

3. Semi-simple rings

We next consider semi-simple modules in more detail.

LEMMA 3.1. *Let R be a ring, let M be a left R -module, and let $\{S_i\}_{i \in I}$ be a finite family of simple submodules the union of which generates M . Then there exists a subset $J \subset I$ such that $M = \bigoplus_{j \in J} S_j$.*

PROOF. We consider a subset $J \subset I$ which is maximal among subsets with the property that the sum of submodules $\sum_{j \in J} S_j \subset M$ is direct. Now, if $i \in I \setminus J$, then $S_i \cap \sum_{j \in J} S_j \neq \{0\}$ or else J would not be maximal. Since S_i is simple, we conclude that $S_i \cap \sum_{j \in J} S_j = S_i$. It follows that $\sum_{j \in J} S_j = M$ as desired. \square

PROPOSITION 3.2. *Let R be a ring and let M be a semi-simple left R -module.*

- (i) *Let Q be a left R -module and let $p: M \rightarrow Q$ be a surjective R -linear map. Then Q is semi-simple and there exists an R -linear map $s: Q \rightarrow M$ such that $p \circ s: Q \rightarrow Q$ is the identity map.*
- (ii) *Let N be a left R -module and let $i: N \rightarrow M$ be an injective R -linear map. Then N is semi-simple and there exists an R -linear map $r: M \rightarrow N$ such that $r \circ i: N \rightarrow N$ is the identity map.*

PROOF. (i) We write $M = \bigoplus_{i \in I} S_i$ as a finite direct sum of simple submodules. Let $J \subset I$ be the subset of indices i such that $p(S_i)$ is non-zero. By Lemma 3.1, we can find a subset $K \subset J$ such that $\bigoplus_{i \in K} p(S_i) = Q$. Let $j: \bigoplus_{i \in K} S_i \rightarrow M$ be the canonical inclusion. Then $p \circ j$ is an isomorphism which shows that Q is semi-simple. Moreover, the composite map $s = j \circ (p \circ j)^{-1}: Q \rightarrow M$ has the desired property that $p \circ s$ is the identity map of Q .

(ii) It follows from (i) that there exists a submodule $P \subset M$ such that the composition $P \rightarrow M \rightarrow M/N$ of the canonical inclusion and the canonical projection is an isomorphism. Now, if $q: M \rightarrow M/P$ is the projection onto the quotient by P , then $q \circ i: N \rightarrow M/P$ is an isomorphism. This shows that N is semi-simple and that the map $r = (q \circ i)^{-1} \circ q: M \rightarrow N$ satisfies that $r \circ i = \text{id}_N$. \square

We fix a ring R and define $\Lambda(R)$ be the set of isomorphism classes of the simple left R -modules that are of the form $S = R/I$ with $I \subset R$ a left ideal.¹ Let S be any simple left R -module. To define the *type* of S , we choose a non-zero element $x \in S$ and consider the R -linear map $p: R \rightarrow S$ defined by $p(a) = ax$. It is surjective, since S is simple, and hence, induces an isomorphism $\bar{p}: R/I \rightarrow S$, where $I = \text{Ann}_R(x)$ is the kernel of p . We now define the type of S to be the isomorphism class $\lambda \in \Lambda(R)$ of R/I . (Exercise: Show that the type of S is well-defined.) We prove that semi-simple left R -modules admit the following canonical *isotypic decomposition*.

PROPOSITION 3.3. *Let R be a ring.*

- (i) *Let M be a semi-simple left R -module, and let $M_\lambda \subset M$ be the submodule generated by the union of all simple submodules of type $\lambda \in \Lambda(R)$. Then*

$$M = \bigoplus_{\lambda \in \Lambda(R)} M_\lambda$$

and M_λ is a direct sum of simple submodules of type λ . In addition, M_λ is zero for all but finitely many $\lambda \in \Lambda(R)$.

¹It is not possible, within standard ZFC set theory, to speak of the isomorphism classes of all simple R -modules or the set thereof. This is the reason that we define $\Lambda(R)$ in this way.

- (ii) Let M and N be semi-simple left R -modules and let $f: M \rightarrow N$ be an R -linear map. Then for every $\lambda \in \Lambda(R)$, $f(M_\lambda) \subset N_\lambda$.

