Configuration spaces

Lars Hesselholt

We first consider the following situation. Suppose that we have N robots that we wish to move around on a factory floor without collisions. For simplicity, we will assume that each robot is a point and that the factory floor is $\mathbb{R}^2$. In particular, there are no obstacles for the robots to avoid. Then the set of all possible ways of positioning the N robots is

$$\tilde{C}_N(\mathbb{R}^2) = \{(x_1, \dots, x_N) \in (\mathbb{R}^2)^N \mid x_i \neq x_j \text{ for } i \neq j\}.$$

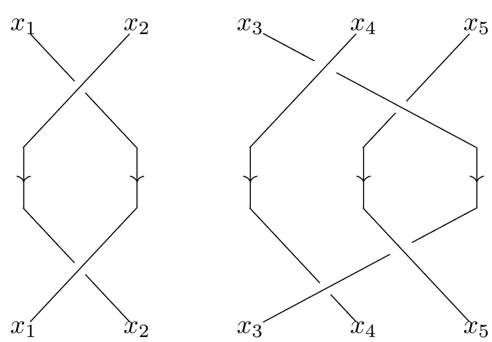
It is an open subset of the topological (vector) space $(\mathbb{R}^2)^N = \mathbb{R}^{2N}$ and is called the *configuration space* of N ordered and non-colliding particles in $\mathbb{R}^2$. We call an element $\xi = (x_1, \dots, x_N) \in \tilde{C}_N(\mathbb{R}^2)$ a configuration.

The movement of the N robots on the factory floor is mirrored by the topology of the space $\tilde{C}_N(\mathbb{R}^2)$. For example, the statements "It is possible to move the N robots from any one configuration ξ to any other configuration ξ'" and "The topological space $\tilde{C}_N(\mathbb{R}^2)$ is path connected" are equivalent. (Exercise: Argue that the statements are true.)

In actual production, the N robots will typically move in a periodic pattern, where they begin at some configuration ξ, then move around on the factory floor, and finally return to the original configuration ξ. This movement is described by a continuous map $F\colon \mathbb{R} \to \tilde{C}_N(\mathbb{R}^2)$ such that $F(t+n) = F(t)$ and $F(n) = \xi$, for all $t \in \mathbb{R}$ and $n \in \mathbb{Z}$, or equivalently, by a continuous map

$$f\colon S^1 \to \tilde{C}_N(\mathbb{R}^2)$$

from the circle $S^1 = \mathbb{R}/\mathbb{Z}$ with the property that $f(\mathbb{Z}) = \xi$. The figure



illustates such a continuous map with $N = 5$. It would not be advisable to let the robots move as illustrated in this figure: The sudden changes of direction would cause the robots to vibrate violently and break apart.

We briefly recall the definition of the fundamental group $\pi_1(X, x)$ of a topological space X with a chosen base-point $x \in X$. Two continuous maps $f, g\colon S^1 \to X$ such that $f(\mathbb{Z}) = g(\mathbb{Z}) = x$ are said to be homotopic, if there exists a continuous map

$$h\colon S^1 \times [0, 1] \to X$$

such that $h(t + \mathbb{Z}, 0) = f(t + \mathbb{Z})$ and $h(t + \mathbb{Z}, 1) = g(t + \mathbb{Z})$, for all $t + \mathbb{Z} \in S^1$, and such that $h(\mathbb{Z}, s) = x$, for all $s \in [0, 1]$. Then $\pi_1(X, x)$ is the set of homotopy classes of continuous maps $f\colon S^1 \to X$ such that $f(\mathbb{Z}) = x$ equipped with the following group structure. If $[f\,]$ denotes the homotopy class of the map f, then the product is defined by $[f\,] * [g] = [f * g]$, where

$$(f * g)(t + \mathbb{Z}) = \begin{cases} f(2t + \mathbb{Z}) & (t \in [0, 1/2]) \\ g(2t - 1 + \mathbb{Z}) & (t \in [1/2, 1]), \end{cases}$$

the inverse is defined by $[f\,]^{-1} = [\bar{f}\,]$, where $\bar{f}(t + \mathbb{Z}) = f(-t + \mathbb{Z})$, and the unit element is the class e of the constant map with value x.

In the case at hand, the fundamental group is Artin's [**1**] pure braid group

$$P_N = \pi_1(\tilde{C}_N(\mathbb{R}^2), \xi).$$

The elements of this (infinite) group represent the essentially different ways in which the N robots can move around periodically from the initial configuration ξ without collisions. We remark that the higher homotopy groups $\pi_q(\tilde{C}_N(\mathbb{R}^2), \xi)$ with $q \geqslant 2$ are all trivial [**3**, Cor 2.3].

