A space X is an *I-cell complex* if the map $\emptyset \to X$ is a relative I-cell complex. In particular, the geometric realization $|X[-]|$ of a simplicial set $X[-]$ is an I-cell complex.

The classes of weak equivalences, Serre fibrations, and Serre cofibrations satisfy the axioms of a Quillen model structure which we now define.

Def (Quillen) A *model category* is a category $\mathcal{E}$ together with three classes of maps called the weak equivalences ($\xrightarrow{\sim}$), fibrations ($\twoheadrightarrow$), and cofibrations ($\rightarrowtail$) that satisfy the following axioms:

M1: All small limits and colimits exist in $\mathcal{E}$.

M2: Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps in $\mathcal{E}$. If two of the maps $f, g, g \circ f$ are weak equivalences, then so is the third.

M3: If f and g are maps in $\mathcal{E}$ such that f is a retract of g, and if g is a weak equivalence, a fibration, or a cofibration, then so is f.

M4: Given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "i"'] & X \arrow[d, two heads, "p"] \\ B \arrow[r, "g"] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

where i is a cofibration, p is a fibration, and one of i and p is a weak equivalence, there exists a map

$$h: B \longrightarrow X$$

such that $f = h \circ i$ and $g = p \circ h$.

M5: Every map f can be factored

$$f = p \circ i = q \circ j$$

where p is a fibration and i a cofibration and a weak equivalence, and where q is a fibration and a weak equivalence and j is a cofibration. //

<u>Thm</u> (Quillen) The category $\mathcal{K}$ with the classes of weak equivalences, Serre fibrations, and Serre cofibrations is a model category.

<u>Proof</u> This is not so easy to prove. We refer to Hovey: "Model categories" for the proof. //

Let $i: A \to B$ and $p: X \to Y$ be maps in a category $\mathcal{E}$. We say that i has the <u>left lifting property</u> (LLP) with respect to p and that p has the <u>right lifting property</u> (RLP) with respect to i if, for every commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}.$$

there exists a map

$$h : B \to X$$

such that $f = h \circ i$ and $g = p \circ h$. If $\mathcal{E}$ is a model category, we say that the map $i : A \to B$ is a trivial cofibration ($\overset{\sim}{\rightarrowtail}$) if i is both a cofibration and a weak equivalence. We say that $p : X \to Y$ is a trivial fibration ($\overset{\sim}{\twoheadrightarrow}$) if p is both a fibration and a weak equivalence.

Prop Let $\mathcal{E}$ be a model category.

(i) The map $i : A \to B$ is a cofibration (resp. trivial cofibration) if and only if it has LLP with respect to every map p that is a trivial fibration (resp. fibration).

(ii) The map $p : X \to Y$ is a fibration (resp. trivial fibration) if and only if it has RLP with respect to every map i that is a trivial cofibration (resp. cofibration).

<u>Proof</u> We prove the first part of (i) and leave the second part of (ii) along with (ii) as an exercise. If $i : A \to B$ is a cofibration, then, by M4, it has LLP with respect to all maps p that are trivial fibrations. To prove the converse, we assume $i : A \to B$ has LLP with respect to trivial fibrations. By M5, we can factor i as

$$i = q \circ j$$

where $j : A \rightarrowtail X$ is a cofibration and $q : X \xrightarrow{\sim} B$ a trivial fibration. By our assumption on i, there exists a map $h : B \to X$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, tail, "j"] \arrow[d, "i"'] & X \arrow[d, two heads, "q", "\sim"'] \\
B \arrow[r, equal] \arrow[ur, dashed, "h"] & B
\end{tikzcd}$$

We may rewrite this diagram in the following way:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "i'"] & A \arrow[r, equal] \arrow[d, tail, "j"] & A \arrow[d, "i'"] \\ B \arrow[r, "h"] \arrow[rr, bend right, "id"'] & X \arrow[r, "q"] & B \end{tikzcd}$$

This shows that i' is a retract of the cofibration j, and hence, by M3, i' is a cofibration. //

It follows that, in a model category, the two of the three classes of weak equivalences, fibrations, and cofibrations determine the third. Given a push-out (resp. pull-back) diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"] & X' \arrow[d, "f'"] \\ Y \arrow[r, "h"] & Y' \end{tikzcd},$$

we say that f' is the <u>cobase-change</u> of f along g (resp. f is the <u>base-change</u> of f' along h).

