18.727: *Topics in algebraic geometry*

Lars Hesselholt, MW 3-4:30, 4-149.

The course introduces schemes and cohomology of sheaves. Topics include smooth morphisms, sheaves of differentials, the spectral sequence from Cech to derived functor cohomology, and, if time permits, application of the general theory to curves and surfaces.

References

[1] D. Mumford, *The red book of varieties and schemes*, Lecture Notes in Mathematics, vol. 1358, Springer-Verlag, 1988.

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R ring $\rightsquigarrow$ $(X, \mathcal{O}_X)$ ringed space

Point set: $X = \operatorname{Spec} R = \{ \mathfrak{p} \subsetneqq R \text{ prime ideal} \}$

$$\begin{array}{ccc} \Psi & & \Psi \\ x & \longleftrightarrow & \mathfrak{p}_x \end{array}$$

$\ulcorner$ $\mathcal{O}_{X,x} := R_{\mathfrak{p}_x}$ local ring at $x \in X$

$\quad \cup \qquad\quad \cup$

$m_x := \mathfrak{p}_x R_{\mathfrak{p}_x}$ maximal ideal

$k(x) := \mathcal{O}_{X,x} / m_x$ residue field at $x \in X$

$$\begin{array}{ccccc} R & \twoheadrightarrow & R/\mathfrak{p}_x & \hookrightarrow & k(x) \\ \Psi & & & & \Psi \\ f & \longmapsto & & & f(x) \end{array}$$

value of f at $x \in X$

$\mathfrak{p}_x \ni f \iff f(x) = 0 \in k(x)$ $\lrcorner$

(Zariski) topology: closed sets, $I \subset R$ ideal

$V(I) = \{ x \in X \mid \forall f \in I : f(x) = 0 \}$

$\quad\;\; = \{ x \in X \mid \mathfrak{p}_x \supset I \}$

$$\left.\begin{array}{l} \bigcap_{\alpha \in A} V(I_\alpha) = V(\sum_{\alpha \in A} I_\alpha) \\ V(I) \cap V(J) = V(I \cap J) = V(IJ) \end{array}\right\} \text{topology}$$

Pf $V(I) \cup V(J) \subset V(I \cap J) \subset V(IJ)$ trivial

$V(I) \cup V(J) \supset V(IJ) \iff X \setminus (V(I) \cup V(J)) \subset X \setminus V(IJ)$

$x \notin V(I) \cup V(J) \iff \exists f \in I, g \in J: f(x) \neq 0 \neq g(x)$

$\Rightarrow \exists f \in I, g \in J: (fg)(x) = f(x) \cdot g(x) \neq 0$

$\Rightarrow x \notin V(IJ)$. //

Note: $V(I) = V(\sqrt{I})$,

$$\sqrt{I} = \bigcap_{\mathfrak{p} \supset I} \mathfrak{p} \overset{\text{lemma}}{=} \{x \in R \mid x^n \in I, \text{ some } n \geq 1\}.$$

Distinguished open:

$$D(f) = \{x \in X \mid f \notin \mathfrak{p}_x\} \qquad f \in R$$

$$= \{x \in X \mid f(x) \neq 0\}$$

forms a basis of the open sets:

$$X \setminus V(I) = \{x \in X \mid \exists f \in I : f(x) \neq 0\}$$

$$= \bigcup_{f \in I} D(f)$$

<u>Lemma</u> $\operatorname{Spec} R = \bigcup_{\alpha \in A} D(f_\alpha) \iff (f_\alpha \mid \alpha \in A) = R$

Pf $\operatorname{Spec} R = \bigcup_{\alpha\in\mathcal{A}} D(f_\alpha) \iff \forall x\in X,\ \exists \alpha\in\mathcal{A}:\ f_\alpha(x)\neq 0$

$\iff \forall x\in X:\ (f_\alpha \mid \alpha\in\mathcal{A}) \not\subseteq \mathfrak{p}_x \iff (f_\alpha \mid \alpha\in\mathcal{A}) = R$

every ideal is contained in a max'l ideal //

Cor Spec R is quasi-compact.

Pf Suppose $\operatorname{Spec} R = \bigcup_{\alpha\in\mathcal{A}} U_\alpha$; can assume $U_\alpha = D(f_\alpha)$; then $1 \in (f_\alpha \mid \alpha\in\mathcal{A})$, say, $1 = g_1 f_{\alpha_1} + \dots + g_n f_{\alpha_n}$. But then $(f_{\alpha_1},\dots,f_{\alpha_n}) = R$, i.e. $\operatorname{Spec} R = \bigcup_{i=1}^{n} U_{\alpha_i}$. //

Exercise In general, $\bigcup_{\alpha\in\mathcal{A}} D(f_\alpha) = X \setminus V(I)$; describe I.

Structure sheaf $\mathcal{O}_X$: $\quad R_f = \{1, f, f^2, \dots\}^{-1} R$

$D(f) \supset D(g) \iff g \in \sqrt{(f)}$

Pf $D(f) \not\supset D(g) \iff \exists x \in X : x \notin D(f)$ and $x \in D(g) \iff$
$\exists x \in X : f \in \mathfrak{p}_x$ and $g \notin \mathfrak{p}_x \iff g \notin \bigcap_{\mathfrak{p} \ni f} \mathfrak{p} = \sqrt{(f)}$. //

$D(f) \supset D(g) \rightsquigarrow$ <u>canonical</u> ring homomorphism

$$R_f \longrightarrow R_g \, , \qquad \frac{a}{f^i} \longmapsto \frac{a\, b^i}{g^{ni}} \, , \quad g^n = b \cdot f$$

This is well-defined, and if $D(f) \supset D(g) \supset D(h)$, the diagram of can. maps commute:

$$\begin{tikzcd} R_f \arrow[rr] \arrow[dr] & & R_g \arrow[dl] \\ & R_h & \end{tikzcd}$$

So if $D(f) = D(g)$, can. iso. $R_f \overset{\sim}{\rightleftarrows} R_g$.

$x \in D(f) \iff f \notin \mathfrak{p}_x \iff f \in R \setminus \mathfrak{p}_x \rightsquigarrow$
can. map

$$R_f \longrightarrow R_{\mathfrak{p}_x} \, , \qquad \frac{a}{f^n} \longmapsto \frac{a}{f^n} \, .$$

Prop Suppose $D(f) = \bigcup_{\alpha \in \mathcal{U}} D(f_\alpha)$. Then the diagram of can. maps

$$R_f \longrightarrow \prod_{\alpha \in \mathcal{U}} R_{f_\alpha} \rightrightarrows \prod_{\alpha, \beta \in \mathcal{U}} R_{f_\alpha f_\beta}$$

is an equalizer.

Show this in two steps:

<u>Lemma 1</u> If $g \in R_f$ has image $0 \in R_{f_\alpha}$, for all $\alpha \in \mathcal{U}$, then $g = 0$.

Pf Write $g = b/f^n$, let $I = \{c \in R \mid cb = 0\}$; the following are equivalent:

(i) $g = 0 \in R_f$

(ii) $\exists\, m \geq 0 : f^m b = 0 \in R$

(iii) $f \in \sqrt{I}$

(iv) $\mathfrak{p}_x \supset I \Rightarrow f \in \mathfrak{p}_x$

(v) $\forall x \in V(I)$, $f(x) = 0$.

Suppose $g \neq 0 \in R_f$; then $\exists\, x \in V(I) : f(x) \neq 0$. Since cover, $x \in D(f_\alpha)$, some $\alpha \in \mathcal{U}$; consider

$$\begin{tikzcd} g \arrow[rr, mapsto] & & 0 \text{ — by assump.} \\ R_f \arrow[rr] \arrow[dr] & & R_{f_\alpha} \arrow[dl] \\ & R_{\mathfrak{p}_x} & \end{tikzcd}$$

$0 \mapsto 0$ in $R_{\mathfrak{p}_x}$

Hence $b = gf^n \longmapsto 0 \in R_{\mathfrak{P}_x}$, i.e.

$\exists\, c \in R \setminus \mathfrak{P}_x : \ c \cdot b = 0$ i.e. $c \in I$.

But $x \in V(I)$ so $\mathfrak{P}_x \supset I$ ↯ //

<u>Lemma 2</u> Let $g_\alpha \in R_{f_\alpha}$, $\alpha \in A$, and suppose that for all $\alpha, \beta \in A$, g_α and g_β have the same image in $R_{f_\alpha f_\beta}$. Then there exists $g \in R_f$ with image $g_\alpha \in R_{f_\alpha}$, $\alpha \in A$.

<u>Pf</u> Consider only $f = 1$. Choose finite subcover, i.e. $f_1, \dots, f_k \in \{ f_\alpha \mid \alpha \in A \}$ s.t. $(f_1, \dots, f_k) = R$. Suppose we can find $g \in R$ with image $g_i \in R_{f_i}$, $1 \le i \le k$. Then the image g_α' of g in R_{f_α} is equal to g_α, $\forall \alpha \in A$. Indeed, g_α and g_α' have the same image in $R_{f_\alpha f_i}$, $1 \le i \le k$, and hence are equal by lemma 1.

To find $g \in R$, write $g_i = \dfrac{b_i}{f_i^n} \in R_{f_i}$, $1 \le i \le k$.

$$\left.\begin{aligned} g_i &= \frac{b_i}{f_i^n} \in R_{f_i} \\ g_j &= \frac{b_j}{f_j^n} \in R_{f_j} \end{aligned}\right\} \longmapsto$$

$$\frac{b_i \cdot f_j^n}{(f_i f_j)^n} = \frac{b_j\, f_i^n}{(f_i f_j)^n} \in R_{f_i f_j}$$

i.e. $\exists\, m_{ij} \geq 0$ s.t.

$$(f_i f_j)^{m_{ij}} \left[b_i f_j^n - b_j f_i^n \right] = 0 \in R.$$

Choose $M \geq m_{ij}$, $1 \leq i, j \leq k$; let $N = M + n$,
$b_i' = b_i f_i^M$; then

$$g_i = \frac{b_i'}{f_i^N} \in R_{f_i}$$

$$b_i' f_j^N = b_j' f_i^N \in R$$

Since $X = \bigcup_{i=1}^{k} D(f_i) = \bigcup_{i=1}^{k} D(f_i^N)$,
$1 \in (f_1^N, \dots, f_k^N)$, say $1 = h_1 f_1^N + \dots + h_k f_k^N$; let

$$g = h_1 b_1' + \dots + h_k b_k' \in R.$$

Then

$$f_j^N g = \sum_{i=1}^{k} f_j^N b_i' h_i = \sum_{i=1}^{k} f_i^N b_j' h_i = b_j'$$

so

$$g \longmapsto \frac{b_j'}{f_j^N} = g_j \in R_{f_j}. \qquad //$$

For $U \subset X$ open, define

$$\Gamma(U, \mathcal{O}_X) \subset \prod_{x \in U} \mathcal{O}_{X,x}$$

to be the tuples $\{s_x\}_{x \in U}$ s.t. there exists a cover $U = \bigcup_{\alpha \in A} D(f_\alpha)$ and elements $s_\alpha \in R_{f_\alpha}$ s.t. for all $\alpha \in A$ and $x \in D(f_\alpha)$, the can. map $R_{f_\alpha} \to R_{\mathfrak{p}_x}$ takes $s_\alpha \mapsto s_x$. If $U \supset V$

$$\begin{array}{ccc} \prod_{x \in U} \mathcal{O}_{X,x} & \xrightarrow{pr} & \prod_{x \in V} \mathcal{O}_{X,x} \\ \cup & & \cup \\ \Gamma(U, \mathcal{O}_X) & \longrightarrow & \Gamma(V, \mathcal{O}_X) \end{array}$$

so have a presheaf $\mathcal{O}_X$ on X.