We mention a related situation. If the N robots are all the same kind, we would not insist that they return to exactly the same configuration $\xi = (x_1, \dots, x_N)$ after each production cycle. Instead, we would ask only that they return to some permutation $\xi_\sigma = (x_{\sigma(1)}, \dots, x_{\sigma(N)})$ after each cycle. This situation may be described as follows. The symmetric group Σ_N acts freely on the space $\tilde{C}_N(\mathbb{R}^2)$ by the rule

$$\sigma \cdot (x_1, \dots, x_N) = (x_{\sigma(1)}, \dots, x_{\sigma(N)}).$$

The orbit space $C_N(\mathbb{R}^2) = \tilde{C}_N(\mathbb{R}^2)/\Sigma_N$ is called the configuration space of N unordered non-colliding particles in $\mathbb{R}^2$. Let $\bar{\xi} = \xi\Sigma_N$ be the orbit through ξ. Then the movement of the N robots described above corresponds to a continuous map

$$f\colon S^1 \to C_N(\mathbb{R}^2)$$

such that $f(\mathbb{Z}) = \bar{\xi}$. In this case, the fundamental group is Artin's [**1**] braid group

$$B_N = \pi_1(C_N(\mathbb{R}^2), \bar{\xi}).$$

By covering space theory, we obtain the following extension of groups

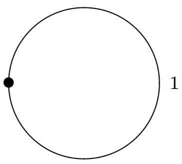
$$1 \to P_N \to B_N \to \Sigma_N \to 1.$$

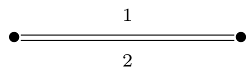
We remark that it was proved only very recently that the group B_N can be embedded as a subgroup of $GL_n(\mathbb{C})$, for some n [**2**]. Our assumption here that robots are points and that they navigate in a workplace without obstacles is of course not very realistic. There is vast literature on robot motion planning in more realistic situations [**5, 6**].

We next consider configuration spaces of a robotic arm or mechanical linkage. We consider only planar mechanical linkages. We begin by making a precise definition. An *oriented graph* $\Gamma_+ = (V, E_+, s, t)$ is a finite set V of vertices, a finite set E_+ of oriented edges, and two functions $s, t \colon E_+ \to V$ called source and target. A map of oriented graphs $f \colon \Gamma_+ \to \Gamma'_+$ is a pair of maps $f_V \colon V \to V'$ and $f_E \colon E_+ \to E'_+$ such that $s' f_E = f_V s$ and $t' f_E = f_V t$. For every oriented graph $\Gamma_+ = (V, E_+, s, t)$, the opposite oriented graph is defined by $\Gamma^*_+ = (V, E_+, t, s)$. A *graph* $\Gamma = (V, E_+, s, t, f)$ is an oriented graph $\Gamma_+ = (V, E_+, s, t)$ and a map $f \colon \Gamma_+ \to \Gamma^*_+$ such that $f_V = \mathrm{id}_V$ and f_E is an involution. The edges of the graph $\Gamma = (V, E_+, s, t, f)$ is defined to be the set E of orbits for the involution f_E on E_+. An *abstract linkage* $L = (\Gamma, V_0, \ell)$ is a graph Γ, a subset $V_0 \subset V$ of the set of vertices called the set of fixed vertices, and a function $\ell \colon E \to (0, \infty)$. A *planar realization* of the abstract linkage L is a map $\phi \colon V \to \mathbb{R}^2$ such that, if the edge e connects the vertices v and v', then

$$d(\phi(v), \phi(v')) = \ell(e).$$

This definition allows for many examples of abstract linkages that have no planar realizations at all. The following two abstract linkages are such examples.



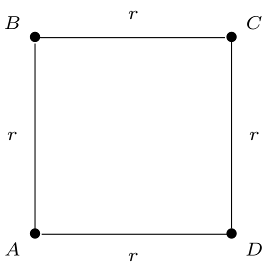


Now let L be an abstract linkage. We will now assume that L has a planar realization. Then, for every fixed vertex v, we choose a point $p(v) \in \mathbb{R}^2$ such that, for every edge e connecting two fixed vertices v and v', we have $d(p(v), p(v')) = \ell(e)$. We define the *configuration space* of the abstract linkage L to be the subspace

$$C(L) \subset (\mathbb{R}^2)^V$$

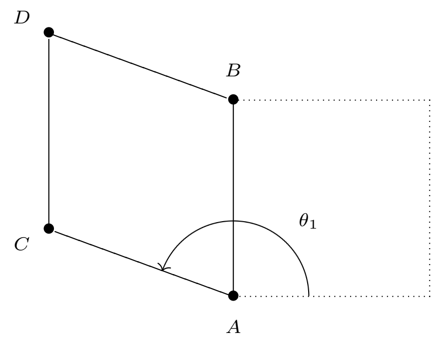
of all planar realizations $\phi \colon V \to \mathbb{R}^2$ such that, for every fixed vertex v, $\phi(v) = p(v)$.

