The Brouwer Fixed Point Theorem

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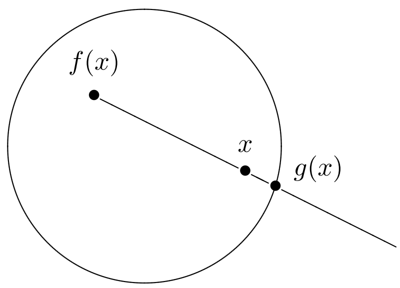
THEOREM 1. *Let $D^n \subset \mathbb{R}^n$ be the unit ball, and let $f\colon D^n \to D^n$ be a continuous map. Then there exists $x \in D^n$ such that $f(x) = x$.*

The topological space D^n may be replaced by any topological space X homeomorphic to D^n. Indeed, if $g\colon X \to D^n$ is a homeomorphism and $h\colon X \to X$ a continuous map, the theorem shows that the composition $f = ghg^{-1}\colon D^n \to D^n$ has a fixed point $x \in D^n$. But then $y = g^{-1}(x) \in X$ is a fixed point for the map h. For example, D^n may be replaced by any rectangle

$$X = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n.$$

The theorem has important applications in economics. It is used to prove that, in economic theories, an equilibrium exists. The theorem, however, does not reveal how to find the equilibrium.

To prove the theorem, we assume that it is false and show that this leads to a contradiction. So let $f\colon D^n \to D^n$ be a continuous map and assume that $f(x) \neq x$, for all $x \in D^n$. We then define $g(x) \in S^{n-1} = \partial D^n$ to be the intersection of S^{n-1} and the half-line beginning at $f(x)$ and passing through x:



In particular, $g(x) = x$, for $x \in S^{n-1}$. It is also easy to see that $g(x)$ is a continuous function of $x \in D^n$. Indeed, by definition, we have

$$g(x) = tx + (1-t)f(x)$$

where t is the unique positive solution to the quadratic

$$\langle tx + (1-t)f(x), tx + (1-t)f(x)\rangle = 1.$$

Here $\langle -, - \rangle$ is the inner product on $\mathbb{R}^n$. In conclusion, given a continuous function $f\colon D^n \to D^n$ with $f(x) \neq x$, for all $x \in D^n$, then we obtain the following commutative diagram of topological spaces and continuous maps.

$$\begin{array}{ccccc} S^{n-1} & \overset{\iota}{\hookrightarrow} & D^n & \xrightarrow{g} & S^{n-1} \\ \downarrow & & \mathrm{id}_{S^{n-1}} & & \uparrow \end{array}$$

We wish to prove that such a diagram cannot exist.

The method of algebraic topology is to construct an algebraic "image" of our topological situation. We now introduce one such "image" and refer to the book [**1**] for a detailed treatment. Let M be a smooth manifold, possibly with boundary, such as $M = D^n$. Then we have the notion of a differential q-form ω on M. The differential $d\omega$ of a differential q-form on M is a differential $(q+1)$-form on M. We say that ω is a *closed* differential q-form, if $d\omega = 0$, and we say that ω is an *exact* differential q-form, if $\omega = d\eta$, for some differential $(q-1)$-form η. The set of all closed differential q-forms on M forms a real vector space, and the set of all exact differential q-forms on M forms a real subspace of this vector space. These vector spaces are both infinite dimensional. However, the quotient vector space

$$H^q_{\mathrm{dR}}(M) = \frac{\{\text{closed differential } q\text{-forms on } M\}}{\{\text{exact differential } q\text{-forms on } M\}}$$

is often a finite dimensional vector space. This is true, for example, if M is compact, but it is not trivial to prove so [**1**, Prop. 9.25]. The vector space $H^q_{\mathrm{dR}}(M)$ is called the qth *de Rham cohomology group* of M. Suppose that $f\colon N \to M$ is a smooth map from the smooth manifold N to the smooth manifold M. Then a differential q-form ω on M gives rise to a differential q-form $f^*\omega$ on N called the pull-back of ω by f. If ω is closed, then also $f^*\omega$ is closed, and if ω is exact, then $f^*\omega$ is exact. Hence, we have a well-defined map $f^*\colon H^q_{\mathrm{dR}}(M) \to H^q_{\mathrm{dR}}(N)$ that takes the class of the closed differential q-form ω to the class of the closed differential q-form $f^*\omega$. The map f^* is a linear map from the real vector space $H^q_{\mathrm{dR}}(M)$ to the real vector space $H^q_{\mathrm{dR}}(N)$. More generally, by using the Weierstrass approximation theorem, one can associate to every continuous map $f\colon N \to M$, a linear map

$$f^*\colon H^q_{\mathrm{dR}}(M) \to H^q_{\mathrm{dR}}(N);$$

see [**1**, pp. 40–41]. This association has the following properties:

(i) $(\mathrm{id}_M)^* = \mathrm{id}_{H^q_{\mathrm{dR}}(M)}$.
(ii) $(f \circ g)^* = g^* \circ f^*$.

