fund. cycle

$$[X] \in \mathbb{Z}_*(X) = \bigoplus_{r \geq 0} \mathbb{Z}_r(X)$$

to be

$$[X] = \sum_{1 \leq i \leq n} \text{length}_{\mathcal{O}_{X, x_i}}(\mathcal{O}_{X, x_i}) \cdot [X_i]$$

This is well-defined since

$\mathcal{O}_{X, x_i}$ is an artinian ring (since

x_i is a generic point). If X

is a closed subscheme of Y ,

we also write $[X]$ for the image of $[X]$ by the canonical incl.

Cubical definition of higher Chow groups. Define

$$\mathbb{Z}^r(k, n) \subset \mathbb{Z}^r((\mathbb{P}_k^1 \setminus \{1\})^n)$$

to be the free abelian group generated by the closed integral subschemes

$$Z \subset (\mathbb{P}_k^1 \setminus \{1\})^n$$

that meet all faces in a closed subset of codimension $\geq r$. The faces $F \subset (\mathbb{P}_k^1 \setminus \{1\})^n$ are defined as follows. Let $0 \leq m \leq n$, let $1 \leq i_1 < \dots < i_m \leq n$, and let $\varepsilon_j \in \{0, \infty\}$, $1 \leq j \leq m$. Then the closed integral subscheme

$$"x_{i_1} = \varepsilon_1, \dots, x_{i_m} = \varepsilon_m"$$

is a face of codim. m .

let $1 \leq i \leq n$, and let $\varepsilon \in \{0, \infty\}$. Then intersection with the effective divisor " $x_i = \varepsilon$ " or $(x_i - \varepsilon)$ defines a face map

$$\partial_i^\varepsilon : \mathbb{Z}^r(k, n) \rightarrow \mathbb{Z}^r(k, n-1)$$

which, in turn, defines a diff.

$$d : \mathbb{Z}^r(k, n) \rightarrow \mathbb{Z}^r(k, n-1)$$

given by

$$d = \sum_{1 \leq i \leq n} (-1)^i (\partial_i^\infty - \partial_i^0).$$

The complex $(\mathbb{Z}^r(k, -), d)$ does not have the correct homology: It must be normalized. Pull-back along the flat morphisms

$$(\mathbb{P}_k^1 - \{1\})^{n-1} \xleftarrow{\sigma^i} (\mathbb{P}_k^1 - \{1\})^n \quad 1 \leq i \leq n$$

$$x_j \mapsto \begin{cases} x_j & 1 \leq j < i \\ x_{j+1} & i \leq j \leq n-1 \end{cases}$$

define degeneracy maps

$$g_i : z^r(k, n-1) \rightarrow z^r(k, n),$$

and we then define the normalized group $\bar{z}^r(k, n)$ to be the cokernel of the total map

$$\bigoplus_{1 \leq i \leq n} z^r(k, n-1) \xrightarrow{\bigoplus g_i} z^r(k, n).$$

The differential on $z^r(k, -)$ induces one on $\bar{z}^r(k, -)$, and we define

$$CH^r(k, n) := H_n(\bar{z}^r(k, -), d).$$

We note that since $z^n(k, n-1) = 0$,

$$CH^n(k, n) = H_n(z^n(k, -), d).$$

$$k^* \xrightarrow{\phi} CH^1(k, 1)$$

$$1 \longmapsto 0$$

$a \neq 1 \longmapsto [a]$ - the class of the effective divisor with local eq. a

To show that ϕ is a group homom., we must show that for $a \in F \setminus \{0, 1\}$,

$$[a] + [\frac{1}{a}] = 0 \in CH^1(k, 1),$$

and for $a, b, ab \in k \setminus \{0, 1\}$,

$$[a] + [b] = [ab] \in CH^1(k, 1).$$

Consider the rational curve

$$x \longmapsto \left(x, \frac{ax - ab}{x - ab} \right)$$

in $(\mathbb{P}_k^1 \setminus \{1\})^2$ for $a, b \in k \setminus \{0, 1\}$.

$$x=0 \Rightarrow \frac{ax - ab}{x - ab} = 1$$

-ii-

$$x = \infty \Rightarrow \frac{ax - ab}{x - ab} = a$$

$$\frac{ax - ab}{x - ab} = 0 \Rightarrow x = b$$

$$\frac{ax - ab}{x - ab} = \infty \Rightarrow x = ab$$

Hence, this curve defines an element $[c] \in \mathbb{Z}^1(k, 2)$. If $ab \neq 1$, then

$$\begin{aligned} d([c]) &= -\partial_1^\infty([c]) + \partial_1^0([c]) \\ &\quad + \partial_2^\infty([c]) - \partial_2^0([c]) \\ &= -[a] + [ab] - [b] \end{aligned}$$

so we get

$$[a] + [b] = [ab] \in CH^1(k; 1).$$

If $ab = 1$, then

$$d([c]) = -[a] - [b]$$

so

$$[a] + [a^{-1}] = 0 \in CH^1(k, 1).$$

So we have a group homom.

$$k^* \xrightarrow{\phi} CH^1(k, 1).$$

To show that for $a \in k \setminus \{0, 1\}$, the product

$$\phi(a) \cdot \phi(1-a) = 0 \in CH^2(k, 2),$$

we must show that the rational point

$$(x-a, y-(1-a)) \in (\mathbb{P}_k^1 - \{1\})^2$$

is the boundary of an element in $Z^2(k, 3)$. Consider the rational curve C :

$$x \mapsto \left(x, 1-x, \frac{a-x}{1-x} \right).$$

$$x = 0 \Rightarrow 1-x = 1$$

$$x = \infty \Rightarrow \frac{a-x}{1-x} = 1$$

$$1-x = 0 \Rightarrow x = 1$$

$$1-x = \infty \Rightarrow \frac{a-x}{1-x} = 1$$

$$\frac{a-x}{1-x} = 0 \Rightarrow x = a, \quad 1-x = 1-a$$

$$\frac{a-x}{1-x} = \infty \Rightarrow x = 0, \quad 1-x = 1$$

This shows that $[C] \in \mathbb{Z}^2(k, \beta)$ and

$$\begin{aligned} d([C]) &= -\partial_1^\infty([C]) + \partial_1^0([C]) + \partial_2^\infty([C]) \\ &\quad - \partial_2^0([C]) - \partial_3^\infty([C]) + \partial_3^0([C]) \\ &= [(x, y) = (a, 1-a)]. \end{aligned}$$

So, the Steinberg relation holds and we have a map of graded rings

$$K_q^M(k) \rightarrow CH^q(k, q).$$

Define a map

$$Z^n(k, n) \longrightarrow K_n^M(k).$$

This group is the free abelian group generated by the set of closed points in $(\mathbb{P}_k^1 \setminus \{1\})^n$. Let $x \in (\mathbb{P}_k^1 \setminus \{1\})^n$ be a closed point with residue field $k(x)$. Then we have a canonical map

$$\text{Spec } k(x) \xrightarrow{f} (\mathbb{P}_k^1 \setminus \{0, 1, \infty\})^n$$

and hence the coordinate functions $x_1, \dots, x_n$ on $(\mathbb{P}_k^1 \setminus \{0, 1, \infty\})^n$ induce functions $\bar{x}_i = f^* x_i$ on $\text{Spec } k(x)$.

