We have seen that the geometric realization functor has a right adjoint functor, and hence, preserves colimits. We will now show that the canonical map

$$|X[-]\times Y[-]| \longrightarrow |X[-]|\times|Y[-]|$$

is a continuous bijection. The inverse map need not be continuous, since the product topology has too few open sets. We will give the product a new topology such that also the inverse map is continuous. Next time, we will see that, from the point of view of homotopy theory, this change is immaterial. We first prove:

<u>Lemma</u> The canonical map

$$p: |\Delta[m][-]\times\Delta[n][-]| \to |\Delta[m][-]|\times|\Delta[n][-]|$$

is a homeomorphism.

<u>Proof</u> We construct the inverse map q. To do so, we need to

triangulate the product $\Delta[r]\times\Delta[s]$ of two topological standard simplices. More precisely, we claim that, given $(u,v)\in\Delta[r]\times\Delta[s]$, there exists unique morphisms $\theta_1\colon[\ell]\to[r]$ and $\theta_2\colon[\ell]\to[s]$ and a unique interior point $w\in\Delta[\ell]$ such that

$$(u,v) = (\theta_{1*}(w), \theta_{2*}(w))$$

Moreover, the map

$$\theta = (\theta_1,\theta_2)\colon[\ell]\to[r]\times[s]$$

is injective. To prove this, we let

$$\Delta'[k] = \{(x_1,\dots,x_k)\in\mathbb{R}^k \mid 0\le x_1\le\dots\le x_k\le 1\}$$

and note the homeomorphism

$$\varepsilon\colon\Delta[k]\to\Delta'[k]$$

defined by

$$x=\sum_{i\in[k]} a_i\cdot i \mapsto \varepsilon(x)=(x_1,\dots,x_k)$$

$$x_j = \sum_{1\le i\le j} a_{i-1} \qquad (1\le j\le k).$$

(The space $\Delta'[k]$ is called the configuration space of k ordered points in $[0,1]$.) We now write

$$\varepsilon(u) = (u_1, \dots, u_r)$$

$$\varepsilon(v) = (v_1, \dots, v_s)$$

and define $z \in \Delta[r+s]$ by

$$\varepsilon(z) = (z_1, z_2, \dots, z_{r+s})$$

where $0 \le z_1 \le \dots \le z_{r+s} \le 1$ is given by re-ordering $u_1, \dots, u_r, v_1, \dots, v_s$ in increasing order. We now choose a map of partially ordered sets

$$\psi : [r+s] \longrightarrow [r] \times [s]$$

such that $\psi(0) = (0,0)$, $\psi(r+s) = (r,s)$, and for all $1 \le i \le r+s$

$$\psi(i) = \begin{cases} \psi(i-1) + (1,0) & \text{if } z_i \in \{u_1, \dots, u_r\} \\ \psi(i-1) + (0,1) & \text{if } z_i \in \{v_1, \dots, v_s\} \end{cases}.$$

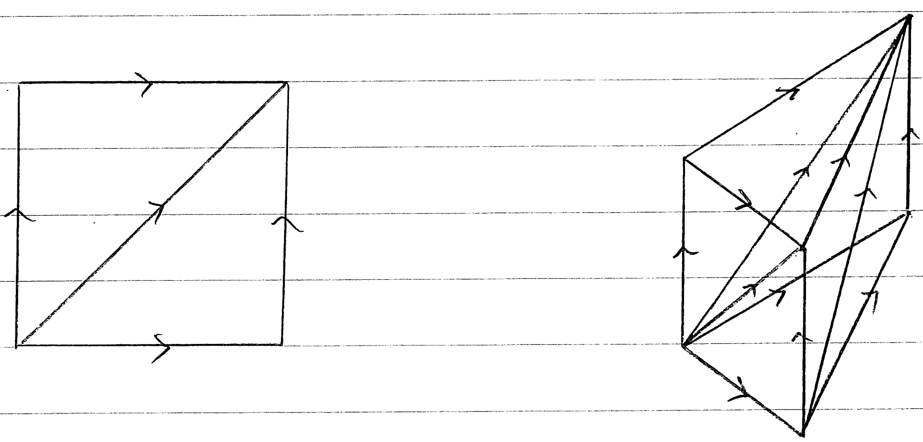
Then

$$\begin{array}{ccc} \Delta[r+s] & \xrightarrow{(\psi_{1*},\,\psi_{2*})} & \Delta[r]\times\Delta[s] \\ \in & & \in \\ z & \longmapsto & (u,v) \end{array}$$