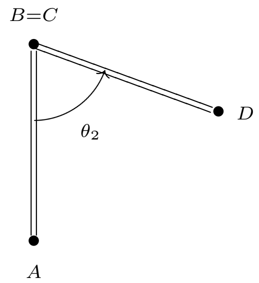
It is now high time to consider an example. We let L be the following abstract linkage with all edges of the same length r and with the vertices A and B fixed.

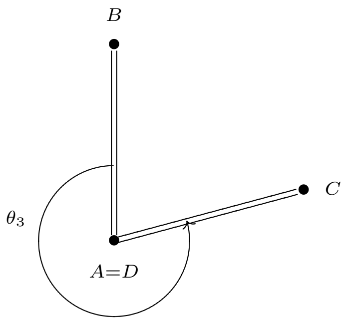


The space $C(L)$ consists of three circles that pairwise intersect in a single point. Indeed, the three circles are parametrized by the angles θ_1, θ_2, and θ_3 indicated by the following figure. The circles parametrized by θ_1 and θ_2 intersect in the point where $\theta_1 = \pi/2$ and $\theta_2 = \pi$, the circles parametrized by θ_1 and θ_3 intersect at $\theta_1 = 3\pi/2$ and $\theta_3 = \pi$, and the circles parametrized by θ_2 and θ_3 intersect at

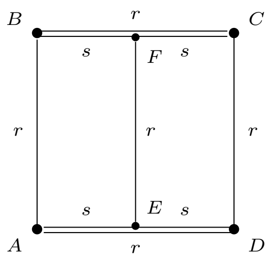
$\theta_2 = \theta_3 = 0$.







The following abstract linkage, where $r = 2s$, is called the rigidified square. Its configuration space consists of a single circle.



In particular, in this case, the configuration space is a smooth manifold. In turns out that the following general theorem holds [**4**, Cor. C]:

Theorem. *Let M be a compact smooth manifold. Then there exists an abstract linkage L such that the configuration space $C(L)$ is diffeomorphic to the disjoint union of a finite number of copies of M.*

We briefly outline the steps in the proof. First, we note that the configuration space $C(L) \subset (\mathbb{R}^2)^V$ naturally has the structure of a real algebraic set. Indeed, we may equivalently define $C(L) \subset (\mathbb{R}^2)^V$ to be the set maps $\phi \colon V \to \mathbb{R}^2$ that satisfy the following polynomial equations: If the edge e connects vertices v and v', then $d(\phi(v), \phi(v'))^2 = \ell(e)^2$, and if v is a fixed vertex, then $\phi(v) = p(v)$. Next, based on a theorem of Nash [**8**, Thm. 1], it can be shown that every compact smooth manifold is diffeomorphic to a compact real algebraic set [**4**, Thm. 2.20]. Hence, it suffices to show that, for every compact real algebraic set X, there exists an abstract linkage L such that $C(L)$ is real analytically isomorphic to the disjoint union of a finite number of a copies of X. This problem, in turn, is closely related to the problem of representing a function

$$f\colon (\mathbb{R}^2)^m \to (\mathbb{R}^2)^n$$

by an abstract linkage L which we now explain. For every vertex v in L, we have the evaluation map $\mathrm{ev}_v\colon C(L) \to \mathbb{R}^2$ defined by $\mathrm{ev}_v(\phi) = \phi(v)$ which reads off the position of the vertex v. We consider a diagram of the form