<u>Lemma</u> $\mathcal{O}_X$ is a sheaf. //

<u>Prop</u> The canonical map $R_f \to \Gamma(D(f), \mathcal{O}_X)$ is an isomorphism.

<u>Pf</u> The map $R_f \to \prod_{x \in D(f)} \mathcal{O}_{X,x}$ is injective by (the proof of) lemma 1, it lands in $\Gamma(D(f), \mathcal{O}_X)$ by defn, and it is surj. by lemma 2. //

F presheaf on X; stalk at $x \in X$:

$$F_x := \varinjlim_{U \ni x} F(U) \quad \text{— coequalizer:}$$

$$\coprod_{U \supset V \ni x} F(U) \rightrightarrows \coprod_{U \ni x} F(U) \longrightarrow \varinjlim_{U \ni x} F(U)$$

ex $X = \operatorname{Spec} R$, $F = \mathcal{O}_X$; then

$$\mathcal{O}_{X,x} = \varinjlim_{U \ni x} \Gamma(U, \mathcal{O}_X) \xleftarrow{\sim} \varinjlim_{D(f) \ni x} \Gamma(D(f), \mathcal{O}_X)$$

$$\xleftarrow{\sim} \varinjlim_{f \notin \mathfrak{p}_x} R_f \xrightarrow{\sim} R_{\mathfrak{p}_x} . \quad //$$

Lemma A map $F \xrightarrow{\varphi} G$ of sheaves on X is an isomorphism if and only if the induced map of stalks $F_x \xrightarrow{\varphi_x} G_x$ is a bijection, $\forall x \in X$.

pf If $F(U) \xrightarrow{\varphi_U} G(U)$ is a bijection, $\forall U \subset X$, clearly $F_x \xrightarrow{\varphi_x} G_x$ is a bijection, $\forall x \in X$. Show the converse.

$F(U) \xrightarrow{\varphi_U} G(U)$ injective: let $s, s' \in F(U)$ and suppose $\varphi_U(s) = \varphi_U(s') \in G(U)$. Then, $\forall x \in U$, these sections have the same image in G_x:

$$\varphi_x(s_x) = \varphi_U(s)_x = \varphi_U(s')_x = \varphi_x(s'_x).$$

Since φ_x 1-1, $s_x = s'_x$, $\forall x \in U$. Hence, there exists $U \supset V_x \ni x$ s.t. $s|_{V_x} = s'|_{V_x}$. But the subsets $\{V_x\}_{x\in U}$ form an open cover of U, and since s, s' are sheaves, we get $s = s' \in F(U)$.

$F(U) \xrightarrow{\varphi_U} G(U)$ surjective: Let $t \in G(U)$ with image $t_x \in G_x$, $x \in U$. Choose $s_x \in F_x$ s.t. $\varphi_x(s_x) = t_x$, $x \in X$; choose $s_{V_x} \in F(V_x)$, $x \in V_x \subset U$, representing s_x. Then $\varphi_{V_x}(s_{V_x})$ and $t|_{V_x}$ both have image $t_x \in G_x$, so by replacing V_x by a smaller open nbhd, we can assume $\varphi_{V_x}(s_{V_x}) = t|_{V_x}$. The subsets V_x, $x \in U$, cover U, and for all $x, x' \in U$,

$$s_{V_x}|_{V_x \cap V_{x'}} = s_{V_{x'}}|_{V_x \cap V_{x'}}.$$

Indeed, both have image $t|_{V_x \cap V_{x'}} \in G(V_x \cap V_{x'})$ and $\varphi_{V_x \cap V_{x'}}$ is injective. Since F is a sheaf, there exists $s \in F(U)$ s.t. $s|_{V_x} = s_{V_x}$, $x \in U$. Hence $\varphi(s)_x = \varphi_x(s_x) = t_x$, so $\varphi(s) = t$. //

A map $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$ of ringed spaces is:

(i) a cts. map $X \xrightarrow{f} Y$;

(ii) for all $V \subset Y$ open, a ring homomorphism

$$\Gamma(V, \mathcal{O}_Y) \xrightarrow{f_V^*} \Gamma(f^{-1}(V), \mathcal{O}_X)$$

s.t. for all $V' \subset V$, the diagr. commutes:

$$\begin{array}{ccc} \Gamma(V, \mathcal{O}_Y) & \xrightarrow{f_V^*} & \Gamma(f^{-1}(V), \mathcal{O}_X) \\ \downarrow \text{res} & & \downarrow \text{res} \\ \Gamma(V', \mathcal{O}_Y) & \xrightarrow{f_{V'}^*} & \Gamma(f^{-1}(V'), \mathcal{O}_X) \end{array}.$$

Induced map of stalks: if $y = f(x)$,

$$\mathcal{O}_{Y,y} \xrightarrow{f_x^*} \mathcal{O}_{X,x}$$

is the composite:

$$\varinjlim_{V \ni y} \Gamma(V, \mathcal{O}_Y) \longrightarrow \varinjlim_{V \ni y} \Gamma(f^{-1}(V), \mathcal{O}_X)$$

$$\xrightarrow{\text{can}} \varinjlim_{U \ni x} \Gamma(U, \mathcal{O}_X) \qquad \left[\begin{array}{c} V \ni f(x) \iff \\ f^{-1}(V) \ni x \end{array} \right]$$

<u>Def</u> A <u>scheme</u> is a ringed space $(X, \mathcal{O}_X)$ for which there exists an open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ s.t. for all $\alpha \in \mathcal{A}$, $(U_\alpha, \mathcal{O}_X|U_\alpha)$ is isomorphic to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$, for some ring R.

$$
\begin{array}{lcl}
\mathcal{O}_{X,x} & = & \text{local ring at } x \in X \quad (\text{why local?}) \\
\cup & & \\
m_x & = & \text{max'l ideal} \\
k(x) & = & \text{residue field at } x \in X
\end{array}
$$

$$
\begin{array}{ccccc}
\Gamma(U, \mathcal{O}_X) & \longrightarrow & \mathcal{O}_{X,x} & \twoheadrightarrow & k(x) \\
\Psi & & & & \Psi \\
a & \longmapsto & & & a(x)
\end{array}
\qquad \text{value of } a \text{ at } x \in X
$$

Def A morphism of schemes is a map of ringed spaces $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$ s.t. for all $x \in X$, the induced map of stalks

$$\mathcal{O}_{Y,y} \xrightarrow{f_x^*} \mathcal{O}_{X,x} \quad , \; y = f(x),$$

is a local homomorphism.

$$
\begin{array}{ccc}
\mathcal{O}_{Y,y} & \xrightarrow{f_x^*} & \mathcal{O}_{X,x} \\
\cup & & \cup \\
m_y & \dashrightarrow & m_x \\
& & \\
k(y) & \xhookrightarrow{\bar{f}_x^*} & k(x)
\end{array}
$$

$f_x^*(m_y) \subset m_x \iff$

$m_y \subset (f_x^*)^{-1}(m_x) \iff$

$m_y = (f_x^*)^{-1}(m_x)$

$$
\begin{array}{ccc}
a \in \Gamma(V, \mathcal{O}_Y) & \xrightarrow{f_V^*} & \Gamma(f^{-1}(V), \mathcal{O}_X) \\
\downarrow & & \downarrow \\
\mathcal{O}_{Y,y} & & \mathcal{O}_{X,x} \\
\downarrow & & \downarrow \\
k(y) & \hookrightarrow & k(x)
\end{array}
$$

$$a(f(x)) = 0 \iff f_V^*(a)(x) = 0$$

$R \xrightarrow{\varphi} S \rightsquigarrow (\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S}) \xrightarrow{f} (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$:

$f(x) = y \underset{\text{def}}{\iff} \mathfrak{p}_y = f^{-1}(\mathfrak{p}_x)$

<u>Claim</u> $f^{-1}(V(I)) = V(\varphi(I)S)$ $\quad (\Rightarrow f$ cts.$)$.

$x \in f^{-1}(V(I)) \iff f(x) \in V(I) \iff \forall a \in I : a(f(x)) = 0$

$\iff \forall a \in I : \varphi(a)(x) = 0 \iff x \in V(\varphi(I)S)$.

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \downarrow & & \downarrow \\ k(x) & \hookrightarrow & k(y) \end{array}$$

If $U = D(a)$, $f^{-1}(U) = D(\varphi(a))$; define

$$R_a \xrightarrow{f_U^*} S_{\varphi(a)}, \quad \frac{c}{a^n} \longmapsto \frac{\varphi(c)}{\varphi(a)^n}$$

Then

$$\begin{array}{ccc} R_a & \xrightarrow{f^*_{D(a)}} & S_{\varphi(a)} \\ \downarrow{\scriptstyle \text{can}} & \circlearrowleft & \downarrow{\scriptstyle \text{can}} \\ R_{ab} & \xrightarrow{f^*_{D(ab)}} & S_{\varphi(ab)} = S_{\varphi(a)\varphi(b)} \end{array}$$

commutes, so map of sheaves.

$$\begin{array}{ccc} \mathcal{O}_{Y,y} & \longrightarrow & \mathcal{O}_{X,x} \\ \| & & \| \\ R_{\mathfrak{p}_y} & \longrightarrow & R_{\mathfrak{p}_x} \\ \cup & & \cup \\ \mathfrak{p}_y R_{\mathfrak{p}_y} & \dashrightarrow & \mathfrak{p}_x R_{\mathfrak{p}_x} \end{array}$$

$y = f(x)$

$\mathfrak{p}_y = f^{-1}(\mathfrak{p}_x)$

so local hom.

Prop The map

$$\begin{array}{ccc} \mathrm{Hom}_{Sch}(X, \operatorname{Spec} R) & \xrightarrow{\sim} & \mathrm{Hom}_{Rng}(R, \Gamma(X, \mathcal{O}_X)) \\ \psi & & \psi \\ f & \longmapsto & f^*_Y \end{array} \qquad Y = \operatorname{Spec} R$$

is a natural bijection.

