

Today, we presents Quillen's Theorem A and Theorem B. We outline the proof of the former. Let

$$f : \mathcal{C} \longrightarrow \mathcal{C}'$$

be a functor between small categories. The induced map of spaces

$$Bf : B\mathcal{C} \longrightarrow B\mathcal{C}'$$

is a weak equivalence if and only if for all $X' \in \mathrm{ob}\,\mathcal{C}'$, the mapping fiber $F(Bf, X')$ is contractible. Quillen's theorems aim to express the homotopy type of the mapping fiber $F(Bf, X')$ in terms of the following categorical data. The under-category $X' \backslash f$ is defined by:

<u>objects</u> Pairs $(X, X' \xrightarrow{u} f(X))$ where X is an object of $\mathcal{C}$ and u a morphism of $\mathcal{C}'$.

<u>morphisms</u> The set of maps in $X' \backslash f$ from (X, u) to (Y, v) is the subset

$\mathrm{Hom}_{X'\backslash f}((X,u),(Y,v)) \subset \mathrm{Hom}_{\mathcal{E}}(X,Y)$

of maps $w : X \longrightarrow Y$ such that

$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[d, equal] & f(X) \arrow[d, "f(w)"] \\ X' \arrow[r, "v"] & f(Y) \end{tikzcd}$$

commutes.

There is a canonical map

$$\alpha_{X'} : B(X'\backslash f) \longrightarrow F(Bf, X')$$

that we now define. The functor

$$pr_1 : X'\backslash f \longrightarrow \mathcal{E}$$

defined by $pr_1(X,u) = X$ induces a map of spaces

$$Bpr_1 : B(X'\backslash f) \longrightarrow B\mathcal{E}.$$

Moreover, there is a natural transformation

$$\begin{tikzcd} X'\backslash f \arrow[r, bend left, "\overline{X'}"] \arrow[r, bend right, "f\circ pr_1"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{E}' \end{tikzcd}$$

from the constant functor with value X' to the functor $f \circ pr_1$ defined by

$$\begin{array}{ccc} \overline{X'}(X,u) & \xrightarrow{\eta(X,u)} & (f \circ pr_1)(X,u) \\ \| & & \| \\ X' & \xrightarrow{u} & f(X) \end{array}.$$

This, in turn, gives rise to a homotopy

$$\begin{tikzcd} B(X'/f) \arrow[r, bend left, "\overline{X'}"] \arrow[r, bend right, "Bf \circ Bpr_1"'] \arrow[r, phantom, "\Downarrow B\eta"] & B\mathcal{E}' \end{tikzcd}$$

from the constant map $\overline{X'}$ to the map $Bf \circ Bpr_1$. The map Bpr_1 and the homotopy $B\eta$ determine the canonical map $\alpha_{X'}$. Finally, if $v : X' \to Y'$ is a morphism in $\mathcal{E}'$, we define the functor

$$v^* : Y' \backslash f \longrightarrow X' \backslash f$$

by $v^*(X,u) = (X, u \circ v)$.

<u>Thm</u> (Quillen's Theorem B) Let $f : \mathcal{E} \to \mathcal{E}'$ be a functor between small categories and assume that for every morphism $v : X' \to Y'$ in $\mathcal{E}'$, the induced map

$$Bv^* : B(Y' \backslash f) \longrightarrow B(X' \backslash f)$$

is a weak equivalence. Then the canonical map

$$\alpha_{X'} : B(X' \backslash f) \longrightarrow F(Bf, X')$$

is a weak equivalence for every $X' \in \mathrm{ob}\,\mathcal{E}'$. /

The following special case of Thm. B is particularly important.

<u>Thm</u> (Quillen's Theorem A) Let $f : \mathcal{E} \to \mathcal{E}'$ be a functor between small categories and suppose that for all $X' \in \mathrm{ob}\,\mathcal{E}'$, $B(X' \backslash f)$ is contractible. Then the map

$$Bf : B\mathcal{E} \longrightarrow B\mathcal{E}'$$

is a weak equivalence. /

To prove Thm. A, we need some results about simplicial spaces. Given a simplicial k-space

