We proceed to better understand the boundary

$$\partial\Delta[n][-] = sk_{n-1}\Delta[n][-] \subset \Delta[n][-].$$

For every $0 \leq i \leq n$, we define

$$d^i : [n-1] \longrightarrow [n]$$

to be the unique order preserving injective map such that

$$\operatorname{im}(d^i) = [n] \setminus \{i\} \subset [n]$$

and note that for all $0 \leq i < j \leq n$,

$$d^j d^i = d^i d^{j-1} : [n-2] \longrightarrow [n].$$

We call the induced map

$$d^i = d^i_* : \Delta[n-1][-] \longrightarrow \Delta[n][-]$$

the i'th *face* map. We show that the boundary $\partial\Delta[n][-]$ admits the following description as a coequalizer.

<u>Lemma</u> The diagram

$$\coprod_{0\le i<j\le n} \Delta[n-2][-] \underset{g}{\overset{f}{\rightrightarrows}} \coprod_{0\le k\le n} \Delta[n-1][-] \xrightarrow{h} \partial\Delta[n][-]$$

where f, g, and h are defined by

$$f \circ in_{ij} = in_i \circ d^{j-1}_*$$

$$g \circ in_{ij} = in_j \circ d^i_*$$

$$h \circ in_k = d^k_*$$

is a coequalizer.

<u>Pf</u> We show that for all $m \ge 0$, the diagr.

$$\coprod_{0\le i<j\le n} \Delta[n-2][m] \underset{g}{\overset{f}{\rightrightarrows}} \coprod_{0\le k\le n} \Delta[n-1][m] \xrightarrow{h} \partial\Delta[n][m]$$

is a coequalizer. If $0 \le i < j \le n$, then $d^j d^i = d^i d^{j-1}$ so $hf = hg$. The subset

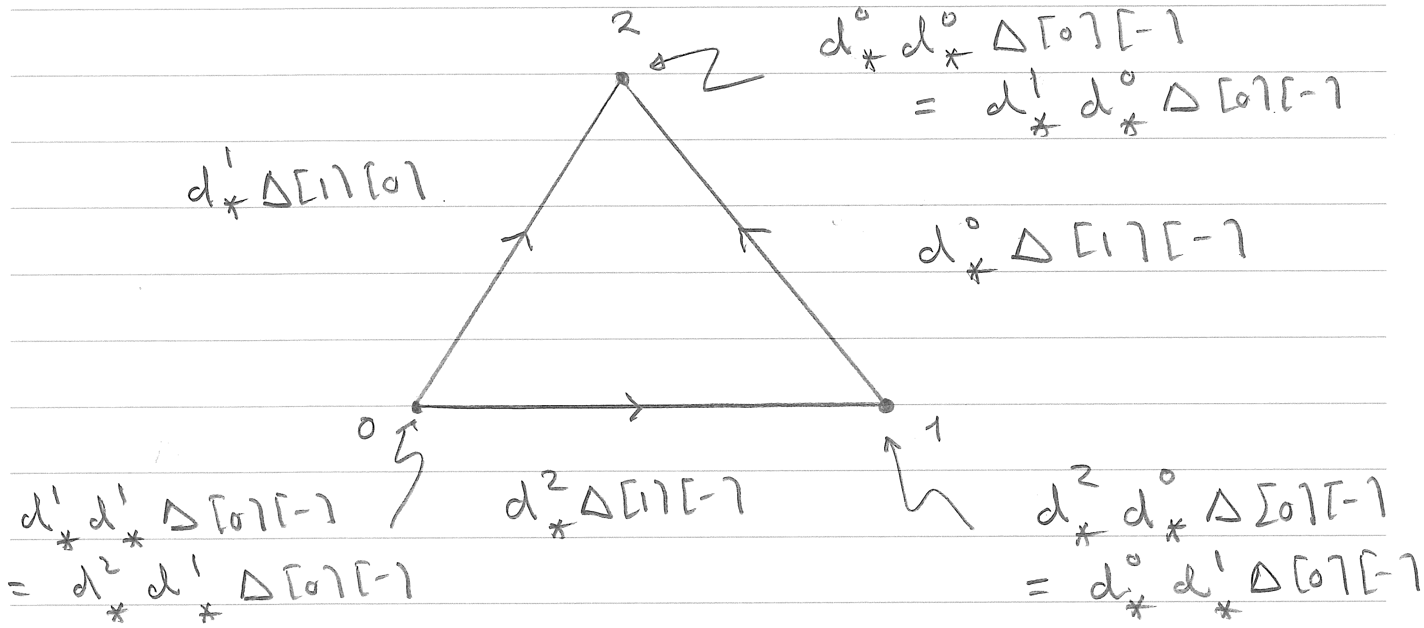
$$\partial\Delta[n][m] \subset \mathrm{Hom}_\Delta([m],[n])$$

consists of all maps that are not surjective. But if the image of the map $\theta : [m] \to [n]$ does not contain $i \in [n]$, then $\theta = d^i \circ \theta'$ with $\theta' : [m] \to [n-1]$.