Pf Injective: show f^*_Y determines f

$$\begin{tikzcd} R \arrow[r, "f^*_Y"] \arrow[d] & \Gamma(X, \mathcal{O}_Y) \arrow[d] \\ \mathcal{O}_{Y,f(x)} \arrow[r, "f^*_x"] \arrow[d] & \mathcal{O}_{X,x} \arrow[d] \\ k(f(x)) \arrow[r, hook] & k(x) \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, mapsto] \arrow[d, mapsto] & f^*_Y(a) \arrow[d, mapsto] \\ a(f(x)) \arrow[r, mapsto] & f^*_Y(a)(x) \end{tikzcd} \quad \Rightarrow$$

$$\mathfrak{p}_{f(x)} = \{a \in R \mid a(f(x)) = 0\} = \{a \in R \mid f^*_Y(a)(x)\}.$$

so f^*_Y determines $X \xrightarrow{f} Y$.

$$\begin{tikzcd} R = \Gamma(Y, \mathcal{O}_Y) \arrow[r, "f^*_Y"] \arrow[d, "\text{localization}"'] & \Gamma(X, \mathcal{O}_X) \arrow[d] \\ R_b = \Gamma(D(b), \mathcal{O}_Y) \arrow[r, "f^*_{D(b)}"] & \Gamma(f^{-1}(D(b)), \mathcal{O}_X) \end{tikzcd}$$

so f^*_Y determines $f^*_{D(b)}$, $\forall b \in R$, hence f^*_V for all $V \subset Y$ open.

Surjective: if $X = \operatorname{Spec} S$, $R \xrightarrow{\varphi} S$ defines $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$ and $f_Y^* = \varphi$. In general, choose $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ open affine cover of X. Then $R \xrightarrow{\varphi} \Gamma(X, \mathcal{O}_X)$ induces

$$\varphi_\alpha : R \to \Gamma(X, \mathcal{O}_X) \xrightarrow{\text{res}} \Gamma(X_\alpha, \mathcal{O}_X)$$

Let $f_\alpha : (X_\alpha, \mathcal{O}_X|_{X_\alpha}) \to (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ be the corresponding morphism. Then f_α and f_β agree on $X_\alpha \cap X_\beta$. For the homomorphisms

$$\begin{tikzcd}
 & \Gamma(X_\alpha, \mathcal{O}_X) \arrow[dr] & \\
R \arrow[ur, "\varphi_\alpha"] \arrow[dr, "\varphi_\beta"'] & & \Gamma(X_\alpha \cap X_\beta, \mathcal{O}_X) \\
 & \Gamma(X_\beta, \mathcal{O}_X) \arrow[ur] &
\end{tikzcd}$$

are equal and determine the corresponding morphisms (by the 1-1 statement). Hence, we can glue the f_α to get f. //

ex The global sections functor is representable:

$$\operatorname{Hom}_{\text{Sch}}(X, \mathbb{A}^1_{\mathbb{Z}}) \xrightarrow{\sim} \operatorname{Hom}_{\text{Rng}}(\mathbb{Z}[x], \Gamma(X, \mathcal{O}_X)) \xrightarrow{\sim} \Gamma(X, \mathcal{O}_X).$$
$$\varphi \longmapsto \varphi(x)$$

18.727: Problem Set 1

Due: No, but you should do it anyway. Really!

1. (The following is proved in Mumford, but try to do it yourself.) Let X be the prime spectrum of a ring. Show that the closure $\{x\}^-$ of the one-point set $\{x\} \subset X$ is an irreducible closed subset, i.e that it cannot be written as the union of two proper closed subsets. Show that x is a generic point of $\{x\}^-$, i.e. that the only closed subset of $\{x\}^-$ which contains x is the whole set. Show that every irreducible closed subset of X is of the form $\{x\}^-$ and that x is its unique generic point. Conclude that the assignment $x \mapsto \{x\}^-$ defines a one-to-one correspondance between the points of X and the irreducible closed subsets of X.

2. Let X be a space, let $x \in X$ be a point, and let E be a set. We define the *skyscraber sheaf* $i_{x*}E$ on X by assigning to $U \subset X$ the set E, if $x \in U$, and a one-point set, if $x \notin U$. Show that $i_{x*}E$ is indeed a sheaf. Then show that the functor $E \mapsto i_{x*}E$ is right adjoint to the functor $F \mapsto F_x$, which to a sheaf of sets F on X assigns the stalk in the point x. (This means that there is natural one-to-one correspondance between maps of sets $F_x \to E$ and maps of sheaves $F \to i_{x*}E$.) Finally, show that the stalk of $i_{x*}E$ at $x' \in X$ is E, if $x' \in \{x\}^-$, and a one-point set, otherwise.

3. Show that $\operatorname{Spec} R$ is connected if and only if R does not contain non-trivial idempotents. (A non-trivial idempotent is an element $e \neq 1$ such that $e^2 = e$.)

4. Let $(X, \mathcal{O}_X)$ be a scheme, let $f \in \Gamma(X, \mathcal{O}_X)$, and define X_f to be the set of points $x \in X$ such that $f(x) \neq 0 \in k(x)$.

(a) If U is an affine open of X, and if $f|_U$ is the image of f in $\Gamma(U, \mathcal{O}_X)$, show that $U \cap X_f = D(f|_U)$. Conclude that $X_f \subset X$ is open.

(b) Assume that X is quasi-compact. Suppose that the restriction of $a \in \Gamma(X, \mathcal{O}_X)$ to $\Gamma(X_f, \mathcal{O}_X)$ is zero. Show that for some $n \geq 0$, $f^n a = 0$.

(c) Assume in addition that X is quasi-separated, i.e. that the intersection of two affine open subsets $U, V \subset X$ is quasi-compact. Let $b \in \Gamma(X_f, \mathcal{O}_X)$. Show that for some $n \geq 0$, $f^n b$ is the restriction of an element of $\Gamma(X, \mathcal{O}_X)$.

(c) Conclude that if X is quasi-compact and quasi-separated, then the restriction induces an isomorphism

$$\Gamma(X, \mathcal{O}_X)_f \xrightarrow{\sim} \Gamma(X_f, \mathcal{O}_X).$$

Let I be an (index) category. An I-<u>diagram</u> in a category C is a functor $I \xrightarrow{X} C$. Sometimes write $\{X_\lambda\}_{\lambda \in I}$.

ex 1) ob $I = \{1, 2, 3, \ldots\}$; one morphism $i \to j$ if $i \le j$; an I-diagram is:

$$X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow \cdots$$

not writing composite morphisms.

2) ob $I = \{\emptyset \neq S \subset \{1, 2, \ldots, n\}\}$; one morphism $S \to S'$ if $S \supset S'$; an I-diagram is a punctured n-cube.

$n = 2$:

$$\begin{tikzcd} X_{12} \arrow[r] \arrow[d] & X_1 \\ X_2 & \end{tikzcd}$$

$n = 3$:

$$\begin{tikzcd} X_{123} \arrow[rr] \arrow[dr] \arrow[dd] & & X_{13} \arrow[dr] \arrow[dd] & \\ & X_{12} \arrow[rr] \arrow[dd] & & X_1 \\ X_{23} \arrow[rr] \arrow[dr] & & X_3 & \\ & X_2 & & \end{tikzcd}$$

3) ob $I = \{*\}$; mor $I = G$ group; an I-diagram is an object $X = X(*)$ with a G-action.

An object $X \in C$ is a <u>colimit</u> of $\{X_\lambda\}_{\lambda \in I}$ if

(1) for all $\lambda \in I$, there is a morphism

$$X_\lambda \xrightarrow{f_\lambda} X$$

s.t. for all $\lambda \xrightarrow{\varphi} \mu \in I$, the diagr.

$$\begin{tikzcd} X_\lambda \arrow[d, dashed, "X_\varphi"'] \arrow[dr, "f_\lambda"] & \\ X_\mu \arrow[r, "f_\mu"'] & X \end{tikzcd}$$

commutes;

(2) X is universal with the property (1).
(this means: given $X_\lambda \xrightarrow{f'_\lambda} X'$, $\lambda \in I$, s.t. $f'_\lambda = f'_\mu \circ X_\varphi$, $\lambda \xrightarrow{\varphi} \mu \in I$, there exists a unique $X \xrightarrow{h} X'$ s.t. $f'_\lambda = h \circ f_\lambda$, $\lambda \in I$.)

<u>Prop</u> Every diagram $\{(X_\lambda, \mathcal{O}_{X_\lambda})\}_{\lambda \in I}$ of ringed spaces has a colimit $(X, \mathcal{O}_X)$.

<u>pf</u> point set:

$$X = \Big(\coprod_{\lambda \in I} X_\lambda\Big) / \sim \;; \quad \sim \text{ eq. rel. gen. by:}$$

for all $\lambda \xrightarrow{\varphi} \mu \in I$. $x \in X_\lambda \sim X_\varphi(x) \in X_\mu$

⌜

$$\coprod_{\lambda \to \mu \in I} X_\lambda \rightrightarrows \coprod_{\lambda \in I} X_\lambda \to X \quad \text{coequalizer}$$

⌟

$$f_\lambda : X_\lambda \xrightarrow{\text{incl}} \coprod_{\lambda \in I} X_\lambda \twoheadrightarrow X .$$

topology:

$$U \subset X \text{ open } \underset{\text{def}}{\iff} f_\lambda^{-1}(U) \subset X_\lambda \text{ open}, \ \forall \lambda \in I .$$

sheaf:

⌜ a map $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$ of ringed spaces is a cts. map $X \xrightarrow{f} Y$ plus a map of sheaves $\mathcal{O}_Y \to f_* \mathcal{O}_X$, where $f_* \mathcal{O}_X$ is the <u>direct image</u> sheaf:

$$\Gamma(V, f_* \mathcal{O}_X) := \Gamma(f^{-1}(V), \mathcal{O}_X) . ⌟$$

$$\Gamma(U, \mathcal{O}_X) \underset{\text{def}}{\longrightarrow} \prod_{\lambda \in I} \Gamma(U, f_{\lambda *} \mathcal{O}_{X_\lambda}) \rightrightarrows \prod_{\lambda \to \mu \in I} \Gamma(U, f_{\lambda *} \mathcal{O}_{X_\lambda})$$

equalizer: $s \in \Gamma(U, \mathcal{O}_X)$ is by def$^{\underline{n}}$:

for all $\lambda \in I$, a section

$$s_\lambda \in \Gamma(U, f_{\lambda *} \mathcal{O}_{X_\lambda}) = \Gamma(f_\lambda^{-1}(U), \mathcal{O}_{X_\lambda})$$

s.t. for all $\lambda \xrightarrow{\varphi} \mu \in I$,

$$\begin{array}{ccc} \Gamma(U, f_{\lambda *} \mathcal{O}_{X_\lambda}) & \xrightarrow{X_{\varphi *}} & \Gamma(U, f_{\mu *} \mathcal{O}_{X_\mu}) \\ \Psi & & \Psi \\ s_\lambda & \longmapsto & s_\mu \end{array}$$

Then $\mathcal{O}_X$ is a sheaf of rings on X.