We proved last time that there there is a unique order preserving map $\iota : [\ell] \to [r+s]$ and a unique interior point $w \in \Delta[\ell]$ such that $\iota_*(w) = z$. Now, the composite

$$\theta = \psi \circ \iota : [\ell] \to [r] \times [s].$$

is the desired map. This proves the claim. The triangulations of $\Delta[1]\times\Delta[1]$ and $\Delta[1]\times\Delta[2]$ look like this



We now define the map

$$q : |\Delta[m][-1]| \times |\Delta[n][-1]| \to |\Delta[m][-1] \times \Delta[n][-1]|$$

as follows: let

$$\xi = (\xi_1, \xi_2) \in |\Delta[m][-1]| \times |\Delta[n][-1]|,$$

and let (a, u), $a : [r] \hookrightarrow [m]$, $u \in \Delta[r]$ interior, and (b, v), $b : [s] \hookrightarrow [n]$, $v \in \Delta[s]$ interior, be the unique non-degenerate representatives of ξ_1 and ξ_2, respectively. The claim gives

$$\theta = (\theta_1, \theta_2) : [\ell] \longrightarrow [r] \times [s]$$

and $w \in \Delta[\ell]$ interior such that $\theta_{1*}(w) = u$ and $\theta_{2*}(w) = v$. Then

$$q(\xi) := \text{class of } ((a,b) \circ \theta, w).$$

We show that q is inverse to p:

$$(pr_1 \circ q)(\xi) = \text{class of } pr_1((a,b)\circ\theta, w)$$

$$= \text{class of } (a \circ \theta_1, w)$$

$$= \text{class of } (a, \theta_{1*}(w))$$

$$= \text{class of } (a, u)$$

and, similarly,

$$(pr_2 \circ q)(\xi) = \text{class of } (b, v)$$

so $p \circ q = \mathrm{Id}$. Next, let

$$\eta \in |\Delta[m][-] \times \Delta[n][-]|$$

have non-degenerate representative $((f,g),x)$, $(f,g) : [\ell] \hookrightarrow [m] \times [n]$, $x \in \Delta[\ell]$ interior, and factor

$$f : [\ell] \xrightarrow{\varphi_1} [r] \xhookrightarrow{a} [m]$$

$$g : [\ell] \xrightarrow{\varphi_2} [s] \xhookrightarrow{b} [n].$$

Then

$$p(\eta) = (\text{class of } (f,x), \text{class of } (g,x))$$

$$= (\text{class of } (a, \varphi_{1*}(x)), \text{class of } (b, \varphi_{2*}(x))).$$

But $x \in \Delta[\ell]$ is interior so

$$q(p(\eta)) := \text{class of } ((a,b) \circ \varphi, x)$$

$$= \text{class of } ((f,g), x).$$

The geometric realization of the simplicial set $X[-]$ is a k-space. Indeed,

$$|X[-]| = \operatorname*{colim}_{\Delta/X[-]} \Delta[-]$$

and $\Delta[n]$ is compact, hence a k-space. From now on, we will always view geometric realization as a functor

$$|-| : \mathrm{Sets}^{\Delta^{op}} \longrightarrow \mathcal{K}.$$

The right-adjoint

$$\mathrm{Sin}(-)[-] : \mathcal{K} \longrightarrow \mathrm{Sets}^{\Delta^{op}}$$

is defined by

$$\mathrm{Sin}(Y)[-] = \mathrm{Hom}_{\mathcal{K}}(\Delta[-], Y).$$

Thm let $X[-]$ and $Y[-]$ be simplicial sets. Then the canonical map

$$|X[-] \times Y[-]| \longrightarrow |X[-]| \times |Y[-]|$$

is an isomorphism in $\mathcal{K}$.

Proof We have the following series of canonical maps in $\mathcal{K}$:

$$|X[-] \times Y[-]|$$

$$\xrightarrow{①} \left|\left(\operatorname*{colim}_{\Delta[m][-]\to X[-]} \Delta[m][-]\right) \times \left(\operatorname*{colim}_{\Delta[n][-]\to Y[-]} \Delta[n][-]\right)\right|$$

$$\xleftarrow{②} \left|\operatorname*{colim}_{\Delta[m][-]\to X[-]} \left(\operatorname*{colim}_{\Delta[n][-]\to Y[-]} (\Delta[m][-] \times \Delta[n][-])\right)\right|$$

