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Let $Y[-]$ be a simplicial set. We say that $y \in Y[k]$ is <u>degenerate</u> if

$$y \in \operatorname{im}(\theta^* : Y[l] \to Y[k])$$

for some $\theta : [k] \to [l]$ with $n < m$. We say that $y \in Y[k]$ is <u>non-degenerate</u> if y is not degenerate. In particular, every $y \in Y[0]$ is non-degenerate.

<u>Lemma</u> (Eilenberg-Zilber) Let $Y[-]$ be a simplicial set and let $y \in Y[k]$. Then there exists a unique map $\theta : [k] \to [l]$ and a unique non-degenerate $z \in Y[l]$ such that

$$y = \theta^*(z).$$

Moreover, the map $\theta : [k] \to [l]$ is surjective.

<u>Proof</u> If $\theta : [k] \to [l]$ is not surjective then any $z \in Y[l]$ with $y = \theta^*(z)$ is necessarily degenerate. So θ is surjective. We first prove uniqueness. So suppose

$$y = \theta^*(z), \quad \theta : [k] \to [l], \quad z \in Y[l]$$

$$y = \theta'^*(z'), \quad \theta' : [k] \to [l'], \quad z' \in Y[l']$$

with z, z' non-degenerate. We must show that $l = l'$, $z = z'$, and $\theta = \theta'$. Since θ and θ' are surjective, there exist maps $\sigma : [l] \to [k]$ and $\sigma' : [l'] \to [k]$ such that $\theta \circ \sigma = \mathrm{id}_{[l]}$ and $\theta' \circ \sigma' = \mathrm{id}_{[l']}$. We now calculate:

$$(\theta' \circ \sigma)^*(z') = \sigma^*(\theta'^*(z')) = \sigma^*(y)$$

$$= \sigma^*(\theta^*(z)) = (\theta \circ \sigma)^*(z) = z$$

$$(\theta \circ \sigma')^*(z) = \sigma'^*(\theta^*(z)) = \sigma'^*(y)$$

$$= \sigma'^*(\theta'^*(z')) = (\theta' \circ \sigma')^*(z') = z'.$$

Since both z and z' are non-degenerate, we conclude that $l = l'$, $z = z'$, and $\theta' \circ \sigma = \theta \circ \sigma' = \mathrm{id}_{[l]}$. But given $i \in [k]$, we can find $\sigma : [l] \to [k]$ with $\theta \circ \sigma = \mathrm{id}_{[l]}$ and $(\sigma \circ \theta)(i) = i$. Therefore,

$$\theta'(i) = \theta'(\sigma(\theta(i))) = \theta'(i).$$

So $\theta = \theta'$ which shows that the pair (θ, z) is uniquely determined by y.

Finally, to prove uniqueness, let $l \leq k$ be minimal with the property that there exists $\theta : [k] \to [l]$ and $z \in Y[l]$ with $y = \theta^*(z)$. Since l is minimal, z is necessarily non-degenerate. //

We consider the full subcategory

$$\Delta_{\leq n} \xhookrightarrow{\iota_n} \Delta$$

with objects all $[m]$ with $m \leq n$. The inclusion functor ι_n gives rise to a (forgetful) functor

$$\mathrm{Sets}^{\Delta_{\leq n}^{op}} \xleftarrow{\iota_n^*} \mathrm{Sets}^{\Delta^{op}}$$

$$Y[-] \circ \iota_n \longleftarrow\!\shortmid \; Y[-]$$

We define a functor

$$\mathrm{Sets}^{\Delta_{\leq n}^{op}} \xrightarrow{F_n} \mathrm{Sets}^{\Delta^{op}}$$

by

$$F_n X[-] = \operatorname*{colim}_{\Delta_{\leq n} / X[-]} \mathrm{Hom}_{\Delta}(-, c_n(-))$$

where

$$\mathrm{Hom}_{\Delta}(-, c_n(-))\left(\mathrm{Hom}_{\Delta_{\leq n}}(-, [m]) \xrightarrow{6_x} X[-]\right)$$

$$= \mathrm{Hom}_{\Delta}(-, c_n([m]))$$

$$= \mathrm{Hom}_{\Delta}(-, [m]) = \Delta[m][-]$$

and

$$\mathrm{Hom}_{\Delta}(-, c_n(-))\left(\begin{tikzcd}[m] \arrow[d, "\theta"] & \mathrm{Hom}_{\Delta_{\leq n}}(-, [m]) \arrow[d, "\theta_*"] \arrow[dr, "6_x"] & \\ {[m']} & \mathrm{Hom}_{\Delta_{\leq n}}(-, [m']) \arrow[r, "6_{x'}"'] & X[-]\end{tikzcd}\right)$$

$$= \left(\mathrm{Hom}_{\Delta}(-, c_n([m])) \xrightarrow{\theta_*} \mathrm{Hom}_{\Delta}(-, c_n([m']))\right)$$

$$= \left(\Delta[m][-] \xrightarrow{\theta_*} \Delta[m'][-]\right).$$

The proof of the following result is an exercise in definitions.

