Pespectives in Mathematical Sciences: Report Problems

Due: Thursday, November 26, in Science Building 1, Room 105.

This problem set concerns the scissors congruence group $P(S^1)$ of the unit circle which we now define. We first define the unit circle to be the set

$$S^1 = \mathbb{R}/2\pi\mathbb{Z}$$

of left cosets of the subgroup $2\pi\mathbb{Z} \subset \mathbb{R}$ equipped with the metric

$$d(\alpha, \beta) = \min\{|x - y| \mid x \in \alpha, y \in \beta\}.$$

So the distance from α to β is the length of the shortest arc between α and β. The group $I(S^1)$ of isometries of S^1 is generated by the rotations $t_\alpha(\beta) = \alpha + \beta$ and the reflections $r_\alpha(\beta) = 2\alpha - \beta$ with $\alpha \in S^1$.

We define a 1-simplex in S^1 to be a tuple $\sigma = (\alpha_0, \alpha_1)$ with $\alpha_0, \alpha_1 \in S^1$ and say that α_0 and α_1 are the vertices of σ. We define $\sigma = (\alpha_0, \alpha_1)$ to be proper if $0 < d(\alpha_0, \alpha_1) < \pi$, and in this case, we define the associated geometric simplex to be the following subset.

$$|\sigma| = \{\beta \in S^1 \mid d(\alpha_0, \beta) \leqslant d(\alpha_0, \alpha_1) \text{ and } d(\beta, \alpha_1) \leqslant d(\alpha_0, \alpha_1)\} \subset S^1$$

We then define a polytope in S^1 to be subset $P \subset S^1$ with the property that there exists a finite number of proper 1-simplices $\sigma_1, \dots, \sigma_N$ in S^1 such that

$$P = \bigcup_{1 \leqslant i \leqslant N} |\sigma_i| \tag{1}$$

and such that for all $1 \leqslant i < j \leqslant N$, the intersection $|\sigma_i| \cap |\sigma_j|$ is either empty or a vertex in both σ_i and σ_j. We say that (1) is a triangulation of P. Finally, we define the scissors congruence group $P(S^1)$ to be the quotient

$$P(S^1) = F(S^1)/R(S^1)$$

of the free abelian group $F(S^1)$ with one generator (P) for every polytope $P \subset S^1$ by the subgroup $R(S^1) \subset F(S^1)$ generated by the following elements (i)–(ii).

(i) For every polytope $P \subset S^1$ and for every triangulation

$$P = \bigcup_{1 \leqslant i \leqslant N} |\sigma_i|,$$

the element

$$(P) - \sum_{1 \leqslant i \leqslant N} (|\sigma_i|).$$

(ii) For every polytope $P \subset S^1$ and every isometry $f \in I(S^1)$, the element

$$(P) - (f(P)).$$

We write $[P] = (P) + R(S^1) \in P(S^1)$ for the class that contains $P \subset S^1$.

(The problems are on the back side.)

Problem 3. Show that there exists a group homomorphism

$$\mathrm{vol}\colon P(S^1) \to \mathbb{R}$$

with the property that for every proper 1-simplex $\sigma = (\alpha_0, \alpha_1)$,

$$\mathrm{vol}([|\sigma|]) = d(\alpha_0, \alpha_1).$$

Problem 4. Show that there exists a group homomorphism

$$\mathrm{arc}\colon \mathbb{R} \to P(S^1)$$

with the property that for all $0 < x < \pi$,

$$\mathrm{arc}(x) = [|(0 + 2\pi\mathbb{Z}, x + 2\pi\mathbb{Z})|].$$

Problem 5. Show that the composite maps

$$\mathrm{vol} \circ \mathrm{arc}\colon \mathbb{R} \to \mathbb{R}$$

$$\mathrm{arc} \circ \mathrm{vol}\colon P(S^1) \to P(S^1)$$

are equal to the respective identity maps.