Universal property: given $(X_\lambda, \mathcal{O}_{X_\lambda}) \xrightarrow{f'_\lambda} (X', \mathcal{O}_{X'})$, $\lambda \in I$, s.t. $f'_\lambda = f'_\mu \circ X_\varphi$, $\lambda \xrightarrow{\varphi} \mu$. Then the set map

$$\coprod_{\lambda \in I} X_\lambda \xrightarrow{\amalg f'_\lambda} X'$$

factors through the eq. rel. $\sim$ to give a set map $X \xrightarrow{h} X'$ s.t. $f'_\lambda = h \circ f_\lambda$, and h is unique with this property. The topology on X is defined s.t. h is continuous. Must show that there is a unique map of sheaves on X',

$$\mathcal{O}_{X'} \longrightarrow h_* \mathcal{O}_X$$

s.t. for all $\lambda \in I$, the diagr. commutes:

$$\begin{array}{ccc} \mathcal{O}_{X'} & \longrightarrow & h_* \mathcal{O}_X \\ \downarrow & & \downarrow \\ f'_{\lambda *} \mathcal{O}_{X_\lambda} & = & h_* f_{\lambda *} \mathcal{O}_{X_\lambda} \end{array}.$$

This is easy. //

ex I discrete $\rightsquigarrow$ disjoint union.

Lemma

$$\begin{tikzcd}
(\operatorname{Spec} R_f, \mathcal{O}_{\operatorname{Spec} R_f}) \arrow[r] \arrow[d, "\sim"] & (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) \\
(D(f), \mathcal{O}_{\operatorname{Spec} R}|_{D(f)}) \arrow[ur] &
\end{tikzcd}$$

Pf $R \xrightarrow{i} R_f$ loc.

$$\operatorname{Spec} R_f \xrightarrow{\sim} D(f) = \{ \mathfrak{p} \subset R \mid \mathfrak{p} \not\ni f \}$$

$$\mathfrak{p} \longmapsto i^{-1}(\mathfrak{p})$$

(inverse : $\mathfrak{p} R_f \longleftarrow\!\shortmid \ \mathfrak{p}$)

$$\begin{tikzcd}
\operatorname{Spec} R_f \arrow[r, "\sim"] & D(f) \\
D(g) \arrow[u, phantom, "\cup"] \arrow[r, dashed, "\sim"] & D(fg) \arrow[u, phantom, "\cup"]
\end{tikzcd} \quad \text{homeomorphism}$$

$$\begin{tikzcd}
\Gamma(D(g), \mathcal{O}_{\operatorname{Spec} R_f}) & \Gamma(D(fg), \mathcal{O}_{\operatorname{Spec} R}) \arrow[l, "\sim"'] \\
(R_f)_g \arrow[u, "\sim"] & R_{fg} \arrow[l, "can", "\sim"'] \arrow[u, "\sim"]
\end{tikzcd}$$

//

Cor Let $(X, \mathcal{O}_X)$ be a scheme, let $U \subset X$ be an open subset. Then $(U, \mathcal{O}_X|_U)$ is a scheme. //

Prop (Gluing) Let $\{X_i\}$ be a family of schemes, for all i,j, let $U_{ij} \subset X_i$ be an open subset, and for all i,j, let $\varphi_{ij} : U_{ij} \longrightarrow U_{ji}$ be an iso. of schemes s.t.

(1) for all i,j, $\varphi_{ij} = \varphi_{ji}^{-1}$;

(2) for all i,j,k, $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$.

Then there exists a scheme X and morphisms $X_i \xrightarrow{f_i} X$ s.t. $f_i = f_j \circ \varphi_{ij}$ on U_{ij}, and X is universal with this property.

pf We have a diagr. of the form

$$\begin{tikzcd} i & j \\ (i,j) \arrow[u] \arrow[r, shift left] & (j,i) \arrow[u] \arrow[l, shift left] \end{tikzcd} \qquad \begin{tikzcd} X_i & X_j \\ U_{ij} \arrow[u, hook] \arrow[r, shift left, "\varphi_{ij}"] & U_{ji} \arrow[u, hook] \arrow[l, shift left, "\varphi_{ji}"] \end{tikzcd}$$

and wish to show that it has a colimit. Let $(X, \mathcal{O}_X)$ be the colimit as ringed spaces. The condition (2) implies that

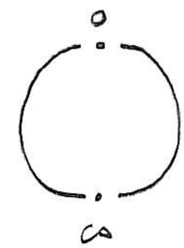
$$(X_i, \mathcal{O}_{X_i}) \xrightarrow[\sim]{f_i} (f(X_i), \mathcal{O}_X|_{f(X_i)}).$$

(The relation defining X is an eq. relation.)

Hence X has an open cover $\{f(X_i)\}$ by schemes so is itself a scheme. //

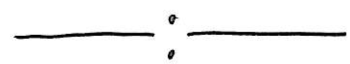
ex 1)

$$\begin{tikzcd}
\operatorname{Spec} k[t,t^{-1}] \arrow[r, hook, "t \,\leftarrow\!\shortmid\, x"] \arrow[d, hook, "t^{-1} \,\leftarrow\!\shortmid\, y"'] & \operatorname{Spec} k[x] \arrow[d, hook] \\
\operatorname{Spec} k[y] \arrow[r, hook] & \mathbb{P}^1_k
\end{tikzcd}$$



2)

$$\begin{tikzcd}
\operatorname{Spec} k[t,t^{-1}] \arrow[r, hook, "t \,\leftarrow\!\shortmid\, x"] \arrow[d, hook, "t \,\leftarrow\!\shortmid\, y"'] & \operatorname{Spec} k[x] \arrow[d, hook] \\
\operatorname{Spec} k[y] \arrow[r, hook] & X
\end{tikzcd}$$



3) $Y = \operatorname{Spec} \mathcal{O}$, $\mathcal{O}$ noetherian local ring;

$X = Y \setminus \{x\}$, $\mathfrak{p}_x = \mathfrak{m} \subset \mathcal{O}$

$\ulcorner$ $V(\mathfrak{p}_x) = \overline{\{x\}}$: $x \in V(I) \Leftrightarrow \mathfrak{p}_x \supset I \Rightarrow V(\mathfrak{p}_x) \subset V(I)$.

$\{x\} = \overline{\{x\}} \Longleftrightarrow \mathfrak{p}_x = \{\mathfrak{p} \subset R \mid \mathfrak{p} \supset \mathfrak{p}_x\} \Leftrightarrow \mathfrak{p}_x$ max'l $\lrcorner$

If $\mathfrak{p}_x = \sqrt{(f_1, \dots, f_n)}$, then

$$\{x\} = V(\mathfrak{p}_x) = \{y \in Y \mid f_1(y) = \dots = f_n(y) = 0\}$$
$$= \bigcap_{i=1}^{n} \{y \in Y \mid f_i(y) = 0\}$$

so

$$X = \bigcup_{i=1}^{n} D(f_i) = \bigcup_{i=1}^{n} \operatorname{Spec} \mathcal{O}_{f_i}$$

Universal property of fiber product:

$$\begin{tikzcd} Z \arrow[dr, dashed, "\exists !"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_S Y \arrow[r, "p"] \arrow[d, "q"] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"] & S \end{tikzcd}$$

<u>Lemma 1</u> If $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, and $S = \operatorname{Spec} C$ are affine, then $X \times_S Y = \operatorname{Spec} A \otimes_C B$.

pf We may assume Z affine; for a morphism from a scheme Z to an affine scheme factors uniquely through the canonical map

$$Z \to \operatorname{Spec} \Gamma(Z, \mathcal{O}_Z).$$

Hence the lemma is equivalent to the statement that the tensor product is characterized by the universal property:

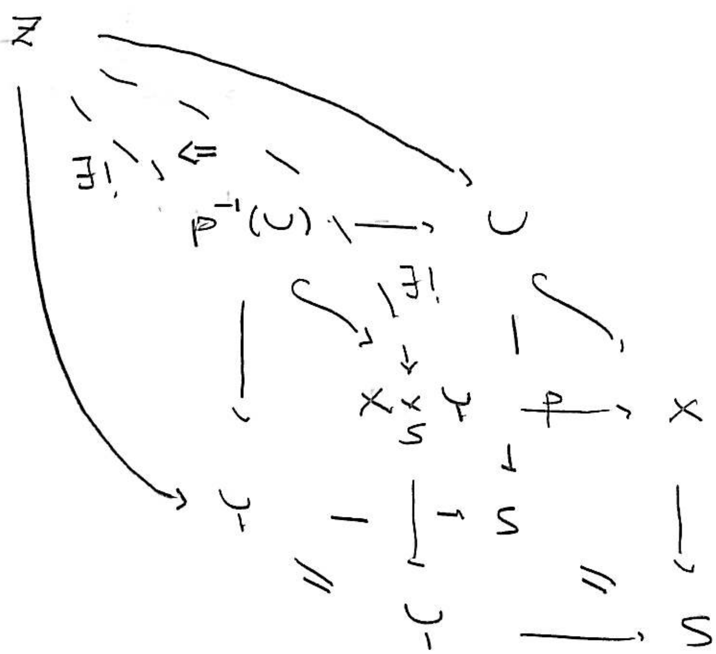
$$\begin{tikzcd} D & & \\ & A \otimes_C B \arrow[ul, dashed, "\exists !"'] & A \arrow[l] \arrow[ull, bend right] \\ & B \arrow[u] \arrow[uul, bend left] & C \arrow[l] \arrow[u] \end{tikzcd}$$

//

Use universal property to show that we can glue fiber products of pieces of affine coverings:

<u>Lemma 2</u> Suppose $X \times_S Y$ exists and let $U \subset X$ be an open subset. Then $p^{-1}(U) = U \times_S Y$.

Pf Since $p^{-1}(U) \subset X \times_S Y$ is open, it has a unique scheme structure s.t. $p^{-1}(U) \hookrightarrow X \times_S Y$ is a morphism. Show $p^{-1}(U)$ has univeral property defining $U \times_S Y$; this follow from the diagr:



<u>Lemma 3</u> Let $\{X_i\}_{i \in I}$ be an open cover of X and suppose that for all $i \in I$, $X_i \times_S Y$ exists. Then $X \times_S Y$ exists.