We say that $H^q_{\mathrm{dR}}(-)$ is a *functor*

$$\left\{\begin{matrix}\text{smooth manifolds}\\ \text{continuous maps}\end{matrix}\right\} \xrightarrow{H^q_{\mathrm{dR}}(-)} \left\{\begin{matrix}\text{real vector spaces}\\ \text{linear maps}\end{matrix}\right\}$$

from the category of smooth manifolds and continuous maps to the category of vector spaces and linear maps. We explain two basic results that make it possible to calculate the groups $H^q_{\mathrm{dR}}(M)$ for every manifold.

Let $f, g\colon N \to M$ be two continuous maps. Then a *homotopy* from f to g is defined to be a continuous map $H\colon N \times [0,1] \to M$ such that $H(x, 0) = f(x)$ and

$H(x,1) = g(x)$, for all $x \in N$. We say that $f, g\colon N \to M$ are *homotopic* maps, if there exists a homotopy from f to g. The following result is [**1**, Thm. 6.8].

PROPOSITION 2. *Let $f, g\colon N \to M$ be two homotopic continuous maps between smooth manifolds with boundary. Then the induced maps*

$$f^* = g^* \colon H^q_{\mathrm{dR}}(M) \to H^q_{\mathrm{dR}}(N)$$

are equal.

We use this result to show:

COROLLARY 3. *The de Rham cohomology groups of the unit ball are given by*

$$H^q_{\mathrm{dR}}(D^n) = \begin{cases} \mathbb{R}\cdot 1 & (q=0) \\ 0 & (q>0) \end{cases}$$

where $1\colon D^n \to \mathbb{R}$ is the constant function with value 1.

PROOF. Let $i\colon \{0\} \to D^n$ be the inclusion of the origin, and let $p\colon D^n \to \{0\}$ be the map that sends every point in D^n to 0. We claim that the induced maps

$$H^q_{\mathrm{dR}}(\{0\}) \underset{i^*}{\overset{p^*}{\rightleftarrows}} H^q_{\mathrm{dR}}(D^n)$$

are the inverses of each other. Indeed, we have $p \circ i = \mathrm{id}_{\{0\}}$, and hence

$$i^* \circ p^* = (p \circ i)^* = (\mathrm{id}_{\{0\}})^* = \mathrm{id}_{H^q_{\mathrm{dR}}(\{0\})}\,.$$

The composition $i \circ p\colon D^n \to D^n$ is equal to the map that takes every $x \in D^n$ to the origin $0 \in D^n$. But the map $H\colon D^n \times [0,1] \to D^n$ defined by $H(x,t) = tx$ is a homotopy from $i \circ p$ to the identity map of D^n. Therefore, by Prop. 2,

$$p^* \circ i^* = (i \circ p)^* = (\mathrm{id}_{D^n})^* = \mathrm{id}_{H^q_{\mathrm{dR}}(D^n)}\,.$$

Now, it follows immediately from the definition that $H^q_{\mathrm{dR}}(\{0\})$ is equal to $\mathbb{R}\cdot 1$, for $q=0$, and is zero, for $q>0$. This shows that $H^q_{\mathrm{dR}}(D^n)$ is as stated. □

The second basic result is the Mayer-Vietoris sequence [**1**, Thm. 5.2, Rem. 9.30]. We remark that an open subset $U \subset M$ of an n-dimensional smooth manifold with boundary is again an n-dimensional smooth manifold with boundary.

PROPOSITION 4. *Let M be a smooth manifold with boundary, and let $U_1, U_2 \subset M$ be two open subsets with $U_1 \cup U_2 = M$. Then there is a long-exact sequence of real vector spaces and linear maps*

$$\cdots \to H^q_{\mathrm{dR}}(M) \xrightarrow{(i_1^*, i_2^*)} H^q_{\mathrm{dR}}(U_1) \oplus H^q_{\mathrm{dR}}(U_2) \xrightarrow{j_1^* - j_2^*} H^q_{\mathrm{dR}}(U_1 \cap U_2) \xrightarrow{\partial} H^{q+1}_{\mathrm{dR}}(M) \to \cdots$$

where $i_s\colon U_s \hookrightarrow M$ and $j_s\colon U_1 \cap U_2 \hookrightarrow U_s$ are the inclusion maps.

Here, we recall, that the sequence $V' \xrightarrow{f} V \xrightarrow{g} V''$ of real vector spaces and linear maps is said to be exact at V, if the kernel of g is equal to the image of f. The long-exact Mayer-Vietoris sequence is exact at every position.