PROOF. We first prove (i) Since M is semi-simple, we can write M as a finite direct sum $M = \bigoplus_{i \in I} S_i$ of simple submodules. If $M'_\lambda = \bigoplus_{i \in I_\lambda} S_i$, where $I_\lambda \subset I$ is the subset of $i \in I$ such that S_i is of type λ , then $M = \bigoplus_{\lambda \in \Lambda(R)} M'_\lambda$ and $M'_\lambda \subset M_\lambda$. We must show that $M_\lambda \subset M'_\lambda$. So let $S \subset M$ be a simple submodule of type λ and let $i \in I$. The composition $f_i: S \rightarrow M \rightarrow S_i$ of the canonical inclusion and the canonical projection is an R -linear map, and since S and S_i are both simple left R -modules, the map f_i is either zero or an isomorphism. If it is an isomorphism, then we have $i \in I_\lambda$, which shows that $S \subset M'_\lambda$, and hence, $M_\lambda \subset M'_\lambda$ as desired. Finally, the finite set I is the disjoint union of the subsets I_λ with $\lambda \in \Lambda(R)$, and hence, all but finitely many of these subsets must be empty.

Next, to prove (ii), we let $S \subset M$ be a simple submodule of type λ . Since S is simple, either $f(S) \subset N$ is zero or else $f|_S: S \rightarrow f(S)$ is an isomorphism of left R -modules. Therefore, $f(M_\lambda) \subset N_\lambda$ as stated. \square

DEFINITION 3.4. A ring R is *semi-simple* if it is semi-simple as a left module over itself. A ring R is *simple* if it is semi-simple and if it has exactly one type of simple modules.

We proceed to prove two theorems that, taken together, constitute a structure theorem for semi-simple rings.

THEOREM 3.5. Let R be a semi-simple ring and let $R = \bigoplus_{\lambda \in \Lambda(R)} R_\lambda$ be the isotypic decomposition of R as a left R -module.

- (i) For every $\lambda \in \Lambda(R)$, the left ideal $R_\lambda \subset R$ is non-zero. In particular, the set of types $\Lambda(R)$ is finite.
- (ii) For every $\lambda \in \Lambda(R)$, the left ideal $R_\lambda \subset R$ is also a right ideal.
- (iii) Let $a, b \in R$ and write $a = \sum_{\lambda \in \Lambda(R)} a_\lambda$ and $b = \sum_{\lambda \in \Lambda(R)} b_\lambda$ with $a_\lambda, b_\lambda \in R_\lambda$. Then $ab = \sum_{\lambda \in \Lambda(R)} a_\lambda b_\lambda$ and $a_\lambda b_\lambda \in R_\lambda$.
- (iv) For every $\lambda \in \Lambda(R)$, the subset $R_\lambda \subset R$ is a ring with respect to the restriction of the addition and multiplication on R , and the identity element is the unique element $e_\lambda \in R_\lambda$ such that $\sum_{\lambda \in \Lambda(R)} e_\lambda = 1$.
- (v) For every $\lambda \in \Lambda(R)$, the ring R_λ is simple.

PROOF. (i) Let S be a simple left R -module of type λ . We choose a non-zero element $x \in S$ and consider again the surjective R -linear map $p: R \rightarrow S$ defined by $p(a) = a \cdot x$. By Proposition 3.2 there exists an R -linear map $s: S \rightarrow R$ such that $p \circ s = \text{id}_S$. But then $s(S) \subset R$ is a simple submodule of type λ . Since R_λ is non-zero, it follows from Proposition 3.3 (i) that $\Lambda(R)$ is a finite set.

(ii) Let $a \in R$ and let $\rho_a: R \rightarrow R$ be the map $\rho_a(b) = ba$ defined by right multiplication by a . It is an R -linear map from the left R -module R to itself. By Proposition 3.3 (ii), we conclude that $\rho_a(R_\lambda) \subset R_\lambda$ which is precisely the statement that $R_\lambda \subset R$ is a right ideal.

(iii) Since $R_\mu \subset R$ is a left ideal, we have $a_\lambda b_\mu \in R_\mu$, and since $R_\lambda \subset R$ is a right ideal, we have $a_\lambda b_\mu \in R_\lambda$. It follows that $a_\lambda b_\mu \in R_\lambda \cap R_\mu$ which is equal to R_λ and $\{0\}$, respectively, as $\lambda = \mu$ and $\lambda \neq \mu$.