<u>Lemma</u> The triple (F_n, ι_n^*, a) where

$$\mathrm{Hom}_{\mathrm{Sets}^{\Delta^{op}}}(F_n X[-], Y[-])$$
$$\xrightarrow{a} \mathrm{Hom}_{\mathrm{Sets}^{\Delta^{op}_{\leq n}}}(X[-], \iota_n^* Y[-])$$

is defined by $(x \in X[m])$

$$a(f)(x) = (f \circ in_{\sigma_x})(id_{[m]})$$

is an adjunction. //

<u>Lemma</u> The unit map

$$X[-] \xrightarrow{\eta} \iota_n^* F_n X[-]$$

is an isomorphism.

<u>Pf</u> We recall the canonical isomorphism

$$\operatorname*{colim}_{\Delta_{\leq n}/X[-]} \mathrm{Hom}_{\Delta_{\leq n}}(-,-) \xrightarrow[\sim]{y} X[-]$$

Since $\Delta_{\leq n} \subset \Delta$ is a full subcategory,

$$\mathrm{Hom}_{\Delta_{\leq n}}(-,-) = \mathrm{Hom}_{\Delta}(\iota_n(-), \iota_n(-))$$

so we have

$$\operatorname*{colim}_{\Delta_{\leq n}/X[-]} \mathrm{Hom}_{\Delta_{\leq n}}(-,-)$$

$$= \operatorname*{colim}_{\Delta_{\leq n}/X[-]} \mathrm{Hom}_{\Delta}(\iota_n(-), \iota_n(-))$$

$$= \iota_n^* F_n X[-].$$

Hence, it suffices to show that the composite map

$$X[-] \xrightarrow{\eta} \iota_n^* F_n X[-] \xrightarrow[\sim]{y} X[-]$$

is equal to the identity map. Now, given $x \in X[m]$, the following diagram commutes by the definition of η and y.

$$\begin{tikzcd}
X[-] \arrow[r, "\eta"] & \iota_n^* F_n X[-] \arrow[r, "y"] & X[-] \\
\mathrm{Hom}_{\Delta_{\leq n}}(-,[m]) \arrow[u, "\sigma_x"] \arrow[r, equal] & \mathrm{Hom}_{\Delta}(\iota_n(-), \iota_n([m])) \arrow[u, "in_{\sigma_x \iota_n}"] \arrow[r, equal] & \mathrm{Hom}_{\Delta_{\leq n}}(-,[m]) \arrow[u, "\sigma_x"]
\end{tikzcd}$$

The lemma follows. //

<u>Thm</u> The counit map

$$F_n i_n^* Y[k] \xrightarrow{\varepsilon} Y[k]$$

is an isomorphism, for $k \leq n$, and a monomorphism, for all k.

<u>Pf</u> We have

$$\begin{tikzcd} i_n^* Y[-] \arrow[r, "\eta i_n^*", "\sim"'] \arrow[rr, bend right, "\mathrm{id}"'] & i_n^* F_n i_n^* Y[-] \arrow[r, "i_n^* \varepsilon"] & i_n^* Y[-] \end{tikzcd}$$

whichs shows that the counit map is an isomorphism for $k \leq n$. So let $y, y' \in F_n i_n^* Y[k]$ with $k > n$ and suppose $\varepsilon(y) = \varepsilon(y') \in Y[k]$. By the Eilenberg-Zilber lemma, there are surjective maps $\theta : [k] \to [\ell]$ and $\theta' : [k] \to [\ell']$ and non-degenerate simplices $z \in F_n i_n^* Y[\ell]$ and $z' \in F_n i_n^* Y[\ell']$ such that

$$y = \theta^*(z) \quad , \quad y' = \theta'^*(z').$$

Now, all simplices in $F_n i_n^* Y[p]$ with $p > n$ are degenerate, so $\ell, \ell' \leq n$. Since

$$F_n \iota_n^* Y[m] \xrightarrow{\varepsilon} Y[m]$$

is an isomorphism for $m \leq n$, we conclude that $\varepsilon(z) \in Y[\ell]$ and $\varepsilon(z') \in Y[\ell']$ are non-degenerate. But

