The notion of homotopy equivalence of two spaces is too fine: It is too difficult for two spaces to be homotopy equivalent. The better notion is that of weak equivalence.

<u>Def</u> A map $f: X \to Y$ of topological spaces is a <u>weak equivalence</u> if, for all $x \in X$ and $n \geq 0$, the map

$$f_*: \pi_n(X, x) \to \pi_n(Y, f(x))$$

is a bijection. //

<u>Rem</u> The long-exact sequence

$$\cdots \to \pi_n(F(f,y), (x,\tilde{y})) \xrightarrow{i_*} \pi_n(X, x) \xrightarrow{f_*} \pi_n(Y, y) \to \cdots$$

shows that f is a weak equivalence if and only if, for all $x \in X$ and $n \geq 0$,

$$\pi_n(F(f,y), (x,\tilde{y})) = 0.$$

We note that, in particular, $F(f,y)$ must be non-empty, for all $y \in Y$. //

The following definition was given by Serre in his thesis. Let $D^n = \{x \in \mathbb{R}^n \mid \|x\| \le 1\}$.

Def A map $p : X \to Y$ of topological spaces is a Serre fibration if, for every $n \geq 0$ and every commutative diagram

$$\begin{tikzcd} D^n \arrow[r, "g"] \arrow[d, "\iota_0"] & X \arrow[d, "p"] \\ D^n \times [0,1] \arrow[r, "h"] & Y \end{tikzcd}.$$

there exists a map

$$\tilde{h} : D^n \times [0,1] \to X$$

such that $g = \tilde{h} \circ \iota_0$ and $h = p \circ \tilde{h}$. //

Rem Since S^n, D^n, $S^n \times [0,1]$, and $D^n \times [0,1]$ are all compact Hausdorff spaces, the map $f : X \to Y$ is a weak equivalence (resp. Serre fibration) if and only if the induced map $f : k(X) \to k(Y)$ of the associated k-spaces is a weak equivalence (resp. Serre fibration). //

Lemma Let $p: X \to Y$ be a Serre fibration. Then, for every $y \in Y$, the map

$$f : p^{-1}(y) \to F(p, y)$$

defined by $f(x) = (x, \bar{y})$ is a weak equivalence.

Proof We must show that, for every $x \in p^{-1}(y)$ and $n \geq 0$, the map

$$f_* : \pi_n(p^{-1}(y), x) \to \pi_n(F(p,y), (x, \bar{y}))$$

is a bijection. We first prove that is surjective. By adjunction, a map

$$\omega : (S^n, \infty) \to (F(p,y), (x, \bar{y}))$$

is equivalent to a commutative diagram of maps

$$\begin{tikzcd} S^n \times \{1\} \cup \{\infty\} \times [0,1] \arrow[r, "\omega'"] \arrow[d, hook, "\iota"] & X \arrow[d, "p"] \\ S^n \times [0,1] \arrow[r, "\omega''"] & Y \end{tikzcd}$$

with $\omega'(\infty, s) = x$ and $\omega''(\infty, s) = y$, for all $s \in [0,1]$, and $\omega''(z, 0) = y$, for all $z \in S^n$. We claim there exists a map

$$\tilde{\omega} : S^n \times [0,1] \longrightarrow X$$

such that $\tilde{\omega} \circ \iota = \omega'$ and $p \circ \tilde{\omega} = \omega''$. It follows, by adjunction, that $\tilde{\omega}$ gives a (base-point preserving) homotopy from $f \circ \eta$ to ω with $\eta : (S^n, \infty) \longrightarrow (p^{-1}(y), x)$ defined by $\eta(z) = \tilde{\omega}(z, 0)$. To prove the claim, we first define

$$\varkappa : D^n \longrightarrow S^n$$

to be the map $\varkappa(x) = \dfrac{\|x\|}{1 - \|x\|} \cdot x$. Then the diagram

$$\begin{tikzcd}
D^n \times \{1\} \cup \partial D^n \times [0,1] \arrow[r, "\varkappa \times \mathrm{id}"] \arrow[d] & S^n \times \{1\} \cup \{\infty\} \times [0,1] \arrow[d] \\
D^n \times [0,1] \arrow[r, "\varkappa \times \mathrm{id}"] & S^n \times [0,1]
\end{tikzcd}$$