(iv) We have already proved in (iii) that the multiplication on R restricts to a multiplication on R_λ . Now, for all $a_\lambda \in R_\lambda$, we have

$$a_\lambda = a_\lambda \cdot 1 = a_\lambda \cdot \left(\sum_{\mu \in \Lambda} e_\mu \right) = \sum_{\mu \in \Lambda} a_\lambda \cdot e_\mu = a_\lambda \cdot e_\lambda$$

and the identity $a_\lambda = e_\lambda \cdot a_\lambda$ is proved analogously. It follows that R_λ is a ring.

(v) Let S_λ be a simple left R -module of type λ . Since $R_\lambda \subset R$, the left multiplication of R on S_λ defines a left multiplication of R_λ on S_λ . To prove that this defines a left R_λ -module structure on S_λ , we must show that $e_\lambda \cdot x = x$, for all $x \in S_\lambda$. We have just proved that $e_\lambda \cdot y = y$, for all $y \in R_\lambda$. Moreover, by Proposition 3.3 (i), we can find an injective R -linear map $f_\lambda: S_\lambda \rightarrow R_\lambda$. Since

$$f_\lambda(e_\lambda \cdot x) = e_\lambda \cdot f_\lambda(x) = f_\lambda(x),$$

we conclude that $e_\lambda \cdot x = x$, for all $x \in S_\lambda$, as desired. We further note that S_λ is a simple left R_λ -module. Indeed, it follows from (iii) that a subset $N \subset S_\lambda$ is an R -submodule if and only if it is an R_λ -submodule. Finally, by Proposition 3.3 (i), the left R -module R_λ is a direct sum $S_{\lambda,1} \oplus \cdots \oplus S_{\lambda,r}$ of simple submodules, all of which are isomorphic to the simple left R -module S_λ . Therefore, also as a left R_λ -module, R_λ is the direct sum $S_{\lambda,1} \oplus \cdots \oplus S_{\lambda,r}$ of submodules, all of which are isomorphic to the simple left R_λ -module S_λ . This shows that R_λ is a semi-simple ring, and we conclude from (i) that every simple left R_λ -module is isomorphic to S_λ . So R_λ is a simple ring. \square

REMARK 3.6. The inclusion map $i_\lambda: R_\lambda \rightarrow R$ is not a ring homomorphism unless $R = R_\lambda$. Indeed, the map i_λ takes the multiplicative identity element $e_\lambda \in R_\lambda$ to the element $e_\lambda \in R$ which is not equal to the multiplicative identity element $1 \in R$ unless $R = R_\lambda$. However, the projection map

$$p_\lambda: R \rightarrow R_\lambda$$

that takes $a = \sum_{\mu \in \Lambda} a_\mu$ with $a_\mu \in R_\mu$ to a_λ is a ring homomorphism. In general, the *product ring* of the family of rings $\{R_\lambda\}_{\lambda \in \Lambda}$ is defined to be the set

$$\prod_{\lambda \in \Lambda} R_\lambda = \{(a_\lambda)_{\lambda \in \Lambda} \mid a_\lambda \in R_\lambda\}$$

with componentwise addition and multiplication. The identity element in the product ring is the tuple $(e_\lambda)_{\lambda \in \Lambda}$, where $e_\lambda \in R_\lambda$ is the identity element. We may now restate Theorem 3.5 (ii)–(v) as saying that the map

$$p: R \rightarrow \prod_{\lambda \in \Lambda(R)} R_\lambda$$

defined by $p(a) = (p_\lambda(a))_{\lambda \in \Lambda}$ is an isomorphism of rings, and that each of the component rings R_λ is a simple ring.

THEOREM 3.7. *The following statements holds.*

- (i) *Let D be a division ring and let $R = M_n(D)$ be the ring of $n \times n$ -matrices. Then R is a simple ring with the left R -module $S = M_{n,1}(D)$ of column n -vectors as its simple module, and the map*

$$\rho: D \rightarrow \text{End}_R(S)^{\text{op}}$$

defined by $\rho(a)(x) = xa$ is a ring isomorphism.

- (ii) Let R be a simple ring and let S be a simple left R -module. Then S is a finite dimensional right vector space over the division ring $D = \text{End}_R(S)^{\text{op}}$ opposite of the ring of R -linear endomorphisms of S , and the map

$$\lambda: R \rightarrow \text{End}_D(S)$$

defined by $\lambda(a)(x) = ax$ is a ring isomorphism.