So the map h is surjective. Finally, if the image of $\theta : [m] \to [n]$ does not contain $\{i, j\}$ with $0 \le i < j \le n$, then

$$\theta = d^j \circ d^i \circ \theta'' = d^i \circ d^{j-1} \circ \theta''$$

with $\theta'' : [m] \to [n-2]$. So the diagram is a coequalizer as stated. //



We define the <u>boundary</u> of the topological standard n-simplex $\Delta[n]$ to be the subspace

$$\partial\Delta[n] = \Delta[n] \setminus \mathrm{Int}(\Delta[n]) \subset \Delta[n]$$

of points that are not interior. Then, similarly, we have:

<u>Lemma</u> The diagram

$$\coprod_{0\le i<j\le n} \Delta[n-2] \overset{f}{\underset{g}{\rightrightarrows}} \coprod_{0\le k\le n} \Delta[n-1] \xrightarrow{h} \partial\Delta[n]$$

where

$$f \circ in_{i,j} = in_i \circ d_*^{j-1}$$

$$g \circ in_{i,j} = in_j \circ d_*^{i}$$

$$h \circ in_k = d_*^k$$

is a coequalizer.

<u>Cor</u> In the diagram of canonical maps

$$\begin{array}{ccc} |\partial\Delta[n][-]| & \longrightarrow & \partial\Delta[n] \\ \big\downarrow & & \big\downarrow \\ |\Delta[n][-]| & \longrightarrow & \Delta[n] \end{array}$$

the horizontal maps are homeomorphisms.

<u>Pf</u> For every $k \geq 0$, we have the canonical homeomorphism

$$|\Delta[k][-]| \xrightarrow{\sim} \Delta[k].$$

Moreover, the following diagram, where the vertical maps are the canonical homeomorphisms, commutes.

$$\begin{tikzcd}
\coprod_{0\le i<j\le n} |\Delta[n-2][-]| \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "\sim"] & \coprod_{0\le k\le n} |\Delta[n-1][-]| \arrow[r, "h"] \arrow[d, "\sim"] & |\Delta[n][-]| \arrow[d, "\sim"] \\
\coprod_{0\le i<j\le n} \Delta[n-2] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \coprod_{0\le k\le n} \Delta[n-1] \arrow[r, "h"] & \Delta[n]
\end{tikzcd}$$

Since geometric realization preserves colimits, the two lemmas show that the right-hand vertical map induces a homeomorphism

$$|\partial\Delta[n][-]| \xrightarrow{\sim} \partial\Delta[n].$$

This completes the proof. //

Let $Y[-]$ be a simplicial set. The inclusion $\mathrm{sk}_{n-1}Y[-] \hookrightarrow \mathrm{sk}_n Y[-]$ gives rise to a continuous map

$$|\mathrm{sk}_{n-1}Y[-]| \to |\mathrm{sk}_n Y[-]|.$$

The following result shows that

this map is the inclusion of a subspace and that $|sk_n Y[-]|$ is obtained from $|sk_{n-1} Y[-]|$ by attaching an n-simplex $\Delta[n]$ along the boundary $\partial\Delta[n]$ for every non-degenerate $y \in Y[n]$.

<u>Thm</u> Let $Y[-]$ be a simplicial set, let $n \geq 0$ be an integer, and let $Y[n]' \subset Y[n]$ be the subset of non-degenerate n-simplices. Then

$$\begin{tikzcd} \coprod_{y \in Y[n]'} \partial\Delta[n] \arrow[r, "\Sigma\sigma_y"] \arrow[d, hook] & {|sk_{n-1} Y[-]|} \arrow[d, hook] \\ \coprod_{y \in Y[n]'} \Delta[n] \arrow[r, "\Sigma\sigma_y"] & {|sk_n Y[-]|} \end{tikzcd}$$

is a push-out diagram of topological spaces.

<u>Pf</u> The push-out diagram of simplicial sets on page 9 gives after realization a push-out diagram of spaces. The previous corollary identifies the left-hand terms as stated. //

Since geometric realization preserves colimits, we have

$$\begin{aligned} &\operatorname{colim}_{n} |\mathrm{sk}_n Y[-]| \\ &\xrightarrow{\sim} |\operatorname{colim}_{n} \mathrm{sk}_n Y[-]| \\ &\xrightarrow{\sim} |Y[-]|. \end{aligned}$$

Cor Let $\xi \in |Y[-]|$. Then there exists a unique non-degenerate simplex $y \in Y[n]'$ and a unique interior point $z \in \Delta[n]$ such that

$$\xi = \text{class of } (y, z).$$

Pf Let $n \geq 0$ be the unique integer such that

$$\xi \in |\mathrm{sk}_n Y[-]| \setminus |\mathrm{sk}_{n-1} Y[-]|.$$

The theorem shows that the map

$$\coprod_{y \in Y[n]'} \operatorname{Int}(\Delta[n]) \xrightarrow{\Sigma \sigma_y} |\mathrm{sk}_n Y[-]| \setminus |\mathrm{sk}_{n-1} Y[-]|$$

is a homeomorphism. We let (y, z) be the unique pair with $\sigma_y(z) = \xi$. Then the corollary follows. //