In other words, we have elements $\bar{x}_1, \dots, \bar{x}_n \in k(x)$ and $k(x) = k(\bar{x}_1, \dots, \bar{x}_n)$. These elements define a symbol

$$\{\bar{x}_1, \dots, \bar{x}_n\} \in K_n^M(k(x)).$$

To get an element of $K_n^M(k)$, we need the norm homomorphism

$$N_{k(k)/k} : K_n^M(k(k)) \rightarrow K_n^M(k).$$

We will construct this later. So the desired map is

$$z^n(k, n) \xrightarrow{\gamma} K_n^M(k)$$

$$[x] \mapsto N_{k(k)/k}(\{\bar{x}_1, \dots, \bar{x}_n\}).$$

We must show that the composite

$$z^n(k, n+1) \xrightarrow{\alpha} z^n(k, n) \xrightarrow{\gamma} K_n^M(k)$$

is equal to zero. The left-hand group is the free abelian group on generators $[C]$ where $C \in (\prod_{i=1}^n \{1\})^{n+1}$ are integral curves that meet the codimension 1 faces " $x_i = 0$ " and " $x_i = \infty$ " in points and that do not meet

the codim. 2 faces. Must show that $\varphi(d([c])) = c$. Will do this later. Assuming this, we get a homeomorphism of graded rings

$$CH^n(k, n) \xrightarrow{\varphi} K_n^M(k).$$

It is clear that the composite

$$K_n^M(k) \xrightarrow{\varphi} CH^n(k, n) \xrightarrow{\varphi} K_n^M(k)$$

is the identity map. Let us first show that the composite

$$CH^n(k, n) \xrightarrow{\varphi} K_n^M(k) \xrightarrow{\varphi} CH^n(k, n)$$

is the identity if $n \leq 1$. The case $n=0$ is trivial, so consider $n=1$. If $x \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ is a k -rational point, then $\varphi(\varphi([x])) = [x]$, so it suffices to show

that for every closed point $x \in \mathbb{P}_k^1 \setminus \{0, 1, \infty\}$, $[x]$ is equivalent in $CH^1(k, 1)$ to a sum of k -rational points. let us write $t = \bar{x}_1$ for the generator of $k(x)$ given by

$$\text{Spec } k(x) \longrightarrow \mathbb{P}_k^1 \setminus \{0, 1, \infty\}$$

$$t \longleftarrow x_1$$

let

$$f(t) = t^d - a_{d-1}t^{d-1} + \dots + (-1)^d a_0$$

be the minimal polynomial of x , i.e.

$$k[t] / (f(t)) \xrightarrow{\sim} k(x).$$

Then

$$k(x)^* \xrightarrow{N_{k(x)/k}} k$$

$$t \longmapsto a_0$$

-ix -

Indeed, a basis of $k[t]$ as a k -vector space is given by the set of elem. $1, t, \dots, t^{d-1}$, and multiplication by t has matrix

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & (-1)^{d-1} a_0 \\ 1 & 0 & 0 & \dots & 0 & (-1)^{d-2} a_1 \\ 0 & 1 & 0 & \dots & 0 & (-1)^{d-3} a_2 \\ 0 & 0 & 1 & \dots & 0 & (-1)^{d-4} a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -a_{d-2} \\ 0 & 0 & 0 & \dots & 1 & a_{d-1} \end{pmatrix}$$

Expanding along the first row we see that the determinant is a_0 as stated.

Consider the curve in $(\mathbb{P}_k^1 \times \mathbb{A}_k^1)^2$ given by the equation

$$C: p(x_1, x_2) = f(x_1) - (x_1 - 1)^{d-1} (x_1 - a_0) x_2.$$

Then

-x-

$$x_1 = 0 \Rightarrow (-1)^d a_0 - (-1)^d a_0 x_2 = 0 \Rightarrow x_2 = 1.$$

$$x_1 = \infty \Rightarrow x_2 = 0. \quad (p(x_1, x_2) \text{ has leading coeff. } (1-x_2) \text{ wrt. } x_1)$$

$$x_2 = 0 \Rightarrow f(x_1) = 0 \quad \swarrow \text{multiplicity one}$$

$$x_2 = \infty \Rightarrow x_1 = 1 \text{ or } x_1 = a_0$$

This shows that

$$\begin{aligned} d([C]) &= -\partial_1^\infty([C]) + \partial_1^0([C]) \\ &\quad + \partial_2^\infty([C]) - \partial_2^0([C]) \\ &= [(x_1, -a_0)] - [(\neq(x_1))] \end{aligned}$$

or

$$[x] = [N_{\text{Nuc}(x)/k}(t)].$$

Assuming that γ is well-defined,
this shows

$$k^* \xrightarrow[\sim]{\varphi} CH^1(k, 1).$$

$$t^2 = t - \{ -1 \} \quad (t = \deg \pi)$$

$$K_*^M(K) \longrightarrow K_*^M(K)[t] / (t^2 - t - \{ -1 \})$$

$$\{ \pi^i u \} \longmapsto \{ \bar{u} \} + i \cdot t$$

$$\{ \pi^{i_1} u_1, \dots, \pi^{i_n} u_n \} \longmapsto \prod_{1 \leq j \leq n} (\{ \bar{u}_j \} + i_j \cdot t)$$

$$\{ \pi^{i_1} u_1, \pi^{i_2} u_2 \} \longmapsto (\{ \bar{u}_1 \} + i_1 t)(\{ \bar{u}_2 \} + i_2 t)$$

$$= \{ \bar{u}_1, \bar{u}_2 \} + \{ \bar{u}_1 \} i_2 t + i_1 t \{ \bar{u}_2 \} + i_1 i_2 t^2$$

$$= \{ \bar{u}_1, \bar{u}_2 \} + (i_2 \{ \bar{u}_1 \} - i_1 \{ \bar{u}_2 \} + i_1 i_2 \{ -1 \}) t$$

Suppose

$$\pi^{i_1} u_1 + \pi^{i_2} u_2 = 1.$$

Consider four cases:

$i_1 > 0$: Then $i_2 = 0$ and $\bar{u}_2 = 1$. So

$$\Theta_{\pi}(\{\pi^{i_1} u_1, \pi^{i_2} u_2\}) = 0.$$

$i_1 = 0, i_2 > 0$: Similar.

$i_1 = i_2 = 0$: Then $\bar{u}_1 + \bar{u}_2 = 1$, so again

$$\Theta_{\pi}(\{\pi^{i_1} u_1, \pi^{i_2} u_2\}) = 0.$$

$i_1 < 0$: Then $i_1 = i_2$ and $\bar{u}_1 + \bar{u}_2 = 0$. So

$$\Theta_{\pi}(\{\pi^{i_1} u_1, \pi^{i_2} u_2\})$$

$$= \{\bar{u}_1, -\bar{u}_1\} + (i_1 \{\bar{u}_1\} - i_1 \{-\bar{u}_1\} + i_1^2 \{-1\})t$$

$$= 0 + (i_1 \{\bar{u}_1\} + i_1 \{-1\} - i_1 \{\bar{u}_1\} + i_1^2 \{-1\})t$$

$$= i_1 (i_1 + 1) \{-1\} t$$

$$= 0$$

since $i_1 (i_1 + 1)$ is even.

The abelian group $K_n^M(K)$ is generated by symbols of the form

$$\{\pi, \dots, \pi, u_{r+1}, \dots, u_n\}$$

with $u_j \in \sigma_K^*$. We calculate

$$\Theta_\pi(\{\pi, \dots, \pi, u_{r+1}, \dots, u_n\})$$

$$= t^r \cdot \{\bar{u}_{r+1}, \dots, \bar{u}_n\}$$

$$= \begin{cases} \{\bar{u}_1, \dots, \bar{u}_n\} & \text{if } r=0 \\ t \cdot \{-1, \dots, -1, \bar{u}_{r+1}, \dots, \bar{u}_n\} & \text{if } r>0. \end{cases}$$

In particular

$$\partial : K_n^M(K) \rightarrow K_{n-1}^M(K)$$

maps $\{\pi, u_2, \dots, u_n\}$ to $\{\bar{u}_2, \dots, \bar{u}_n\}$ and $\{u_1, \dots, u_n\}$ to zero. Hence, ∂ does not depend on the choice of uniformizer π .