Here, in (ii), the ring $\text{End}_R(S)^{\text{op}}$ is a division ring by Schur's lemma.

PROOF. (i) We have proved in Lemma 2.12 that S is a simple R -module. Now, let $e_i \in M_{1,n}(D)$ be the row vector whose i th entry is 1 and whose remaining entries are 0. Then the map $f: S \oplus \cdots \oplus S \rightarrow R$, where there are n summands S , defined by $f(v_1, \dots, v_n) = v_1 e_1 + \cdots + v_n e_n$ is an isomorphism of left R -modules. Indeed, in the $n \times n$ -matrix $v_i e_i$, the i th column is v_i and the remaining columns are zero. This shows that R is a semi-simple ring. By Theorem 3.5 (i), we conclude that every simple left R -module is isomorphic to S . Hence, the ring R is simple.

It is readily verified that the map ρ is a ring homomorphism. Now, the kernel of ρ is a two-sided ideal in the division ring D , and hence, is either zero or all of D . But $\rho(1) = \text{id}_S$ is not zero, so the kernel is zero, and hence the map ρ is injective. It remains to show that ρ is surjective. So let $f: S \rightarrow S$ be an R -linear map. We must show that there exists $a \in D$ such that for all $y \in S$, $f(y) = ya$. To this end, we fix a non-zero element $x \in S$ and choose a matrix $P \in R$ such that $Px = x$ and such that $PS = xD \subset S$. Since f is R -linear, we have

$$f(x) = f(Px) = Pf(x) \in xD$$

which shows that $f(x) = xa$ with $a \in D$. Now, given any $y \in S$, we can find a matrix $A \in R$ such that $Ax = y$. Again, since f is R -linear, we have

$$f(y) = f(Ax) = Af(x) = Axa = ya$$

as desired. This shows that ρ is surjective, and hence, an isomorphism.

(ii) Since R is a simple ring with simple left R -module S , there exists an isomorphism of left R -modules $f: S^n \rightarrow R$ from the direct sum of a finite number n of copies of S onto R . We now have ring isomorphisms

$$R^{\text{op}} \xrightarrow{\sim} \text{End}_R(R) \xrightarrow{\sim} \text{End}_R(S^n) \xrightarrow{\sim} M_n(\text{End}_R(S)) = M_n(D^{\text{op}})$$

where the left-hand isomorphism is given by Remark 2.6, the middle isomorphism is induced by the chosen isomorphism f , and the right-hand isomorphism takes the endomorphism g to the matrix of endomorphisms (g_{ij}) with the endomorphism g_{ij} defined to be the composition $g_{ij} = p_i \circ g \circ i_j$ of the inclusion $i_j: S \rightarrow S^n$ of the j th summand, the endomorphism $g: S^n \rightarrow S^n$, and the projection $p_i: S^n \rightarrow S$ on the i th summand. It follows that we have a ring isomorphism

$$R \xrightarrow{\sim} M_n(D^{\text{op}})^{\text{op}} \xrightarrow{\sim} M_n((D^{\text{op}})^{\text{op}}) = M_n(D)$$

given by the composition of the isomorphism above and the isomorphism that takes the matrix A to its transpose ${}^t A$. This shows that the simple ring R is isomorphic to the simple ring $M_n(D)$ we considered in (i). Therefore, it suffices to show that the map λ is an isomorphism in this case. But this is precisely the statement of Corollary 2.5, so the proof is complete. \square

EXERCISE 3.8. Let D be a division ring, let $R = M_n(D)$, and let $S = M_{n,1}(D)$. We view S as a left R -module and as a right D -vector space.

- (1) Let $x \in S$ be a non-zero vector. Show that there exists a matrix $P \in R$ such that $PS = xD \subset S$. (Hint: Try $x = e_1$ first.)
- (2) Let $x, y \in S$ be non-zero vectors. Show that there exists a matrix $A \in R$ such that $Ax = y$.