<u>Exercise</u> Use the proposition to show that the cobase-change of a cofibration (resp. trivial cofibration) is a cofibration (resp. trivial cofibration) and that the base-change of a fibration (resp. trivial fibration) is a fibration (resp. trivial fibration). //

<u>Def</u> Let $\mathcal{C}$ be a model category. A <u>cylinder</u> <u>object</u> for $A \in \operatorname{ob}\mathcal{C}$ is a diagram of the form

$$\begin{array}{ccc} & \mathrm{Cyl}(A) & \\ {\scriptstyle d^0+d^1}\nearrow & & \searrow{\scriptstyle \sigma}\ {\scriptstyle\sim} \\ A \amalg A & \xrightarrow{\nabla} & A \end{array}.$$

A <u>path</u> <u>object</u> for $B \in \operatorname{ob}\mathcal{C}$ is a diagram

$$\begin{array}{ccc} & \mathrm{Path}(B) & \\ {\scriptstyle (d_0,d_1)}\swarrow & & \nwarrow{\scriptstyle s}\ {\scriptstyle\sim} \\ B \times B & \xleftarrow{\Delta} & B \end{array}.$$

A <u>left</u> <u>homotopy</u> from $f : A \to B$ to $g : A \to B$ is a diagram

$$\begin{tikzcd} & Cyl(A) \arrow[dr, "h"] & \\ A \amalg A \arrow[ur, "d^0+d^1"] \arrow[rr, "f+g"] & & B \end{tikzcd}$$

and a <u>right</u> <u>homotopy</u> from f to g is a diagram

$$\begin{tikzcd} & Path(B) \arrow[dl, "(d_0, d_1)"'] & \\ B \times B & & A \arrow[ul, "k"'] \arrow[ll, "(f,g)"'] \end{tikzcd}$$

//

<u>Example</u> If $\mathcal{E} = \mathcal{K}$, then

$$\begin{tikzcd} & Path(B) = \underline{Hom}_{\mathcal{K}}([0,1], B) \arrow[dl, "(ev_0, ev_1)"'] & \\ B \times B & & B \arrow[ul, "s", "\sim"'] \arrow[ll, "\Delta"'] \end{tikzcd}$$

with $s(b) = \bar{b}$ is a path object for B. Hence right homotopy is the usual notion of homotopy. However, in general,

$$\begin{tikzcd} & A \times [0,1] \arrow[dr, "6 = pr_1"] & \\ A \amalg A \arrow[ur, "4_0 + 4"] \arrow[rr, "\nabla"] & & A \end{tikzcd}$$

is not a cylinder object, because the map $i_0 + i_1$ may fail to be a Serre cofibration. //

<u>Def</u> Let $\mathcal{C}$ be a model category, and let $\emptyset$ and $*$ denote the initial and terminal objects. The object A is called <u>cofibrant</u> if $\emptyset \rightarrowtail A$ is a cofibration. The object B is called <u>fibrant</u> if $B \twoheadrightarrow *$ is a fibration.

<u>Example</u> In $\mathcal{C} = \mathcal{K}$, every object is fibrant and the cofibrant objects are the retracts of I-cell complexes. //

<u>Prop</u> Let $\mathcal{C}$ be a model category, let A be a cofibrant object, and let B be a fibrant object. Then both left homotopy and right homotopy are equivalence relations on the set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and these two equivalence relations are equal.

<u>Proof</u> See Hovey, Prop. 1.2.5. //

Thm (Whitehead) Let $\mathcal{E}$ be a model category, and let $f : X \to Y$ be a map between objects that are both fibrant and cofibrant. Then f is a weak equivalence if and only if f is a homotopy equivalence.

Proof Suppose first that f is a trivial fibration. Since Y is cofibrant,

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, tail] & X \arrow[d, two heads, "f", "\sim"'] \\ Y \arrow[r, equal] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

show that there exists $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$. To show that $g \circ f$ is left homotopic to id_X, let

$$\begin{tikzcd} & \mathrm{Cyl}(X) \arrow[dr, "\sigma", "\sim"'] & \\ X \amalg X \arrow[ur, tail, "d^0 + d^1"] \arrow[rr, "\nabla"] & & X \end{tikzcd}$$

be a cylinder object for X. Then the diagram

$$\begin{tikzcd}
X \amalg X \arrow[r, "g\circ f + id"] \arrow[d, "d^0 + d^1"'] & X \arrow[d, two heads, "f", "\sim"'] \\
Cyl(X) \arrow[r, "f\circ\sigma"] \arrow[ur, dashed, "h"] & Y
\end{tikzcd}$$

gives the desired left homotopy. The dual argument shows that trivial cofibrations, too, are homotopy equivalences. Finally, let $f : X \xrightarrow{\sim} Y$ be a weak equivalence. We use M5 to factor f as

$$\begin{tikzcd}
& Z \arrow[dr, two heads, "p", "\sim"'] & \\
X \arrow[ur, tail, "i"] \arrow[rr, "f", "\sim"'] & & Y
\end{tikzcd}$$

By M2, also i is a weak equivalence. Moreover, Z is cofibrant, since X is cofibrant and i a cofibration, and Z is fibrant, since Y is fibrant and p a fibration. Hence, i and p are both homotopy equivalences, and therefore, f is a homotopy equivalence.

We refer to Hovey, Prop. 1.2.8 for the proof that every homotopy equivalence is a weak equivalence. //

<u>Ex</u> If $\mathcal{C} = \mathcal{X}$, the theorem states that a map $f : X \to Y$ between two retracts of I-cell complexes is a homotopy equivalence if and only if it is a weak equivalence. //