Pf Let $U_{ij} = p_i^{-1}(X_i \cap X_j) \subset X_i \times_S Y$. Then U_{ij} and U_{ji} both represent $(X_i \cap X_j) \times_S Y$. Let $\varphi_{ij} : U_{ij} \to U_{ji}$ be the can. iso. Then

$$\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$$

and since $U_{ij} \cap U_{ik}$, $U_{ji} \cap U_{jk}$ and $U_{ki} \cap U_{kj}$ all repr. $(X_i \cap X_j \cap X_k) \times_S Y$, the diagram

$$\begin{tikzcd} U_{ij} \cap U_{ik} \arrow[rr, "\varphi_{ij}"] \arrow[dr, "\varphi_{ik}"'] & & U_{ji} \cap U_{jk} \arrow[dl, "\varphi_{jk}"] \\ & U_{ki} \cap U_{kj} & \end{tikzcd}$$

commutes. So we can glue the schemes $X_i \times_S Y$ to get a scheme $X \times_S Y$. Check universal property: given

$$\begin{tikzcd} Z \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\ & X \times_S Y \arrow[r] \arrow[d] & X \arrow[d] \\ & Y \arrow[r] & S \end{tikzcd}$$

Let $Z_i = f^{-1}(X_i)$. Then $\{Z_i\}_{i \in I}$ is an open cover of Z; hence we can get Z by gluing the Z_i along $Z_i \cap Z_j$. Let $h_i : Z_i \longrightarrow X \times_S Y$ be the composite

$$Z_i \xrightarrow{\exists !} X_i \times_S Y \hookrightarrow X \times_S Y .$$

Then h_i and h_j agree on $Z_i \cap Z_j$. So by the universal property of the colimit there is a unique morphism

$$Z \xrightarrow{h} X \times_S Y$$

whose restriction to Z_i is h_i. Finally,

$$\begin{tikzcd}
Z \arrow[dr, "h"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& X \times_S Y \arrow[r, "p"] \arrow[d, "q"] & X \arrow[d] \\
& Y \arrow[r] & S
\end{tikzcd}$$

commutes: for $p \circ h|_{Z_i} = f_i$, and f is the unique morphism with this property. (This again is the universal property of the colimit.)

//

<u>Lemma 4</u> If S is affine, then $X \times_S Y$ exists.

Pf Let $\{X_i\}_{i \in I}$ and $\{Y_j\}_{j \in J}$ be affine open covers of X and Y. Then by lemma 1, $X_i \times_S Y_j$ exists for all $i \in I$, $j \in J$. Hence, by lemma 3, $X \times_S Y_j$ exists, for all $j \in J$. Again, by lemma 3, $X \times_S Y$ exists. //

Prop $X \times_S Y$ exists.

Pf Cover S by affine open $\{S_i\}_{i \in I}$; let $X_i = f^{-1}(S_i)$, $Y_i = g^{-1}(S_i)$. Then $X_i \times_{S_i} Y_i$ exists by lemma 4. But the diagram

$$
\begin{tikzcd}
Z \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & & \\
& X_i \times_{S_i} Y_i \arrow[r] \arrow[d] & X_i \arrow[d] \arrow[dr, equal] & \\
& Y_i \arrow[r] \arrow[dr, equal] & S_i \arrow[dr, hook] & X_i \arrow[d] \\
& & Y_i \arrow[r] & S
\end{tikzcd}
$$

shows that $X_i \times_S Y_i = X_i \times_{S_i} Y_i$.

Glue these to get $X \times_S Y$. //

Prop If $X \xrightarrow{f} S$ is surjective and $S' \xrightarrow{g} S$ is any morphism then $X \times_S S' \longrightarrow S'$ is surjective.

Pf First: a morphism

$$\mathrm{Spec}\, k \longrightarrow X \quad , \quad k \text{ field},$$

determines and is determined by a point $x \in X$ and a homomorphism $k(x) \hookrightarrow k$.

Let $s' \in S$, $s = g(s')$; pick $x \in S$ s.t. $x \in X$. Let Ω be a field s.t.

$$\begin{tikzcd} & \Omega & \\ k(s') \arrow[ur, dash] & & k(x) \arrow[ul, dash] \\ & k(s) \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

then

$$\begin{tikzcd} \mathrm{Spec}\,\Omega \arrow[rrr] \arrow[dr, dashed, "\exists !"] \arrow[ddd, dash] & & & \mathrm{Spec}\, k(x) \arrow[dl] \arrow[ddd] \\ & X \times_S S' \arrow[r] \arrow[d] & X \arrow[d] & \\ & S' \arrow[r] & S & \\ \mathrm{Spec}\, k(s') \arrow[ur] \arrow[rrr] & & & \mathrm{Spec}\, k(s) \arrow[ul] \end{tikzcd}$$

//

Prop' "surjective" = "universally surjective".

ex "injective" $\neq$ "universally injective"

$$\begin{array}{l} L \\ \mid G \quad \text{galois} \\ K \end{array} \qquad L \underset{K}{\otimes} L \xrightarrow{\sim} \prod_{\sigma \in G} L \qquad \Rightarrow$$

$$a \otimes b \longmapsto (a\sigma(b))_{\sigma \in G}$$

$$\begin{tikzcd} \operatorname{Spec} L \underset{\operatorname{Spec} K}{\times} \operatorname{Spec} L \arrow[d] & \coprod_{\sigma \in G} \operatorname{Spec} L \arrow[l, "\sim"'] \arrow[d, "\text{fold}"] \\ \operatorname{Spec} L \arrow[r, equal] & \operatorname{Spec} L \end{tikzcd}$$

So $\operatorname{Spec} L \longrightarrow \operatorname{Spec} K$ is injective but not universally injective. //

Def A universally injective morphism $X \xrightarrow{f} Y$ is called <u>radicial</u>.

ex The canonical morphism

$$\operatorname{Spec} k(x) \longrightarrow X$$

is radicial. (One shows that it is a monomorphism in the category of ringed spaces.) //

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart.}"] & X \arrow[d, "f"] \\ \operatorname{Spec} k(y) \arrow[r] & Y \end{tikzcd}$$ fiber at $y \in Y$.

<u>Lemma</u> The can. map of spaces $X_y \to f^{-1}(y)$ is a homeomorphism.

<u>pf</u> The map is injective since $\operatorname{Spec} k(y) \to Y$ is radicial; it is surjective by the proof of the prop. on page 6, It remains to show that the subspace topology on $f^{-1}(y)$ correspond to the topology on X_y. See EGA I, 3.6.1. //

<u>ex</u> Intersection with multiplicities:

$$\begin{tikzcd} \operatorname{Spec} k[x]/(x^2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart.}"] & \operatorname{Spec} k[x] \arrow[d] \\ \operatorname{Spec} k[x,y]/(y-x^2) \arrow[r] & \operatorname{Spec} k[x,y] \end{tikzcd}$$

i.e, the curves "$y=0$" and "$y=x^2$" meet in the point $x=y=0$ with multiplicity 2. //

Prop Let $X \xrightarrow{f} \operatorname{Spec} R$ be a morphism. The following are equivalent:

(i) X has a finite covering $\{U_i\}$ by affine open subsets s.t. $\Gamma(U_i, \mathcal{O}_X)$ is a f.g. R-algebra;

(ii) X is quasi-cpt, and for every open affine subset $U \subset X$, $\Gamma(U, \mathcal{O}_X)$ is a f.g. R-algebra.

⌈Say f is a morphism of finite type.⌋

pf (ii) ⇒ (i) trivial; prove (i) ⇒ (ii)

Since $U_i = \operatorname{Spec} R_i$ is quasi-cpt, so is X.

Let $U = \operatorname{Spec} S \subset X$ be affine open. If $f_i \in R_i$ then $R_i[\frac{1}{f_i}]$ is also f.g. $/R$. And $(U_i)_{f_i} = \operatorname{Spec} R_i[\frac{1}{f_i}]$ form a basis for the topology on X. Hence U is a finite (since quasi-cpt) union of affine open $U_i' = \operatorname{Spec} R_i'$ with R_i' f.g. $/R$.

Write $\rho_i : S = \Gamma(U, \mathcal{O}_X) \to \Gamma(U_i', \mathcal{O}_X) = R_i'$. If $f \in S$, $U_f \cap U_i' = (U_i')_{\rho_i(f)}$. The U_f form a basis for the topology on U, and $U_i' \subset U$ is quasi-cpt. So U_i' is a finite union of U_f. And if $U_f \subset U_i'$,

$$\Gamma(U_f, \mathcal{O}_X) = \Gamma((U_i')_{\rho_i(f)}, \mathcal{O}_X) = R_i'\left[\frac{1}{\rho_i(f)}\right]$$

is f.g. / R. Hence U is a finite union of distinguished open subsets U_{f_i} s.t. $\Gamma(U_{f_i}, \mathcal{O}_X) = S_{f_i}$ is f.g. / R.

For each i, pick a finite set of generators of S_{f_i} / R. Clearing denominators, we can assume that these are in S. Let $\tilde{S} \subset S$ be the subring generated by these generators, for all i. Then $\tilde{S}$ is f.g. / R and for all i,

$$\tilde{S}_{f_i} \xrightarrow{\sim} S_{f_i}.$$

Since the U_{f_i} cover U, the f_i generate the unit ideal, i.e. there exists $g_i \in S$:

$$1 = \sum f_i g_i .$$

Let $S' \subset S$ be the subring generated by $\tilde{S}$ and the g_i. Then S' f.g. / R.

Claim $S' \xrightarrow{\sim} S$.

Let $\alpha \in S$, in $S_{f_i} \xleftarrow{\sim} S'_{f_i}$ can write

$$\alpha = \frac{\beta_i}{f_i^{n_i}}, \quad \beta_i \in S', \quad \text{i.e.}$$

$$f_i^{k_i}(f_i^{n_i}\alpha - \beta_i) = 0 \in S', \quad \text{some } k_i \geq 0.$$

Pick $N \geq 0$ s.t. $f_i^N \alpha \in S'$, for all i. We have

$$\alpha = 1 \cdot \alpha = 1^k \cdot \alpha = (\Sigma f_i g_i)^k \alpha$$

and by choosing $k \gg 0$, each summand of $(\Sigma f_i g_i)^k$ will be divisible by f_i^N, for some i. Hence $\alpha \in S$. //

<u>Cor</u> If X is affine and $U \subset X$ an affine open subset then $\Gamma(U, \mathcal{O}_X)$ is f.g. / $\Gamma(X, \mathcal{O}_X)$. //

Consider morphisms of finite type:

$$X \xrightarrow{f} \operatorname{Spec} k \quad , \ k \text{ field.}$$

For all $x \in X$, we have a can. map

$$k \longrightarrow k(x)$$

given as the composite

$$k \xrightarrow{\tilde{f}} \Gamma(X, \mathcal{O}_X) \longrightarrow \mathcal{O}_{X,x} \longrightarrow k(x).$$

<u>Lemma</u> A point $x \in X$ is closed if and only if $k \hookrightarrow k(x)$ is a finite extension.

<u>pf</u> We can assume X is affine. For let $\{U_i\}$ be an affine open cover. Then $Z \subset X$

is closed iff $Z \cap U_i \subset U_i$ is closed, $\forall i$.
So $X = \operatorname{Spec} R$, R f.g. $/k$. Then

$$R/\mathfrak{p}_x \hookrightarrow k(x)$$

and $x \in X$ is closed if and only if this is an iso.

Suppose $x \in X$ closed. Then $k(x)$ is a f.g. k-algebra which is also a field. But then $k(x)$ is a finite ext. of k by Noether normalization.

Conversely, if $k(x)$ is finite $/k$, then so is $R/\mathfrak{p}_x \subset k(x)$. But an artinian k-alg. which is an integral domain is a field: in every artinian ring, the Jacobson radical is nilpotent with semi-simple quotient. //

<u>Cor</u> A point $x \in X$ is closed if and only if it is locally closed.

<u>ex</u> This is not true in general. In $\operatorname{Spec} \mathbb{Z}_p$, the open point is locally closed but not closed.

