Duality

We introduce the classical notions of an inner product on a real vector space and a hermitian inner product on a complex vector space. To more properly explain these notions, we will again consider vector spaces over a general skew field.

Let F be a skew field and let V be a right F-vector space. The dual space of V is the left F-vector space given by the set of all F-linear maps $f\colon V \to F$ equipped with the vector addition and left scalar multiplication given by

$$\begin{aligned}(f+g)(v) &= f(v)+g(v)\\ (a\cdot f)(v) &= a\cdot f(v).\end{aligned}$$

Here $f, g\colon V \to F$ are F-linear maps, $v \in V$, and $a \in F$. Moreover, in the bottom line, the product $a \cdot f(v)$ is the product in F. Let us check that if $f\colon V \to F$ is an F-linear map and $a \in F$, then the map $a \cdot f\colon V \to F$ again is F-linear. So let $v, w \in V$, and let $b \in F$. We have

$$\begin{aligned}(a\cdot f)(v+w) &= a\cdot f(v+w) = a\cdot(f(v)+f(w)) = (a\cdot f(v)) + (a\cdot f(w))\\ &= (a\cdot f)(v) + (a\cdot f(w))\\ (a\cdot f)(v\cdot b) &= a\cdot f(v\cdot b) = a\cdot(f(v)\cdot b) = (a\cdot f(v))\cdot b\\ &= (a\cdot f)(v)\cdot b\end{aligned}$$

as required. We note that the zero vector in V^* is the zero map $0\colon V \to F$.

Exercise 1. Let F be a skew field and let $\varphi\colon V \to W$ be an F-linear map between right F-vector spaces. The dual map of φ is the F-linear map

$$W^* \xrightarrow{\varphi^*} V^*$$

defined by $\varphi^*(g)(v) = g(\varphi(v))$, where $g \in W^*$ and $v \in V$. Verify that the dual map φ^* is an F-linear map between the left F-vector space W^* and V^*.

Proposition 2. *Let F be a skew field, let V be a right F-vector space, and let V^* be the dual left F-vector space. If the finite family $(v_1, \dots, v_n)$ is a basis of V, then the finite family $(v_1^*, \dots, v_n^*)$, where*

$$v_i^*(v_1a_1 + \dots + v_na_n) = a_i,$$

is a basis of V^*; it is called the dual basis of $(v_1, \dots, v_n)$.

Proof. If $f \in V^*$ and $v = v_1a_1 + \dots + v_na_n \in V$, then

$$f(v) = f(\sum_{i=1}^n v_ia_i) = \sum_{i=1}^n f(v_i)a_i = \sum_{i=1}^n f(v_i)v_i^*(v) = (\sum_{i=1}^n f(v_i)\cdot v_i^*)(v),$$

which shows that $(v_1^*, \dots, v_n^*)$ generates V^*. Moreover, if $b_1v_1^* + \dots + b_nv_n^*$ is the zero map $0\colon V \to F$, then for all $1 \leqslant j \leqslant n$, we have

$$b_j = (\sum_{i=1}^n b_iv_i^*)(v_j) = 0(v_j) = 0,$$

which shows that $(v_1^*, \dots, v_n^*)$ is also linearly independent. □

We wish to compare the left F-vector space V^* to the right F-vector space V and in preparation introduce the following notion.

Definition 3. Let F be a skew field. A map $\sigma\colon F \to F$ is an anti-involution if it has the following properties:

(I1) For all $a, b \in F$, $\sigma(a+b) = \sigma(a) + \sigma(b)$.
(I2) For all $a, b \in F$, $\sigma(a \cdot b) = \sigma(b) \cdot \sigma(a)$.
(I3) For all $a \in F$, $\sigma(\sigma(a)) = a$.

Example 4. (1) If F is a field, then $\mathrm{id}_F\colon F \to F$ is an anti-involution, since for all $a, b \in F$, we have $a \cdot b = b \cdot a$. In particular, the identity map $\mathrm{id}_{\mathbb{R}}\colon \mathbb{R} \to \mathbb{R}$ is an anti-involution, as is $\mathrm{id}_{\mathbb{C}}\colon \mathbb{C} \to \mathbb{C}$.