$$X[-] : \Delta^{op} \longrightarrow \mathcal{K},$$

we define its geometric realization to be the quotient space

$$|X[-]| = \left(\coprod_{n \geq 0} X[n] \times \Delta[n] \right) / \sim$$

where $\sim$ is the equivalence relation generated by

$$(\theta^* x, z) \in X[m] \times \Delta[m]$$
$$\wr$$
$$(x, \theta_* z) \in X[n] \times \Delta[n]$$

for all $\theta : [m] \to [n]$, $x \in X[n]$, and $z \in \Delta[n]$. Note that $X[n]$ is not necessarily a discrete space. Given a bi-simplicial set

$$X[-,-] : \Delta^{op} \times \Delta^{op} \longrightarrow \mathrm{Sets},$$

we define three simplicial k-spaces as follows:

$$[m] \longmapsto |[n] \longmapsto X[m,n]|$$

$$[n] \longmapsto |[m] \longmapsto X[m,n]|$$

$$[k] \longmapsto X[k,k].$$

The following theorem is extremely useful.

<u>Thm 1</u> Let $X[-,-]$ be a bi-simplicial set. Then there are canonical natural homeomorphisms

$$|[m] \longmapsto |[n] \longmapsto X[m,n]||$$

$$\cong |[k] \longmapsto X[k,k]|$$

$$\cong |[n] \longmapsto |X[m,n]||.$$ /

We give the proof of Thm 1 later in these notes. It is similar to the proof that geometric realization preserves finite products.

The next result is also very useful.

<u>Thm 2</u> Let

$$X[-,-] \longrightarrow Y[-,-]$$

be a map of simplicial sets and assume that for all $m \geq 0$, the induced map

$$|[n] \longmapsto X[m,n]|$$

$$\longrightarrow |[n] \longmapsto Y[m,n]|$$

is a weak equivalence. Then the induced map

$$|[m] \longmapsto |[n] \longmapsto X[m,n]||$$

$$\longrightarrow |[m] \longmapsto |[n] \longmapsto Y[m,n]||$$

is a weak equivalence.

<u>Proof of Thm. A</u> If $X[-]$ is a simplicial set, we define bi-simplicial sets $XL[-,-]$ and $XR[-,-]$ by

$$\begin{tikzcd} \Delta^{op} \times \Delta^{op} \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_2"'] \arrow[rr, bend left, "{XL[-,-]}"] \arrow[rr, bend right, "{XR[-,-]}"'] & \Delta^{op} \arrow[r, "{X[-]}"] & \mathrm{Sets} \end{tikzcd}$$

We define $T(f)[-,-]$ to be the bi-simplicial set whose (m,n)-simplices are pairs of diagrams

$$(Y_m \to \cdots \to Y_0 \to f(X_0),\ X_0 \to \cdots \to X_n)$$

in $\mathcal{E}'$ and $\mathcal{E}$, respectively. The bi-simplicial structure maps (θ^*, η^*) are defined in the unique way that makes the maps

$$N(\mathcal{E}'^{op})L[-,-] \xleftarrow{p_1} T(f)[-,-] \xrightarrow{p_2} N(\mathcal{E})R[-,-]$$

given by

$$p_1(Y_m \to \cdots \to Y_0 \to f(X_0),\ X_0 \to \cdots \to X_n)$$

$$= Y_m \to \cdots \to Y_0$$

$$p_2(Y_m \to \cdots \to Y_0 \to f(X_0),\ X_0 \to \cdots \to X_n)$$

$$= X_0 \to \cdots \to X_n$$

maps of bi-simplicial sets. We now show that both p_1 and p_2 induce weak equivalences of geometric realizations. (For a bi-simplicial set

$X[-,-]$, we write $|X[-,-]|$ for the space $|[k] \mapsto X[k,k]|$.) For p_1, we realize first in the n-direction:

$$\begin{array}{ccc} |[n] \mapsto T(f)[m,n]| & \xrightarrow{p_1} & N(\mathcal{E}'^{op})[m] \\ \| & & \| \\ \coprod_{Y_m \to \cdots \to Y_0} B(Y_0 \backslash f) & \longrightarrow & \coprod_{Y_m \to \cdots \to Y_0} * \end{array}$$

