We now show that the geometric realization functor

$$|-| : \mathrm{Sets}^{\Delta^{op}} \longrightarrow \mathcal{T}$$

from the category of simplicial sets to the category of topological spaces has a right adjoint functor

$$\mathrm{Sin}(-)[-] : \mathcal{T} \longrightarrow \mathrm{Sets}^{\Delta^{op}}.$$

To define it, we recall the functor

$$\Delta[-] : \Delta \longrightarrow \mathcal{T}$$

that to $[n] \in \mathrm{ob}\,\Delta$ associates the topological space

$$\Delta[n] = \mathrm{conv}([n]) \subset \mathbb{R}^{[n]}.$$

Then we define

$$\mathrm{Sin}(Y)[-] = \mathrm{Hom}_{\mathcal{T}}(\Delta[-], Y).$$

So the set $\mathrm{Sin}(Y)[n]$ consists of all continuous maps from the standard simplex $\Delta[n]$ to Y.

Prop There is an adjunction

$$(|-|, \mathrm{Sin}(-)[-], a)$$

from the category $\mathcal{T}$ of topological spaces to the category $\mathrm{Sets}^{\Delta^{op}}$ of simplicial sets where

$$\mathrm{Hom}_{\mathcal{T}}(|X[-]|, Y) \xrightarrow{a_{(X[-], Y)}} \mathrm{Hom}_{\mathrm{Sets}^{\Delta^{op}}}(X[-], \mathrm{Sin}(Y)[-])$$

is defined by

$$a_{(X[-], Y)}(f)(x)(z) = f\begin{pmatrix} \text{class of} \\ (x, z) \end{pmatrix}$$

Proof The inverse map is defined as follows. Let $x \in X[n]$ and $z \in \Delta[n]$. Then

$$a^{-1}_{(X[-], Y)}(g)\begin{pmatrix} \text{class of} \\ (x, z) \end{pmatrix}$$

$$= g[n](x)(z) .$$

It is a well-defined map, since $g[-]$ is a map of simplicial sets. //

We define the <u>simplicial standard</u> <u>n-simplex</u> to be the simplicial set

$$\Delta[n][-] = \mathrm{Hom}_{\Delta}([-],[n]).$$

It is equal to the nerve of the category $[n]$. The following lemma is (a special case of) the Yoneda lemma.

<u>Lemma</u> (Yoneda) Let $X[-]$ be a simplicial set. Then the map

$$\mathrm{Hom}_{\mathrm{Sets}^{\Delta^{op}}}(\Delta[n][-], X[-]) \xrightarrow{y} X[n]$$

defined by

$$y(f[-]) = f[n](\mathrm{id}_{[n]})$$

is a bijection.

<u>Proof</u> Let $\theta \in \Delta[n][m]$. The diagram

$$\begin{array}{ccc} \Delta[n][n] & \xrightarrow{f[n]} & X[n] \\ \downarrow \theta^* & & \downarrow \theta^* \\ \Delta[n][m] & \xrightarrow{f[m]} & X[m] \end{array}$$

commutes because $f[-]$ is a simplicial map. Hence,

$$f[m](\theta) = f[m](\theta^*(id_{[n]}))$$

$$= \theta^*(f[n](id_{[n]})) = \theta^*(y(f[-]))$$

which shows that the map $f[-]$ is uniquely determined by the element $y(f[-])$. //

If $x \in X[n]$, we write

$$\sigma_x[-] : \Delta[n][-] \longrightarrow X[-]$$

for the unique map of simplicial sets such that

$$\sigma_x[n](id_{[n]}) = x.$$

Composition with $\theta : [n] \to [n']$ defines a map of simplicial sets

$$\theta[-] : \Delta[n][-] \longrightarrow \Delta[n'][-]$$

and every map between these simplicial sets is of this form.

Let $X[-]$ be a simplicial set. We define the category $\mathbb{\Delta}/X[-]$ of simplices as follows: The objects are all maps of simplicial sets

$$\sigma_x[-] : \Delta[n][-] \longrightarrow X[-]$$

for some $n \geq 0$. A morphism from $\sigma_x[-] : \Delta[n][-] \to X[-]$ to $\sigma_{x'}[-] : \Delta[n'][-] \to X[-]$ is a map $\theta : [n] \to [n']$ such that the following diagram commutes

$$\begin{tikzcd} \Delta[n][-] \arrow[dr, "{\sigma_x[-]}"] \arrow[dd, "{\theta[-]}"'] & \\ & X[-] \\ \Delta[n'][-] \arrow[ur, "{\sigma_{x'}[-]}"'] & \end{tikzcd}$$

There is an obvious functor

$$\Delta[-][-] : \mathbb{\Delta}/X[-] \longrightarrow \mathrm{Sets}^{\mathbb{\Delta}^{op}}$$

defined by

$$\Delta[-][-]\left(\sigma_x[-] : \Delta[n][-] \longrightarrow X[-]\right)$$

$$= \Delta[n][-]$$

$$\Delta[-][-]\left(\begin{tikzcd} \Delta[n][-] \arrow[d, "\theta[-]"] \arrow[dr, "\sigma_x[-]"] & \\ \Delta[n'][-] \arrow[r, "\sigma_{x'}[-]"'] & X[-] \end{tikzcd}\right)$$

$$= \theta[-].$$

Prop The maps $\sigma_x[-]$ define a map

$$\operatorname*{colim}_{\Delta/X[-]} \Delta[-][-] \longrightarrow X[-]$$

and this map is an isomorphism of simplicial sets.