Prop Let $Z \subset X$ be closed. Then the closed points are dense in Z.

Pf If not, we can find an open $U \subset X$ s.t. $U \cap Z$ does not contain a closed pt. Replacing U by a smaller open subset, we can assume that $U = \operatorname{Spec} R$ is affine. Then $Z \cap U = V(I)$, $I \subset R$. Pick $\mathfrak{p}_x \supset I$ be a max'l ideal. Then $x \in U \cap Z$ and is closed in U. But then $x \in X$ is locally closed, hence closed. ↯ //

Change notation: morphism of f. type

$$X^* \longrightarrow \operatorname{Spec} k \quad , \; k \text{ field},$$

let $X \subset X^*$ be the subspace of closed points w. subspace topology. Then

$$U^* \longmapsto U^* \cap X$$

defines a 1-1 correspondance between open subsets of X^* and of X. Indeed, if $U \subset X$ is open, let

$$U^* = X^* \setminus (\overline{X \setminus U}).$$

(i.e. X^* and X define the same topos.)

Define sheaf of rings on X:

$$\Gamma(U^* \cap X, \mathcal{O}_X) := \Gamma(U^*, \mathcal{O}_{X^*}).$$

In general, if X is a scheme, $U \subset X$ an affine open subset, and $f \in \Gamma(U, \mathcal{O}_X)$, then

$$f(x) = 0 \in k(x),\ \forall x \in U \iff f \text{ nilpotent}$$

Indeed, $f(x) = 0$, $\forall x \in U = \operatorname{Spec} R$, iff $f \in \mathfrak{p}_x$, $\forall x \in U$, iff $f \in \bigcap_{\mathfrak{p} \subset R} \mathfrak{p} = \{g \in R \mid g \text{ nilpotent}\}$.

Prop The following are equivalent:

(i) for all $U \subset X$ open, $\Gamma(U, \mathcal{O}_X)$ does not have any nilpotent elements;

(ii) there exists an affine open cover $\{U_i\}_{i \in I}$ of X s.t. for all $i \in I$, $\Gamma(U_i, \mathcal{O}_X)$ does not have any nilpotent elements;

(iii) for all $x \in X$, $\mathcal{O}_{X,x}$ does not have any nilpotent elements.

Pf Only (iii) $\Rightarrow$ (i) needs proof. But

$$\Gamma(U, \mathcal{O}_X) \longrightarrow \prod_{x \in U} \mathcal{O}_{X,x}$$

is injective. $\square$

If X^* is reduced, we can think of $f \in \Gamma(U, \mathcal{O}_X)$ as a function on U whose value at $x \in U$

lie in $k(x)$. If further $k = \bar{k}$, the can. map $k \hookrightarrow k(x)$ is an isomorphism, so we can really regard $f \in \Gamma(U, \mathcal{O}_X)$ as a k-valued function on U. If further X^*, or equivalently X, is irreducible, $(X, \mathcal{O}_X)$ is a variety in the classical sense. However, we will use this term more broadly:

Def A variety is a reduced and irreducible scheme of finite type over a field.

Let X be a space and let $X^\sim$ be the cat. of sheaves of sets on X. If $f : X \to Y$ is a continuous map, the direct image functor

$$X^\sim \xrightarrow{f_*} Y^\sim$$

has a left adjoint, the inverse image

$$X^\sim \xleftarrow{f^*} Y^\sim,$$

i.e.,

$$\mathrm{Hom}_{X^\sim}(f^*F, G) \cong \mathrm{Hom}_{Y^\sim}(F, f_*G).$$

ex (i) If $U \xhookrightarrow{j} X$ is the inclusion of an open subset, $j^*F(V) = F(j(V))$.

(ii) If $\{x\} \xhookrightarrow{i_x} X$ is the inclusion of a point $x \in X$, $i_x^* F = F_x$. ($i_{x*}E$ is the skyscraper sheaf of E at x.).

Note In general,

$$(f^*F)_x = i_x^* f^* F = (f \circ i_x)^* F$$

$$= i_{f(x)}^* F = F_{f(x)}. \quad //$$

Let $(X, \mathcal{O}_X)$ be a scheme. If $U \xhookrightarrow{j} X$ is open, then $(U, j^*\mathcal{O}_X)$ is again a scheme. But if

$Y \xhookrightarrow{i} X$ is closed, then $(Y, i^*\mathcal{O}_X)$ is not necessarily a scheme:

ex $Y = \operatorname{Spec} \mathbb{F}_p \xhookrightarrow{i} \operatorname{Spec} \mathbb{Z}_p = X$. Then Y is a open-point space $Y = \{x\}$

$$\Gamma(Y, i^*\mathcal{O}_X) = \mathcal{O}_{X,x} = \mathbb{Z}_p .$$

If $(Y, i^*\mathcal{O}_X)$ were a scheme it would have to be $\operatorname{Spec} \mathbb{Z}_p$. And it is not. //

Fact (will prove this soon) $X^\sim$ has all limits and colimits. The functor $F \longmapsto \Gamma(U, F)$ preserves limits; the functor $F \longmapsto F_x$ preserves colimits and finite limits.

<u>Lemma</u> A map $F \to G$ in $X^\sim$ is an iso. (resp. mono., resp. epi) if and only if for all $x \in X$, $F_x \to G_x$ is an iso. (resp. a mono, resp. an epi.).

pf We have already proved the statement for iso's. By def$^{\underline{n}}$ $F \to G$ is an epi if given $G \rightrightarrows H$ s.t. the two composites $F \rightrightarrows H$ are equal implies $G \rightrightarrows H$ equal.

$$\begin{tikzcd} F \arrow[r] & G \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift right, "i_1"'] \arrow[d, shift left, "i_2"] & H \\ & G \amalg_F G \arrow[ur, dashed, "\exists !"'] & \end{tikzcd}$$

so $\quad F \longrightarrow G$ epi $\iff$

$$G \underset{i_2}{\overset{i_1}{\rightrightarrows}} G \amalg_F G \iff$$

$$G \amalg_F G \xrightarrow[\sim]{\nabla} G .$$

And $G \amalg_F G \xrightarrow{\nabla} G$ is an iso. if and only if for all $x \in X$,

$$G_x \amalg_{F_x} G_x = (G \amalg_F G)_x \xrightarrow{\nabla} G_x$$

is an iso. This, in turn, happens if and only if $F_x \to G_x$ epi, $\forall x \in X$. //

Warning $F \twoheadrightarrow G \not\Rightarrow F(U) \twoheadrightarrow G(U)$. //

Def A morphism $Y \xrightarrow{f} X$ of schemes is a closed immersion if

(i) f induces a homeomorphism of Y onto a closed subset of X;

(ii) $\mathcal{O}_X \xrightarrow{f^*} f_* \mathcal{O}_Y$ is surjective.

Note Since f is closed and injective:

$$(f_* \mathcal{O}_Y)_x = \begin{cases} \mathcal{O}_{Y,y} & , \text{ if } x = f(y), \\ 0 & , \text{ if } x \notin f(Y). \end{cases}$$ //

Prop Let R be a ring and $I \subset R$ an ideal. Then the morphism

$$\operatorname{Spec} R/I \xrightarrow{f} \operatorname{Spec} R$$

induced from the proj. $R \xrightarrow{\pi} R/I$ is a closed immersion.

Pf There is a 1-1 correspondance between ideals of R/I and ideals of R containing I; i.e.

$$\begin{tikzcd}
\operatorname{Spec} R/I \arrow[rr, "f"] \arrow[dr, "\sim"] & & \operatorname{Spec} R \\
V(J/I) \arrow[u, phantom, "\cup"] \arrow[dr, "\sim"] & V(I) \arrow[ur, hook] & \\
& V(J) \arrow[u, phantom, "\cup"] &
\end{tikzcd} \qquad \mathfrak{p} \longmapsto \pi^{-1}(\mathfrak{p})$$

so f is a homeomorphism onto the closed subset $V(I)$.

$$\begin{tikzcd}
Y \arrow[r, hook, "f"] & X \\
y \arrow[u, phantom, "\in"] \arrow[r, mapsto] & x \arrow[u, phantom, "\in"]
\end{tikzcd} \qquad \mathfrak{p}_y = \mathfrak{p}_x / I$$

Since $R \to R_{\mathfrak{p}_x}$ is flat:

$$\begin{tikzcd}
0 \arrow[r] & I \otimes_R R_{\mathfrak{p}_x} \arrow[r] \arrow[d, "\sim"] & R \otimes_R R_{\mathfrak{p}_x} \arrow[r] \arrow[d, "\sim"] & R/I \otimes_R R_{\mathfrak{p}_x} \arrow[r, "\underline{flat}"] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & I \cdot R_{\mathfrak{p}_x} \arrow[r] & R_{\mathfrak{p}_x} \arrow[r] \arrow[d, phantom, "\parallel"] & (R/I)_{\mathfrak{p}_y} \arrow[r] \arrow[d, phantom, "\parallel"] & 0 \\
& & \mathcal{O}_{X,x} \arrow[r] & \mathcal{O}_{Y,y} &
\end{tikzcd}$$

surj //

Thm Let $X = \operatorname{Spec} R$, let $Y \xhookrightarrow{i} X$ be a closed immersion, and let $I = \ker(R \xrightarrow{i^*} \Gamma(Y, \mathcal{O}_Y))$. Then the induced morphism is an iso:

$$Y \xrightarrow{\sim} \operatorname{Spec} R/I.$$

pf Can assume w.l.o.g. that $I = \{0\}$, so

$$R \xhookrightarrow{i^*} \Gamma(Y, \mathcal{O}_Y)$$

is injective.