$$\xleftarrow{③} \left|\operatorname*{colim}_{\substack{\Delta[m][-]\to X[-] \\ \Delta[n][-]\to Y[-]}} (\Delta[m][-] \times \Delta[n][-])\right|$$

$$\xleftarrow{④} \operatorname*{colim}_{\substack{\Delta[m][-]\to X[-] \\ \Delta[n][-]\to Y[-]}} |\Delta[m][-] \times \Delta[n][-]|$$

$$\xrightarrow{⑤} \operatorname*{colim}_{\substack{\Delta[m][-]\to X[-] \\ \Delta[n][-]\to Y[-]}} (|\Delta[m][-]| \times |\Delta[n][-]|)$$

$$\xrightarrow{⑥} \operatorname*{colim}_{\Delta[m][-]\to X[-]} \left(\operatorname*{colim}_{\Delta[n][-]\to Y[-]} (|\Delta[m][-]| \times |\Delta[n][-]|)\right)$$

$$\xrightarrow{⑦} \Big(\operatorname*{colim}_{\Delta[m][-] \to X[-]} |\Delta[m][-]| \Big) \times \Big(\operatorname*{colim}_{\Delta[n][-] \to Y[-]} |\Delta[n][-]| \Big)$$

$$\xrightarrow{⑧} |X[-]| \times |Y[-]| .$$

The maps ① and ⑧ are isomorphisms by a proposition proved in last lecture (p. 81). The maps ② and ⑦ are isomorphism because product with a simplicial set (resp. a k-space) commutes with colimits in $\mathbf{Sets}^{\Delta^{op}}$ (resp. in $\mathcal{K}$). The maps ③ and ⑥ are isomorphisms, since the colimit of a product diagram

$$\mathcal{X} \times \mathcal{Y} : I \times J \longrightarrow \mathcal{E}$$

is the product of the colimits of the two diagrams. (Exercise.) The map ④ is an isomorphism since geometric realization preserves colimits. Finally, the map ⑤ is an isomorphism by the lemma proved at the begining of the lecture. We leave as an exercise to show that the composition of these isomorphisms is the canonical map. //

A pointed k-space is a pair (X, x) of a k-space X and a point $x \in X$. The point x is called the base-point. A map $f: (X,x) \to (Y,y)$ is a continuous map $f: X \to Y$ such that $f(x) = y$. We write $\mathcal{K}_*$ for the category of pointed k-spaces. The forgetful functor $U: \mathcal{K}_* \to \mathcal{K}$ defined by $U(X,x) = X$ has a left-adjoint functor $(-)_+ : \mathcal{K} \to \mathcal{K}_*$ that to the k-space X associates the pointed k-space $(X_+, +)$, where $X_+ = X \sqcup \{+\}$ is the disjoint union of X and a base-point $+$. It follows that U preserves limits: The limit of the diagram $i \in I \mapsto (X_i, x_i) \in \mathcal{K}_*$ is

$$\lim_{i \in I} (X_i, x_i) := (\lim_{i \in I} X_i, (x_i)).$$

The functor $(-)_+$ preserves colimits, but the functor U does not. The colimit of $i \in I \mapsto (X_i, x_i)$ is the quotient space

$$\operatorname*{colim}_{i \in I} (X_i, x_i) := ((\operatorname*{colim}_{i \in I} X_i) / B, \overline{B})$$

obtained from the colimit of the

diagram $i \in I \mapsto X_i \in \mathcal{K}$ by collapsing the subspace

$$B = \{ in_i (x_i) \mid i \in I \}$$

to a single point $\overline{B}$. The point $\overline{B}$ is the base-point of the colimit. For example, the coproduct of (X,x) and (Y,y) is the quotient space

$$(X,x) \vee (Y,y) := ((X \amalg Y)/\{x,y\}, \overline{\{x,y\}})$$

obtained from $X \amalg Y$ by identifying $x \in X$ and $y \in Y$ to a single point $\overline{\{x,y\}}$. The coproduct is called the <u>wedge sum</u> of (X,x) and (Y,y). There is a canonical map

$$(X,x) \vee (Y,y) \xrightarrow{h} (X,x) \times (Y,y)$$

defined by the diagram

$$\begin{tikzcd}
 & (X,x) \vee (Y,y) \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "g"] & \\
(X,x) & (X,x) \times (Y,y) \arrow[l, "pr_1"'] \arrow[r, "pr_2"] & (Y,y)
\end{tikzcd}$$