Proof It is clear that the maps $\sigma_x[-]$ define a map as stated. To show that it is an isomorphism, we show that $X[-]$ satisfies the defining properties (i') and (ii') of the colimit. So let $Y[-]$ be a simplicial set and suppose that, for every object

$$\sigma_x[-]: \Delta[n][-] \longrightarrow X[-]$$

of $\Delta / X[-]$, we are given a map of simplicial sets

$$\alpha_x[-] : \Delta[n][-] \longrightarrow Y[-]$$

such that, for every morphism

$$\begin{tikzcd}
\Delta[n][-] \arrow[dr, "\sigma_x[-]"] \arrow[dd, "\theta[-]"'] & \\
& X[-] \\
\Delta[n'][-] \arrow[ur, "\sigma_{x'}[-]"'] &
\end{tikzcd}$$

of $\Delta / X[-]$, the diagram

$$\begin{tikzcd}
\Delta[n][-] \arrow[dr, "\alpha_x[-]"] \arrow[dd, "\theta[-]"'] & \\
& Y[-] \\
\Delta[n'][-] \arrow[ur, "\alpha_{x'}[-]"'] &
\end{tikzcd}$$

commutes. We must show that there exists a unique map of simplicial sets

$$k[-] : X[-] \longrightarrow Y[-]$$

such that, for every $\sigma_x[-] : \Delta[n][-] \to X[-]$,

$$k[-] \circ in_{\sigma_x}[-] = \alpha_x[-].$$

This equation requires us to define the map $k[n] : X[n] \to Y[n]$ by

$$k[n](x) = \alpha_x[n](id_{[n]}).$$

We must show that these maps form a map of simplicial sets, i.e. that, for every $\theta : [n] \to [n']$, the following diagram commutes

$$\begin{tikzcd} X[n] \arrow[r, "k[n]"] & Y[n] \\ X[n'] \arrow[u, "\theta^*"] \arrow[r, "k[n']"] & Y[n'] \arrow[u, "\theta^*"'] \end{tikzcd}.$$

So let $x' \in X[n']$ and $x = \theta^*(x') \in X[n]$. Then

$$\theta^*(k[n'](x')) := \theta^*(\alpha_{x'}[n'](id_{[n']}))$$

$$= \alpha_{x'}[n](\theta^*(id_{[n']})) \qquad (\alpha_{x'}[-] \text{ simpl.})$$

$$= \alpha_{x'}[n](id_{[n']} \circ \theta)$$

$$= \alpha_{x'}[n](\theta \circ id_{[n]})$$

$$= (\alpha_{x'}[n] \circ \theta[n])(id_{[n]})$$

$$= \alpha_x[n](id_{[n]})$$

$$= \alpha_{\theta^*(x')}[n](id_{[n]})$$

$$=: k[n](\theta^*(x')).$$

This completes the proof. //

<u>Cor</u> The maps $|\sigma_x[-]|$ give rise to a homeomorphism

$$\operatorname*{colim}_{\Delta/X[-]} |\Delta[n][-]| \longrightarrow |X[-]|.$$

<u>Proof</u> This is the composition

$$\operatorname*{colim}_{\Delta/X[-]} |\Delta[n][-]| \longrightarrow |\operatorname*{colim}_{\Delta/X[-]} \Delta[n][-]|$$

$$\longrightarrow |X[-]|.$$

The first map is a homeomorphism since $|-|$ has a right adjoint. //

There is also a functor

$$\Delta[-] : \underline{\Delta} / X[-] \longrightarrow \mathcal{T}$$

defined by

$$\Delta[-]\left(\sigma_x[-] : \Delta[n][-] \longrightarrow X[-]\right)$$

$$= \Delta[n]$$

$$\Delta[-]\left(\begin{tikzcd} \Delta[n][-] \arrow[d, "\theta[-]"] \arrow[dr, "\sigma_x[-]"] & \\ \Delta[n'][-] \arrow[r, "\sigma_{x'}[-]"'] & X[-] \end{tikzcd}\right)$$

$$= \theta_* .$$

We define $\sigma_x : \Delta[n] \longrightarrow |X[-]|$ to be the composite map

$$\Delta[n] \xrightarrow{\xi} |\Delta[n][-]| \xrightarrow{|\sigma_x[-]|} |X[-]|$$

$$z \longmapsto \text{class of } (\mathrm{id}_{[n]}, z)$$

Then we have:

Prop The maps σ_x define a map

$$\operatorname*{colim}_{\mathbb{\Delta}/X[-]} \Delta[-] \longrightarrow |X[-]|$$

and this map is a homeomorphism.

Proof This is really a reformulation of the definition of $|X[-]|$ that we gave earlier. //

Cor The map

$$\xi : \Delta[n] \longrightarrow |\Delta[n][-]|$$

is a homeomorphism.

Proof The object

$$\sigma_{\mathrm{id}_{[n]}}[-] : \Delta[n][-] \longrightarrow \Delta[n][-]$$

is an initial object of $\mathbb{\Delta}/\Delta[n][-]$. Therefore,

$$\mathrm{in}_{\sigma_{\mathrm{id}_{[n]}}[-]} : \Delta[n] \longrightarrow \operatorname*{colim}_{\mathbb{\Delta}/\Delta[n][-]} \Delta[-]$$

is a homeomorphism. //