The resulting map is a weak equivalence by the assumption that the spaces $B(Y_0 \backslash f)$ be contractible for all $Y_0 \in \mathrm{ob}\,\mathcal{E}'$. Hence, by Thm. 1 and Thm. 2, we may conclude that

$$|T(f)[-,-]| \xrightarrow{|p_1|} B(\mathcal{E}'^{op})$$

is a weak equivalence. For p_2, we realize instead in the m-direction:

$$\begin{array}{ccc} |[m] \mapsto T(f)[m,n]| & \xrightarrow{p_2} & N(\mathcal{E})[n] \\ \| & & \| \\ \coprod_{X_0 \to \cdots \to X_n} B(f(X_0) \backslash id_{\mathcal{E}'^{op}}) & \longrightarrow & \coprod_{X_0 \to \cdots \to X_n} * \end{array}$$

The resulting map is a weak equivalence since $f(X_0)\backslash \mathrm{id}_{\mathcal{E}'^{op}}$ has the initial object $(f(X_0), \mathrm{id})$ such that $B(f(X_0)\backslash \mathrm{id})$ is contractible. By Thm. 1 and Thm. 2, we conclude that

$$|T(f)[-,-]| \xrightarrow{|p_2|} B(\mathcal{E})$$

is a weak equivalence. Finally, the diagram

$$\begin{array}{ccccc}
B(\mathcal{E}'^{op}) & \underset{\sim}{\xleftarrow{p_1}} & |T(f)[-,-]| & \underset{\sim}{\xrightarrow{p_2}} & B(\mathcal{E}) \\
\| & & \downarrow f' & & \downarrow Bf \\
B(\mathcal{E}'^{op}) & \underset{\sim}{\xleftarrow{p_1}} & |T(\mathrm{id}_{\mathcal{E}'})[-,-]| & \underset{\sim}{\xrightarrow{p_2}} & B(\mathcal{E}')
\end{array}$$

shows that Bf is a weak equivalence as stated. //

We proceed to prove Theorem 2. Geometric realization is a functor

$$|-| : \mathcal{X}^{\Delta^{op}} \longrightarrow \mathcal{X}.$$

It has the right adjoint functor

$$\underline{\mathrm{Sin}}(-)[-] : \mathcal{X} \longrightarrow \mathcal{X}^{\Delta^{op}}$$

defined by

$$\underline{\mathrm{Sin}}(Y)[-] = \underline{\mathrm{Hom}}_{\mathcal{K}}(\Delta[-], Y).$$

We note that $\underline{\mathrm{Sin}}(Y)[n]$ is equal to the set $\mathrm{Sin}(Y)[n]$ but now equipped with the function space topology and not the discrete topology. We next define a model structure on $\mathcal{K}^{\Delta^{op}}$ called the Reedy model structure and show that $(|-|, \underline{\mathrm{Sin}}, a)$ is a Quillen adjunction.

Let $X[-]$ be a simplicial k-space. We define the n'th <u>latching space</u> by

$$L_n X = \operatorname*{colim}_{\substack{\theta : [n] \to [p] \\ p < n}} X[p]$$

and the n'th <u>matching space</u> by

$$M_n X = \lim_{\substack{\theta : [m] \to [n] \\ m < n}} X[m].$$

There are canonical natural maps

$$L_n X \longrightarrow X[n] \longrightarrow M_n X.$$

Thm The category $\mathcal{K}^{\Delta^{op}}$ of simplicial spaces has a model structure where the map $f : X[-] \to Y[-]$ is a weak equivalence if and only if the maps $f : X[n] \to Y[n]$ are weak equivalences in $\mathcal{K}$, for all $n \geq 0$; a (trivial) cofibration if and only if, the induced maps

$$X[n] \coprod_{L_n X} L_n Y \longrightarrow Y[n]$$

are (trivial) cofibrations in $\mathcal{K}$, for all $n \geq 0$; a (trivial) fibration if and only if the induced maps

$$X[n] \longrightarrow Y[n] \times_{M_n Y} M_n X$$

are (trivial) fibrations in $\mathcal{K}$, for all $n \geq 0$.

Proof See Hovey, pages 120 – 126. The proof uses the lemma above. //

We now prove:

Prop Geometric realization

$$|-|: \mathcal{T}^{\Delta^{op}} \longrightarrow \mathcal{T}$$

is a left Quillen functor from the category of simplicial k-spaces with the Reedy model structure to the category of k-spaces with the Serre model structure.