COROLLARY 5. *The de Rham cohomology groups of the unit sphere are given by*

$$H^q_{\mathrm{dR}}(S^0) = \begin{cases} \mathbb{R}\cdot 1 \oplus \mathbb{R}\cdot \xi_0 & (q=0) \\ 0 & (q>0), \end{cases}$$

for $m = 0$, and by

$$H^q_{\mathrm{dR}}(S^m) = \begin{cases} \mathbb{R}\cdot 1 & (q=0) \\ \mathbb{R}\cdot \xi_m & (q=m) \\ 0 & \textit{otherwise}, \end{cases}$$

for $m > 0$, where $1\colon S^m \to \mathbb{R}$ is the constant function with value 1, and where ξ_m is a class that will be defined in the proof.

PROOF. The proof is by induction on $m \geqslant 0$. First, if $m = 0$, then

$$S^0 = \{x \in \mathbb{R} \mid \|x\| = 1\} = \{-1, +1\}.$$

Hence, it follows immediately from the definition that

$$H^q_{\mathrm{dR}}(S^0) = \begin{cases} \mathbb{R}\cdot 1_{(+1)} \oplus \mathbb{R}\cdot 1_{(-1)} & (q=0) \\ 0 & (q>0), \end{cases}$$

where $1_{(+1)}\colon S^0 \to \mathbb{R}$ (resp. $1_{(-1)}\colon S^0 \to \mathbb{R}$) denotes the function whose value at $+1$ (resp. -1) is 1 and whose value at -1 (resp. $+1$) is 0. We define $\xi_0 = 1_{(+1)} - 1_{(-1)}$. Since $1 = 1_{(+1)} + 1_{(-1)}$, the statement for $m = 0$ follows.

To prove the induction step, we assume that the statement has been proved for $m-1$ and prove it for m. We let $U_+, U_- \subset S^m$ be the open subsets

$$U_+ = S^m \smallsetminus \{(0,\dots,0,+1)\}$$
$$U_- = S^m \smallsetminus \{(0,\dots,0,-1)\}$$

obtained by removing the north and south pole, respectively. Then, from Prop. 4, we obtain the long-exact sequence

$$\begin{aligned} 0 \to{} & H^0_{\mathrm{dR}}(S^m) \to H^0_{\mathrm{dR}}(U_+) \oplus H^0_{\mathrm{dR}}(U_-) \to H^0_{\mathrm{dR}}(U_+ \cap U_-) \\ \xrightarrow{\partial}{} & H^1_{\mathrm{dR}}(S^m) \to H^1_{\mathrm{dR}}(U_+) \oplus H^1_{\mathrm{dR}}(U_-) \to H^1_{\mathrm{dR}}(U_+ \cap U_-) \\ & \vdots \\ \xrightarrow{\partial}{} & H^q_{\mathrm{dR}}(S^m) \to H^q_{\mathrm{dR}}(U_+) \oplus H^q_{\mathrm{dR}}(U_-) \to H^q_{\mathrm{dR}}(U_+ \cap U_-) \to \cdots \end{aligned} \tag{6}$$

We first evaluate the groups $H^q_{\mathrm{dR}}(U_\pm)$. Let $i_+\colon \{(0,\dots,0,-1)\} \to U_+$ be the inclusion of the south pole, and let $p_+\colon U_+ \to \{(0,\dots,0,-1)\}$ be the map that takes every point in U_+ to the south pole. Then $p_+ \circ i_+$ is equal to the identity map of $\{(0,\dots,0,-1)\}$, and $i_+ \circ p_+$ is homotopic to the identity map of U_+. Indeed,

$$H_+\colon U_+ \times [0,1] \to U_+$$

defined by

$$H_+(x,t) = \frac{tx + (1-t)(0,\dots,0,-1)}{\|tx + (1-t)(0,\dots,0,-1)\|}$$

is well-defined and provides a homotopy from $i_+ \circ p_+$ to id_{U_+}. Therefore, we may argue, as in the proof of Cor. 3, that

$$H^q_{\mathrm{dR}}(U_+) = \begin{cases} \mathbb{R} \cdot 1_+ & (q = 0) \\ 0 & (q > 0), \end{cases}$$

where $1_+ \colon U_+ \to \mathbb{R}$ is the constant function with value 1. Similarly,

$$H^q_{\mathrm{dR}}(U_-) = \begin{cases} \mathbb{R} \cdot 1_- & (q = 0) \\ 0 & (q > 0), \end{cases}$$

where $1_- \colon U_- \to \mathbb{R}$ is the constant function with value 1. Next, we consider

$$U_+ \cap U_- = S^m \smallsetminus \{(0, \dots, 0, -1), (0, \dots, 0, +1)\}.$$

We define $i \colon S^{m-1} \to U_+ \cap U_-$ and $p \colon U_+ \cap U_- \to S^{m-1}$ by

$$i(x_1, \dots, x_m) = (x_1, \dots, x_m, 0)$$

$$p(x_1, \dots, x_m, x_{m+1}) = \frac{(x_1, \dots, x_m)}{\sqrt{x_1^2 + \dots + x_m^2}}.$$