(2) The complex conjugation map $\sigma\colon \mathbb{C} \to \mathbb{C}$ defined by

$$\sigma(a + ib) = a - ib$$

is an anti-involution. Here $a, b \in \mathbb{R}$.

(3) The quaternionic conjugation map $\sigma\colon \mathbb{H} \to \mathbb{H}$ defined by

$$\sigma(a + ib + jc + kd) = a - ib - jc - kd$$

is an anti-involution. Here $a, b, c, d \in \mathbb{R}$.

So let F be a skew field and let $\sigma\colon F \to F$ be an anti-involution. If $(W, +, \cdot\,)$ is a left F-vector space, then the triple $(W, +, \star)$, where for $w \in W$ and $a \in F$,

$$w \star a = \sigma(a) \cdot w,$$

is a right F-vector space. Indeed, if $w \in W$ and $a, b \in F$, then we have

$$(w \star a) \star b = \sigma(b) \cdot (\sigma(a) \cdot w) = (\sigma(b) \cdot \sigma(a)) \cdot w = \sigma(a \cdot b) \cdot w = w \star (a \cdot b)$$

which shows that (V1) holds, and the remaining axioms are verified analogously. We abbreviate and write W^{σ} instead of $(W, +, \star)$.

Exercise 5. Let F be a skew field and let $\sigma\colon F \to F$ be an anti-involution. Let V and W be two finite dimensional right F-vector spaces and let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

be the matrix of an F-linear map $\varphi\colon V \to W$ with respect to bases $(v_1, \dots, v_n)$ of V and $(w_1, \dots, w_m)$ of W. Show that the matrix of the dual map

$$W^{*,\sigma} \xrightarrow{\varphi^*} V^{*,\sigma}$$

with respect to the dual bases $(w_1^*, \dots, w_m^*)$ of $W^{*,\sigma}$ and $(v_1^*, \dots, v_n^*)$ of $V^{*,\sigma}$ is the conjugate transpose matrix

$$A^* = \begin{bmatrix} \sigma(a_{11}) & \sigma(a_{21}) & \cdots & \sigma(a_{m1}) \\ \sigma(a_{12}) & \sigma(a_{22}) & \cdots & \sigma(a_{m2}) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma(a_{1n}) & \sigma(a_{2n}) & \cdots & \sigma(a_{mn}) \end{bmatrix}.$$

We stress that $W^{*,\sigma}$ and $V^{*,\sigma}$ are right F-vector spaces.

Corollary 6. *Let F be a skew field, let $\sigma\colon F \to F$ be an anti-involution, and let V be a right F-vector space. In this situation, the map*

$$V \xrightarrow{\eta} (V^{*,\sigma})^{*,\sigma}$$

defined by $\eta(v)(f) = \sigma(f(v))$ is well-defined and F-linear. It is an isomorphism if the dimension of V is finite.

Proof. To prove that the map η is well-defined, we must show that for all $v \in V$, the map $\eta(v)\colon V^{*,\sigma} \to F$ is F-linear. Now, for all $f, g \in V^{*,\sigma}$ and $a \in F$,

$$\begin{aligned}\eta(v)(f+g) &= \sigma((f+g)(v)) = \sigma(f(v)+g(v)) = \sigma(f(v)) + \sigma(g(v)) \\ &= \eta(v)(f) + \eta(v)(g) \\ \eta(v)(f \star a) &= \eta(v)(\sigma(a)\cdot f) = \sigma((\sigma(a)\cdot f)(v)) = \sigma(\sigma(a)\cdot f(v)) \\ &= \sigma(f(v))\cdot\sigma(\sigma(a)) = \eta(v)(f)\cdot a\end{aligned}$$