Then There is an exact sequence

$$0 \rightarrow K_n^M(F) \rightarrow K_n^M(F[t]) \xrightarrow{\partial} \bigoplus_{m \in F[t]} K_{n-1}^M(F[t]/m) \rightarrow 0$$

where $\partial = \bigoplus \partial_m$. //

Let $v_\infty : F(t)^* \rightarrow \mathbb{Z}$ be the valuation with parameter t^{-1} , i.e.

$$v_\infty(f) = -\deg(f).$$

Then v_∞ vanishes on $K_n^M(F)$, so

there are unique maps

$$N_m : K_*^M(F[t]/m) \rightarrow K_*^M(F)$$

s.t.

$$-\partial_{v_\infty} = \sum_m N_m \partial_m.$$

Letting

$$N_\infty = id : K_*^M(F) \rightarrow K_*^M(F)$$

we can restate the definition of the norm maps N_v by the Weil reciprocity formula:

$$\sum_v N_v \circ \partial_v = 0$$

where the sum runs over all discrete valuations on $F(t)/F$.

Def Let E be a finite extension of F generated by a single element $a \in E$. Define

$$N_{a/F} := K_*^M(E) \rightarrow K_*^M(F)$$

to be the norm map associated with minimal ideal of a/F :

$$m = \ker (F[t] \rightarrow E)$$

$$t \mapsto a \quad //$$

Lemma (Projection formula). Let $E = F(a)$.

Then the norm map

$$N_{a/F} : K_*^M(E) \rightarrow K_*^M(F)$$

is a map of $K_*^M(F)$ - modules. "

Cor If $E = F(a)$ has degree d , then the kernel of the canonical map

$$K_*^M(F) \rightarrow K_*^M(E)$$

is annihilated by d .

Pf Indeed,

$$K_*^M(F) \rightarrow K_*^M(E) \xrightarrow{N_{a/F}} K_*^M(F)$$

is multiplication by

$$N_{a/F}(1) = d \in K_0^M(F) = \mathbb{Z}. \quad "$$

Def Let $E = F(a_1, \dots, a_r)$ be a finite field extension of F . The norm maps

$$N_{E/F} : K_*^M(E) \rightarrow K_*^M(F)$$

is defined to be the composition

$$N_{E/F} = N_{a_1/F} \circ \dots \circ N_{a_r/F(a_1, \dots, a_{r-1})}$$

of the norm maps defined above. //

Thm (Kato) The norm map $N_{E/F}$ is well-defined.

Lemma If E is a normal extension of F , of prime degree $[E:F] = p$, then the norm map

$$N_{E/F} = N_{a/F} : K_*^M(E) \rightarrow K_*^M(F)$$

does not depend on the choice of $a \in E$ with $E = F(a)$.

Pf (Outline) Reduce to the case where every finite extension of F is of degree a power of p . Use that in this case, $K_n^M(E)$ is generated by symbols

$$\{x_1, y_2, \dots, y_n\}$$

with $x \in E$ and $y_2, \dots, y_n \in F$. But then, by the projection formula,

$$N_{a/F}(\{x, y_2, \dots, y_n\})$$

$$= N_{a/F}(\{x\}) \cdot \{y_2, \dots, y_n\}$$

$$= \{N_{E/F}(x), y_2, \dots, y_n\}$$

is independent of $a \in E$. //

lemma let K be a CDVF and L a finite normal extension of degree p . let F and E be the residue fields of K and L , resp. Then the diagram

$$\begin{array}{ccc} K_n^M(L) & \xrightarrow{\quad} & K_{n-1}^M(E) \\ \downarrow N_{L/K} & & \downarrow N_{E/F} \\ K_n^M(K) & \xrightarrow{\quad} & K_{n-1}^M(F) \end{array}$$

commutes.

Pf let $z = \{z_1, \dots, z_n\} \in K_n^M(L)$. Then, for every prime p , there exists a finite extension K'/K of degree prime to p s.t. the image z' of z in $K_n^M(L')$ is a sum of elem. $\{x', y'_2, \dots, y'_n\}$ with $x' \in L'^*$ and $y'_2, \dots, y'_n \in K'^*$. So we may assume that $z = \{x, y_2, \dots, y_n\}$

with $x \in L^*$ and $y_1, \dots, y_n \in K^*$. In this case, we calculate

$$N_{E/F}(\partial(z)) = \partial(N_{L/K}(z)).$$

Suppose, for instance, that $x = \pi_L$ is a uniformizer in L and that L/K is totally ramified so $E = F$. Then π_L satisfies a Eisenstein equation

$$\pi^e - a_{d-1}\pi^{e-1} + \dots + (-1)^d a_0$$

with $a_i \in m_K$, and $a_0 \notin m_K^2$.

Then

$$N_{L/K}(\pi_L) = a_0$$

is a uniformizer of K . Suppose also $y_1, \dots, y_n \in \mathcal{O}_K$. Then we have

Now by definition

$$N_{a/F}(N_{E'/F'}(x)) = -\partial_{\infty}(N_{E(t)/F(t)}(\gamma))$$

$$N_{E/F}(N_{a/E}(x)) = -N_{E/F}(\partial_{\infty}(\gamma)) //$$

Def Let $v: F(t)^* \rightarrow \mathbb{Z}$ be a discrete valuation, and let $w: E(t)^* \rightarrow \mathbb{Z}$ range over the extensions of v to $E(t)$. Then the following diagram commutes.

$$\begin{array}{ccc}
 K_n^M(E(t)) & \xrightarrow{\oplus \partial_w} & \bigoplus_w K_{n-1}^M(k(w)) \\
 \downarrow N_{E(t)/F(t)} & & \downarrow \sum N_{k(w)/k(v)} \\
 K_n^M(F(t)) & \xrightarrow{\partial_v} & K_{n-1}^M(k(v))
 \end{array}$$

Hence

$$\begin{aligned}
 & \partial_v (N_{E(t)/F(t)}(\gamma)) \\
 &= \sum_w N_{k(w)/k(v)} (\partial_w(\gamma)) \\
 &= \begin{cases} N_{E'/F'}(x) & \text{if } v = v_x \\ N_{E/F}(\partial_\infty(\gamma)) & \text{if } v = \infty \\ 0 & \text{else} \end{cases}
 \end{aligned}$$

$$\partial (N_{L/K} (\{ \pi_L, \gamma_1, \dots, \gamma_n \}))$$

$$= \partial (\{ N_{L/K}(\pi_L), \gamma_2, \dots, \gamma_n \})$$

$$= \{ \bar{\gamma}_2, \dots, \bar{\gamma}_n \}$$

since $N_{L/K}(\pi_L)$ is a uniformizer in K .

$$N_{E/F} (\partial (\{ \pi_L, \gamma_2, \dots, \gamma_n \}))$$

$$= N_{E/F} (\{ \bar{\gamma}_2, \dots, \bar{\gamma}_n \})$$

$$= \{ \bar{\gamma}_2, \dots, \bar{\gamma}_n \}$$

since $E = F$.