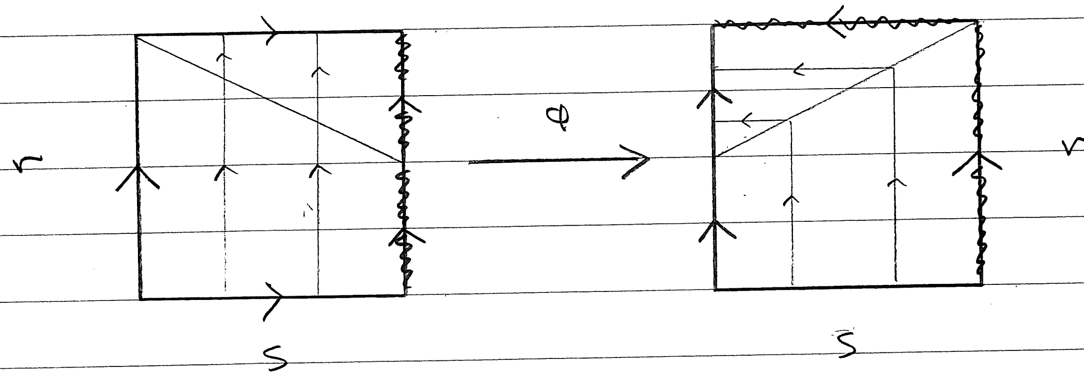
is a push-out. Hence, it will suffice to find a map $\tilde{k} : D^n \times [0,1] \longrightarrow X$ that makes the following diagram commute.

$$
\begin{tikzcd}
D^n \times \{1\} \cup \partial D^n \times [0,1] \arrow[r, "\omega' \circ (r \times id)"] \arrow[d, hook] & X \arrow[d, "f"] \\
D^n \times [0,1] \arrow[r, "\omega'' \circ (r \times id)"'] \arrow[ur, dashed, "\tilde{k}"] & Y
\end{tikzcd}
$$

We choose a homeomorphism

$$\varphi : [0,1] \times [0,1] \longrightarrow [0,1] \times [0,1]$$

as indicated by the picture



We also define the quotient map

$$c : \partial D^n \times [0,1] \times [0,1] \longrightarrow D^n \times [0,1]$$

by $c(z, r, s) = (rz, s)$. Then the unique map Φ that makes the diagram commute

$$\begin{tikzcd}
\partial D^n \times [0,1) \times [0,1] \arrow[r, "id \times \varphi"] \arrow[d, "c"] & \partial D^n \times [0,1) \times [0,1] \arrow[d, "c"] \\
D^n \times [0,1] \arrow[r, "\Phi"] & D^n \times [0,1]
\end{tikzcd}$$

is a homeomorphism, and the following diagram commutes

$$\begin{tikzcd}
D^n \times \{1\} \arrow[r, "\Phi"] \arrow[d, hook] & D^n \times \{1\} \cup \partial D^n \times [0,1] \arrow[d, hook] \\
D^n \times [0,1] \arrow[r, "\Phi"] & D^n \times [0,1]
\end{tikzcd}$$

Hence, the problem at hand is reduced to finding $\tilde{h} : D^n \times [0,1] \to X$ that makes the following diagram commute:

$$\begin{tikzcd}
D^n \times \{1\} \arrow[r, "w' \circ (r \times id) \circ \Phi"] \arrow[d, hook] & X \arrow[d, "p"] \\
D^n \times [0,1] \arrow[r, "w'' \circ (r \times id) \circ \Phi"] \arrow[ur, dashed, "\tilde{h}"] & Y
\end{tikzcd}$$

But this map $\tilde{h}$ exists by the definition of a Serre fibration. Hence, f_* is onto.

