We continue with some more of the abstract homotopy theory connected with model categories. Let $\mathcal{E}$ be a model category. We define the homotopy category $\mathrm{Ho}(\mathcal{E})$ as follows: The objects are

$$\mathrm{ob}\,\mathrm{Ho}(\mathcal{E}) = \mathrm{ob}(\mathcal{E}).$$

To define $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{E})}(X,Y)$, we choose, for every object X, two factorizations

$$\begin{tikzcd} & Q(X) \arrow[dr, two heads, "\sim", "p_X"'] & \\ \emptyset \arrow[ur, hook] \arrow[rr] & & X \end{tikzcd} \qquad \begin{tikzcd} & R(X) \arrow[dr, two heads] & \\ X \arrow[ur, hook, "\sim", "i_X"'] \arrow[rr] & & * \end{tikzcd}$$

We refer to $Q(X)$ as a cofibrant replacement of X and $R(X)$ as a fibrant replacement. We note that $RQ(X)$ is both fibrant and cofibrant:

$$\begin{tikzcd} & RQ(X) \arrow[dr, two heads] & \\ Q(X) \arrow[ur, hook, "\sim"] \arrow[rr] \arrow[dr, two heads, "\sim"] & & * \\ \emptyset \arrow[u, hook] \arrow[r] & X & \end{tikzcd}$$

We then define

$$\mathrm{Hom}_{Ho(\mathcal{C})}(X, Y)$$

$$= \mathrm{Hom}_{\mathcal{C}}(RQ(X), RQ(Y))/\sim$$

to be the set of equivalence classes for the common equivalence relation of left or right homotopy.

We choose, for every map $f : X \to Y$ in $\mathcal{C}$, two liftings

$$\begin{tikzcd}
\emptyset \arrow[r, tail] \arrow[d, tail] & & Q(Y) \arrow[d, two heads, "\sim"', "p_Y"] \\
Q(X) \arrow[urr, dashed, "Q(f)"] \arrow[r, two heads, "p_X", "\sim"'] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, tail, "\sim"', "i_X"] & Y \arrow[r, tail, "i_Y", "\sim"'] & R(Y) \arrow[d, two heads] \\
R(X) \arrow[urr, dashed, "R(f)"] \arrow[rr, two heads] & & *
\end{tikzcd}$$

Any two liftings $Q(f)$ and $Q(f)'$ of $f \circ p_X$ are left homotopic, and any two lifts $R(f)$ and $R(f)'$ of $i_Y \circ f$ are right homotopic; see [Quillen, Lemma 7]. In particular, we have

$$Q(f \circ g) \overset{\ell}{\sim} Q(f) \circ Q(g)$$

$$R(f \circ g) \overset{r}{\sim} R(f) \circ R(g)$$

where $\overset{\ell}{\sim}$ and $\overset{r}{\sim}$ indicates left and right homotopy. Hence, there is a well-defined functor

$$\gamma : \mathcal{C} \longrightarrow Ho(\mathcal{C})$$

with $\gamma(X) = X$ and $\gamma(f) =$ class of $RQ(f)$.

Thm Let $\mathcal{C}$ be a model category. Then the functor $\gamma : \mathcal{C} \to Ho(\mathcal{C})$ satisfies:

(i) For every weak equivalence f in $\mathcal{C}$, $\gamma(f)$ is an isomorphism in $Ho(\mathcal{C})$.

(ii) If $F : \mathcal{C} \to \mathcal{A}$ is a functor such that, for every weak equivalence f in $\mathcal{C}$, $F(f)$ is an isomorphism in $\mathcal{A}$, then there exists a unique functor

$$\Theta : \mathrm{Ho}(\mathcal{E}) \longrightarrow \mathcal{A}$$

such that $F = \Theta \circ \gamma$.

<u>Proof</u> Part (i) follows from Whitehead's theorem. To prove (ii), we define

$$\Theta : \mathrm{Ho}(\mathcal{E}) \longrightarrow \mathcal{A}$$

as follows: $\Theta(X) = F(X)$, and if

$$\begin{array}{ccc} \alpha & \in & \mathrm{Hom}_{\mathrm{Ho}(\mathcal{E})}(X, Y) \\ \| & & \| \\ \text{class of } f & \in & \mathrm{Hom}_{\mathcal{E}}(RQ(X), RQ(Y))/\sim \end{array}$$

then we define

$$\begin{tikzcd}
F(X) \arrow[r, "\Theta(\alpha)"] & F(Y) \\
F(Q(X)) \arrow[u, "\sim", "F(p_X)"'] \arrow[d, "\sim"', "F(i_{Q(X)})"] & F(Q(Y)) \arrow[u, "\sim", "F(p_Y)"'] \arrow[d, "\sim"', "F(i_{Q(Y)})"] \\
F(RQ(X)) \arrow[r, "F(f)"] & F(RQ(Y))
\end{tikzcd}$$

