(∅134

$SS^* = E \implies$

$\det(SS^*) = \det(E) = 1$

$\shortparallel$

$\det(S)\det(S^*)$

$\shortparallel$

$\det(S)\det(S)^*$

$\shortparallel$

$|\det(S)|^2$

So $|\det(S)| = 1.$

$(ST)^* = T^* S^*$ (∅135

$= T^{-1} S^{-1} = (ST)^{-1}$

Find orthogonal basis (v_1, v_2) for subspace $U \subset \mathbb{R}^3$ generated by (a_1, a_2): By Gram-Schmidt,

$$v_1 = a_1$$

$$v_2 = a_2 - a_1 \langle a_1, a_1 \rangle^{-1} \langle a_1, a_2 \rangle$$

$$= \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \cdot \frac{1}{9} \cdot 9 = \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix}$$

So orthogonal projection of a on U is the vector

$$P_U(a) = v_1 \langle v_1, v_1 \rangle^{-1} \langle v_1, a \rangle$$
$$+ v_2 \langle v_2, v_2 \rangle^{-1} \langle v_2, a \rangle$$

$$= \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \cdot \frac{1}{9} \cdot 18 + \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix} \cdot \frac{1}{18} \cdot 9$$

$$= \frac{1}{2} \begin{pmatrix} 4 \\ 11 \\ 5 \end{pmatrix}$$

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$$A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{pmatrix}$$

(a) $\chi_A(t) = \det(A - tE)$

$$= \det\begin{pmatrix} 3-t & 1 & -1 \\ 1 & 3-t & -1 \\ -1 & -1 & 5-t \end{pmatrix} \begin{matrix} +(3-t)R_3 \\ +R_3 \\ \end{matrix}$$

$$= \det\begin{pmatrix} 0 & -(2-t) & (5-t)(3-t)-1 \\ 0 & 2-t & 4-t \\ -1 & -1 & 5-t \end{pmatrix} \begin{matrix} +R_2 \\ \\ \end{matrix}$$

$$= \det\begin{pmatrix} 0 & 0 & (5-t)(3-t)+3-t \\ 0 & 2-t & 4-t \\ -1 & -1 & 5-t \end{pmatrix}$$

$$= \det\begin{pmatrix} 0 & 0 & (6-t)(3-t) \\ 0 & 2-t & 4-t \\ -1 & -1 & 5-t \end{pmatrix}$$

$$= -\det\begin{pmatrix} -1 & -1 & 5-t \\ 0 & 2-t & 4-t \\ 0 & 0 & (6-t)(3-t) \end{pmatrix}$$

$$= (2-t)(3-t)(6-t).$$

So three distinct eigenvalues

$\lambda_1 = 2, \ \lambda_2 = 3, \ \lambda_3 = 6.$

(b) Find basis of eigenvectors.

$$(A - 2E)x = 0$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 3 \end{pmatrix} \begin{matrix} \\ +(-1)R_1 \\ +R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{matrix} \\ \\ \times \frac{1}{2} \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} +R_3 \\ \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

General solution is

$$x = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot t$$

so

$$b_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

$$(A - 3E)\,x = 0$$

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{array}{l} \\ + R_3 \\ \\ \end{array}$$

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & -1 & 2 \end{pmatrix} \begin{array}{l} + R_2 \\ \\ + (-1) R_2 \end{array}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \times (-1) \\ \times (-1) \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

General sol. is

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot t$$

so

$$b_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

$$(A - 6E)\, x = 0$$

$$\begin{pmatrix} -3 & 1 & -1 \\ 1 & -3 & -1 \\ -1 & -1 & -1 \end{pmatrix} \begin{matrix} + (-3) R_3 \\ + R_3 \\ \end{matrix}$$

$$\begin{pmatrix} 0 & 4 & 2 \\ 0 & -4 & -2 \\ -1 & -1 & -1 \end{pmatrix} \begin{matrix} \times (1/4) \\ \\ \times (-1) \end{matrix}$$

$$\begin{pmatrix} 0 & 1 & 1/2 \\ 0 & -4 & -2 \\ 1 & 1 & 1 \end{pmatrix} \begin{matrix} \\ + 4 \cdot R_1 \\ + (-1) R_1 \end{matrix}$$

$$\begin{pmatrix} 0 & 1 & 1/2 \\ 0 & 0 & 0 \\ 1 & 0 & 1/2 \end{pmatrix} \quad \text{Ga}$$

$$\begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

General solution

$$x = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix} \cdot t$$

or

$$x = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \cdot t$$

So

$$b_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}.$$

Since eigenvalues distinct, (b_1, b_2, b_3) is an orthogonal basis, hence, an orthonormal basis!