as required. Next, to prove that η is F-linear, we must show that for all $v, w \in V$ and $a \in F$, we have $\eta(v+w) = \eta(v) + \eta(w)$ and $\eta(v\cdot a) = \eta(v) \star a$. These are equalities between F-linear maps from $V^{*,\sigma}$ to F, and the calculation

$$\begin{aligned}\eta(v+w)(f) &= \sigma(f(v+w)) = \sigma(f(v)+f(w)) = \sigma(f(v)) + \sigma(f(w)) \\ &= \eta(v)(f) + \eta(w)(f) = (\eta(v)+\eta(w))(f) \\ \eta(v\cdot a)(f) &= \sigma(f(v\cdot a)) = \sigma(f(v)\cdot a) = \sigma(a)\cdot\sigma(f(v)) \\ &= (\sigma(a)\cdot\eta(v))(f) = (\eta(v)\star a)(f)\end{aligned}$$

shows that the respective maps take the same value at every $f \in V^{*,\sigma}$, and hence, are equal. Finally, suppose that V is finite dimensional. To show that the F-linear map $\eta\colon V \to (V^{*,\sigma})^{*,\sigma}$ is an isomorphism, we let $(v_1, \dots, v_n)$ be a basis of V and show that the image family $(\eta(v_1), \dots, \eta(v_n))$ is a basis of $(V^{*,\sigma})^{*,\sigma}$. More precisely, we show that the latter family is equal to the family $(v_1^{**}, \dots, v_n^{**})$, which is a basis of $(V^{*,\sigma})^{*,\sigma}$ by Proposition 2. Now, to show that $\eta(v_j) = v_j^{**}$, we calculate that

$$\eta(v_j)(v_i^*) = \sigma(v_i^*(v_j)) = \sigma(\delta_{ij}) = \delta_{ij} = v_j^{**}(v_i^*),$$

for all $1 \leqslant i \leqslant n$, where δ_{ij} is the Kronecker symbol. This completes the proof. □

The F-linear map $\eta\colon V \to (V^{*,\sigma})^{*,\sigma}$ in Corollary 6 is *canonical* in the sense that it is defined by a formula that does not require additional structure on the right F-vector space V. By contrast, there is no canonical map between V and $V^{*,\sigma}$ and a comparison of these two right F-vector spaces depends on additional structure.

Definition 7. Let F be a skew field, let $\sigma\colon F \to F$ be an anti-involution, and let V be a right F-vector space. A σ-hermitian form on V is an F-linear map

$$V \xrightarrow{\varphi} V^{*,\sigma}$$

with the property that the composite F-linear map

$$V \xrightarrow{\eta} (V^{*,\sigma})^{*,\sigma} \xrightarrow{\varphi^*} V^{*,\sigma}$$

and the map $\varphi\colon V \to V^{*,\sigma}$ are equal. A σ-hermitian form $\varphi\colon V \to V^{*,\sigma}$ is said to be non-degenerate if the map $\varphi\colon V \to V^{*,\sigma}$ is an isomorphism.

We spell out Definition 7 in terms that are easier for the human brain to process. Any map $\varphi\colon V \to V^{*,\sigma}$ determines a map $\langle -,-\rangle\colon V \times V \to F$ defined by

$$\langle v, w\rangle = \varphi(v)(w).$$

Conversely, a map $\langle -,-\rangle\colon V \times V \to F$ determines a map $\varphi\colon V \to V^{*,\sigma}$ defined by the same formula read in reverse, provided that the following hold:

(H1) For all $v, w_1, w_2 \in V$, $\langle v, w_1 + w_2\rangle = \langle v, w_1\rangle + \langle v, w_2\rangle$.
(H2) For all $v, w \in V$ and $b \in F$, $\langle v, w \cdot b\rangle = \langle v, w\rangle \cdot b$.

Indeed, the properties (H1)–(H2) precisely express that for every $v \in V$, the induced map $\varphi(v) = \langle v, -\rangle\colon V \to F$ is F-linear and hence an element of $V^{*,\sigma}$.

