Pespectives in Mathematical Sciences: Report Problems

Due: Monday, January 27, 2014, in Science Building 1, Room 105.

Problem 1. Let R be a (not necessarily commutative) ring. Show that the map

$$(-)^t \colon M_n(R)^{\mathrm{op}} \to M_n(R^{\mathrm{op}})$$

that takes a matrix $A = (a_{ij})$ to its transpose $A^t = (a_{ji})$ is a ring homomorphism between the indicated rings. Conclude that this map is an isomorphism of rings.

Problem 2. Let D be a division ring, let $R = M_n(D)$, and let $S = M_{n,1}(D)$. We view S as a left R-module and as a right D-vector space.

(i) Let $x \in S$ be a non-zero vector. Show that there exists a matrix $P \in R$ such that $PS = xD \subset S$. (Hint: Try $x = e_1$ first.)
(ii) Let $x, y \in S$ be non-zero vectors. Show that there exists a matrix $A \in R$ such that $Ax = y$.

Problem 3. Let k be a field, let Λ be a set, and let R be the product ring

$$R = \prod_{\lambda \in \Lambda} k = \{(a_\lambda)_{\lambda \in \Lambda} \mid a_\lambda \in k\}.$$

Let $I_\lambda \subset R_\lambda$ be the kernel of the projection map $p_\lambda \colon R \to k$ that takes $a = (a_\lambda)_{\lambda \in \Lambda}$ to a_λ, and let $S_\lambda = R/I_\lambda$ considered as a left R-module.

(i) Show that for every $\lambda \in \Lambda$, the left R-module S_λ is simple.
(ii) Let $\lambda, \mu \in \Lambda$ and suppose that there exists an isomorphism of left R-modules $f \colon S_\lambda \to S_\mu$. Show that $\lambda = \mu$.
(iii) Conclude that the cardinality of the set $\Lambda(R)$ of types of simple left R-modules is greater than or equal to the cardinality of the set Λ.

Problem 4. Let R be a commutative ring and let $\mathfrak{p} \subset R$ be a proper ideal. Show that the following (i)–(ii) are equivalent.

(i) For all elements $a, b \in R$, $ab \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.
(ii) For all ideals $\mathfrak{a}, \mathfrak{b} \subset R$, $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$ implies $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$.

Problem 5. Show that $\mathbb{Z}$ is a Dedekind ring. (Hint: Show that every ideal in $\mathbb{Z}$ is a principal ideal and use Example 4.3.)