//

lemma let E be a finite normal extension of F of prime degree.

let $F' = F(\alpha)$ be a finite ext.

and let $E' = E(\alpha)$. Then

$$N_{\alpha/F} \circ N_{E'/F'} = N_{E/F} \circ N_{\alpha/E}$$

Lemma Let $E = F(a)$ be a finite ext. of F , and let $\pi \in F[x]$ be the minimal polynomial of $a \in E$. Let F' be an extension field of F , and let

$$\pi = \prod \pi_i^{u_i}$$

be the decomposition of π in $F'[x]$. Let $E_i = F'[x]/(\pi_i)$, and let a_i be the image of x in E_i . Then the following diagram commutes.

$$\begin{array}{ccc} K_n^M(E) & \xrightarrow{(e_i)} & \bigoplus_i K_n^M(E_i) \\ \downarrow N_{E/F} & & \downarrow \sum N_{E_i/F'} \end{array}$$

$$K_n^M(F) \longrightarrow K_n^M(F')$$

Pf Let v be a discrete valuation on $F(t)$, and let w range over

the extensions of v to $F'(t)$. If $v = v_\infty$ then $w = w_\infty$ is the only ext. with uniformizer t^{-1} . If $v \neq v_\infty$ then the minimal polynomial π_v of $k(v)$ over F decomposes as

$$\pi_v = \prod_{w/v} \pi_w^{e(w/v)}$$

in $F'[t]$. We prove that the diagram on the next page commutes. The unnamed maps are the maps induced by the canonical inclusion maps. The left-hand square commutes by functoriality, and the commutativity of the middle square follows from:

$$\begin{array}{ccc} K_n^M(F'(t)) & \xrightarrow{(2w)} & \bigoplus_{w/v} K_{n-1}^M(k(w)) \\ \uparrow & & \uparrow (c_{w/v}) \\ K_n^M(F(t)) & \xrightarrow{2_v} & K_{n-1}^M(k(v)) \end{array}$$

[illegible]

which is immediate from the def.

Since the rows in the diagram are exact, it follows that there exists a map

$$h: K_{n-1}^M(F) \rightarrow K_{n-1}^M(F')$$

making the right-hand square commute. In particular, the diagr.

$$\begin{array}{ccc} K_{n-1}^M(k(w_\infty)) & \xrightarrow{N_{w_\infty}} & K_{n-1}^M(F') \\ \uparrow & & \uparrow h \\ K_{n-1}^M(k(v_\infty)) & \xrightarrow{N_{v_\infty}} & K_{n-1}^M(F) \end{array}$$

commutes. But N_{w_∞} and N_{v_∞} are

both the identity map, so h is

the map induced by the canonical inclusion of F in F' . //

lemma let K be a complete discrete valuation field and let L be a finite normal extension of K of prime degree p . let k_L and k_K be the residue fields of L and K . Then the following diagr. comm.

$$\begin{array}{ccc} K_n^M(L) & \xrightarrow{\partial_L} & K_{n-1}^M(k_L) \\ \downarrow N_{L/K} & & \downarrow N_{k_L/k_K} \\ K_n^M(K) & \xrightarrow{\partial_K} & K_{n-1}^M(k_K) \end{array}$$

Pf We saw last time that it suffices to prove this for symbols of the form $\{x, y_2, \dots, y_n\}$ with $x \in L^*$ and $y_2, \dots, y_n \in K^*$. In that case, one can calculate directly that the diagram commutes. For example,

suppose L/K is totally ramified,
and let π_L and π_K be uniformizers
of L and K , resp. Then

$$\pi_L^p + \Theta(\pi_L) \pi_K = 0$$

where $\Theta(x) \in \mathcal{O}_K[x]$ is a polynomial
of degree $< p$ s.t. $\Theta(0) \in \mathcal{O}_K^*$. It
follows that $\Theta(\pi_L) \in \mathcal{O}_L^*$ and that

$$N_{L/K}(\pi_L) = (-1)^p \Theta(0) \pi_K$$

$$\pi_K = -\Theta(\pi_L)^{-1} \pi_L^p.$$

We show that

$$\partial_L(\{a, b\}) = \partial_K(N_{L/K}(\{a, b\})).$$

We may assume $a = \pi_K^i u \in K^*$
and $b = \pi_L^j v \in L^*$. Then

$$\{a, b\} = \{\pi_K^i u, \pi_L^j v\}$$

$$= \{ (-\Theta(\pi_L)^{-1} \pi_L^P)^i u, \pi_L^j v \}$$

$$= i j \{ -1, \pi_L \} - i j \{ \Theta(\pi_L), \pi_L \} + p i j \{ \pi_L, \pi_L \}$$

$$+ j \{ u, \pi_L \} + i \{ -1, v \} - i \{ \Theta(\pi_L), v \} + p i \{ \pi_L, v \}$$

$$+ \{ u, v \}$$

which by ∂_L is mapped to

$$(p+1) i j \{ -1 \} - i j \{ \overline{\Theta(0)} \} + j \{ \overline{u} \} - p i \{ \overline{v} \}.$$

On the other hand,

$$N_{L/K}(\{a, b\}) = \{a, N_{L/K}(b)\}$$

$$= \{ \pi_K^i u, ((-1)^P \Theta(0) \pi_K)^j N_{L/K}(v) \}$$

$$= p i j \{ \pi_K, -1 \} + i j \{ \pi_K, \Theta(0) \} + i j \{ \pi_K, \pi_K \}$$

$$+ i \{ \pi_K, N_{L/K}(v) \} + p j \{ u, -1 \} + j \{ u, \Theta(0) \}$$

$$+ j \{ u, \pi_K \} + \{ u, N_{L/K}(v) \}$$

which by ∂_K is mapped to

$$(p+1)ij\{-1\} - ij\{\overline{\theta(0)}\} - i\{\overline{N_{L/K}(w)}\} + j\{\overline{w}\}.$$

We must show that the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}_L^* & \longrightarrow & k^* \\ \downarrow N_{L/K} & & \downarrow p \\ \mathcal{O}_K^* & \longrightarrow & k^* \end{array}$$

let $1, \pi_L, \pi_L^2, \dots, \pi_L^{p-1}$ be our basis of L as a K -vector space and write $v = v_0 + v_1 \pi_L + \dots + v_{p-1} \pi_L^{p-1}$. Then the matrix of mult. by v is

$$\begin{array}{cccccc} v_0 & * & * & \dots & * \\ v_1 & v_0 & * & \dots & * \\ v_2 & v_1 & v_0 & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{p-1} & v_{p-2} & v_{p-3} & \dots & v_0 \end{array}$$

where $* \in m_K$. It follows that

$$\overline{N_{L/K}(v)} = (\bar{v})^p$$

as desired. //

Exercise Let L/K be unramified with residue extension k_L/k_K . Show

$$\begin{aligned} N_{k_L/k_K}(\partial_L(\{a, b\})) \\ = \partial_K(N_{L/K}(\{a, b\})) . \end{aligned} //$$

Lemma Let E be a finite normal extension of F of prime degr. p , let $F' = F(a)$ be a finite ext. and let $E' = E(a)$. Then

$$\begin{array}{ccc} K_n^M(E') & \xrightarrow{N_{a/E}} & K_n^M(E) \\ \downarrow N_{E'/F'} & & \downarrow N_{E/F} \\ K_n^M(F') & \xrightarrow{N_{a/F}} & K_n^M(F) \end{array}$$

commutes.

