2009.12.11

Let $f : (X, x) \to (Y, y)$ be a map of pointed k-spaces. We let

$$f_* : \pi_n(X, x) \to \pi_n(Y, y)$$

be the induced map defined by

$$f_*([\omega]) = [f \circ \omega].$$

The map f_* is a map of pointed sets for $n \geq 0$, and a group homomorphism for $n \geq 1$. We will show that f_* fits in a long exact sequence

$$\cdots \to \pi_n(F(f, y), (x, \bar{y})) \xrightarrow{i_*} \pi_n(X, x)$$

$$\xrightarrow{f_*} \pi_n(Y, y) \xrightarrow{-\delta} \pi_{n-1}(F(f, y), (x, \bar{y})) \to \cdots$$

of homotopy groups. The pointed space $(F(f, y), (x, \bar{y}))$ is called the mapping fiber of f over y. To define it, we first define the path space of (Y, y) to be the pointed mapping space

$$(P(Y,y),\bar{y}) = (\underline{\mathrm{Hom}}_{\mathcal{T}_*}(([0,1],0),\bar{y})).$$

It is a contractible space. Indeed, a homotopy from the constant map $\bar{y}$ to the identity map is given by

$$h : (P(Y,y),\bar{y}) \wedge ([0,1],0) \to (P(Y,y),\bar{y})$$

$$h(\sigma,s)(t) = \sigma(s\cdot t).$$

We now define the <u>mapping fiber</u> of f over y to be the pull-back

$$\begin{tikzcd}
(F(f,y),(x,\bar{y})) \arrow[r, "i"] \arrow[d, "i'"] & (X,x) \arrow[d, "f"] \\
(P(Y,y),\bar{y}) \arrow[r, "ev_1"] & (Y,y)
\end{tikzcd}$$

where $ev_1(\sigma) = \sigma(1)$.

<u>Lemma</u> The sequence

$$\pi_n(F(f,y),(x,\bar{y})) \xrightarrow{i_*} \pi_n(X,x) \xrightarrow{f_*} \pi_n(Y,y)$$

is exact.

Proof The composition $f_* \circ i_*$ is equal to the composition

$$\pi_n(F(f,y),(x,\bar{y})) \xrightarrow{i'_*} \pi_n(P(Y,y),\bar{y}) \xrightarrow{w_{1*}} \pi_n(Y,y)$$

and $\pi_n(P(Y,y),\bar{y}) = \{[\bar{y}]\}$. Hence, $f_* \circ i_*$ is the constant map $[\bar{y}]$.

Let $[\omega] \in \pi_n(X,x)$ and suppose that $f_*([\omega]) = [\bar{y}]$. Then there exists a homotopy

$$h : (S^n,\infty) \wedge ([0,1],0) \longrightarrow (Y,y)$$

from the constant map $\bar{y}$ to the map $f \circ \omega$. Hence, we have a commutative diagram

$$\begin{tikzcd}
(S^n,\infty) \arrow[r, "\omega"] \arrow[d, "i_1"'] & (X,x) \arrow[d, "f"] \\
(S^n,\infty) \wedge ([0,1],0) \arrow[r, "h"] & (Y,y)
\end{tikzcd}$$

We adjoin h to get the map

$$a(h) : (S^n,\infty) \longrightarrow (P(Y,y),\bar{y})$$

Then the diagram

$$\begin{tikzcd}
(S^n, \infty) \arrow[r, "\omega"] \arrow[d, "a(h)"] & (X, x) \arrow[d, "f"] \\
(P(Y,y), \bar{y}) \arrow[r, "ev_1"] & (Y, y)
\end{tikzcd}$$

commutes. The induced map

$$(\omega, a(h)) : (S^n, \infty) \longrightarrow (F(f,y), (x, \bar{y}))$$

satisfies $i \circ (\omega, a(h)) = \omega$. Hence, $i_*([(\omega, a(h))]) = [\omega]$, so the kernel of f_* is equal to the image of i_*. ||

