2009. 12. 04

We consider the maps

$$\nu : (S^1, \infty) \longrightarrow (S^1, \infty) \vee (S^1, \infty)$$

$$\iota : (S^1, \infty) \longrightarrow (S^1, \infty)$$

defined by

$$\nu(t) = \begin{cases} in_1(t - t^{-1}) & (t \in (0, \infty)) \\ in_2(t - t^{-1}) & (t \in (-\infty, 0)) \\ \infty & (t \in \{0, \infty\}) \end{cases}$$

$$\iota(t) = \begin{cases} -t & (t \in \mathbb{R}) \\ \infty & (t = \infty) \end{cases}$$

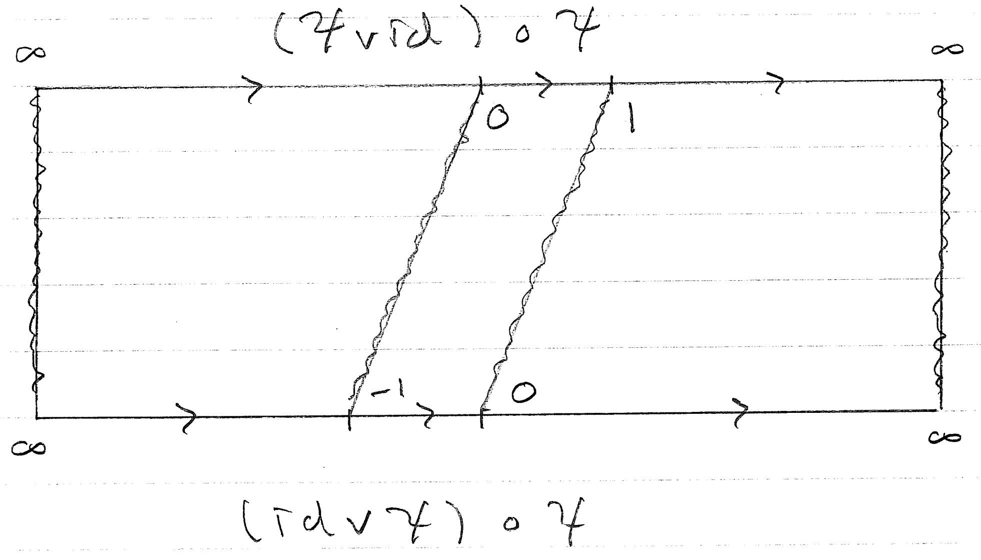
The following diagrams in $\mathcal{K}_*$ are homotopy commutative

$$\begin{tikzcd} (S^1, \infty) \arrow[r, "\nu"] \arrow[d, "\nu"] & (S^1, \infty) \vee (S^1, \infty) \arrow[d, "\nu \vee id"] \\ (S^1, \infty) \vee (S^1, \infty) \arrow[r, "id \vee \nu"] & (S^1, \infty) \vee (S^1, \infty) \vee (S^1, \infty) \end{tikzcd}$$

$$\begin{tikzcd}
 & (S^1,\infty) \arrow[dl, "id"'] \arrow[d, "\psi"] \arrow[dr, "id"] & \\
(S^1,\infty) & (S^1,\infty)\vee(S^1,\infty) \arrow[l, "\bar\infty + id"'] \arrow[r, "id+\bar\infty"] & (S^1,\infty)
\end{tikzcd}$$

$$\begin{tikzcd}
 & (S^1,\infty)\vee(S^1,\infty) \arrow[r, "\psi\vee id"] & (S^1,\infty)\vee(S^1,\infty) \arrow[dr, "\nabla"] & \\
(S^1,\infty) \arrow[ur, "\psi"] \arrow[rrr, "\bar\infty"] \arrow[dr, "\psi"'] & & & (S^1,\infty) \\
 & (S^1,\infty)\vee(S^1,\infty) \arrow[r, "id\vee\psi"] & (S^1,\infty)\vee(S^1,\infty) \arrow[ur, "\nabla"'] &
\end{tikzcd}$$

The following picture explains the homotopy between the two compositions in the first diagram:



We define the <u>fundamental group</u> of the pointed k-space (X, x) to be the set

$$\pi_1(X, x) = [(S^1, \infty), (X, x)]$$

with the following group structure: The product of the classes of the maps $f : (S^1, \infty) \to (X, x)$ and $g : (S^1, \infty) \to (X, x)$ is the class of the map

$$f * g : (S^1, \infty) \longrightarrow (X, x)$$

defined to be the composition

$$(S^1, \infty) \xrightarrow{\psi} (S^1, \infty) \vee (S^1, \infty) \xrightarrow{f+g} (X, x) ;$$

the inverse of the class of f is the class of the composite map

$$(S^1, \infty) \xrightarrow{c} (S^1, \infty) \xrightarrow{f} (X, x) ;$$

and the unit element $e \in \pi_1(X, x)$ is the class of the constant map $\bar{x}$.

Similarly, we can define two multiplications $*$ and $*'$ on the pointed set

$$\pi_2(X,x) = [(S^2,\infty),(X,x)]$$

$$\xleftarrow{\sim} [(S^1,\infty)\wedge(S^1,\infty),(X,x)]$$

To define $*$, we first recall that the functor $-\wedge(Z,z)$ has a right adjoint, and hence, preserves colimits. In particular, the canonical map

$$(X,x)\wedge(Z,z) \vee (Y,y)\wedge(Z,z)$$

$$\longrightarrow ((X,x)\vee(Y,y))\wedge(Z,z)$$

is a homeomorphism. We write

$$\rho : ((X,x)\vee(Y,y))\wedge(Z,z)$$

$$\longrightarrow (X,x)\wedge(Z,z) \vee (Y,y)\wedge(Z,z)$$

for the inverse homeomorphism and call it the right distributivity homeomorphism. Now, let

$$f, g : (S^1,\infty)\wedge(S^1,\infty) \longrightarrow (X,x)$$

be two maps. Then we define

$$f * g : (S^1,\infty)\wedge(S^1,\infty) \longrightarrow (X,x)$$

to be the composite map

$$(S^1,\infty)\wedge(S^1,\infty)$$

$$\xrightarrow{\nu\wedge \mathrm{id}} ((S^1,\infty)\vee(S^1,\infty))\wedge(S^1,\infty)$$

$$\xrightarrow{\rho} (S^1,\infty)\wedge(S^1,\infty)\vee(S^1,\infty)\wedge(S^1,\infty)$$

$$\xrightarrow{f+g} (X,x).$$

The product $*$ is the induced map of homotopy classes of maps.

To define $*'$, we first define

$$\gamma : (X,x)\wedge(Y,y) \longrightarrow (Y,y)\wedge(X,x)$$

to be the homeomorphism given by

$$\gamma(\text{class of } (x',y')) = \text{class of } (y',x').$$

We then define the left distributivity homeomorphism

$$\lambda : (X,x) \wedge ((Y,y) \vee (Z,z)) \longrightarrow (X,x)\wedge(Z,z) \vee (X,x)\wedge(Y,y)$$

to be the composition

$$(X,x) \wedge ((Y,y) \vee (Z,z))$$
$$\xrightarrow{\gamma} ((Y,y) \vee (Z,z)) \wedge (X,x)$$
$$\xrightarrow{\rho} (Y,y)\wedge(X,x) \vee (Z,z)\wedge(X,x)$$
$$\xrightarrow{\gamma \vee \gamma} (X,x)\wedge(Y,y) \vee (X,x)\wedge(Z,z).$$

Then with f and g as before, we define the map

$$f *' g : (S^1,\infty)\wedge(S^1,\infty) \longrightarrow (X,x)$$

to be the composition

$$(S^1,\infty) \wedge (S^1,\infty)$$
$$\xrightarrow{\mathrm{id}\wedge\psi} (S^1,\infty) \wedge ((S^1,\infty) \vee (S^1,\infty))$$
$$\xrightarrow{\lambda} (S^1,\infty)\wedge(S^1,\infty) \vee (S^1,\infty)\wedge(S^1,\infty)$$
$$\xrightarrow{f+g} (X,x).$$