REMARK 3.9. The center of a ring R is the subring $Z(R) \subset R$ of all elements $a \in R$ with the property that for all $b \in R$, $ab = ba$; it is a commutative ring. The center $k = Z(D)$ of the division ring D is a field, and it is not difficult to show that also $Z(M_n(D)) = k \cdot I_n$. It is possible for a division ring D to be of infinite dimension over the center k . However, one can show that if D is of finite dimension d over k , then $d = m^2$ is a square and every maximal subfield $E \subset D$ has dimension m over k . For example, the center of the division ring of quaternions \mathbb{H} is the field of real numbers \mathbb{R} and the complex numbers $\mathbb{C} \subset \mathbb{H}$ is a maximal subfield.

It is now high time that we see an example of a semi-simple ring. In general, if k is a commutative ring and G a group, then the group ring $k[G]$ is defined to be the free k -module with basis G and with multiplication

$$\left(\sum_{g \in G} a_g g\right) \cdot \left(\sum_{g \in G} b_g g\right) = \sum_{g \in G} \left(\sum_{\substack{h, k \in G \\ hk = g}} a_h b_k\right) g.$$

We note that $G \subset k[G]$ as the set of basis elements; the unit element $e \in G$ is also the multiplicative unit element in the ring $k[G]$. Moreover, the map $\eta: k \rightarrow k[G]$ defined by $\eta(a) = a \cdot e$ is ring homomorphism. If M is a left $k[G]$ -module, we also say that M is a k -linear representation of the group G .

Let k be a field and let $\eta: \mathbb{Z} \rightarrow k$ be the unique ring homomorphism. We define the characteristic of k to be the unique non-negative integer $\text{char}(k)$ such that $\ker(\eta) = \text{char}(k)\mathbb{Z}$. For example, the fields \mathbb{Q} , \mathbb{R} , and \mathbb{C} have characteristic zero while, for every prime number p , the field $\mathbb{Z}/p\mathbb{Z}$ has characteristic p .

EXERCISE 3.10. Let k be a field. Show that $\text{char}(k)$ is either zero or a prime number, and that every integer n not divisible by $\text{char}(k)$ is invertible in k .

THEOREM 3.11 (Maschke's theorem). *Let k be a field and let G be finite group whose order is not divisible by the characteristic of k . Then the group ring $k[G]$ is a semi-simple ring.*

PROOF. We show that every left $k[G]$ -module M of finite dimension m over k is a semi-simple left $k[G]$ -module. The proof is by induction on m ; the basic case $m = 1$ follows from Example 2.11, since a left $k[G]$ -module of dimension 1 over k is simple as a left k -module, and hence, also as a left $k[G]$ -module. So we let M be a left $k[G]$ -module of dimension $m > 1$ over k and assume, inductively, that every left $k[G]$ -module of smaller dimension is semi-simple. We must show that M is semi-simple. If M is simple, we are done. If M is not simple, there exists a non-zero proper submodule $N \subset M$. We let $i: N \rightarrow M$ be the inclusion and choose a k -linear map $\sigma: M \rightarrow N$ such that $\sigma \circ i = \text{id}_N$. The map σ is not necessarily $k[G]$ -linear. However, we claim that the map $s: M \rightarrow N$ defined by

$$s(x) = \frac{1}{|G|} \sum_{g \in G} g\sigma(g^{-1}x)$$

is $k[G]$ -linear and satisfies $s \circ i = \text{id}_N$. Indeed, s is k -linear and if $h \in G$, then

$$\begin{aligned} s(hx) &= \frac{1}{|G|} \sum_{g \in G} g\sigma(g^{-1}hx) = \frac{1}{|G|} \sum_{g \in G} hh^{-1}g\sigma(g^{-1}hx) \\ &= \frac{1}{|G|} \sum_{k \in G} hk\sigma(k^{-1}x) = hs(x) \end{aligned}$$

which shows that s is $k[G]$ -linear. Moreover, we have

$$\begin{aligned} (s \circ i)(x) &= \frac{1}{|G|} \sum_{g \in G} g\sigma(g^{-1}i(x)) = \frac{1}{|G|} \sum_{g \in G} g\sigma(i(g^{-1}x)) \\ &= \frac{1}{|G|} \sum_{g \in G} gg^{-1}x = x \end{aligned}$$

which shows that $s \circ i = \text{id}_N$. This proves the claim. Now, let P be the kernel of s . The claim shows that M is equal to the direct sum of the submodules $N, P \subset M$. But N and P both have dimension less than m over k , and hence, are semi-simple by the induction hypothesis. This shows that M is semi-simple as desired. \square

EXAMPLE 3.12 (Cyclic groups). To illustrate the theory above, we determine the structure of the group rings $\mathbb{C}[C_n]$, $\mathbb{R}[C_n]$, and $\mathbb{Q}[C_n]$, where C_n is a cyclic group of order n . Theorem 3.11 shows that the three rings are semi-simple rings, and their structure are given by Theorems 3.5 and 3.7 once we identify the corresponding sets of types of simple modules; we proceed to do so. We choose a generator $g \in C_n$ and a primitive n th root of unity $\zeta_n \in \mathbb{C}$.