Proof We may instead show that the right adjoint functor

$$\underline{\mathrm{Sin}}(-)[-]: \mathcal{T} \longrightarrow \mathcal{T}^{\Delta^{op}}$$

takes (trivial) Serre fibrations to (trivial) Reedy fibrations. Recall that $p: X[-] \to Y[-]$ is a (trivial) Reedy fibration if the induced map

$$X[n] \longrightarrow Y[n] \times_{M_n Y} M_n X$$

is a (trivial) Serre fibration for all $n \geq 0$. Now, we calculate:

$$M_n \underline{\mathrm{Sin}}(Z) := \lim_{\substack{\eta : [m] \to [n] \\ m < n}} \underline{\mathrm{Sin}}(Z)[m]$$

$$= \lim_{\substack{\eta : [m] \to [n] \\ m < n}} \underline{\mathrm{Hom}}_{\mathcal{T}}(\Delta[m], Z)$$

$$\xleftarrow{\sim} \underline{\mathrm{Hom}}_{\mathcal{T}}\Big(\operatorname*{colim}_{\substack{\eta : [m] \to [n] \\ m < n}} \Delta[m], Z\Big)$$

$$\xleftarrow{\sim} \underline{\mathrm{Hom}}_{\mathcal{T}}(\partial\Delta[m], Z).$$

Hence, given $p : Z \to W$, we have a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Sin}}(Z)[n] \arrow[r] & \underline{\mathrm{Sin}}(W)[n] \times_{M_n \underline{\mathrm{Sin}}(W)} M_n \underline{\mathrm{Sin}}(Z) \\
\underline{\mathrm{Hom}}(\Delta[n], Z) \arrow[r] \arrow[u, "\sim"'] & \underline{\mathrm{Hom}}(\Delta[n], W) \times_{\underline{\mathrm{Hom}}(\partial\Delta[n], W)} \underline{\mathrm{Hom}}(\partial\Delta[n], Z) \arrow[u, "\sim"']
\end{tikzcd}$$

where the vertical maps are homeomorphisms and where the horizontal maps are induced by p. By SM7, which we proved last, the lower horizontal map is a (trivial) fibration if p is a (trivial) fibration. The proposition follows. //

Cor Let $f : X[-] \longrightarrow Y[-]$ be a map between Reedy cofibrant simplicial k-spaces, and assume that for all $n \geq 0$,

$$f : X[n] \longrightarrow Y[n]$$

is a weak equivalence. Then

$$|f| : |X[-]| \longrightarrow |Y[-]|$$

is a weak equivalence.

Proof Indeed, the left Quillen functor $|-|$ preserves weak equivalences between cofibrant objects. //

To prove Thm. 2, it remains to show that if $X[-,-]$ is a bi-simplicial set, then the simplicial k-space

$$[m] \longmapsto |[n] \longmapsto X[m,n]|$$

is Reedy cofibrant. To this end, we first prove the following lemma.

Lemma Let $i : A[-] \to X[-]$ be a map of simplicial sets such that for all $n \geq 0$, $i : A[n] \to X[n]$ is injective. Then

$$|i| : |A[-]| \longrightarrow |X[-]|$$

is a Serre cofibration.

Proof We define

$$I' = \{ \partial\Delta[n][-] \hookrightarrow \Delta[n][-] \mid n \geq 0 \}$$

$$|I'| = \{ \partial\Delta[n] \hookrightarrow \Delta[n] \mid n \geq 0 \}.$$

Then the geometric realization of an I'-cofibration is an $|I'|$-cofibration. Moreover, there are commutative diagrams

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, hook] \arrow[d, "\sim"] & \Delta[n] \arrow[d, "\sim"] \\ \partial D^{n-1} \arrow[r, hook] & D^n \end{tikzcd}$$

with the vertical maps homeomorphisms. So the class of $|I'|$-cofibrations is equal to the class of Serre cofibrations.

Let $i : A[-] \to X[-]$ be a map of simplicial sets. We claim that the following are equivalent:

(i) For all $n \geq 0$, $i : A[n] \to X[n]$ is injective.

(ii) The map $i : A[-] \to X[-]$ is a relative I'-cell complex.