Proposition 8. *Let F be a skew field, let $\sigma\colon F \to F$ be an anti-involution, and let V be a right F-vector space. A map $\varphi\colon V \to V^{*,\sigma}$ is a σ-hermitian form if and only if the induced map $\langle -,-\rangle\colon V \times V \to F$ defined by*

$$\langle v, w\rangle = \varphi(v)(w)$$

satisfies both (H1)–(H2) *above and the following* (H3)–(H5):

(H3) *For all $v_1, v_2, w \in V$, $\langle v_1 + v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle$.*
(H4) *For all $a \in F$ and $v, w \in V$, $\langle v \cdot a, w\rangle = \sigma(a) \cdot \langle v, w\rangle$.*
(H5) *For all $v, w \in V$, $\langle w, v\rangle = \sigma(\langle v, w\rangle)$.*

Proof. First, for all $v_1, v_2, w \in V$, the identity

$$\varphi(v_1 + v_2)(w) = \varphi(v_1)(w) + \varphi(v_2)(w)$$

is equivalent to the identity

$$\langle v_1 + v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle;$$

and for all $a \in F$ and $v, w \in V$, the identity

$$\varphi(v \cdot a)(w) = (\varphi(v) \star a)(w) = (\sigma(a) \cdot \varphi(v))(w) = \sigma(a) \cdot \varphi(v)(w)$$

is equivalent to the identity

$$\langle v \cdot a, w\rangle = \sigma(a) \cdot \langle v, w\rangle.$$

This proves the that the map $\varphi\colon V \to V^{*,\sigma}$ is a F-linear if and only if the map induced $\langle -,-\rangle\colon V \times V \to F$ satisfies (H1)–(H4). Finally, the composite map

$$V \xrightarrow{\eta} (V^{*,\sigma})^{*,\sigma} \xrightarrow{\varphi^*} V^{*,\sigma},$$

by definition, is given by

$$(\varphi^* \circ \eta)(v)(w) = \eta(v)(\varphi(w)) = \sigma(\varphi(w)(v)).$$

Therefore, we conclude that for all $v, w \in V$, the identity

$$\varphi(v)(w) = (\varphi^* \circ \eta)(v)(w)$$

is equivalent to the identity

$$\langle v, w\rangle = \sigma(\langle w, v\rangle)$$

as desired. This completes the proof. □

In the following, we will abuse language and also say that a map

$$V \times V \xrightarrow{\langle -,- \rangle} F$$

satisfying the properties (H1)–(H5) in Proposition 8 is a σ-hermitian form on V. We also say that the pair $(V, \langle -,- \rangle)$ is a σ-hermitian space over F, and we say that $v, w \in V$ are orthogonal if $\langle v, w \rangle = 0$. A σ-hermitian space $(V, \langle -,- \rangle)$ is said to be anisotropic if $\langle v, v \rangle = 0$ implies that $v = 0$. The proof of the following result is known as Gram-Schmidt orthogonalization.

Proposition 9. *Let F be a skew field and let $\sigma \colon F \to F$ be an anti-involution. Every anisotropic σ-hermitian space $(V, \langle -,- \rangle)$ of finite dimension over F admits a basis $(v_1, \dots, v_n)$ such that the vectors $v_1, \dots, v_n$ are pairwise orthogonal.*

Proof. Let $(w_1, \dots, w_n)$ be any basis of the right F-vector space V. We claim that the family of vectors $(v_1, \dots, v_n)$ defined, recursively, by

$$v_j = w_j - \sum_{1 \leqslant k < j} v_k \cdot \langle v_k, v_k \rangle^{-1} \cdot \langle v_k, w_j \rangle$$

is a basis of V and that the vectors $v_1, \dots, v_n$ are pairwise orthogonal. First, by induction on $1 \leqslant j \leqslant n$, we see that $(v_1, \dots, v_j)$ and $(w_1, \dots, w_j)$ generate the same subspace of V, which shows that $(v_1, \dots, v_n)$ is a basis. Next, to show that for $1 \leqslant i < j \leqslant n$, we have $\langle v_i, v_j \rangle = 0$, we proceed by induction on j, the case $j = 1$ being trivial. To prove the induction step, we assume that the statement has been proved for $1 \leqslant k < j$ and calculate that for all $1 \leqslant i < j$,