We may apply the lemma to the map

$$i : (F(f,y), (x,\bar{y})) \longrightarrow (X, x).$$

It turns out that the mapping fiber $(F(i,x), ((x,\bar{y}), \bar{x}))$ is homotopy equivalent to the loop space $(\Omega(Y,y), \bar{y})$. To see this, we first let

$$\tau : ([-1,1]/\{-1,1\}, \overline{\{-1,1\}}) \longrightarrow (S^1, \infty)$$

be the homeomorphism defined by

$$\tau(t) = \begin{cases} t/(1-t^2) & (t \in (-1,1)) \\ \infty & (t \in \{-1,1\}) \end{cases}$$

It induces the homeomorphism

$$(\Omega(Y,y),\bar{y}) = (\underline{\mathrm{Hom}}_{\mathcal{X}_*}((S^1,\infty),(Y,y)),\bar{y})$$
$$\xrightarrow[\sim]{\tau^*} (\underline{\mathrm{Hom}}_{\mathcal{X}_*}(([-1,1]/\{-1,1\},\overline{\{-1,1\}}),(Y,y)),\bar{y}).$$

Now we have the pull-back diagram

$$\begin{tikzcd} (\Omega(Y,y),\bar{y}) \arrow[r, "\rho_-"] \arrow[d, "\rho_+"] & (P(Y,y),\bar{y}) \arrow[d, "ev_1"] \\ (P(Y,y),\bar{y}) \arrow[r, "ev_1"] & (Y,y) \end{tikzcd}$$

where

$$\rho_-(\omega)(t) = \tau^*(\omega)(t-1) \qquad t \in [0,1]$$

$$\rho_+(\omega)(t) = \tau^*(\omega)(1-t) \qquad t \in [0,1]$$

On the other hand, the mapping fiber of i over $(x,\bar{y})$ is defined to be the pull-back

$$\begin{tikzcd}
(F(i,x),((x,\bar{y}),\bar{x})) \arrow[r, "j"] \arrow[d, "j'"] & (F(f,y),(x,\bar{y})) \arrow[d, "i"] \\
(P(X,x),\bar{x}) \arrow[r, "ev_1"] & (X,x)
\end{tikzcd}$$

and the mapping fiber of f over y is defined to be the pull-back

$$\begin{tikzcd}
(F(f,y),(x,\bar{y})) \arrow[r, "i'"] \arrow[d, "i"] & (P(Y,y),\bar{y}) \arrow[d, "ev_1"] \\
(X,x) \arrow[r, "f"] & (Y,y)
\end{tikzcd}$$

It follows that we have a pull-back

$$\begin{tikzcd}
(F(i,x),((x,\bar{y}),\bar{x})) \arrow[r, "i' \circ j"] \arrow[d, "j'"] & (P(Y,y),\bar{y}) \arrow[d, "ev_1"] \\
(P(X,x),\bar{x}) \arrow[r, "f \circ ev_1"] & (Y,y)
\end{tikzcd}$$

We now define

$$\alpha : (F(i,x),((x,\bar{y}),\bar{x})) \longrightarrow (\Omega(Y,y),\bar{y})$$

to be the unique map such that

$$\rho_- \circ \alpha = i' \circ j$$

$$\rho_+ \circ \alpha = f_* \circ j'$$

where $f_* : (P(X,x), \bar{x}) \to (P(Y,y), \bar{y})$ is the map induced by f. We define

$$\beta : (\Omega(Y,y), \bar{y}) \to (F(i,x), L(x,\bar{y}), \bar{x}))$$

to be the unique map such that

$$(i' \circ j \circ \beta)(\omega)(t) = \tau^*(\omega)(2t-1), \quad t \in [0,1]$$