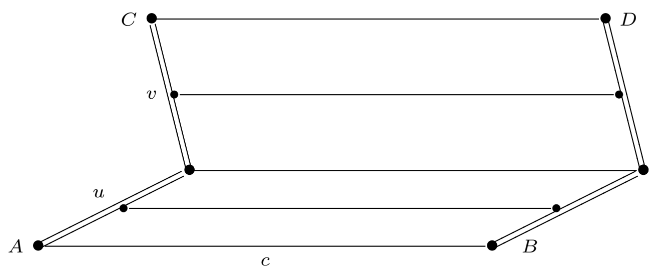
$$\begin{array}{ccc} & C(L) & \\ {}^{s}\swarrow & & \searrow^{t} \\ (\mathbb{R}^2)^m & \xrightarrow{\quad f \quad} & (\mathbb{R}^2)^n \end{array}$$

where s is the evaluation map at m of the vertices, which we call the input vertices, and t is the evaluation map at n of the vertices, which we call the output vertices. We say that this diagram represents f around the point $p \in (\mathbb{R}^2)^m$, if there exists an open neighborhood $p \in U \subset (\mathbb{R}^2)^m$ such that restricted diagram

$$\begin{array}{ccc} & s^{-1}(U) & \\ {}^{s}\swarrow & & \searrow^{t} \\ U & \xrightarrow{\quad f \quad} & (\mathbb{R}^2)^n \end{array}$$

commutes and such that $s\colon s^{-1}(U) \to U$ is a locally analytically trivial covering. This definition will soon become clearer when we consider an example. We will actually consider three examples.

We first consider the translator linkage built from two ridigified parallelograms with side lengths $u > v$ and c.



If we fix the vertices A and B at $(0, 0)$ and $(c, 0)$ and let C be the input vertex and D the output vertex, then the translator linkage represents the function

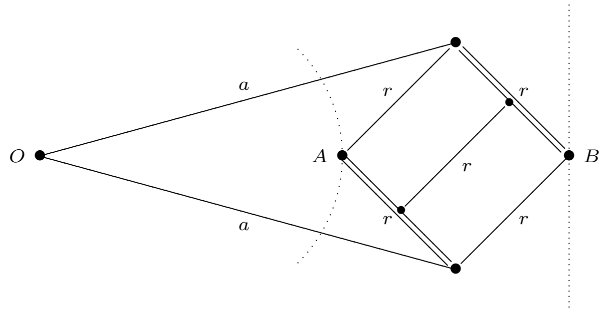
$$f(z) = z + c$$

on the open annuli $\{z \in \mathbb{C} \mid u - v < |z| < u + v\}$. Here we identify $(x, y) \in \mathbb{R}^2$ with $x + iy \in \mathbb{C}$. Similarly, if we fix A and B at $(-c, 0)$ and $(0, 0)$ and let D be the input vertex and C the output vertex, then the translator linkage represents the function

$$f(z) = z - c$$

on the same open annulus. In both cases, the map $s : s^{-1}(U) \to U$ is a two-to-one covering. (Exercise: Show this.)

We next consider the famous inversor linkage constructed by Peaucellier [**9**] and Lipkin [**7**], independently, around 1870.

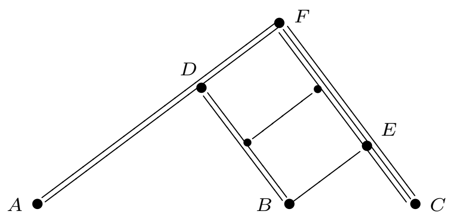


We fix O at the origin of $\mathbb{C}$ and let A be the input vertex and B the output vertex. Then the Peaucellier-Lipkin linkage represents the function

$$f(z) = t^2/\bar{z}$$

on the open annulus $U = \{z \in \mathbb{C} \mid a - r < |z| < t\}$, where $t^2 = a^2 - r^2$. In particular, if we let move A along a circle that passes through O, then B will move along a straight line perpendicular to the line through O and the center of the circle. This was the original purpose of the inversor linkage: to transform circular motion into straight-line motion and vice versa. The map $s\colon s^{-1}(U) \to U$ again is a two-to-one covering. The non-trivial deck transformation acts by reflection of the rectified square in the line through A and B.

The final example, we consider, is the rigidified pantograph constructed by Scheiner in 1603. We will use it to represent several functions.



We assume that $u = \ell(AD) > v = \ell(EF)$ and that $\ell(AF) = \lambda u$ and $\ell(CF) = \lambda v$, where $\lambda > 1$. First, if we fix A at the origin of $\mathbb{C}$ and let B be the input vertex and C the output vertex, then the pantograph represents the function

$$f(z) = \lambda z$$

on the open annulus $\{z \in \mathbb{C} \mid \lambda u - v < |z| < \lambda u + v\}$. Second, if we fix A at the origin of $\mathbb{C}$ and let C be the input vertex and B the output vertex, then the pantograph represents the function

$$f(z) = \lambda^{-1} z$$

on the open annulus $\{z \in \mathbb{C} \mid \lambda(u - v) < |z| < \lambda(u + v)\}$. Third, if we let $\lambda = 2$, fix B at the origin of $\mathbb{C}$, and let A be the input vertex and C the output vertex, then the pantograph represents the function