$$\begin{aligned} \langle v_i, v_j \rangle &= \Big\langle v_i, w_j - \sum_{1 \leqslant k < j} v_k \cdot \langle v_k, v_k \rangle^{-1} \cdot \langle v_k, w_j \rangle \Big\rangle \\ &= \langle v_i, w_j \rangle - \sum_{1 \leqslant k < j} \langle v_i, v_k \rangle \cdot \langle v_k, v_k \rangle^{-1} \cdot \langle v_k, w_j \rangle \\ &= \langle v_i, w_j \rangle - \langle v_i, v_i \rangle \cdot \langle v_i, v_i \rangle^{-1} \cdot \langle v_i, w_j \rangle = 0 \end{aligned}$$

as desired. Here the second equality holds by (H2) and the third equality holds by the inductive hypothesis. □

We now specialize to $F = \mathbb{R}$ and $\sigma = \mathrm{id}_{\mathbb{R}}$. In this case, an anisotropic σ-hermitian space $(V, \langle -,- \rangle)$ is said to be a real inner product space if for all $v \in V$,

$$\langle v, v \rangle \geqslant 0.$$

This inequality is meaningful, since $\mathbb{R}$ is an ordered field, as is the square root

$$\|v\| = \langle v, v \rangle^{1/2}.$$

We say that $\|-\| \colon V \to \mathbb{R}$ is the norm associated with the inner product $\langle -,- \rangle$. The following extremely useful result is called the Cauchy-Schwarz inequality.

Proposition 10. *If $(V, \langle -,- \rangle)$ is a real inner product space, then*

$$|\langle v, w \rangle| \leqslant \|v\| \cdot \|w\|$$

for all $v, w \in V$.

Proof. Fixing $v, w \in V$, we have that for all $x \in \mathbb{R}$,

$$\langle v \cdot x + w, v \cdot x + w \rangle \geqslant 0.$$

Indeed, this is immediate from the definition of a real inner product space. Using properties (H1)–(H5) and that multiplication in $\mathbb{R}$ is commutative, we have

$$\langle v, v \rangle x^2 + 2\langle v, w \rangle x + \langle w, w \rangle \geqslant 0$$

for all $x \in \mathbb{R}$. Therefore, by the quadratic formula, we conclude that

$$\Delta = (2\langle v, w \rangle)^2 - 4\langle v, v \rangle \langle w, w \rangle \leqslant 0,$$

from which the statement follows by simple manipulations. □

An inner product $\langle -, - \rangle$ on a real vector space V is the mathematical structure that encodes the geometric notions of length of vectors and angles between them. The following result, called the triangle inequality, justifies the interpretation of the norm $\|-\|$ associated with $\langle -, - \rangle$ as a length measure.

Corollary 11. *If $(V, \langle -, - \rangle)$ is a real inner product space, then*

$$\|v + w\| \leqslant \|v\| + \|w\|$$

for all $v, w \in V$. Here $\|-\|$ is the norm associated with $\langle -, - \rangle$.

Proof. For all $v, w \in V$, we have

$$\begin{aligned} \|v + w\|^2 &= \langle v + w, v + w \rangle \\ &= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle \\ &= \|v\|^2 + 2\langle v, w \rangle + \|w\|^2 \\ &\leqslant \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 \\ &= (\|v\| + \|w\|)^2, \end{aligned}$$

where the inequality holds by Cauchy-Schwarz. Taking square roots, the stated inequality follows. □

If $(V, \langle -, - \rangle)$ is a real inner product space, then we also use the Cauchy-Schwarz inequality to define the angle between two vectors $v, w \in V$ to be the unique real number $0 \leqslant \theta \leqslant \pi$ with the property that

$$\cos \theta = \frac{\langle v, w \rangle}{\|v\|\|w\|}.$$

We next prove the following result, which we refer to as saying that every finite dimensional real inner product space admits an orthonormal basis.