$$\begin{tikzcd}
(e_1, e_2, e_3) \arrow[rrr, dashed, "A"] & & & (e_1, e_2, e_3) \\
& \mathbb{R}^3 \arrow[r, "f"] & \mathbb{R}^3 & \\
& \mathbb{R}^3 \arrow[r, "f"] \arrow[u, "id"] & \mathbb{R}^3 \arrow[u, "id"] & \\
(b_1, b_2, b_3) \arrow[uuu, dashed, "P"] \arrow[rrr, dashed, "D"] & & & (b_1, b_2, b_3) \arrow[uuu, dashed, "P"']
\end{tikzcd}$$

$$P = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{pmatrix}$$

Diagram shows

$$D = P^{-1} A P$$

and since P is an orthogonal matrix,

$$P^{-1} = P^{*}.$$

Also,

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

(a)

$$A = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 5 & -1 \\ 1 & -1 & 5 \end{pmatrix}$$

$$\chi_A(t) = \det(A - tE)$$

$$= \det\begin{pmatrix} 5-t & 1 & 1 \\ 1 & 5-t & -1 \\ 1 & -1 & 5-t \end{pmatrix} \begin{matrix} +(-(5-t))R_3 \\ +(-1)R_3 \\ \end{matrix}$$

$$= \det\begin{pmatrix} 0 & 6-t & 1-(5-t)^2 \\ 0 & 6-t & -(6-t) \\ 1 & -1 & 5-t \end{pmatrix} \begin{matrix} +(-1)R_2 \\ \\ \end{matrix}$$

$$= \det\begin{pmatrix} 0 & 0 & 1-(5-t)^2+6-t \\ 0 & 6-t & -(6-t) \\ 1 & -1 & 5-t \end{pmatrix}$$

$$= \det\begin{pmatrix} 0 & 0 & -(6-t)(3-t) \\ 0 & 6-t & -(6-t) \\ 1 & -1 & 5-t \end{pmatrix}$$

$$= (3-t)(6-t)^2$$

$$(A - 3E)x = 0$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{matrix} +(-2)R_3 \\ +(-1)R_3 \\ \end{matrix}$$

$$\begin{pmatrix} 0 & 3 & -3 \\ 0 & 3 & -3 \\ 1 & -1 & 2 \end{pmatrix} \begin{matrix} \times 1/3 \\ +(-1)R_1 \\ \end{matrix}$$

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & -1 & 2 \end{pmatrix} \begin{matrix} \\ \\ +R_1 \end{matrix}$$

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

General solution

$$x = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \cdot t \rightsquigarrow b_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$(A - 6E)x = 0$$

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \begin{array}{l} \times(-1) \\ + R_1 \\ + R_1 \end{array}$$

$$\begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

General solution:

$$x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} t_1 + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} t_2$$

Basis for eigenspace corresponding to eigenvalue $\lambda = 6$:

$$\left(v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \ v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right)$$

Use Gram-Schmidt to find orthogonal basis

$$(v_1' = v_1, \ v_2').$$

$$v_2' = v_2 - v_1 \cdot \langle v_1, v_1 \rangle^{-1} \langle v_1, v_2 \rangle$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \frac{1}{2} \cdot 1 = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Normalize to get orthonormal basis :

$$\left(b_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \; b_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right)$$

$$\begin{tikzcd}
(e_1, e_2, e_3) \arrow[rrr, "A"] & & & (e_1, e_2, e_3) \\
& \mathbb{R}^3 \arrow[r, "f"] & \mathbb{R}^3 & \\
& \mathbb{R}^3 \arrow[r, "f"] \arrow[u, "id"] & \mathbb{R}^3 \arrow[u, "id"] & \\
(b_1, b_2, b_3) \arrow[rrr, "D"] \arrow[uuu, "P"] & & & (b_1, b_2, b_3) \arrow[uuu, "P"']
\end{tikzcd}$$

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$P = \begin{pmatrix} -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{pmatrix}$$

$$A = P^{-1} D P.$$

(b) The desired matrix is

$$P^{-1} = P^*$$

$$= \begin{pmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{pmatrix}$$

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(a) $A = \begin{pmatrix} -a & 2+2a \\ -a & 1+2a \end{pmatrix}$

$$\chi_A(t) = \det(A - tE)$$

$$= \det \begin{pmatrix} -a-t & 2+2a \\ -a & 1+2a-t \end{pmatrix}$$

$$= (-a-t)(1+2a-t) + a(2+2a)$$

$$= -a - \cancel{2a^2} + at - t - 2at + t^2$$

$$+ 2a + \cancel{2a^2}$$

$$= t^2 - (1+a)t + a$$

$$\Delta = (-(1+a))^2 - 4a$$

$$= 1 + 2a + a^2 - 4a$$

$$= (1-a)^2$$

So if $a \neq 1$, there are two eigenvalues

$$\lambda = \frac{1+a \pm (1-a)}{2}$$

$$= 1, \ a \ ;$$

and if $a=1$, then $\lambda=1$ is only eigenvalue.

(b) If $a \neq 1$, then $\mathbb{R}^2$ automatically has a basis of eigenvectors for A, so A is diagonalizable. If $a=1$, then we need to calc.:

$$(A-E)x=0$$

$$\begin{pmatrix} -2 & 4 \\ -2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$$

General solution

$$x = \begin{pmatrix} 1 \\ -2 \end{pmatrix} t.$$

So, no the eigenspace corresponding to $\lambda = 1$ has dimension 1.