Show $Y \xrightarrow{i} X$ onto: since Y is homeomorphic to the closed subset $i(Y) \subset X$ of a quasi-cpt. space, Y is quasi-cpt. So can cover Y by a finite number of affine opens, $U_j = \operatorname{Spec} S_j$, $1 \le j \le k$. Let $i(Y) = V(J) \subset X$. Show $J \subset \sqrt{\{0\}}$, i.e. if $s \in J$, then s is nilpotent.

$$s \in J \implies \forall x \in V(J):\ s(x) = 0$$

$$\implies \forall y \in Y:\ (i^* s)(y) = 0.$$

So $i^* s|_{U_j} \in \Gamma(U_j, \mathcal{O}_Y) = S_j$ is nilpotent. Choose N s.t. $(i^* s)^N|_{U_j} = 0$, $1 \le j \le k$. Then $(i^* s)^N = 0 \in \Gamma(Y, \mathcal{O}_Y)$. But

$$(i^* s)^N = i^*(s^N),$$

and since i^* 1-1, $s^N = 0 \in R$. So i is is a homeomorphism.

$$\begin{array}{ccc} Y & \xrightarrow[\sim]{i} & X \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ y & \longmapsto & x \end{array}$$

$$\mathcal{O}_{Y,y} \longleftarrow \mathcal{O}_{X,x} = R_{\beta_x}$$

If not injective, we can find $a \in R$ which maps to zero in $\mathcal{O}_{Y,y}$. Wish to show $a = 0 \in \mathcal{O}_{X,x}$, i.e. to find $b \in R \setminus \beta_x$ s.t. $ab = 0 \in R$. Let

$$U = \{ y' \in Y \mid (i^* a)_{y'} = 0 \} \subset Y.$$

Then $U \subset Y$ is open! For if a section $s \in \Gamma(Y, \mathcal{O}_Y)$ maps to zero in $\mathcal{O}_{Y,y'}$, then there exists an open $y' \in V \subset Y$ s.t. $s|_V = 0$. Since i is a homeomorphism, $i(U) \subset X$ is an open nbhd of $x \in X$. So exists $f \in R \setminus \beta_x$ s.t. $x \in D(f) \subset i(U)$. And then

$$\begin{array}{ccc} y \in & Y_{i^*(f)} & \subset U \\ & \parallel & \\ & \{ y' \in Y \mid (i^* f)(y') = 0 \} & \end{array}$$

Let σ_j be the image of $i^*(f)$ in $\Gamma(U_j, \mathcal{O}_Y) = S_j$. Then

$$Y_{i^*(f)} \cap U_j = (U_j)_{\sigma_j}$$

Now consider:

$$\begin{tikzcd}
i^*a \in \Gamma(Y, \mathcal{O}_Y) \arrow[dd, mapsto, "Y_{i^*f} \subset U"'] \arrow[d] \arrow[drr, bend left, mapsto] & & \\
\Gamma(U_j, \mathcal{O}_Y) = S_j \arrow[d] & \ni & \alpha_j \arrow[d, mapsto] \\
0 \in \Gamma((U_j)_{\sigma_j}, \mathcal{O}_Y) = (S_j)_{\sigma_j} & \ni & 0
\end{tikzcd}$$

Choose $n \geq 0$ s.t. $\alpha_j \cdot \sigma_j^n = 0 \in S_j$, $1 \leq j \leq k$. Then $i^*a \cdot (i^*f)^n \in \Gamma(Y, \mathcal{O}_Y)$ maps to zero in $\Gamma(U_j, \mathcal{O}_Y)$, $1 \leq j \leq k$; hence $i^*a \cdot (i^*f)^n = 0$. So $i^*(a \cdot f^n) = 0$, and since i^* is 1-1, $a \cdot f^n = 0 \in R$. //

Cor Let $Y \xrightarrow{i} X$ be a morphism of schemes. The following are equivalent:

(i) i is a closed immersion;

(ii) for every affine open $U \subset X$, $i^{-1}(U) \subset Y$ is affine and

$$\Gamma(U, \mathcal{O}_X) \xrightarrow{i^*} \!\!\!\!\to \Gamma(i^{-1}(U), \mathcal{O}_Y).$$

(iii) there exists an affine open cover $\{U_j\}$ of X s.t. $i^{-1}(U_j) \subset Y$ is affine and

$$\Gamma(U_j, \mathcal{O}_X) \xrightarrow{i^*} \!\!\!\!\to \Gamma(i^{-1}(U_j), \mathcal{O}_Y).$$

pf (i) $\Rightarrow$ (ii) closed immersions are preserved under base change, and

$$\begin{tikzcd} i^{-1}(U) \arrow[r, "\text{open imm.}"] \arrow[d, "\text{cl. imm.}"'] \arrow[dr, phantom, "\text{Cart.}"] & Y \arrow[d, "\text{cl. imm.}"] \\ U \arrow[r, "\text{open imm.}"'] & X \end{tikzcd}$$

so thm implies (ii).

(ii) $\Rightarrow$ (iii) is trivial, and (iii) $\Rightarrow$ (i) is the proposition. //

Let $Y \overset{i}{\hookrightarrow} X$ be a closed immersion. This defines a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$,

$$\Gamma(U, \mathcal{I}) = \ker(\Gamma(U, \mathcal{O}_X) \to \Gamma(U, i_* \mathcal{O}_Y)).$$

<u>Lemma</u> Let $Y = \operatorname{Spec} R/I \hookrightarrow \operatorname{Spec} R = X$, and let $f \in R$. Then

$$\Gamma(D(f), \mathcal{I}) = I \cdot R_f \subset R_f.$$

pf Since $R \to R_f$ is flat:

$$\begin{tikzcd} 0 \arrow[r, "\text{flat}"] & I \otimes_R R_f \arrow[r] \arrow[d, "\sim"] & R \otimes_R R_f \arrow[r] \arrow[d, "\sim"] & R/I \otimes_R R_f \arrow[r] \arrow[d, "\sim"] & 0 \\ 0 \arrow[r] & I \cdot R_f \arrow[r] & R_f \arrow[r] & (R/I)_{\pi(f)} \arrow[r] & 0 \end{tikzcd}$$

//

Cor Let $(X, \mathcal{O}_X)$ be a scheme, $\mathcal{I} \subset \mathcal{O}_X$ a sheaf of ideals, and let

$$Y = \{x \in X \mid \mathcal{I}_x \neq \mathcal{O}_{X,x}\} \xhookrightarrow{i} X .$$

Let $\mathcal{O}_Y = i^*(\mathcal{O}_X/\mathcal{I})$. Then $(Y, \mathcal{O}_Y)$ is a scheme if and only if

$$(*) \begin{cases} \text{for all } y \in X, \text{ there exists } y \in U \subset X \\ \text{open and } s_\alpha \in \Gamma(U, \mathcal{I}) \text{ s.t. for all } x \in U, \\ \text{the images of } s_\alpha \text{ gen. } \mathcal{I}_x \subset \mathcal{O}_{X,x} \end{cases}$$

Pf Suppose $(*)$ is satisfied and choose $U = \operatorname{Spec} R$ affine. Then

$$(U \cap Y, \mathcal{O}_Y) = \operatorname{Spec} R/(s_\alpha) .$$

Indeed, for all $x \in U$, $\mathcal{O}_{X,x}/\mathcal{I}_x = R_{\mathfrak{p}_x}/(s_\alpha)$.

Conversely, take $y \in U \subset X$ affine, and let s_α be gen. of $\Gamma(U, \mathcal{I})$ over $\Gamma(U, \mathcal{O}_X)$. //

This shows that there is a 1-1 corresp.:

$$\{\text{closed subschemes of } (X, \mathcal{O}_X)\}$$
$$\updownarrow$$
$$\{\text{\underline{quasi}-\underline{coherent} ideals of } \mathcal{O}_X\}$$

2/28/01

Def An S-scheme X is <u>separated</u> if the diagonal

$$X \overset{\Delta}{\hookrightarrow} X \times_S X$$

is a closed immersion.

Lemma X/S is separated if and only if $\Delta(X) \subset X \times_S X$ is a closed subset.

Pf

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[rr, bend right, "\mathrm{id}"'] & X \times_S X \arrow[r, "p_i"] & X \end{tikzcd}$$

$\Rightarrow$ Δ homeo. onto $\Delta(X) \subset X \times_S X$

$$\begin{tikzcd} X \arrow[r, "\Delta"] & X \times_S X \arrow[r, "p_i"] & X \\ x \arrow[r, mapsto] & y \arrow[r, mapsto] & x \\ \mathcal{O}_{X,x} & \mathcal{O}_{X \times_S X, y} \arrow[l, "\Delta^*"'] & \mathcal{O}_{X,x} \arrow[l, "p_i^*"'] \arrow[ll, bend left, "\mathrm{id}"] \end{tikzcd}$$

//

Prop If X and S are both affine then X/S is separated.

Pf $S = \operatorname{Spec} B$, $X = \operatorname{Spec} A \overset{\Delta}{\hookrightarrow} \operatorname{Spec} A \otimes_B A = X \times_S X$ <~>

$A \leftarrow A \otimes_B A$ multiplication; surjective. //

<u>Lemma</u> A point $x \in X \times_S X$ is in $\Delta(X)$ if and only if $p_1(x) = p_2(x)$ $(= y)$ and $p_1^* = p_2^* : k(y) \xrightarrow{\sim} k(x)$.

<u>pf</u> Suppose $x \in \Delta(X)$:

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[rr, bend left, "\mathrm{id}"] & X \times_S X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \\ y \arrow[r, mapsto] & x \arrow[r, mapsto] & y \end{tikzcd}$$

$$\begin{tikzcd} k(y) & k(x) \arrow[l, "\Delta^*"'] & k(y) \arrow[l, shift right, "p_1^*"'] \arrow[l, shift left, "p_2^*"] \arrow[ll, bend left, "\mathrm{id}"] \end{tikzcd}$$

Conversely:

$$\begin{tikzcd} & \operatorname{Spec} k(x) \arrow[dl, dashed] \arrow[d, "(p_1^*, p_2^*)"] \\ \operatorname{Spec} k(y) \arrow[r, "\Delta"] \arrow[d] & \operatorname{Spec} k(y) \times_S \operatorname{Spec} k(y) \arrow[d] \\ X \arrow[r, "\Delta"] & X \times_S X \end{tikzcd}$$

//

<u>Cor</u> $\Delta(X) \subset X \times_S X$ is locally closed.

<u>pf</u>

$$\begin{tikzcd} X \arrow[r, "f"] & S \\ x \arrow[r, mapsto] & y \end{tikzcd}$$

pick: $y \in V \subset S$ aff- open

$x \in U \subset f^{-1}(V)$ — " —

Then $\Delta(x) \in U \times_V U = U \times_S U \subset X \times_S X$. The lemma shows $\Delta(U) = \Delta(X) \cap (U \times_S U)$. And the prop. shows that $\Delta(U) \subset U \times_V U$ is closed. //

<u>Lemma</u> Let $Y \underset{g}{\overset{f}{\rightrightarrows}} X$ be two S-morphisms. Then

$$\{y \in Y \mid f(y) = g(y),\ f^* = g^* : k(x) \to k(y)\} \qquad x = f(y)$$
$$= (f,g)_S^{-1}(\Delta(X)).$$

<u>Pf</u> Suppose $f(y) = g(y) = x$, $f^* = g^* : k(y) \hookleftarrow k(x)$.

$$\begin{tikzcd}
& \operatorname{Spec} k(y) \arrow[dl, dashed] \arrow[dr] \arrow[dd] & \\
\operatorname{Spec} k(x) \arrow[rr] \arrow[dd] & & \operatorname{Spec} k(x) \times_S \operatorname{Spec} k(x) \arrow[dd] \\
& Y \arrow[dr, "(f,g)_S"] & \\
X \arrow[rr, "\Delta"] & & X \times_S X
\end{tikzcd}$$

so $(f,g)_S(y) \in \Delta(X)$. Converse: exercise. //

<u>Cor</u> An S-scheme X is separated if and only if for every pair of S-morphisms $f, g : Y \to X$,

$$\{y \in Y \mid f(y) = g(y),\ f^* = g^* : k(y) \to k(x)\}$$

is a closed subset of Y.