Addendum 12. *Let $(V, \langle -, - \rangle)$ be a real inner product space. If the real vector space V is finite dimensional, then it admits a basis $(u_1, \dots, u_n)$ such that*

$$\langle u_i, u_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

for all $1 \leqslant i, j \leqslant n$.

Proof. By Gram-Schmidt, we know that V admits a basis $(v_1, \dots, v_n)$ such that the vectors $v_1, \dots, v_n$ are pairwise orthogonal. Therefore, if we define

$$u_i = v_i \cdot \|v_i\|^{-1}$$

then the family $(u_1, \dots, u_n)$ again is a basis of V; the vectors $v_1, \dots, v_n$ are pairwise orthogonal; and, in addition, we have for all $1 \leqslant i \leqslant n$ that

$$\langle u_i, u_i \rangle = \langle v_i \cdot \|v_i\|^{-1}, v_i \cdot \|v_i\|^{-1} \rangle = \|v_i\|^{-1} \cdot \langle v_i, v_i \rangle \cdot \|v_i\|^{-1} = 1,$$

completing the proof. □

Proposition 13. *Let $(V, \langle -, - \rangle)$ be a real inner product space. If the real vector space V has finite dimension, then the hermitian form $\langle -, - \rangle$ is non-degenerate.*

Proof. We wish to prove that the induced map

$$V \xrightarrow{\varphi} V^{*,\sigma}$$

is an isomorphism. Here $\sigma = \mathrm{id}_{\mathbb{R}}$. But if $(u_1, \dots, u_n)$ is an orthonormal basis of V, then the family $(\varphi(u_1), \dots, \varphi(u_n))$ is equal to the dual basis $(u_1^*, \dots, u_n^*)$ of $V^{*,\sigma}$. In particular, the map φ is an isomorphism as desired. □

We next consider the case, where $F = \mathbb{C}$ and $\sigma \colon \mathbb{C} \to \mathbb{C}$ is complex conjugation. If $(V, \langle -, - \rangle)$ is a σ-hermitian space, then for every $v \in V$, we have

$$\sigma(\langle v, v \rangle) = \langle v, v \rangle,$$

which shows that the $\langle v, v \rangle$ is a real number. We say that an anisotropic σ-hermitian space $(V, \langle -, - \rangle)$ over $\mathbb{C}$ is an hermitian inner product space if for every $v \in V$,

$$\langle v, v \rangle \geqslant 0.$$

If $\langle -, - \rangle$ is an hermitian inner product on a complex vector space V, then we define the associated norm norm to be the map $\|-\| \colon V \to \mathbb{R}$ defined by

$$\|v\| = \langle v, v \rangle^{1/2}.$$

The modulus of a complex number $a + ib$ is the non-negative real number

$$|a + ib| = (\sigma(a + ib) \cdot (a + ib))^{1/2} = (a^2 + b^2)^{1/2},$$

and with this notion in hand, we have the Cauchy-Schwarz inequality for complex hermitian inner product spaces in the following form.

Proposition 14. *If $(V, \langle -, - \rangle)$ is a complex hermitian inner product space, then*

$$|\langle v, w \rangle| \leqslant \|v\| \cdot \|w\|$$

for all $v, w \in V$.