Consider the following diagram

$$\begin{array}{ccccc}
 K_n^M(E(t)) & \longrightarrow & \bigoplus_{w|v} K_n^M(E(t)_w) & \xrightarrow{\oplus \partial_w} & \bigoplus_{w|v} K_{n-1}^M(k(w)) \\
 \downarrow N_{E(t)/F(t)} & & \downarrow \sum N_{E(t)_w/F(t)_v} & & \downarrow \sum N_{k(w)/k(v)} \\
 K_n^M(F(t)) & \longrightarrow & K_n^M(F(t)_v) & \xrightarrow{\partial_v} & K_{n-1}^M(k(v))
 \end{array}$$

The left-hand square commutes by a previous lemma, since

$$\frac{E(t) \otimes_{F(t)} F(t)_v}{F(t)} \xrightarrow{\sim} \prod_{w|v} E(t)_w$$

so the minimal polynomial π of a generator α of $E(t)$ over $F(t)$ decomposes as

$$\pi = \prod_{w|v} \pi_w$$

in $F(t)_v[x]$. The right-hand square commutes by the previous lemma. In particular, we have the

following commutative diagram:

$$\begin{array}{ccc}
 K_n^M(E(t)) & \xrightarrow{(\partial_w)} & \bigoplus_{w|v} K_{n-1}^M(k(w)) \\
 \downarrow N_{E(t)/F(t)} & & \downarrow \sum N_{k(w)/k(v)} \\
 K_n^M(F(t)) & \xrightarrow{\partial_v} & K_{n-1}^M(k(v))
 \end{array}$$

Now let $\pi \in F[t]$ and $\pi' \in E[t]$ be the minimal polynomials of a over F and E , respectively. Given $x \in K_n^M(E')$, we choose $y \in K_{n+1}^M(E(t))$ such that $\partial_{w_{\pi'}}(y) = x$ and $\partial_w(y) = 0$ if $w \neq w_{\pi'}, w_{\infty}$. Then, by definition,

$$N_{a/E}(x) = -\partial_{w_{\infty}}(y).$$

We claim that

$$N_{a/F}(N_{E'/F'}(x)) = -\partial_{v_{\infty}}(N_{E(t)/F(t)}(y)).$$

Indeed, the commutative diagram

above shows that

$$\partial_v (N_{E(t)/F(t)}(y)) = \sum_{w|v} N_{k(w)/k(v)} (\partial_w(y))$$

$$= \begin{cases} N_{E'/F'}(x) & \text{if } v = v_K \\ N_{E/F}(\partial_{w_\infty}(y)) & \text{if } v = v_\infty \\ 0 & \text{else} \end{cases}$$

so the claim follows from the def. of $N_{A/F}$. Applying the diagram again with $v = v_\infty$, we find

$$\begin{aligned} N_{A/F}(N_{E'/F'}(x)) &:= -\partial_{v_\infty}(N_{E(t)/F(t)}(y)) \\ &= -N_{E/F}(\partial_{w_\infty}(y)) =: N_{E/F}(N_{A/E}(x)). \end{aligned}$$

The lemma follows.

Thm (Kato) let F be a field, and let $E = F(a_1, \dots, a_n)$ be a finite extension of F . Then the homom.

$$N_{E/F} : K_g^M(E) \rightarrow K_g^M(F)$$

defined to be the composite

$$N_{a_1/F} \circ N_{a_2/F(a_1)} \circ \dots \circ N_{a_n/F(a_1, \dots, a_{n-1})}$$

does not depend on the choice of generators $a_1, \dots, a_n$.

Lemma Let F be a field, and let E be a finite normal extension of F of prime degree p . Then

$$N_{E/F} : K_n^M(E) \rightarrow K_n^M(F)$$

is well-defined.

pf Since $[E:F] = p$ is prime, there exists $a \in E$ s.t. $E = F(a)$. Suppose that also $E = F(b)$. We show that

$$\xi = N_{a/F} - N_{b/F}$$

is zero. We first show that the composite

$$K_n^M(E) \xrightarrow{\xi} K_n^M(F) \xrightarrow{j} K_n^M(E)$$

is zero. The extension E/F is either separable, and hence Galois, or purely inseparable. If E/F is Galois with Galois group G , then

$$E \otimes_F E \xrightarrow{\sim} \prod_{\sigma \in G} E$$

$$x \otimes y \mapsto (x\sigma(y))_{\sigma \in G}.$$

The minimal polynomial $\pi \in F[t]$ of $a \in E$ decomposes in $E[t]$ as

$$\pi = \prod_{\sigma \in G} (t - \sigma(a)).$$

Hence, by the lemma proved last time, we have a commutative diagr.

$$\begin{array}{ccc} K_n^M(E) & \longrightarrow & \prod_{\sigma \in G} K_n^M(E) \\ \downarrow N_{E/F} & & \downarrow \Delta \\ K_n^M(F) & \xrightarrow{j_*} & K_n^M(E) \end{array}$$

where the component of the top map indexed by $\sigma \in G$ is the map of Milnor K -groups induced from

$$\begin{array}{ccc} F[t]/(\pi) & \longrightarrow & E[t]/(t - \sigma(a)) \\ \downarrow \sim & & \downarrow \\ E & \xrightarrow{\sigma} & E \\ t & \longmapsto & t \\ \downarrow & & \downarrow \\ a & & \sigma(a) \end{array}$$

This shows that the composite

$$K_n^M(E) \xrightarrow{N_{E/F}} K_n^M(F) \xrightarrow{j_*} K_n^M(E)$$

is given by

$$z \longmapsto \sum_{\sigma \in G} \sigma_*(z)$$

Since this expression is independent of $a \in E$, we see that $j_* \circ \delta = 0$ as desired, if E/F is Galois.

If E/F is purely inseparable, then $\text{char}(F) = \text{char}(E) = p$ and

$$E = F[t] / (t^p - \alpha)$$

where $\alpha = a^p \in F$. Hence

$$E \otimes_F E = E[t] / (t^p - \alpha)$$

$$= E[t] / (t^p - a^p)$$

$$= E[t] / ((t - a)^p)$$

so, by the lemma proved last time, we have a commutative diagram.

$$\begin{array}{ccc} K_n^M(E) & \xrightarrow{p} & K_n^M(E) \\ \downarrow N_{a/F} & & \parallel \\ K_n^M(F) & \xrightarrow{j_*} & K_n^M(E) \end{array}$$

Since $j_* \circ N_{a/F} = p$ is independent of $a \in E$, we again conclude that $j_* \circ \delta = 0$.

Since $N_{a/F} \circ j_* = p$, the kernel of j_* consist of p -torsion elements of $K_n^M(F)$. Let F' be a finite extension of F of degree prime to p , and let $E' = EF'$. Then E'/F' is again a normal extension of degree p . Indeed, $E \cap F' = F$. Moreover, the kernel of the canonical map

$$K_n^M(F) \xrightarrow{i_*} K_n^M(F')$$

consists of $[F':F]$ -torsion elements.

It follows that $\delta(x) = 0$ in

$K_n^M(F)$ if and only if $i_*(\delta(x)) = 0$ in $K_n^M(F')$. Therefore, we can

replace F by the compositum in an algebraic closure of F of all finite extensions of F of degree prime to p . Equivalently, we can assume that F does not have any finite extensions of degree prime to p . In this case, one knows from [Bass-Tate, Ch. I, Cor. 5.3] that $K_n^M(E)$ is generated by symbols of the form

$$\{x, y_2, \dots, y_n\}$$

with $x \in E^*$ and $y_2, \dots, y_n \in F^*$. By the projection formula

$$\begin{aligned} N_{a/F}(\{x, y_2, \dots, y_n\}) \\ = \{N_{a/F}(x), y_2, \dots, y_n\} \end{aligned}$$

and $N_{a/F}(x) = N_{E/F}(x)$ does not depend on $a \in E$. This shows that $\delta(x) = 0$ as stated. //

Thm (Weil) Let $f, g \in k(t)^*$. Then

$$\sum_v N_{k(v)/k} (\partial_v (\{f, g\})) = 0.$$

Rem Hence, for $n=1$, $N_v = N_{k(v)/k}$.