To prove that f_* is injective, let

$$\eta, \eta' : (S^n, \infty) \to (p^{-1}(y), x)$$

be two maps, and let

$$h : (S^n, \infty) \wedge ([0,1]_+, +) \to (F(p,y), (x, \bar{y}))$$

be a homotopy from $f \circ \eta$ to $f \circ \eta'$. We must show that there exists a homotopy

$$k : (S^n, \infty) \wedge ([0,1]_+, +) \to (p^{-1}(y), x)$$

from η to η'. By adjunction, the homotopy h is equivalent to a commutative diagram

$$\begin{tikzcd} S^n \times \{1\} \times [0,1] \cup \{\infty\} \times [0,1] \times [0,1] \arrow[r, "h'"] \arrow[d, "\iota"] & X \arrow[d, "p"] \\ S^n \times [0,1] \times [0,1] \arrow[r, "h''"] & Y \end{tikzcd}$$

where $h'(\infty, s, t) = x$ and $h''(\infty, s, t) = y$, for all $s, t \in [0,1]$, $h''(z, s, 0) = h''(z, s, 1) = y$, for all $z \in S^n$ and $s \in [0,1]$, and $h''(z,0,t) = y$, for all $t \in [0,1]$. We see as before that, since p is a Serre fibration, there exists

$$\tilde{h} : S^n \times [0,1] \times [0,1] \to X$$

such that $h' = \tilde{h} \circ \iota$ and $h'' = p \circ \tilde{h}$. Then $k(z,t) = \tilde{h}(z,0,t)$ is the desired homotopy. //

<u>Cor</u> Let $p : X \to Y$ be a Serre fibration, let $x \in X$, and let $y = p(x)$. Then there is a long-exact sequence

$$\cdots \to \pi_{n+1}(Y,y) \xrightarrow{\delta'} \pi_n(p^{-1}(y),x) \xrightarrow{j} \pi_n(X,x) \xrightarrow{p_*} \pi_n(Y,y) \to \cdots$$

where $j : p^{-1}(y) \to X$ is the inclusion.

<u>Proof</u> We have

$$\begin{array}{ccccccc} \cdots \to \pi_{n+1}(Y,y) & \xrightarrow{\delta} & \pi_n(F(p,y),(x,\bar{y})) & \xrightarrow{i} & \pi_n(X,x) & \xrightarrow{p_*} & \pi_n(Y,y) \to \cdots \\ & & \sim \uparrow f_* & & & & \\ & & \pi_n(p^{-1}(y),x) & & & & \end{array}$$

and $j = i \circ f$. We define $\delta' = f_*^{-1} \circ \delta$. //

We next define a class of maps in $\mathcal{K}$ called the Serre cofibrations. Let

$$I = \{ \partial D^n \rightarrowtail D^n \mid n \geq 0 \}$$

be the set of the standard inclusions of the boundary ∂D^n of D^n in D^n. We note that $\partial D^0 = \emptyset$. We refer to I as the set of generating cofibrations.

Def (i) The map $f : A \to B$ is a <u>relative</u> <u>I-cell complex</u> if there exists a sequence of maps

$$A = B_0 \xrightarrow{i_0} B_1 \xrightarrow{i_1} B_2 \xrightarrow{i_2} \cdots$$

together with push-out diagrams

$$\begin{tikzcd} \coprod_\alpha U_\alpha \arrow[r, "\Sigma \varphi_\alpha"] \arrow[d, "\coprod g_\alpha"'] & B_{m-1} \arrow[d, "i_{m-1}"] \\ \coprod_\alpha V_\alpha \arrow[r, "\Sigma \psi_\alpha"] & B_m \end{tikzcd}$$

with $g_\alpha : U_\alpha \to V_\alpha$ in I, and maps

$$j_m : B_m \longrightarrow B$$

such that $j_0 = f$, $j_m \circ i_{m-1} = j_{m-1}$, and such that the induced map

$$j : \operatorname{colim}_m B_m \longrightarrow B$$

is a homeomorphism.

(ii) The map $f : A \to B$ is a <u>Serre cofibration</u> (or I-cofibration) if there exists a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "id_A"] & A' \arrow[r] \arrow[d, "f'"] & A \arrow[d, "f"] \\
B \arrow[r] \arrow[rr, bend right, "id_B"'] & B' \arrow[r] & B
\end{tikzcd}$$

where $f' : A' \to B'$ is a relative I-cell complex. //

We refer to the commutative diagram in (ii) by saying that the map $f : A \to B$ is a <u>retract</u> of the map $f' : A' \to B'$.