This will prove the lemma. It is clear that (ii) implies (i). So we assume (i) and construct

$$A[-] = X_0[-] \xrightarrow{i_1} X_1[-] \xrightarrow{i_2} \cdots$$

together with push-out diagrams

$$\begin{array}{ccc} \coprod_\alpha \partial\Delta[n_\alpha][-] & \longrightarrow & X_{n-1}[-] \\ \big\downarrow & & \big\downarrow {\scriptstyle i_n} \\ \coprod_\alpha \Delta[n_\alpha][-] & \longrightarrow & X_n[-] \end{array}$$

and maps $j_n : X_n[-] \to X[-]$ such that $j_0 = i$, $j_n \circ i_n = j_{n-1}$, and

$$j : \operatorname{colim}_n X_n[-] \longrightarrow X[-]$$

is an isomorphism. We construct $X_n[-]$ and the maps i_n and j_n by induction on $n \geq 0$. Let $X_0[-] = A[-]$ and $j_0 = i$. We assume that $X_n[-]$ has been constructed together with an injective map $j_n : X_n[-] \to X[-]$ such that

$$j_n : X_n[k] \longrightarrow X[k]$$

is a bijection, for $0 \leq k < n$. We define $S_n \subset X[n]$ to be the set of non-degenerate simplices in $X[n]$ that are not in the image of $j_n : X_n[n] \to X[n]$. We then define $X_{n+1}[-]$, i_{n+1}, and j_{n+1} by the push-out

$$\begin{tikzcd}
\coprod_{x \in S_n} \partial\Delta[n][-] \arrow[r, "\Sigma \sigma_x"] \arrow[d] & X_n[-] \arrow[d, "i_{n+1}"] \arrow[ddr, bend left, "j_n"] & \\
\coprod_{x \in S_n} \Delta[n][-] \arrow[r] \arrow[drr, bend right, "\Sigma \sigma_x"'] & X_{n+1}[-] \arrow[dr, "j_{n+1}"] & \\
& & X[-]
\end{tikzcd}$$

Then $j_{n+1} : X_{n+1}[k] \to X[k]$ is surjective, for $0 \leq k < n+1$, by construction, and injective, for all $k \geq 0$, since j_n is injective and since we have only added non-degenerate simplices not in the image of j_n. This shows (ii). //

<u>Cor</u> Let $X[-,-]$ be a bi-simplicial set. Then the simplicial k-spaces

$$[m] \longmapsto |[n] \longmapsto X[m,n]|$$

$$[n] \longmapsto |[m] \longmapsto X[m,n]|$$

are Reedy cofibrant.

<u>Proof</u> Let $Y[-] = |[q] \longmapsto X[-,q]|$. Then

$$\begin{aligned} L_n Y &= \operatorname*{colim}_{\substack{[n] \twoheadrightarrow [p] \\ p<n}} Y[p] \\ &= \operatorname*{colim}_{\substack{[n] \twoheadrightarrow [p] \\ p<n}} |[q] \longmapsto X[p,q]| \\ &\xrightarrow{\sim} |[q] \longmapsto \operatorname*{colim}_{\substack{[n] \twoheadrightarrow [p] \\ p<n}} X[p,q]| \\ &= |[q] \longmapsto L_n X[-,q]| \end{aligned}$$

Moreover, the map $L_n Y \to Y[n]$ is identified with the realization of the map

$$L_n X[-,-] \longrightarrow X[n,-].$$

But the map

$$L_n X[-,q] \longrightarrow X[n,q]$$

is equal to the inclusion of the degenerate n-simplices in $X[-,q]$. In particular, it is injective. By the lemma, $L_n Y \to Y[n]$ is a Serre cofibration as desired. //

The proof of Theorem 2 is now complete. The purpose of our long digression into homotopy theory was to prove this theorem.