$$\theta^*(\varepsilon(z)) = \varepsilon(x) = \varepsilon(x') = \theta'^*(\varepsilon(z'))$$

so by the uniqueness statement in the Eilenberg-Zilber lemma, we conclude that $\ell = \ell'$, $z = z'$, and $\theta = \theta'$. Hence,

$$x = \theta^*(z) = \theta'^*(z') = x'$$

so ε is injective as stated. //

Def Let $Y[-]$ be a simplicial set and $n \geq 0$ an integer. The n-<u>skeleton</u>

$$\mathrm{sk}_n Y[-] \subset Y[-]$$

is the image of the monomorphism

$$F_n \iota_n^* Y[-] \xrightarrow{\varepsilon} Y[-].$$

We define

$$\partial\Delta[n][-] = \mathrm{sk}_{n-1}\Delta[n][-] \subset \Delta[n][-]$$

and call it the <u>boundary</u> of $\Delta[n][-]$.

<u>Prop</u> Let $Y[-]$ be a simplicial set, $n \geq 0$ an integer, and $Y[n]' \subset Y[n]$ the subset of non-degenerate n-simplices. Then the diagram

$$\begin{tikzcd} \coprod_{y \in Y[n]'} \partial\Delta[n][-] \arrow[r, "\Sigma \sigma_y"] \arrow[d, hook] & \mathrm{sk}_{n-1} Y[-] \arrow[d, hook] \\ \coprod_{y \in Y[n]'} \Delta[n][-] \arrow[r, "\Sigma \sigma_y"] & \mathrm{sk}_n Y[-] \end{tikzcd}$$

is a push-out.

<u>Pf</u> We claim that for all $m \leq n$,

$$\begin{tikzcd} \coprod_{y \in Y[n]'} \partial\Delta[n][m] \arrow[r, "\Sigma \sigma_y"] \arrow[d, hook] & \mathrm{sk}_{n-1} Y[m] \arrow[d, hook] \\ \coprod_{y \in Y[n]'} \Delta[n][m] \arrow[r, "\Sigma \sigma_y"] & \mathrm{sk}_n Y[m] \end{tikzcd}$$

is a push-out diagram of sets. Indeed, for $m < n$, the vertical maps are isomorphisms, so the diagram is a push-out in this case. For $m = n$,

$$sk_n Y[n] \setminus sk_{n-1} Y[n] = Y[n]'$$

and

$$\coprod_{y \in Y[n]'} \Delta[n][n] \setminus \coprod_{y \in Y[n]'} \partial\Delta[n][n] \xrightarrow{\Sigma \sigma_y} sk_n Y[n] \setminus sk_{n-1} Y[n]$$

is a bijection since

$$\Delta[n][n] \setminus \partial\Delta[n][n] = \{ id_{[n]} \}.$$

It follows that the diagram

$$\begin{tikzcd}
\iota_n^* \Big(\coprod_{y \in Y[n]'} \partial\Delta[n][-] \Big) \arrow[r, "\Sigma \sigma_y"] \arrow[d, hook] & \iota_n^* sk_{n-1} Y[-] \arrow[d, hook] \\
\iota_n^* \Big(\coprod_{y \in Y[n]'} \Delta[n][-] \Big) \arrow[r, "\Sigma \sigma_y"] & \iota_n^* sk_n Y[-]
\end{tikzcd}$$

is a push-out in $\mathrm{Sets}^{\Delta_{\leq n}^{op}}$. But the functor F_n preserves colimits, so

$$\begin{tikzcd}
F_n \iota_n^* \Big(\coprod_{y \in Y[n]'} \partial\Delta[n][-] \Big) \arrow[r, "\Sigma\sigma_y"] \arrow[d, hook] & F_n \iota_n^* \, \mathrm{sk}_{n-1} Y[-] \arrow[d, hook] \\
F_n \iota_n^* \Big(\coprod_{y \in Y[n]'} \Delta[n][-] \Big) \arrow[r, "\Sigma\sigma_y"] & F_n \iota_n^* \, \mathrm{sk}_n Y[-]
\end{tikzcd}$$

is a push-out diagram of simplicial sets. Finally, the counit map ε gives an isomorphism from this diagram onto the diagram in the statement. //

We remark that the canonical map

$$\operatorname*{colim}_n \mathrm{sk}_n Y[-] \longrightarrow Y[-]$$

is an isomorphism. Indeed, given $k \geq 0$, the canonical map

$$\mathrm{sk}_n Y[k] \longrightarrow Y[k]$$

is an isomorphism for all $n \geq k$.