$$(j' \circ \beta)(\omega) = \bar{x}.$$

There are homotopies

$$\alpha \circ \beta \simeq id$$

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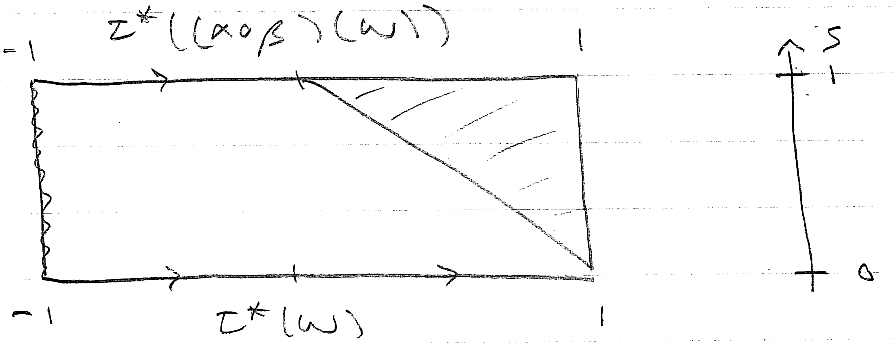
We give the first homotopy and leave it as an exercise to write down the second homotopy. We have

$$\tau^*((\alpha\circ\beta)(\omega))(t) = \begin{cases} \tau^*(\omega)(2t+1) & (-1\le t\le 0) \\ \eta & (0\le t\le 1) \end{cases}$$

so a homotopy h from $\tau^*(\omega)$ to $\tau^*((\alpha\circ\beta)(\omega))$ is given by

$$h(\tau^*(\omega), s)(t) = \begin{cases} \tau^*(\omega)(st+s+t) & (-1\le t\le \frac{1-s}{1+s}) \\ \eta & (\frac{1-s}{1+s}\le t\le 1) \end{cases}$$

The following picture illustrates h:



We define the boundary map

$$\partial : (\Omega(Y,y), \bar{y}) \longrightarrow (F(f,y), (x,\bar{y}))$$

to be the composition $j\circ\beta$. Then the lemma shows that the maps

$$(\Omega(Y,y),\bar{y}) \xrightarrow{\partial} (F(f,y),(x,\bar{y})) \xrightarrow{i} (X,x) \xrightarrow{f} (Y,y)$$

induce an exact sequence

$$\pi_n(\Omega(Y,y),\bar{y}) \xrightarrow{\partial_*} \pi_n(F(f,y),(x,\bar{y})) \xrightarrow{i_*} \pi_n(X,x) \xrightarrow{f_*} \pi_n(Y,y).$$

of homotopy groups. Hence, we have the exact sequence

$$\pi_{n+1}(Y,y) \xrightarrow{\delta} \pi_n(F(f,y),(x,\bar{y})) \xrightarrow{i_*} \pi_n(X,x) \xrightarrow{f_*} \pi_n(Y,y)$$

where δ is the composite

$$\pi_{n+1}(Y,y) \xrightarrow[\simeq]{\sigma^{-1}} \pi_n(\Omega(Y,y),\bar{y}) \xrightarrow{\partial_*} \pi_n(F(f,y),(x,\bar{y})).$$

We leave it as an exercise to show that the composition

$$(\Omega(X,x),\bar{x}) \xrightarrow{\partial'} (F(i,x),((x,\bar{y}),\bar{x})) \xrightarrow[\simeq]{\alpha} (\Omega(Y,y),\bar{y})$$

of the boundary map ∂' corresponding to the map i and the homotopy equivalence α is homotopic to the composition

$$(\Omega(X,x),\bar{x}) \xrightarrow{\iota^*} (\Omega(X,x),\bar{x}) \xrightarrow{f_*} (\Omega(Y,y),\bar{y})$$

of the map induced by the inversion ι of the circle and the map induced by f. It follows that we have a long-exact sequence of homotopy groups

$$
\begin{array}{c}
\vdots \\
\downarrow \\
\pi_{n+2}(Y, y) \\
\downarrow{\scriptstyle -\delta} \\
\pi_{n+1}(F(f,y), (x,\bar{y})) \\
\downarrow{\scriptstyle -i_*} \\
\pi_{n+1}(X, x) \\
\downarrow{\scriptstyle -f_*} \\
\pi_{n+1}(Y, y) \\
\downarrow{\scriptstyle \delta} \\
\pi_n(F(f,y), (x,\bar{y})) \\
\downarrow{\scriptstyle i_*} \\
\pi_n(X, x) \\
\downarrow{\scriptstyle f_*} \\
\pi_n(Y, y) \\
\downarrow{\scriptstyle -\delta} \\
\vdots
\end{array}
$$