We must show that $\theta(\alpha)$ only depends on α and not on the chosen map f. So suppose both f and g represent α. Then f and g are left homotopic:

$$\begin{tikzcd}
RQ(X) \amalg RQ(X) \arrow[rr, "f+g"] \arrow[d, "\nabla"'] \arrow[drr, hook, "d^0+d^1"] & & RQ(Y) \\
RQ(X) & & Cyl(RQ(X)) \arrow[ll, "\sigma"', "\sim"] \arrow[u, "h"']
\end{tikzcd}$$

Now,

$$F(\sigma) \circ F(d^0) = F(\sigma \circ d^0)$$

$$= F(id_{RQ(X)}) = id_{F(RQ(X))}$$

and since $F(\sigma)$ is an isomorphism, we have $F(d^0) = F(\sigma)^{-1}$. Similarly, $F(d^1) = F(\sigma)^{-1}$, so $F(d^0) = F(d^1)$. This shows that

$$\begin{array}{ccccc}
F(f) & = & F(h \circ d^0) & = & F(h) \circ F(d^0) \\
 & & & & \| \\
F(g) & = & F(h \circ d^1) & = & F(h) \circ F(d^1)
\end{array}$$

//

<u>Rem</u> Whitehead's theorem actually shows that the map f in $\mathcal{C}$ is a weak equivalence if and only if $\gamma(f)$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$. //

<u>Lemma</u> Let $\mathcal{C}$ and $\mathcal{D}$ be two model categories, and let (F, G, α) be an adjunction from $\mathcal{C}$ to $\mathcal{D}$. Then the following are equivalent:

(i) F preserves cofibrations and trivial cofibrations.

(ii) G preserves fibrations and trivial fibrations.

<u>Pf</u> We show (i) $\Rightarrow$ (ii). Let $p: X \to Y$ be a fibration in $\mathcal{D}$. We wish to show that $G(p): G(X) \to G(Y)$ is a fibration in $\mathcal{C}$. This is equivalent to showing that, in every diagram of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i", "\sim"'] & G(X) \arrow[d, "G(p)"] \\ B \arrow[r, "g"] \arrow[ur, dashed, "h"] & G(Y) \end{tikzcd}$$

the lifting h exists. By adjunction, this is equivalent to the statement that, in every diagram

$$\begin{tikzcd} F(A) \arrow[r, "f'"] \arrow[d, "F(i)"'] & X \arrow[d, "p"] \\ F(B) \arrow[r, "g'"] \arrow[ur, dashed, "h'"] & Y \end{tikzcd}$$

where $i : A \xrightarrow{\sim} B$ is a trivial cofibration, the lifting h' exists. By assumption $F(i) : F(A) \xrightarrow{\sim} F(B)$ is a trivial cofibration, so h' exists. //

<u>Def</u> Suppose that (F, G, a) satisfies the equivalent statements (i) and (ii) of the lemma. Then we say that that F is a <u>left Quillen functor</u>, that G is a <u>right Quillen functor</u>, and that (F, G, a) is a <u>Quillen adjunction</u>.

<u>Lemma</u> A left Quillen functor preserves weak equivalences between cofibrant objects. A right Quillen functor preserves weak equivalences

between fibrant objects.

Proof Let $f : A \xrightarrow{\sim} B$ be a weak equivalence between cofibrant objects. We factor

$$
\begin{tikzcd}
 & C \arrow[dr, "q", "\sim"'] & \\
A \amalg B \arrow[ur, tail, "j"] \arrow[rr, "f+id"] & & B
\end{tikzcd}
$$

and consider the push-out diagram

$$
\begin{tikzcd}
\emptyset \arrow[r, tail] \arrow[d, tail] & A \arrow[d, tail, "i_1"] \arrow[ddr, bend left, "\sim"] \arrow[dddrr, bend left=40, "f", "\sim"'] & & \\
B \arrow[r, tail, "i_2"] \arrow[drr, "\sim"] \arrow[ddrrr, bend right, "id_B"', "\sim"] & A \amalg B \arrow[dr, tail, "j"] & & \\
 & & C \arrow[dr, "q", "\sim"'] & \\
 & & & B
\end{tikzcd}
$$