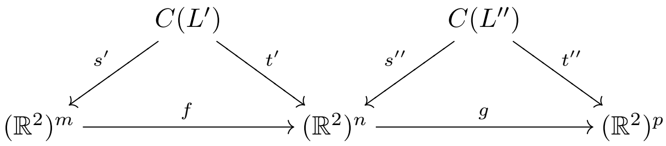
$$f(z) = -z$$

on the open annulus $\{z \in \mathbb{C} \mid u - v < |z| < u + v\}$. Finally, we again let $\lambda = 2$ but, this time, we do not fix any vertices. We let A and C be the input vertices and B the output vertices. Then the pantograph represents the function

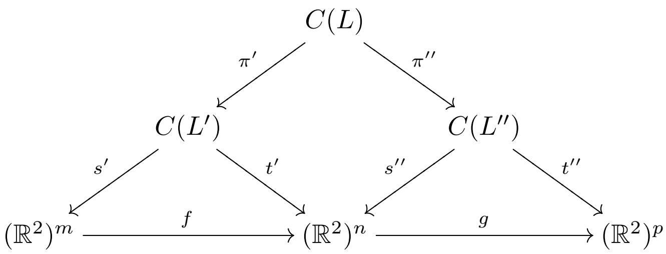
$$f(z, w) = (z + w)/2$$

on the open annulus $\{(z, w) \in \mathbb{C}^2 \mid 2(u - v) < |z - w| < 2(u + v)\}$. (Exercise: In each cases, how many sheets does the covering $s\colon s^{-1}(U) \to U$ have?)

Composing the six functions represented by the three elementary linkages above, we can obtain every polynomial with real coefficients. Given representations of the functions f and g by abstract linkages L' and L'', we wish to find an abstract linkage L that represents the composition $f \circ g$. Consider the following diagram



We define an abstract linkage $L = (\Gamma, V, \ell)$ as follows. Identifying the output vertices in Γ' with the input vertices in Γ'', we obtain the graph Γ. The functions $\ell'\colon V' \to (0, \infty)$ and $\ell''\colon V'' \to (0, \infty)$ give rise to a function $\ell\colon V \to (0, \infty)$. Finally, we define the fixed vertices in L to be the union of the fixed vertices in L' and the fixed vertices in L''. We then have the larger diagram



where the maps π' and π'' are induced from the inclusions of L' and L'' in L. Moreover, the middle square is a fiber square. (More precisely, the middle square is a fiber square of affine schemes, where the configuration spaces are viewed as the affine schemes defined by equations $d(\phi(v), \phi(v'))^2 = \ell(e)^2$, for every edge, and $\phi(v) = p(v)$, for every fixed vertex.) Therefore, if L' represents f on U and L'' represents g on V, then L represents $f \circ g$ on $U \cap f^{-1}(V)$. By using this, one can prove that every polynomial with real coefficients can be represented by an abstract linkage. This, in turn, makes it possible to prove the theorem.

References

[1] E. Artin, *Theorie der Zöpfe*, Abh. Math. Sem. Hamburg **4** (1926), 47–72.
[2] S. Bigelow, *Braid groups are linear*, J. Amer. Math. Soc. **14** (2001), 471–486.
[3] E. Fadell and L. Neuwirth, *Configuration spaces*, Math. Scand. **10** (1962), 111–118.
[4] M. Kapovich and J. J. Millson, *Universality theorems for configuration spaces of planar linkages*, Topology **41** (2002), 1051–1107.
[5] D. Koditschek and E. Rimon, *Robot navigation functions on manifolds with boundary*, Adv. in Appl. Math. **11** (1990), 412–442.
[6] J.-C. Latombe, *Robot motion planning*, Kluwer Academic Press, Boston, 1991.

[7] L. Lipkin, *Dispositif articulé pour la transformation rigoureuse du mouvement circulaire en mouvement rectilinge*, Revue anivers. des Mines et de la Métalurgie de Liège **30** (1871), 149–150.
[8] J. F. Nash, *Real algebraic manifolds*, Ann. of Math. **56** (1952), 405–421.
[9] A. Peaucellier, *Note sur une question de géometrie de compas*, Nouv. Ann. de Math. **12** (1873), 71–73.

Nagoya University, Nagoya, Japan

E-mail address: larsh@math.nagoya-u.ac.jp