Algebraic Topology: Problem Set 1

Due: Friday, November 6.

Problem 1. Let I be a small (index) category and let $\mathscr{C}$ be a category in which all small colimits exist. Show that for every diagram

$$\mathscr{X} : I \to \mathscr{C},$$

the following diagram is a coequalizer.

$$\coprod_{\alpha\colon i\to i'} \mathscr{X}(i) \overset{f}{\underset{g}{\rightrightarrows}} \coprod_{i} \mathscr{X}(i) \xrightarrow{h} \operatorname*{colim}_{I} \mathscr{X}$$

Here the left-hand coproduct is indexed by the set of all morphisms $\alpha\colon i \to i'$ in I and the middle coproduct is indexed by the set of all objects i in I. The three morphisms f, g, and h are defined by as follows:

$$\begin{aligned} f \circ \mathrm{in}_{\alpha\colon i\to i'} &= \mathrm{in}_i \\ g \circ \mathrm{in}_{\alpha\colon i\to i'} &= \mathrm{in}_{i'} \circ X(\alpha) \\ h \circ \mathrm{in}_i &= \mathrm{in}_i \end{aligned}$$

(Hint: Let X be the coequalizer of the morphisms f and g. Show that X satisfies the defining properties (i) and (ii) of the colimit.)

Problem 2. Let I be a small category, Sets be the category of sets, and $\mathrm{Sets}^{I^{\mathrm{op}}}$ the category whose objects are functors $\mathscr{X} : I^{\mathrm{op}} \to \mathrm{Sets}$ and whose morphisms are natural transformations.

(i) Show that for every functor $\mathscr{X} : I^{\mathrm{op}} \to \mathrm{Sets}$, the map

$$y\colon \mathrm{Hom}_{\mathrm{Sets}^{I^{op}}}(\mathrm{Hom}_I(-,i), \mathscr{X}(-)) \to \mathscr{X}(i)$$

defined by $y(f) = f(\mathrm{id}_i)$ is an isomorphism.

(ii) Show that for every functor $\mathscr{X} : I^{\mathrm{op}} \to \mathrm{Sets}$, the canonical map

$$\operatorname*{colim}_{I/\mathscr{X}} \mathrm{Hom}_I(-,-) \to \mathscr{X}(-)$$

is an isomorphism. Here the category $I/\mathscr{X}$ has objects all natural transformations

$$\theta\colon \mathrm{Hom}_I(-,i) \to \mathscr{X}(-)$$

and a morphism from $\theta\colon \mathrm{Hom}_I(-,i) \to \mathscr{X}(-)$ to $\theta'\colon \mathrm{Hom}_I(-,i') \to \mathscr{X}(-)$ is a morphism $\alpha\colon i \to i'$ in I such that the $\theta' \circ \alpha_* = \theta$. The functor

$$\mathrm{Hom}_I(-,-)\colon I/\mathscr{X} \to \mathrm{Sets}^{I^{\mathrm{op}}}$$

takes the object $\theta\colon \mathrm{Hom}_I(-,i) \to \mathscr{X}(-)$ to $\mathrm{Hom}_I(-,i)$ and takes the morphism $\alpha\colon i \to i'$ to the morphism $\alpha_*\colon \mathrm{Hom}_I(-,i) \to \mathrm{Hom}_I(-,i')$.