Applying the left Quillen functor F, we get the following (push-out) diagram

$$\begin{tikzcd}
\emptyset \arrow[r, tail] \arrow[d, tail] & F(A) \arrow[d, tail, "F(i_1)"] \arrow[ddr, tail, bend left, "\sim"] \arrow[dddrr, bend left, "F(f)"] & & \\
F(B) \arrow[r, tail, "F(i_2)"] \arrow[drr, tail, bend right, "\sim"'] \arrow[ddrrr, bend right, "\mathrm{id}", "\sim"'] & F(A \sqcup B) \arrow[dr, tail] & & \\
& & F(C) \arrow[dr, "F(g)"] & \\
& & & F(B)
\end{tikzcd}$$

We see that $F(g)$, and hence, $F(f)$ are weak equivalences as desired. //

The left Quillen functor $F : \mathcal{C} \to \mathcal{C}'$ gives rise to a functor

$$LF : Ho(\mathcal{C}) \to Ho(\mathcal{C}')$$

called the <u>total</u> <u>left</u> <u>derived</u> <u>functor</u> defined as follows. We may define a functor

$$F' : \mathcal{C} \to Ho(\mathcal{C}')$$

by

$$\begin{tikzcd} F'(X) \arrow[d, "F'(f)"] \arrow[r, phantom, "="] & \gamma'(F(Q(X))) \arrow[d, "\gamma'(F(Q(f)))"] \\ F'(Y) \arrow[r, phantom, "="] & \gamma'(F(Q(Y))) \end{tikzcd}$$

It takes weak equivalences in $\mathcal{E}$ to isomorphisms in $Ho(\mathcal{E}')$, so by the universal property of γ, we get the functor LF such that

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\gamma"] \arrow[dr, "F'"] & \\ Ho(\mathcal{E}) \arrow[r, "LF"'] & Ho(\mathcal{D}) \end{tikzcd}$$

We note that the diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r, "F"] \arrow[d, "\gamma"] & \mathcal{E}' \arrow[d, "\gamma'"] \\ Ho(\mathcal{E}) \arrow[r, "LF"] & Ho(\mathcal{E}') \end{tikzcd}$$

does <u>not</u> commute. However, there

is a natural transformation

$$\varepsilon : LF \circ \gamma \longrightarrow \gamma' \circ F$$

defined by the map

$$\gamma'(F(Q(X))) \xrightarrow{\gamma'(F(p_X))} \gamma'(F(X)).$$

Moreover, $LF \circ \gamma$ is best approximation to $\gamma' \circ F$ from the left in the following sense: If $g : Ho(\mathcal{E}) \to Ho(\mathcal{E}')$ is a functor and $\xi : g \circ \gamma \to \gamma' \circ F$ a natural transformation, there exists a unique natural transformation $\theta : g \longrightarrow LF$ such that the diagram

$$\begin{tikzcd}
g \circ \gamma \arrow[d, "\theta \circ \gamma"'] \arrow[dr, "\xi"] & \\
LF \circ \gamma \arrow[r, "\varepsilon"'] & \gamma' \circ F
\end{tikzcd}$$

commutes. The total right derived functor

$$RG : Ho(\mathcal{E}') \longrightarrow Ho(\mathcal{E})$$

of the right Quillen functor $G : \mathcal{C}' \to \mathcal{C}$ is defined similarly.

<u>Prop</u> (Quillen's SM7) The following equivalent statements hold in $\mathcal{K}$:

(i) Let $i : A \rightarrowtail X$ and $j : B \rightarrowtail Y$ be two cofibrations. Then the induced map

$$A \times Y \coprod_{A \times B} X \times B \longrightarrow X \times Y$$

is a cofibration which is trivial if i or j is trivial.

(ii) Let $j : B \rightarrowtail Y$ and $p : Z \twoheadrightarrow W$ be a cofibration and a fibration. Then the induced map

$$\underline{\mathrm{Hom}}(Y, Z) \longrightarrow \underline{\mathrm{Hom}}(Y, W) \times_{\underline{\mathrm{Hom}}(B, W)} \underline{\mathrm{Hom}}(B, Z)$$

is a fibration which is trivial if j or p is trivial.