We first consider the complex group ring $\mathbb{C}[C_n]$. For every $0 \leq k < n$, we define the left $\mathbb{C}[C_n]$ -module $\mathbb{C}(\zeta_n^k)$ to be the sub- \mathbb{C} -vector space $\mathbb{C}(\zeta_n^k) \subset \mathbb{C}$ spanned by the elements ζ_n^{ki} with $0 \leq i < n$ and with the module structure defined by

$$\left(\sum_{i=0}^{n-1} a_i g^i \right) \cdot z = \sum_{i=0}^{n-1} a_i \zeta_n^{ki} z.$$

The left $\mathbb{C}[C_n]$ -module $\mathbb{C}(\zeta_n^k)$ is simple. For as a \mathbb{C} -vector space, $\mathbb{C}(\zeta_n^k) = \mathbb{C}$, and therefore has no non-trivial proper submodules. Suppose that $f: \mathbb{C}(\zeta_n^k) \rightarrow \mathbb{C}(\zeta_n^l)$ is a $\mathbb{C}[C_n]$ -linear isomorphism. Then we have

$$\zeta_n^k f(1) = f(\zeta_n^k) = f(g \cdot 1) = g \cdot f(1) = \zeta_n^l f(1),$$

where the first and third equalities follow from $\mathbb{C}[C_n]$ -linearity. Since $f(1) \neq 0$, we conclude that $k = l$. So the n simple left $\mathbb{C}[C_n]$ -modules $\mathbb{C}(\zeta_n^k)$, $0 \leq k < n$, are pairwise non-isomorphic. Therefore, Theorem 3.5 (i) implies that

$$\mathbb{C}[C_n] = \bigoplus_{k=0}^{n-1} \mathbb{C}(\zeta_n^k)$$

as a left $\mathbb{C}[C_n]$ -module. The endomorphism ring $\text{End}_{\mathbb{C}[C_n]}(\mathbb{C}(\zeta_n^k))$ is isomorphic to the field \mathbb{C} for all $0 \leq k < n$.

We next consider the real group ring $\mathbb{R}[C_n]$. Again, for $0 \leq k < n$, we define the left $\mathbb{R}[C_n]$ -module $\mathbb{R}(\zeta_n^k)$ to be the sub- \mathbb{R} -vector space $\mathbb{R}(\zeta_n^k) \subset \mathbb{C}$ spanned by

the elements ζ_n^{ki} with $0 \leq i < n$ and with the module structure defined by

$$\left(\sum_{i=0}^{n-1} a_i g^i\right) \cdot z = \sum_{i=0}^{n-1} a_i \zeta_n^{ki} z.$$

The left $\mathbb{R}[C_n]$ -module $\mathbb{R}(\zeta_n^k)$ is simple. For if $z, z' \in \mathbb{R}(\zeta_n^k)$ are two non-zero elements, then there exists $\omega \in \mathbb{R}[C_n]$ with $\omega \cdot z = z'$. The dimension of $\mathbb{R}(\zeta_n^k)$ as an \mathbb{R} -vector space is either 1 or 2 according as $\zeta_n^k \in \mathbb{R}$ or $\zeta_n^k \notin \mathbb{R}$. Moreover, we find that the left $\mathbb{R}[C_n]$ -modules $\mathbb{R}(\zeta_n^k)$ and $\mathbb{R}(\zeta_n^l)$ are isomorphic if and only if the complex numbers ζ_n^k and ζ_n^l are conjugate. Again, from Theorem 3.5 (i), we conclude that, as a left $\mathbb{R}[C_n]$ -module,

$$\mathbb{R}[C_n] = \bigoplus_{k=0}^{\lfloor n/2 \rfloor} \mathbb{R}(\zeta_n^k).$$

Here $\lfloor x \rfloor$ is the largest integer less than or equal to x . The ring $\text{End}_{\mathbb{R}[C_n]}(\mathbb{R}(\zeta_n^k))$ is isomorphic to \mathbb{R} , if $k = 0$ or $k = n/2$, and is isomorphic to \mathbb{C} , otherwise.