Pf Since $\{f, f\} = \{f, -1\}$ and since $k(t)$ is a UFD, we may assume that f and g are relatively prime polynomials. Since $\partial_v (\{f, g\}) = 0$, if $v(f) = v(g) = 0$, we then have

$$\begin{aligned} & \sum_v N_{k(v)/k} (\partial_v (\{f, g\})) \\ &= \partial_\infty (\{f, g\}) + \sum_{v(g) > 0} N_{k(v)/k} (\partial_v \{f, g\}) \\ & \quad + \sum_{v(f) > 0} N_{k(v)/k} (\partial_v (\{f, g\})) \\ &= \partial_\infty (\{f, g\}) + \left(\frac{f}{g}\right) - \left(\frac{g}{f}\right) \end{aligned}$$

where

$$\begin{aligned} \left(\frac{f}{g}\right) &:= \sum_{v(g) \geq 0} N_{k(v)/k}(\partial_v(\{f, g\})) \\ &= \sum_{g(\alpha_v) = 0} N_{k(v)/k}(v(g) \cdot \{f(\alpha_v)\}) \end{aligned}$$

where $k[t] \rightarrow k(v)$ maps $t \mapsto \alpha_v$.

let $\bar{k}$ be an algebraic closure of k .

In $\bar{k}[t]$ we can write

$$f(t) = a(t - \alpha_1) \cdots (t - \alpha_n)$$

$$g(t) = b(t - \beta_1) \cdots (t - \beta_m).$$

We claim that (switching to multiplicative notation)

$$\left(\frac{f}{g}\right) = \prod_{1 \leq j \leq m} f(\beta_j)$$

$$= a^m \prod_{1 \leq j \leq m} \prod_{1 \leq i \leq n} (\alpha_i - \beta_j).$$

If g is constant, then both sides

equal 1. If $g = \pi_v$, we have

$$\left(\frac{f}{\pi_v}\right) = N_{k(v)/k}(f(\alpha_v)).$$

The images of α_v under the possible emb. of $k(v)$ in $\bar{k}$ are $\beta_1, \dots, \beta_m$, so we get

$$\left(\frac{f}{\pi_v}\right) = \prod_{1 \leq j \leq m} f(\beta_j)$$

as stated. It follows that

$$\left(\frac{f}{g}\right) \left(\frac{a}{f}\right)^{-1} = (-1)^{mn} a^m b^{-n}.$$

Finally, we have

$$f = a \cdot t^n \cdot u$$

$$g = b \cdot t^m \cdot v$$

where $u, v \in (1 + \mathcal{O}_{v_0}) \subset \mathcal{O}_{v_0}^\times$, so written additively

$$\partial_\omega (\{f, g\})$$

$$= \partial_\omega (\{a, b\}) + \partial_\omega (\{a, t^m\}) + \partial_\omega (\{a, v\})$$

$$+ \partial_\omega (\{t^n, b\}) + \partial_\omega (\{t^n, t^m\}) + \partial_\omega (\{t^n, v\})$$

$$+ \partial_\omega (\{u, b\}) + \partial_\omega (\{u, t^m\}) + \partial_\omega (\{u, v\})$$

$$= 0 + (-m) \{a\} + 0$$

$$- (-n) \{b\} + (-n)(-m) \{-1\} + 0$$

$$+ 0 + 0 + 0$$

$$= (-m) \{a\} + n \{b\} + mn \{-1\}$$

$$= \{(-1)^{mn} a^{-m} b^n\} \quad //$$

Rem Here we have used the def'n of ∂_v where

$$\partial_v (\{a, \pi_v\}) = \{\bar{a}\} \quad //$$

Thm (Sudlin) let k be a field, let $k(t)$ be the field of rational fcts. in one variable over k , and let F be a finite extension of $k(t)$. Then for every $x \in K_n^M(F)$,

$$\sum_w N_{k(v)/k} (d_w(x)) = 0.$$

Here the sum ranges over the discrete valuations of F that are trivial on k .

Pf We know that this is true if $F = k(t)$. The general case is deduced from the following result: let F be a finite extension of $k(t)$ and L a finite extension of F . let v be a discrete valuation on F/k , and let w/v range over all discrete valuations on L/k that extend v . Then the following diagram commutes:

$$\begin{array}{ccc} K_n^M(L) & \xrightarrow{(d_w)} & \bigoplus_{w|v} K_{n-1}^M(k(w)) \\ \downarrow N_{L/F} & & \downarrow \sum N_{k(w)/k(v)} \\ K_n^M(F) & \xrightarrow{d_v} & K_{n-1}^M(k(v)) \end{array}$$

Since the completion F_v is separable over F , this, in turn, follows from the lemma proved last time and from the following extension of a result of Kato: let K be a complete discrete valuation field and K' a finite extension, and let k and k' be the residue fields of K and K' , resp. Then the diagram

$$\begin{array}{ccc} K_n^M(K') & \xrightarrow{\partial'} & K_{n-1}^M(k') \\ \downarrow N_{K'/K} & & \downarrow N_{k'/k} \\ K_n^M(K) & \xrightarrow{\partial} & K_{n-1}^M(k) \end{array}$$

commutes.

11

We call the map

$$\tilde{f} : \tilde{Z}^n(k, n) \rightarrow K_n^M(k).$$

The domain is the free abelian group generated by the set of closed points in $(\mathbb{P}_k^1 \setminus \{0, 1, \infty\})^n$.

Let x be such a closed point with residue field $k(x)$. We have

$$\text{Spec } k(x) \xrightarrow{f} (\mathbb{P}_k^1 \setminus \{0, 1, \infty\})^n,$$

and hence the coordinate functions $x_1, \dots, x_n$ on $(\mathbb{P}_k^1 \setminus \{0, 1, \infty\})^n$ induce functions $\bar{x}_i = f^*(x_i)$ on $\text{Spec } k(x)$.

So $\bar{x}_1, \dots, \bar{x}_n \in k(x)$ and $k(x) = k(\bar{x}_1, \dots, \bar{x}_n)$. These elements define

$$\{\bar{x}_1, \dots, \bar{x}_n\} \in K_n^M(k(x)),$$

and $\tilde{f}$ is then defined by

$$\tilde{Z}^n(k, n) \xrightarrow{\tilde{f}} K_n^M(k)$$

$$[x] \mapsto N_{k(x)/k}(\{ \bar{x}_1, \dots, \bar{x}_n \}).$$

We wish to show that the

Composite map

$$z^n(k, n+1) \xrightarrow{d} z^n(k, n) \xrightarrow{\tilde{\gamma}} K_n^M(k)$$

is zero such that we obtain an induced map

$$\gamma: CH^n(k, n) \rightarrow K_n^M(k).$$

The group $z^n(k, n+1)$ is the free abelian group on generators $[C]$ where $C \subset (\mathbb{P}_{k^*}^1 \times \{0\})^{n+1}$ are integral curves that meet the codim. 1 faces " $x_i = 0$ " and " $x_i = \infty$ " in closed points and that do not meet the codim. 2 faces.