<u>Proof of Thm. 1</u> We first consider the case

$$X[-,-] = \Delta[r][-] \times \Delta[s][-].$$

We have the following canonical homeomorphisms

$$|[m] \mapsto |[n] \mapsto \Delta[r][m] \times \Delta[s][n]||$$

$$\cong |[m] \mapsto \Delta[r][m] \times |[n] \mapsto \Delta[s][n]||$$

$$\cong |[m] \mapsto \Delta[r][m]| \times |[n] \mapsto \Delta[s][n]|$$

$$\cong |[k] \mapsto \Delta[r][k] \times \Delta[s][k]|.$$

Here the third homeomorphism is the combinatorial lemma which we proved earlier on p. 87. The first and second homeomorphism are both cases of the following: let $Y[-]$ be a simplicial k-space and Z a fixed k-space. Then

$$|Y[-]| \times Z \cong |Y[-] \times Z|.$$

Indeed, this follows from the prop. on p. 73 and the canonical homeomorphism between the adjoints:

$$\underline{\mathrm{Sin}}(\underline{\mathrm{Hom}}_{\mathcal{K}}(Z,-))[-]$$

$$\cong \underline{\mathrm{Hom}}_{\mathcal{K}}(Z, \underline{\mathrm{Sin}}(-)[-]).$$

Next, let $X[-,-]$ be a general

bi-simplicial set. Then, as in the prop. on p. 81, we have a canonical isomorphism of bi-simplicial sets

$$\operatorname*{colim}_{\Delta[r][-]\times\Delta[s][-]\to X[-,-]} \Delta[r][-]\times\Delta[s][-] \xrightarrow{\cong} X[-,-].$$

It follows that we have canonical and natural isomorphisms

$$\big|\,[m]\mapsto\big|\,[n]\mapsto X[m,n]\,\big|\,\big|$$

$$\xleftarrow{\cong}\big|\,[m]\mapsto\big|\,[n]\mapsto \operatorname*{colim}_{\Delta[r][-]\times\Delta[s][-]\to X[-,-]} \Delta[r][m]\times\Delta[s][n]\,\big|\,\big|$$

$$\xleftarrow{\cong} \operatorname*{colim}_{\Delta[r][-]\times\Delta[s][-]\to X[-,-]} \big|\,[m]\mapsto\big|\,[n]\mapsto\Delta[r][m]\times\Delta[s][n]\,\big|\,\big|$$

$$\cong \operatorname*{colim}_{\Delta[r][-]\times\Delta[s][-]\to X[-,-]} \big|\,[k]\mapsto\Delta[r][k]\times\Delta[s][k]\,\big|$$

$$\xrightarrow{\cong}\big|\,[k]\mapsto \operatorname*{colim}_{\Delta[r][-]\times\Delta[s][-]\to X[-,-]} \Delta[r][k]\times\Delta[s][k]\,\big|$$

$$\xrightarrow{\cong}\big|\,[k]\mapsto X[k,k]\,\big|.$$

Here the first and last homeomorphisms are induced from the isomorphism of bi-simplicial set on top of p. 204. The second and fourth homeomorphisms are induced by the canonical maps and are homeomorphisms because geometric realization has a right adjoint. In the case of the fourth homeomorphism, we also use that the functor

$$\mathrm{Sets}^{\Delta^{op}\times\Delta^{op}} \xrightarrow{\Delta_*} \mathrm{Sets}^{\Delta^{op}}$$

that takes a bi-simplicial set to the associated diagonal simplicial set has a right adjoint given by the right Kan extension

$$\mathrm{Sets}^{\Delta^{op}} \xrightarrow{\Delta^!} \mathrm{Sets}^{\Delta^{op}\times\Delta^{op}}.$$

See e.g. MacLane: "Categories for the working mathematician," Chap. X. Finally, the third homeomorphism is induced by the homeomorphisms

$$|[m] \mapsto |[n] \mapsto \Delta[r][m]\times\Delta[s][n]||$$

$$\cong |[k] \mapsto \Delta[r][k]\times\Delta[s][k]|$$

which are natural with respect to maps in $\Delta \times \Delta$. This proves the first homeomorphism in the statement of Thm. 1. The second homeomorphism is proved similarly. //

Finally, we mention that an argument similar to the proof of Thm. 2 proves the following; see [Hovey, Lemma 5.2.6].

<u>Lemma</u> (Gluing lemma) Let $\mathcal{C}$ be a model category, and let

$$\begin{array}{ccccc} X & \longleftarrow & A & \longrightarrow & B \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ X' & \longleftarrow & A' & \longrightarrow & B' \end{array}$$

be a diagram of cofibrant objects such that the left-hand horizontal maps are cofibrations and the vertical maps weak equivalences. Then the induced map of push-outs

$$X \amalg_A B \longrightarrow X' \amalg_{A'} B'$$

is a weak equivalence. //