Then $p \circ i = \mathrm{id}_{S^{m-1}}$ and $i \circ p$ is homotopic to the identity map of $U_+ \cap U_-$. Indeed, a homotopy group $i \circ p$ to the identity map $U_+ \cap U_-$ is given by the map

$$H \colon (U_+ \cap U_-) \times [0, 1] \to U_+ \cap U_-$$

defined by

$$H(x_1, \dots, x_m, x_{m+1}, t) = \frac{(x_1, \dots, x_m, t x_{m+1})}{\sqrt{x_1^2 + \dots + x_m^2 + t^2 x_{m+1}^2}}.$$

Therefore, we conclude from Prop. 2 that

$$p^* \colon H^q_{\mathrm{dR}}(S^{m-1}) \to H^q_{\mathrm{dR}}(U_+ \cap U_-)$$

is an isomorphism. The value of the right-hand group is given by the inductive hypothesis.

We now return to the long-exact sequence (6). We consider the cases $m = 1$ and $m > 1$ separatedly. Suppose first that $m = 1$. Then the sequence takes the form

$$0 \to H^0_{\mathrm{dR}}(S^1) \to \mathbb{R} \cdot 1_+ \oplus \mathbb{R} \cdot 1_- \to \mathbb{R} \cdot 1_{(+1)} \oplus \mathbb{R} \cdot 1_{(-1)} \xrightarrow{\partial} H^1_{\mathrm{dR}}(S^1) \to 0$$

and the middle map takes both 1_+ and 1_- to $1_{(+1)} - 1_{(-1)} = \xi_0$. Therefore,

$$H^q_{\mathrm{dR}}(S^1) = \begin{cases} \mathbb{R} \cdot 1 & (q = 0) \\ \mathbb{R} \cdot \xi_1 & (q = 1) \\ 0 & (q > 1), \end{cases}$$

where $1 \colon S^1 \to \mathbb{R}$ is the constant function, and where $\xi_1 = \partial(1)$. Finally, for $m > 1$, the long-exact sequence (6) shows that

$$H^q_{\mathrm{dR}}(S^m) = \begin{cases} \mathbb{R} \cdot 1 & (q = 0) \\ \mathbb{R} \cdot \xi_m & (q = m) \\ 0 & \text{otherwise}, \end{cases}$$

where $\xi_m = \partial(p^*(\xi_{m-1}))$. □

We now complete the proof of Thm. 1. Assuming the theorem to be false, we have obtained the following commutative diagram of smooth manifolds with boundary and continuous maps:

$$\begin{array}{ccccc} S^{n-1} & \xhookrightarrow{\iota} & D^n & \xrightarrow{g} & S^{n-1} \\ \big\lfloor & & \text{id} & & \big\rfloor\!\uparrow \end{array}$$

We apply the functor $H^{n-1}_{\mathrm{dR}}(-)$ to this diagram. Then we obtain a commutative diagram of real vector spaces and linear maps. For $n > 1$, Cor. 3 and 5 show that this diagram takes the form:

$$\begin{array}{ccccc} \mathbb{R}\cdot\xi_{n-1} & \longleftarrow & 0 & \longleftarrow & \mathbb{R}\cdot\xi_{n-1} \\ \uparrow\!\big\lfloor & & \text{id} & & \big\rfloor \end{array}$$

The composition of the top horizontal maps takes ξ_{n-1} to $0 \cdot \xi_{n-1}$ while the lower horizontal map takes ξ_{n-1} to $1 \cdot \xi_{n-1}$. This is a contraction. For $n = 1$, Cor. 3 and 5 show that the induced diagram of de Rham cohomology groups takes the form:

$$\begin{array}{ccccc} \mathbb{R}\cdot 1 \oplus \mathbb{R}\cdot\xi_0 & \longleftarrow & \mathbb{R}\cdot 1 & \longleftarrow & \mathbb{R}\cdot 1 \oplus \mathbb{R}\cdot\xi_0 \\ \uparrow\!\big\lfloor & & \text{id} & & \big\rfloor \end{array}$$

This again gives a contraction. Indeed, the image of the composition of the top horizonal maps is a proper subspace of the vector space $\mathbb{R}\cdot 1 \oplus \mathbb{R}\cdot\xi_0$, while the image of the identity map is the full vector space $\mathbb{R}\cdot 1 \oplus \mathbb{R}\cdot\xi_0$. Hence, we conclude that the theorem cannot be false. Therefore, it is true.

References

[1] I. Madsen and J. Tornehave, *From calculus to cohomology. De Rham cohomology and characteristic classes*, Cambridge University Press, Cambridge, 1997.

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