where the map f collapses Y to the base-point and the map g collapses X to the base-point. We define the smash product of (X, x) and (Y, y) to be the pointed k-space

$$(X, x) \wedge (Y, y)$$

$$= \left(X \times Y / h(X \vee Y),\ \overline{h(X \vee Y)} \right)$$

$$= \left(\frac{X \times Y}{\{x\} \times Y \cup X \times \{y\}},\ \overline{\{x\} \times Y \cup X \times \{y\}} \right)$$

obtained from $X \times Y$ by collapsing the subspace $h(X \vee Y)$ to a point. We also define the mapping space

$$\underline{\mathrm{Hom}}_{\mathcal{K}_*}((X, x), (Y, y)) \subset \underline{\mathrm{Hom}}_{\mathcal{K}}(X, Y)$$

to be the subspace of all maps $f : X \to Y$ such that $f(x) = y$ with the constant map $\bar{y} : (X, x) \to (Y, y)$, $\bar{y}(x') = y$, as the base-point. The canonical homeomorphism

$$\underline{\mathrm{Hom}}_{\mathcal{K}}(X \times Y, Z) \xrightarrow{a} \underline{\mathrm{Hom}}_{\mathcal{K}}(X, \underline{\mathrm{Hom}}_{\mathcal{K}}(Y, Z))$$

induces the homeomorphism

$$\underline{\mathrm{Hom}}_{\mathcal{T}_*}((X,x)\wedge(Y,y),(Z,z))$$
$$\xrightarrow{a} \underline{\mathrm{Hom}}_{\mathcal{T}_*}((X,x),\underline{\mathrm{Hom}}_{\mathcal{T}_*}((Y,y),(Z,z))).$$

defined by

$$a(f)(x')(y') = f(\text{class of } (x',y')).$$

In particular, the functor $-\wedge(Y,y)$ is left adjoint to $\underline{\mathrm{Hom}}_{\mathcal{T}_*}((Y,y),-)$.

We define the n-sphere to be the pointed k-space

$$(S^n,\infty) = ((\mathbb{R}^n)^+,\infty)$$

given by the one-point compactification of $\mathbb{R}^n$ with the point ∞ as the base-point. So S^n is the set $\mathbb{R}^n \amalg \{\infty\}$ and the subset $U\subset S^n$ is open if either $U\subset\mathbb{R}^n$ is open or $\infty\in U$ and $S^n\setminus U = K$ with $K\subset\mathbb{R}^n$ compact. There is a canonical homeomorphism

$$(S^{m+n}, \infty) \xrightarrow{\sim} (S^m, \infty) \wedge (S^n, \infty)$$

that takes $(x,y) \in \mathbb{R}^m \times \mathbb{R}^n$ to the class of (x,y) and the base-point ∞ to the base-point $\infty \times S^n \cup S^m \times \infty$.

Let $(X,*)$ be a pointed k-space. We define the suspension and loop space of $(X,*)$ by

$$\Sigma(X,*) = (X,*) \wedge (S^1, \infty)$$

$$\Omega(X,*) = \underline{\mathrm{Hom}}_{\mathcal{K}_*}((S^1,\infty),(X,*)).$$

Then Σ is left adjoint to Ω and we have canonical homeomorphisms

$$(S^{n+1}, \infty) \xrightarrow{\sim} \Sigma(S^n, \infty).$$

A homotopy from $f : (X,*) \to (Y,y)$ to $g : (X,*) \to (Y,y)$ is a commutative diagram

$$\begin{tikzcd}
(X,x) \arrow[d, "i_0"] \arrow[drr, "f"] & & \\
(X,x)\wedge([0,1]_+,+) \arrow[rr, "h"] & & (Y,y) \\
(X,x) \arrow[u, "i_1"'] \arrow[urr, "g"'] & &
\end{tikzcd}$$

where $i_\varepsilon(x') =$ class of (x', ε). If a homotopy from f to g exists, we say that f and g are <u>homotopic</u> and write $f \sim g$. This defines an equivalence relation on the set of maps from (X,x) to (Y,y), and we define

$$[(X,x),(Y,y)] = \mathrm{Hom}_{\mathcal{T}_*}((X,x),(Y,y))/\sim$$

to be the pointed set of equivalence classes. The base-point of this set is the class of the constant map $\bar{y} : (X,x) \to (Y,y)$. If $f : (X,x) \to (Y,y)$ is homotopic to $\bar{y} : (X,x) \to (Y,y)$, we say that f is <u>null-homotopic</u>.