Finally, we consider the rational group ring $\mathbb{Q}[C_n]$. For all $0 \leq k < n$, we define the left $\mathbb{Q}[C_n]$ -module $\mathbb{Q}(\zeta_n^k)$ to be the sub- \mathbb{Q} -vector space $\mathbb{Q}(\zeta_n^k) \subset \mathbb{C}$ spanned by the elements ζ_n^{ki} with $0 \leq i < n$ and with the module structure defined by

$$\left(\sum_{i=0}^{n-1} a_i g^i\right) \cdot z = \sum_{i=0}^{n-1} a_i \zeta_n^{ki} z.$$

Again, $\mathbb{Q}(\zeta_n^k)$ is a simple left $\mathbb{Q}[C_n]$ -module, since given $z, z' \in \mathbb{Q}(\zeta_n^k)$, there exists an element $\omega \in \mathbb{Q}[C_n]$ with $\omega \cdot z = z'$. Moreover, the simple left $\mathbb{Q}[C_n]$ -modules $\mathbb{Q}(\zeta_n^k)$ and $\mathbb{Q}(\zeta_n^l)$ are isomorphic if and only if

$$\{\zeta_n^{ki} \mid 0 \leq i < n\} = \{\zeta_n^{li} \mid 0 \leq i < n\}$$

as subsets of \mathbb{C} . If this subset has d elements, then d divides n and

$$\{\zeta_n^{ki} \mid 0 \leq i < n\} = \{\zeta_d^i \mid 0 \leq i < d\}$$

with $\zeta_d \in \mathbb{C}$ a primitive d th root of unity. Let $\mathbb{Q}(\zeta_d) \subset \mathbb{C}$ be the left $\mathbb{Q}(\zeta_d)$ -module defined by the sub- \mathbb{Q} -vector space $\mathbb{Q}(\zeta_d) \subset \mathbb{C}$ spanned by the ζ_d^i with $0 \leq i < d$ and with the left $\mathbb{Q}[C_n]$ -module structure defined by

$$\left(\sum_{i=0}^{n-1} z_i g^i\right) \cdot z = \sum_{i=0}^{n-1} a_i \zeta_d^i z.$$

In this case, we have a $\mathbb{Q}[C_n]$ -linear isomorphism

$$f: \mathbb{Q}(\zeta_d) \rightarrow \mathbb{Q}(\zeta_n^k)$$

given by the unique \mathbb{Q} -linear map that takes ζ_d^i to ζ_n^{ki} . One may show, following Gauss, that the dimension of $\mathbb{Q}(\zeta_d)$ as a \mathbb{Q} -vector space is equal to the number $\varphi(d)$ of the integers $1 \leq i \leq d$ that are relatively prime to d . Moreover, since

$$\sum_{d|n} \varphi(d) = n$$

we conclude from Theorem 3.5 (i) that the simple left $\mathbb{Q}[C_n]$ -modules $\mathbb{Q}(\zeta_d)$ with d a divisor of n represent all types of simple left $\mathbb{Q}[C_n]$ -modules. Therefore,

$$\mathbb{Q}[C_n] = \bigoplus_{d|n} \mathbb{Q}(\zeta_d)$$

as a left $\mathbb{Q}[C_n]$ -module. We note that $\mathbb{Q}(\zeta_d) \subset \mathbb{C}$ is a subfield, the d th cyclotomic field over \mathbb{Q} . The endomorphism ring $\text{End}_{\mathbb{Q}[C_n]}(\mathbb{Q}(\zeta_d))^{\text{op}}$ is isomorphic to the field $\mathbb{Q}(\zeta_d)$ for every divisor d of n .

REMARK 3.13 (Modular representation theory). If the characteristic of the field k divides the order of the group G , then the group ring $k[G]$ is not semi-simple, and it is a very difficult problem to understand the structure of this ring. For example, if \mathbb{F}_p is the field with p elements and \mathfrak{S}_p is the symmetric group on p letters, then the structure of the ring $\mathbb{F}_p[\mathfrak{S}_p]$ is understood only for a few primes p .