Let $p: \tilde{C} \rightarrow C$ be the normalization of C . Then $\tilde{C}$ is a smooth curve over k , and p is finite. Suppose that x is a closed point in the intersection of C and the divisor " $x_i = \varepsilon$ ". This is equivalent to the statement that

$$\text{ord}_x: k(C)^* \rightarrow \mathbb{Z}$$

takes a non-zero value at the regular function $x_i - \varepsilon \in k(C)$.
We recall that by definition

$$\text{ord}_x(f) = \text{length}_{\mathcal{O}_{C,x}} (\mathcal{O}_{C,x} / f \cdot \mathcal{O}_{C,x})$$

if $f \in \mathcal{O}_{C,x}$ and $\text{ord}_x(f/g) = \text{ord}_x(f) - \text{ord}_x(g)$. One has

$$\text{ord}_x(f) = \sum_{\tilde{x}/x} [k(\tilde{x}) : k(x)] \cdot \text{ord}_{\tilde{x}}(f)$$

where the sum ranges over all closed points of $\tilde{C}$ that lie over x [Fulton, Ex. A.3.1].

The curve $\tilde{C}$ is a quasi-projective curve; let $P(\tilde{C})$ be the projective closure. Then $P(\tilde{C})$ is a smooth projective curve and

$$\bar{P} : P(\tilde{C}) \rightarrow (\mathbb{P}^1)^{n+1}$$

is finite. The map $\bar{P}$ is given by $n+1$ rational functions $g_1, \dots, g_{n+1}$ on $P(\tilde{C})$. Moreover, none of these are identically 0 or ∞ , and if $w \in \tilde{C}$ is a closed point and $g_i(w) \in \{0, \infty\}$, then $g_j(w) \notin \{0, \infty\}$ for $i \neq j$. We have a symbol

$$\{g_1, \dots, g_{n+1}\} \in K_{n+1}^M(k(\tilde{C}))$$

and closed points on $P(\tilde{C})$ determine and are determined by valuations w on $k(\tilde{C})$ that are trivial on k . We claim that

$$\tilde{\chi}(\partial([C]))$$

$$= \sum_{w \in \tilde{C}} N_{k(w)/k}(\partial_w(\{g_1, \dots, g_{n+1}\}))$$

and we wish to know that this is equal to the sum

$$\sum_{w \in P(\tilde{C})} N_{k(w)/k}(\partial_w(\{g_1, \dots, g_{n+1}\})) = 0.$$

But if $w \in P(\tilde{C}) \setminus \tilde{C}$ is a point at infinity, then we must have $g_i(w) = 1$, for some i , or else $\tilde{p} : \tilde{C} \rightarrow (\mathbb{P}^1 \setminus \{1\})^{n+1}$ would not be proper. But if $g_i(w) = 1$, then

$$\partial_w(\{g_1, \dots, g_{n+1}\}) = 0,$$

so the two sums are identical.

$$\tilde{C} \longrightarrow (\mathbb{P}^1 \setminus \{1\})^{n+1}$$

$$\downarrow P \qquad \parallel$$

$$C \hookrightarrow (\mathbb{P}^1 \setminus \{1\})^{n+1}$$

$x \in C$: closed point

Define "order of vanishing at x "

$$\text{ord}_x : k(C)^* \longrightarrow \mathbb{Z}$$

by

$$\text{ord}_x(f) = \text{length}_{\mathcal{O}_{C,x}} (\mathcal{O}_{C,x} / f \cdot \mathcal{O}_{C,x}),$$

if $f \in \mathcal{O}_{C,x}$ is regular at x , and

$$\text{ord}_x(f) = \text{ord}_x(a) - \text{ord}_x(b),$$

if $f = a/b$, where $a \in \mathcal{O}_{C,x}$ and $b \in \mathcal{O}_{C,x} \cap k(C)^*$. One has

$$\text{ord}_x(f) = \sum_{\tilde{x} \in \mathbb{P}^1(x)} [k(\tilde{x}) : k(x)] \cdot \text{ord}_{\tilde{x}}(f);$$

see [Fulton, Ex. A-3.1].

By definition,

$$(\partial_i^\omega - \partial_i^0)([C]) = - \sum_{x \in C} \text{ord}_x(x_i) \cdot [x].$$

The curve does not meet any codim. 2 face of $(\mathbb{P}^1, \{i\})^{n+1}$. So if $\text{ord}_x(x_i) \neq 0$, then $\text{ord}_x(x_j) = 0$, for all $1 \leq j \leq n+1$ with $j \neq i$. Now, if $\tilde{x} \in \tilde{C}$ with $p(\tilde{x}) = x \in C$, then

$$\begin{array}{ccc} \mathcal{O}_{C,x} & \longrightarrow & \mathcal{O}_{\tilde{C},\tilde{x}} \\ \downarrow & & \downarrow \\ k(x) & \hookrightarrow & k(\tilde{x}) \end{array}$$

and

$$K_n^M(k(x)) \xrightarrow{j_{k(\tilde{x})/k(x)}^*} K_n^M(k(\tilde{x}))$$

$$\begin{array}{ccc} & N_{k(x)/k} & \\ \swarrow & & \searrow \\ & K_n^M(k) & \end{array} \quad \begin{array}{ccc} & [k(\tilde{x}) : k(x)] \cdot N_{k(\tilde{x})/k} & \\ \swarrow & & \searrow \\ & K_n^M(k) & \end{array}$$

We have

$$\tilde{\varphi}(\partial([C])) = \sum_{1 \leq i \leq n+1} (-1)^i \tilde{\varphi}((\partial_i^\infty - \partial_i^0)([C]))$$

$$= \sum_{x \in C} \sum_{1 \leq i \leq n+1} (-1)^{i-1} \text{ord}_x(x_i) \cdot N_{k(x)/k}([x])$$

and, for every $x \in C$, at most one summand $i = i(x)$ is non-zero. So we have

$$\tilde{\varphi}(\partial([C])) = \sum_{x \in C} (-1)^i \tilde{\varphi}((\partial_i^\infty - \partial_i^0)([C]))$$

$$= \sum_{x \in C} (-1)^{i-1} \text{ord}_x(x_i) \cdot N_{k(x)/k}(\{\bar{x}_1, \dots, \hat{\bar{x}}_i, \dots, \bar{x}_{n+1}\})$$

$$= \sum_{\tilde{x} \in \tilde{C}} (-1)^{i-1} \text{ord}_{\tilde{x}}(x_i) \cdot [k(\tilde{x}) : k(x)] \cdot N_{k(x)/k}(\{\bar{x}_1, \dots, \hat{\bar{x}}_i, \dots, \bar{x}_{n+1}\})$$

$$= \sum_{\tilde{x} \in \tilde{C}} (-1)^{i-1} \text{ord}_{\tilde{x}}(x_i) \cdot N_{k(\tilde{x})/k}(\{\bar{x}_1, \dots, \hat{\bar{x}}_i, \dots, \bar{x}_{n+1}\})$$

$$= \sum_{\tilde{x} \in \tilde{C}} N_{k(\tilde{x})/k}(\partial_{\tilde{x}}(\{x_1, \dots, x_{n+1}\}))$$

where $x_1, \dots, x_{n+1} \in k(\tilde{C})$ are the

rational functions on $\tilde{C}$ given by the coordinate functions on $(\mathbb{P}^1 \setminus \{1\})^{n+1}$. Since

$$\tilde{C} \rightarrow (\mathbb{P}^1 \setminus \{1\})^{n+1}$$

is proper, the projective closure

$$P(\tilde{C}) \rightarrow (\mathbb{P}^1)^{n+1}$$

has the property that the fct $x_1, \dots, x_{n+1} \in k(P(\tilde{C})) = k(\tilde{C}) = k(C)$ are all regular at $P(\tilde{C}) \setminus \tilde{C}$. So

$$\tilde{\chi}(\partial(C))$$

$$= \sum_{\tilde{x} \in \tilde{C}} N_{k(\tilde{x})/k}(\partial_{\tilde{x}}(\{x_1, \dots, x_{n+1}\}))$$

$$= \sum_{\tilde{x} \in P(\tilde{C})} N_{k(\tilde{x})/k}(\partial_{\tilde{x}}(\{x_1, \dots, x_{n+1}\}))$$

$$= 0$$

by Saito reciprocity.