The commutativity of the diagram on the next page shows that $*$ and $*'$ satisfy the identity

$$(f *' g) * (f' *' g') = (f * f') *' (g * g').$$

Indeed, the left-hand side is equal to the composition of the maps on the left-hand side of the diagram and the right-hand side is equal to the composition of the maps on the right-hand side of the diagram. We have omitted to write base-points in the diagram. The sub-diagrams ① and ② commute by naturality of the maps ρ and λ, respectively. The homeomorphism τ is defined by

$$\begin{tikzcd}
(X,x) \arrow[r, "in_1"] \arrow[dr, "in_2"'] & (X,x) \vee (Y,y) \arrow[d, "\tau"] & (Y,y) \arrow[l, "in_2"'] \arrow[dl, "in_1"] \\
& (Y,y) \vee (X,x) &
\end{tikzcd}$$

One checks directly that ③ commutes. It is clear that ④ commutes. The under-bar has no mathematical meaning.

$$
\begin{tikzcd}[column sep=small]
(S^1 \vee S^1) \wedge \underline{S}^1 \arrow[d, "\rho"'] & & S^1 \wedge \underline{S}^1 \arrow[ll, "\psi \wedge id"'] \arrow[rr, "id \wedge \psi"] \arrow[d, "\psi \wedge \psi"] & & S^1 \wedge (\underline{S}^1 \vee \underline{S}^1) \arrow[d, "\lambda"] \\
S^1 \wedge \underline{S}^1 \vee S^1 \wedge \underline{S}^1 \arrow[dr, "id \wedge \psi \vee id \wedge \psi"] \arrow[rr, phantom, "1"] & & (S^1 \vee S^1) \wedge (\underline{S}^1 \vee \underline{S}^1) \arrow[dl, "\rho", "\sim"'] \arrow[dr, "\lambda"', "\sim"] \arrow[rr, phantom, "2"] & & S^1 \wedge \underline{S}^1 \vee S^1 \wedge \underline{S}^1 \arrow[dl, "\psi \wedge id \vee \psi \wedge id"'] \\
& S^1 \wedge (\underline{S}^1 \vee \underline{S}^1) \vee S^1 \wedge (\underline{S}^1 \vee \underline{S}^1) \arrow[d, "\lambda \vee \lambda", "\sim"'] & {} \arrow[d, phantom, "3"] & (S^1 \vee S^1) \wedge \underline{S}^1 \vee (S^1 \vee S^1) \wedge \underline{S}^1 \arrow[d, "\rho \vee \rho", "\sim"'] & \\
& S^1 \wedge \underline{S}^1 \vee S^1 \wedge \underline{S}^1 \vee S^1 \wedge \underline{S}^1 \vee S^1 \wedge \underline{S}^1 \arrow[rr, "id \vee \tau \vee id"] \arrow[dr, "f+g+f'+g'"'] & {} \arrow[d, phantom, "4"] & S^1 \wedge \underline{S}^1 \vee S^1 \wedge \underline{S}^1 \vee S^1 \wedge \underline{S}^1 \vee S^1 \wedge \underline{S}^1 \arrow[dl, "f+f'+g+g'"] & \\
& & X & &
\end{tikzcd}
$$

Let us write $[f]$ for the homotopy class of the map f, and let $e = \bar{x}$ be the constant map. The formula on page 113 shows that

$$[f * g] = [(f *' e) * (e *' g)]$$

$$= [(f * e) *' (e * g)] = [f *' g].$$

and

$$[g * f] = [(e *' g) * (f *' e)]$$

$$= [(e * f) *' (g * e)] = [f *' g].$$

Hence, $*$ and $*'$ define the <u>same</u> group structure on $\pi_2(X, x)$ and this group structure is <u>abelian</u>. For all $n \geq 0$, we define

$$\pi_n(X, x) = [(S^n, \infty), (X, x)].$$

It is a pointed set, if $n = 0$, a group, if $n = 1$, and an abelian group, if $n \geq 2$. We define

$$\pi_n((\Omega(X, x), \bar{x}) \xrightarrow{\sim} \pi_{n+1}(X, x)$$

to be the isomorphism given by the composition

$$[(S^n,\infty), \Omega(X,x)]$$

$$\xleftarrow{\sim} [\Sigma(S^n,\infty), (X,x)]$$

$$\xrightarrow{\sim} [(S^{n+1},\infty), (X,x)]$$

of the adjunction isomorphism and the canonical isomorphism.