Proof. We fix $v, w \in V$. If $v = 0$, then there is nothing to prove, so we may assume that $v \neq 0$. We now use the Gram-Schmidt process and write v as a sum

$$v = w\langle w, w \rangle^{-1} \langle w, v \rangle + (v - w\langle w, w \rangle^{-1} \langle w, v \rangle)$$

of two orthogonal vectors. Writing $w' = v - w\langle w, w \rangle^{-1} \langle w, v \rangle$, we have

$$\begin{aligned}
\langle v, v \rangle &= \big\langle w\langle w, w \rangle^{-1} \langle w, v \rangle, w\langle w, w \rangle^{-1} \langle w, v \rangle \big\rangle + \langle w', w' \rangle \\
&= \sigma(\langle w, v \rangle)\sigma(\langle w, w \rangle^{-1})\langle w, w \rangle \langle w, w \rangle^{-1} \langle w, v \rangle + \langle w', w' \rangle \\
&\geqslant \sigma(\langle w, v \rangle)\sigma(\langle w, w \rangle^{-1})\langle w, w \rangle \langle w, w \rangle^{-1} \langle w, v \rangle \\
&= \sigma(\langle w, v \rangle)\langle w, v \rangle \langle w, w \rangle^{-1},
\end{aligned}$$

or equivalently,

$$|\langle v, w\rangle|^2 \leqslant \|v\|^2 \cdot \|w\|^2.$$

Taking square roots, the statement follows. □

Corollary 15. *If $(V, \langle -, -\rangle)$ is a complex hermitian inner product space, then*

$$\|v + w\| \leqslant \|v\| + \|w\|$$

for all $v, w \in V$. Here $\|-\|$ is the norm associated with $\langle -, -\rangle$.

Proof. This is proved as for real inner product spaces but using, in addition to the Cauchy-Schwarz inequality, the following inequality

$$\sigma(a + ib) + (a + ib) = 2a \leqslant 2(a^2 + b^2)^{1/2} = 2|a + ib|^2,$$

valid for every complex number $a + ib$. □

We also see as before that every finite dimensional complex hermitian inner product space $(V, \langle -, -\rangle)$ admits an orthonormal basis and that the hermitian form $\langle -, -\rangle$ is non-degenerate. Finally, we note that the big advantage of an orthonormal basis is that it is easy to determine the coordinates of a vector with respect to the basis. Indeed, we have the following observation.

Proposition 16. *Let $(V, \langle -, -\rangle)$ be either a real inner product space or a complex hermitian inner product space and let $(u_1, \dots, u_n)$ be an orthonormal basis of V. In this situation, the following identity holds for every $v \in V$:*

$$v = u_1\langle u_1, v\rangle + u_2\langle u_2, v\rangle + \cdots + u_n\langle u_n, v\rangle.$$

Proof. Since $(u_1, \dots, u_n)$ is a basis of V, we can write v uniquely as

$$v = u_1a_1 + u_2a_2 + \cdots + u_na_n$$

where $(a_1, \dots, a_n)$ is a family in either $\mathbb{R}$ or $\mathbb{C}$, depending on the situation. But

$$\langle u_i, v\rangle = \langle u_i, u_1a_1 + \cdots + u_na_n\rangle = a_i,$$

since $(u_1, \dots, u_n)$ is an orthonormal basis. □

Example 17. We again let $F = \mathbb{R}$ and σ and give an example of an anisotropic σ-hermitian space $(V, \langle -, -\rangle)$ which is non-degenerate but which is not a real inner product space. This is the Minkowski space $(V, \langle -, -\rangle)$ with $V = \mathbb{R}^n$ and

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

The standard basis $(e_1, \dots, e_n)$ is an orthogonal basis, but $\langle e_i, e_i\rangle$ is equal to either -1 or $+1$ as $i = 1$ and $i > 1$, so it is not an orthonormal basis. In fact, Sylvester's inertia theorem shows that Minkowski space does not admit an orthonormal basis.

Remark 18. Our treatment of complex hermitian inner product spaces extends mutatis mutandis to quaternionic hermitian inner product spaces with the latter defined as follows. Let $F = \mathbb{H}$ be the quaternion skew field and let $\sigma\colon \mathbb{H} \to \mathbb{H}$ be quaternionic conjugation. A quaternionic hermitian inner product space is an anisotropic σ-hermitian space $(V, \langle -, -\rangle)$ over $\mathbb{H}$ such that for all $v \in V$, the real number $\langle v, v\rangle$ is non-negative.