The composite map

$$K_n^M(k) \xrightarrow{\phi} CH^n(k, n) \xrightarrow{\gamma} K_n^M(k)$$

is the identity and the image of ϕ consists of the classes in $CH^n(k, n)$ that are represented by k -rational points $x \in (\mathbb{P}^1 \setminus \{0, 1, \infty\})^n$. So to prove that also

$$CH^n(k, n) \xrightarrow{\gamma} K_n^M(k) \xrightarrow{\phi} CH^n(k, n)$$

is an isomorphism, we must show that for every closed point $x \in (\mathbb{P}^1 \setminus \{0, 1, \infty\})^n$, the class $[x]$ represented by x is equal to a sum of classes repr. by k -rational points. We define

$$Fil_j CH^n(k, n) \subset CH^n(k, n)$$

to be the subgroup generated by the classes $[x]$ where $x \in (\mathbb{P}^1 \setminus \{0, 1, \infty\})^n$ is a closed point s.t. $k(x)$ is generated by the first j coordinate functions $x_1, \dots, x_j$ at x ,

-ii-

$$k(x) = k(x_1, \dots, x_j).$$

Then

$$\text{Fil}_0 CH^n(k, n) = \text{im}(\phi)$$

$$\text{Fil}_n CH^n(k, n) = CH^n(k, n).$$

We show :

Lemma Let $1 \leq j \leq n$ and suppose that k is infinite. Then the canonical inclusion

$$\text{Fil}_{j-1} CH^n(k, n) \hookrightarrow \text{Fil}_j CH^n(k, n)$$

is the identity map.

Prf Let $[x]$ be a generator on the right-hand side. Let

$$x_1, \dots, x_n \in k(x)^*$$

be the image of the coord. fcts. on $(\mathbb{P}^1 \setminus \{0, 1, \infty\})^n$. The point x is given by n polynomial equations over k of the form

-iii-

$$p_1(t_1) = 0$$

$$\vdots$$

$$p_j(t_1, \dots, t_j) = 0$$

$$p_{j+1}(t_1, \dots, t_j) = t_{j+1}$$

$$\vdots$$

$$p_n(t_1, \dots, t_j) = t_n$$

We can choose p_i , $1 \leq i \leq j$, s.t.

$p_i(x_1, \dots, x_{i-1}, t_i)$ is the monic irreducible polynomial of $k(x_1, \dots, x_j)$ over $k(x_1, \dots, x_{i-1})$. Let

$$d = [k(x_1, \dots, x_j) : k(x_1, \dots, x_{j-1})]$$

$$= \deg_{t_j} p(t_1, \dots, t_j).$$

If $d=1$, we are done, so assume $d \geq 2$. For $j+1 \leq i \leq n$,

$$p_i(x_1, \dots, x_j) = x_i,$$

so the euclidian algorithm in

$k(x_1, \dots, x_{j-1})[t_j]$, we can assume that for $j+1 \leq i \leq n$,

$$\deg_{t_j} p_i(t_1, \dots, t_j) < d.$$

We shall prove that, modulo $\text{Fil}_{j-1} CH^n(k, n)$, the class $[x]$ is equal to a sum of classes $[x_\alpha]$ s.t. the corresponding degrees d_α satisfy $d_\alpha < d$. The lemma then follows by induction on d .

Define $a_0(t_1, \dots, t_{j-1}) \in k[t_1, \dots, t_{j-1}]$ by

$$p_j(t_1, \dots, t_{j-1}, 0) = (-1)^d a_0(t_1, \dots, t_{j-1}).$$

Then

$$a_0(x_1, \dots, x_{j-1}) = N(x_j),$$

where $N = N_{k(x_1, \dots, x_j)/k(x_1, \dots, x_{j-1})}$. Define

$$q(t_1, \dots, t_j, u) \in k[t_1, \dots, t_j, u]$$

by

$p_j(x_1, \dots, x_{j-1}, t_j)$ is the irreducible pol. of x_j over $k(x_1, \dots, x_{j-1})$ and since $x_j \in k(x)^*$ (because $d \geq 2$), we have $q_j(x_1, \dots, x_{j-1}) \neq 0$. Hence, C does not meet " $t_j = 0$ ". Since $t_1, \dots, t_j$ are not ∞ on C , C also does not meet " $t_i = \infty$ ", $j+1 \leq i \leq n$. We have chosen C s.t. it meets " $u=0$ " at x , " $u=\infty$ " at the point

$$(x_1, \dots, x_{j-1}, N(x_j), \dots, p_i(x_1, \dots, x_{j-1}, N(x_j)), \dots).$$

in $\text{Fil}_{j-1} CH^n(k, n)$ and " $t_i = 0$ ", $j+1 \leq i \leq n$, in points

$$(x_{\alpha,1}, \dots, x_{\alpha,j})$$

for which $d_\alpha < d$. (Here we use that p_i , $j+1 \leq i \leq n$, has t_j -degree strictly smaller than the t_j -degree of p_j and that since the eq. $g(t_1, \dots, t_k, u) = 0$ can be solved for u as a rational function over k in $x_1, \dots, x_{j-1}$ and the $x_{\alpha,j}$.) We note that C meets " $t_i = 0$ " in points, since $x_i \in k(x)^*$.

So we would be done, if C does not meet any codim. 2 faces.

However, in general, it does, so we need a moving lemma to avoid this problem. These problems separate in two parts:

(1) There exist $j+1 \leq i < i' \leq n$ s.t. $p_i(x_1, \dots, x_{j-1}, t_j)$ and $p_{i'}(x_1, \dots, x_{j-1}, t_j)$ are not relatively prime polynomials in $\mathbb{K}(x_1, \dots, x_{j-1})[t_j]$.

(2) There exists $j+1 \leq i \leq n$ s.t.

$$p_i(x_1, \dots, x_{j-1}, N(x_j)) = 0.$$

-v-

$$q(t_1, \dots, t_j, u)$$

$$= p_j(t_1, \dots, t_j) - (t_j - 1)^{d-1} (t_j - a_0(t_1, \dots, t_{j-1}))u,$$

and let C be the curve in $(\mathbb{A}^1 - \{1\})^{n+1}$ given by the n equations

$$p_1(t_1) = 0$$

$\vdots$

$$p_{j-1}(t_1, \dots, t_{j-1}) = 0$$

$$q(t_1, \dots, t_j, u) = 0$$

$$p_{j+1}(t_1, \dots, t_j) = t_{j+1}$$

$\vdots$

$$p_n(t_1, \dots, t_j) = t_n.$$

The curve C does not meet the faces $t_i = 0$ or $t_i = \infty$, for $1 \leq i \leq j-1$, since $x_i \in k(x)^*$, for $1 \leq i \leq n$. As a polynomial in t_j , $q(t_1, \dots, t_j, u)$ has leading coeff. $1-u$ and constant term $(-1)^{d-1} a_0(t_1, \dots, t_{j-1})(1-u)$. The first statement shows that C does not meet " $t_j = \infty$ ". Since