If Y is a set, then the functor

$$- \times Y : \mathrm{Sets} \longrightarrow \mathrm{Sets}$$

has the right adjoint functor

$$\mathrm{Hom}_{\mathrm{Sets}}(Y, -) : \mathrm{Sets} \longrightarrow \mathrm{Sets}.$$

The adjunction bijection

$$\mathrm{Hom}_{\mathrm{Sets}}(X \times Y, Z) \xrightarrow{a} \mathrm{Hom}_{\mathrm{Sets}}(X, \mathrm{Hom}_{\mathrm{Sets}}(Y, Z))$$

is defined by

$$a(f)(x)(y) = f(x, y).$$

It follows that for every diagram of sets $X : I \to \mathrm{Sets}$, the canonical map

$$\operatorname*{colim}_i (X_i \times Y) \longrightarrow (\operatorname*{colim}_i X_i) \times Y$$

is a bijection. If Y is a topological space, the functor

$$- \times Y : \mathcal{T} \longrightarrow \mathcal{T}$$

in general does not have a right adjoint. We will instead consider a full subcategory $\mathcal{K} \subset \mathcal{T}$ where, for $Y \in \mathcal{K}$, the functor

$$- \times Y : \mathcal{K} \longrightarrow \mathcal{K}$$

does have a right adjoint. Let X be a topological space. We define the subset $U \subset X$ to be <u>compactly open</u> if, for every continuous map $f : K \to X$ with K compact and Hausdorff, $f^{-1}(U) \subset K$ is open. So open subsets $V \subset X$ are compactly open, but compactly open subsets $U \subset X$ are not necessarily open. We say that X is a <u>k-space</u> if every compactly open subset $U \subset X$ is an open subset and define $\mathcal{K}$ to be category whose objects are all k-spaces and whose morphisms are all continuous maps. The inclusion functor

$$i : \mathcal{K} \longrightarrow \mathcal{T}$$

has a right adjoint functor

$$k : \mathcal{T} \longrightarrow \mathcal{K}$$

that takes the topological space X to the k-space $k(X)$ defined to be the set X with the (new) topology where $U \subset k(X)$ is open if and only if $U \subset X$ is compactly open. It is clear that $k \circ i$ is the identity functor of $\mathcal{K}$. Since the functor $i : \mathcal{K} \to \mathcal{T}$ has a right adjoint, it preserves colimits. Hence, if $\mathcal{X} : I \to \mathcal{K}$ is a diagram of k-spaces, then the colimit

$$\operatorname*{colim}_{I}{}^{\mathcal{T}} (i \circ \mathcal{X})$$

is a k-space, and

$$i(\operatorname*{colim}_{I}{}^{\mathcal{K}} \mathcal{X}) = \operatorname*{colim}_{I}{}^{\mathcal{T}} (i \circ \mathcal{X}) .$$

The functor $i : \mathcal{K} \to \mathcal{T}$, however, does not preserve limits. We define

$$\lim_{I}^{\mathcal{K}} \mathfrak{X} := k\left(\lim_{I}^{\mathcal{T}} (i \circ \mathfrak{X})\right);$$

this is the limit in $\mathcal{K}$ of the diagram $\mathfrak{X} : I \to \mathcal{K}$. In particular, the product in $\mathcal{K}$ of the two k-spaces X and Y is the k-space

$$X \times Y := k(i(X) \times i(Y)).$$

Now, if Y and Z are two k-spaces, we define the k-space

$$\underline{\mathrm{Hom}}_{\mathcal{K}}(Y, Z)$$

to be the set

$$\mathrm{Hom}_{\mathcal{K}}(Y, Z) := \mathrm{Hom}_{\mathcal{T}}(i(Y), i(Z))$$

with the k-space topology associated (by applying the functor k) to the topology with subbasis given by the set subsets

$$N(h, U) = \{ f : Y \to Z \mid f(h(K)) \subset U \},$$

where $h: K \to Y$ is continuous and K compact Hausdorff and $U \subset Z$ is open. The following holds:

<u>Prop</u> Let X, Y, and Z be k-spaces. Then the map

$$\underline{\mathrm{Hom}}_{\mathcal{K}}(X \times Y, Z) \xrightarrow{a} \underline{\mathrm{Hom}}_{\mathcal{K}}(X, \underline{\mathrm{Hom}}_{\mathcal{K}}(Y, Z))$$

defined by

$$a(f)(x)(y) = f(x, y)$$

is a homeomorphism.

<u>Proof</u> This is proved in Gaunce Lewis' thesis. I have a copy for anyone interested. //

In particular, if Y is a k-space, then the functor

$$- \times Y : \mathcal{K} \to \mathcal{K}$$

has the right adjoint functor

$$\underline{\mathrm{Hom}}_{\mathcal{K}}(Y, -) : \mathcal{K} \to \mathcal{K}.$$