<u>Proof</u> We first argue that (i) and (ii) are equivalent. Recall that a map is a cofibration (resp. a trivial cofibra-

tion) if and only if it has LLP with respect to trivial fibrations (resp. fibrations). Similarly, a map is a fibration (resp. trivial fibration) if and only if it has RLP with respect to trivial cofibrations (resp. cofibrations). Now, by adjointness, the commutative diagram

$$\begin{tikzcd} A\times Y \sqcup_{A\times B} X\times B \arrow[r] \arrow[d] & Z \arrow[d] \\ X\times Y \arrow[r] \arrow[ur, dashed] & W \end{tikzcd}$$

determines and is determined by the commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(Y,Z) \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Hom}}(Y,W) \times_{\underline{\mathrm{Hom}}(B,W)} \underline{\mathrm{Hom}}(B,Z) \end{tikzcd}$$

It follows that (i) and (ii) are equivalent as stated.

Next, suppose both $i : \partial D^m \rightarrowtail D^m$ and $j : \partial D^n \rightarrowtail D^n$ are generating cofibrations. Then we have a homeomorphism

$$
\begin{tikzcd}
\partial D^m \times D^n \coprod_{\partial D^m \times \partial D^n} D^m \times \partial D^n \arrow[r] \arrow[d, "\cong"] & D^m \times D^n \arrow[d, "\cong"] \\
\partial D^{m+n} \arrow[r, tail] & D^{m+n}
\end{tikzcd}
$$

which shows that the upper horizontal map is a cofibration as desired. In a similar manner, if $i : D^m \overset{\sim}{\rightarrowtail} D^m \times [0,1]$ is a trivial cofibration and $j : \partial D^n \rightarrowtail D^n$ a cofibration, the homeomorphism

$$
\begin{tikzcd}
D^m \times D^n \coprod_{D^m \times \partial D^n} D^m \times [0,1] \times \partial D^n \arrow[r] \arrow[d, "\cong"', "\Phi"] & D^m \times [0,1] \times D^n \arrow[d, "\cong"', "\Phi"] \\
D^{m+n} \arrow[r, tail, "\sim"] & D^{m+n} \times [0,1]
\end{tikzcd}
$$

shows that the upper horizontal map is a trivial cofibration. So (i) holds if both i and j are generating (trivial) cofibrations.

It follows that, if $p: Z \xrightarrow{\sim} W$ is a trivial fibration, a lifting exists in the diagram

$$\begin{tikzcd}
\partial D^m \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(D^n, Z) \arrow[d] \\
D^m \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Hom}}(D^n, W) \times_{\underline{\mathrm{Hom}}(\partial D^n, W)} \underline{\mathrm{Hom}}(\partial D^n, Z)
\end{tikzcd}$$

But then a lifting exists in the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"] & \underline{\mathrm{Hom}}(D^n, Z) \arrow[d] \\
X \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Hom}}(D^n, W) \times_{\underline{\mathrm{Hom}}(\partial D^n, W)} \underline{\mathrm{Hom}}(\partial D^n, Z)
\end{tikzcd}$$

Indeed, this follow immediately from the definition of Serre cofibrations. (Alternatively, the existence of a lifting in the upper diagram shows that the right-hand vertical map is a trivial fibration. But then a lift-

ing exists in the lower diagram.)
We conclude that a lifting exists in the diagram

$$\begin{tikzcd}
A \times D^n \sqcup_{A \times \partial D^n} X \times \partial D^n \arrow[r] \arrow[d] & Z \arrow[d, "p", "\sim"'] \\
X \times D^n \arrow[r] \arrow[ur, dashed] & W
\end{tikzcd},$$

and hence, in the diagram

$$\begin{tikzcd}
\partial D^n \arrow[r] \arrow[d, tail] & \underline{\mathrm{Hom}}(X, Z) \arrow[d] \\
D^n \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Hom}}(X, W) \times_{\underline{\mathrm{Hom}}(A, W)} \underline{\mathrm{Hom}}(A, Z)
\end{tikzcd}$$

But then a lifting exists in

$$\begin{tikzcd}
B \arrow[r] \arrow[d, tail] & \underline{\mathrm{Hom}}(X, Z) \arrow[d] \\
Y \arrow[r] & \underline{\mathrm{Hom}}(X, W) \times_{\underline{\mathrm{Hom}}(A, W)} \underline{\mathrm{Hom}}(A, Z)
\end{tikzcd},$$

and hence, in

$$\begin{tikzcd} A\times Y \amalg_{A\times B} X\times B \arrow[r] \arrow[d] & Z \arrow[d, "p", "\sim"'] \\ X\times Y \arrow[r] \arrow[ur, dashed] & W \end{tikzcd}$$

which shows that the left-hand vertical map is a cofibration as stated. A similar argument shows that it is a trivial cofibration if i or j is a trivial cofibration. This proves the proposition.