$\underline{\text{ex}}$

$$\begin{tikzcd}
\operatorname{Spec} k[t,t^{-1}] \arrow[r, hook, "t \leftarrow\!\shortmid\, x"] \arrow[d, hook, "y \mapsto t"'] & \operatorname{Spec} k[x] \arrow[d, hook] & \\
\operatorname{Spec} k[y] \arrow[r, hook] & X \arrow[r, "\text{not separated}"] & \operatorname{Spec} k
\end{tikzcd}$$

$$\begin{tikzcd}
 & \operatorname{Spec} k[x] \arrow[d, hook] \\
Y = \operatorname{Spec} k[z] \arrow[ur, "\sim", "t \leftarrow\!\shortmid\, x"'] \arrow[dr, "\sim"', "z \updownarrow y"] \arrow[r, shift left] \arrow[r, shift right] & X \\
 & \operatorname{Spec} k[y] \arrow[u, hook]
\end{tikzcd}$$

$$\{ y \in Y \mid f(y) = g(y) \text{ and } f^* = g^* : k(x) \hookrightarrow k(y) \}$$

$$= \operatorname{Spec} k[z, z^{-1}] \hookrightarrow Y \quad - \quad \text{not closed.}$$

Prop If X is separated and $U, V \subset X$ are affine open, then $U \cap V$ is affine and

$$\Gamma(U, \mathcal{O}_X) \otimes \Gamma(V, \mathcal{O}_X) \to \Gamma(U \cap V, \mathcal{O}_X)$$

is surjective.

Pf Cover X by $U_i = \operatorname{Spec} R_i$. Then $U_i \times U_i$ cover $\Delta(X) \subset X \times X$ closed. But $U_i = \Delta^{-1}(U_i \times U_i)$ and

$$\begin{tikzcd} \Gamma(U_i, \mathcal{O}_X) \arrow[d, equal] & \Gamma(U_i \times U_i, \mathcal{O}_X) \arrow[l, "\Delta^*"'] \arrow[d, equal] \\ R_i & R_i \otimes R_i \arrow[l, two heads] \end{tikzcd}$$

$U \times V \subset X \times X$ open aff. w. ring $\Gamma(U, \mathcal{O}_X) \otimes \Gamma(V, \mathcal{O}_X)$

$$\begin{tikzcd} U \times V \arrow[r, phantom, "\subset"] & X \times X \\ U \cap V \arrow[u] \arrow[r, phantom, "\subset"] & X \arrow[u, "\Delta"'] \end{tikzcd}$$

$$\Gamma(U, \mathcal{O}_X) \otimes \Gamma(V, \mathcal{O}_X) \to \Gamma(U \cap V, \mathcal{O}_X)$$

since cl. imm.

V valuation ring w. quotient field K and residue field k.

$$s = \operatorname{Spec} k \hookrightarrow \operatorname{Spec} V \hookleftarrow \operatorname{Spec} K = \eta$$

Lemma If $X \to S$ is separated, then in every diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \operatorname{Spec} V \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

at most one lift exists.

Pf let $f, g : \operatorname{Spec} V \to X$ be two lifts. Then $(f,g)_S : \operatorname{Spec} V \to X \times_S X$ maps the generic pt. $\eta \in \operatorname{Spec} V$ into $\Delta(X)$, and since $\Delta(X) \subset X$ closed, the image of $(f,g)_S$ lies in $\Delta(X)$. I.e. $f(s) = g(s) = x$

Must show $f(\eta) = g(\eta) = z$

$$f^* = g^* : \mathcal{O}_{X,x} \to V .$$

Let $Z = \{z\}^- \subset X$; then Z is an irreducible closed subset and $z \in Z$ is the generic pt.

⌈A closed subset $Z \subset X$ of a scheme X can be given the <u>reduced induced</u> scheme str. If $X = \operatorname{Spec} R$, $Z = V(I)$, then this is:

$$Z = \operatorname{Spec} R/\sqrt{I} \qquad (\text{note } R/\sqrt{I} \text{ reduced})$$

These glue. A morphism $Y \xrightarrow{f} X$ with Y reduced and $f(Y) \subset Z$ factors through the closed imm. $Z \hookrightarrow X$ of Z w. r.i.s.s. $\lrcorner$

We give Z the reduced induced scheme structure. Then the function field of Z is $k(z)$. We now have

$$
\begin{tikzcd}
\mathcal{O}_{X,x} \arrow[rr] \arrow[dr] & & V \arrow[dd, dash] \\
& \mathcal{O}_{Z,z} \arrow[ur, dashed] \arrow[dd, dash] & \\
& & K \\
& k(z) \arrow[ur] &
\end{tikzcd}
$$

Thus, also f^*, g^* are determined by $\operatorname{Spec} K \to X$, so are equal. //

$C \xrightarrow{y} C^\wedge$ fully faithful $\qquad X \mapsto \mathrm{Hom}_C(-,X)$

i.e.

$$\mathrm{Hom}_C(X,Y) \xrightarrow[\sim]{y} \mathrm{Hom}_{C^\wedge}(X,Y)$$

Pf This map is:

$$X \xrightarrow{f} Y \rightsquigarrow \mathrm{Hom}_C(-,X) \xrightarrow{f_*} \mathrm{Hom}_C(-,Y)$$

surjective: a natural transformation

$$\mathrm{Hom}_C(-,X) \xrightarrow{\alpha} \mathrm{Hom}_C(-,Y)$$

is determined by $a(\mathrm{id}_X) \in \mathrm{Hom}_C(X,Y)$:

$$\begin{tikzcd}
\mathrm{id}_X \in \mathrm{Hom}_C(X,X) \arrow[r, "a"] \arrow[d, "g^*"] & \mathrm{Hom}_C(X,Y) \arrow[d, "g^*"] \\
g \in \mathrm{Hom}_C(Z,X) \arrow[r, "a"] & \mathrm{Hom}_C(Z,Y)
\end{tikzcd}$$

$$a(g) = g^*(a(\mathrm{id}_X))$$

injective: $f = f_*(\mathrm{id}_X)$

~~Lemma~~ $\cancel{\mathrm{Sch} \xrightarrow{y} \mathrm{Sch}^\wedge}$

$\cancel{\mathrm{Aff}^\wedge}$

~~Pf Every X in Sch is a colimit of U_i in Aff. So if Y is in Sch:~~

$$\cancel{\mathrm{Hom}_{\mathrm{Sch}}(X,Y) = \lim_i \mathrm{Hom}_{\mathrm{Sch}}(U_i,Y)}$$

Lemma $\quad \mathrm{Sch} \hookrightarrow \widehat{\mathrm{Aff}}$ fully faithful

$$Y \longmapsto \mathrm{Hom}_{\mathrm{Sch}}(-, Y)$$

Pf Enough to show that the functor

$$\mathrm{Hom}_{\mathrm{Sch}}(-, Y) : \mathrm{Sch}^{op} \longrightarrow \mathrm{Sets}$$

is determined by its restriction to the subcategory $\mathrm{Aff}^{op} \subset \mathrm{Sch}^{op}$. But every X in Sch is a colimit of U_α in Aff. So

$$\mathrm{Hom}_{\mathrm{Sch}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Sch}}(\operatorname*{colim}_\alpha U_\alpha, Y)$$

$$\xrightarrow{\sim} \lim_\alpha \mathrm{Hom}_{\mathrm{Sch}}(U_\alpha, Y). \qquad //$$

Of course $\widehat{\mathrm{Aff}} = \mathrm{Funct}(\mathrm{Rng}, \mathrm{Sets})$. E.g.

$$\mathbb{A}^1_{\mathbb{Z}} = \mathrm{Spec}\, \mathbb{Z}[x] \rightsquigarrow \quad R \longmapsto R \quad \text{(underlying set)}$$

$$\mathbb{A}^1_{\mathbb{Z}} \setminus \{0\} = \mathrm{Spec}\, \mathbb{Z}[x^{\pm 1}] \rightsquigarrow R \longmapsto R^\times \quad \text{(units in } R\text{)}$$

Both take value in groups:

$$\mathbb{G}_a(R) = (R, +)$$

$$\mathbb{G}_m(R) = (R^\times, \cdot)$$

i.e. $\mathbb{Z}[x]$ resp. $\mathbb{Z}[x^{\pm 1}]$ are Hopf algebras over $\mathbb{Z}$:

$$\mathbb{Z}[x] \xrightarrow{\Delta} \mathbb{Z}[x] \otimes \mathbb{Z}[x] \qquad x \longmapsto 1 \otimes x + x \otimes 1$$

$$\mathbb{Z}[x^{\pm 1}] \longrightarrow \mathbb{Z}[x^{\pm 1}] \otimes \mathbb{Z}[x^{\pm 1}] \qquad x \longmapsto x \otimes x$$

Recall that the Yoneda embedding is fully faithful

$$Sch/_S \hookrightarrow Funct((Sch/_S)^{op}, Sets)$$

$$X/_S \longmapsto X(-)_S := Hom_{Sch/_S}(-, X).$$

Call $X(K)_S$ the set of K-valued points in X/S.
We can use this fact as follows:

<u>Lemma</u> The square of S-schemes

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

is cartesian if and only if for every S-scheme K, the induced square of K-valued points

$$\begin{array}{ccc} X'(K)_S & \longrightarrow & X(K)_S \\ \downarrow & & \downarrow \\ Y'(K)_S & \longrightarrow & Y(K)_S \end{array}$$

is a cartesian square of sets.

pf The top square is cartesian if and only if

$$X' \longrightarrow X \underset{Y}{\times} Y'$$

is an isomorphism. Since the Yoneda embedding is faithful, this happens if and only if, for every $K \in Sch/S$, the induced map

$$X'(K)_S \longrightarrow (X \times_Y Y')(K)_S \xrightarrow{\sim} X(K)_S \times_{Y(K)_S} Y'(K)_S$$

is a bijection. And by def$^{\underline{n}}$ of fiber prod.
this happens if and
only if the lower square is cartesian. //

Cor Let $S \to T$ be a morphism and let X, Y be S-schemes. Then the sq. is cartesian:

$$\begin{tikzcd}
X \times_S Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & X \times_T Y \arrow[d] \\
S \times_S S \arrow[r] & S \times_T S \\
S \arrow[u, "\Delta_{S/S}", "\sim"'] \arrow[ur, "\Delta_{S/T}"'] &
\end{tikzcd}$$

Pf The corresponding square of K-valued points, K a T-scheme, is obviously cartesian. //

Cor If $S \to T$ is separated, and if X, Y are two S-schemes, then the canonical map

$$X \times_S Y \longrightarrow X \times_T Y$$

is a closed immersion.

Pf Since $S \to T$ sep, $S \xrightarrow{\Delta_{S/T}} S \times_T S$ is a cl. imm. Use cartesian sq. above + cl. imm. preserved under base change. //

Prop If $f: X \to Y$ and $g: Y \to Z$ are separated then so is $gf: X \to Z$.

pf In the commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\Delta_{X/Z}"] \arrow[dr, "\Delta_{X/Y}"'] & & X \times_Z X \\ & X \times_Y X \arrow[ur, "i"'] & \end{tikzcd},$$

$\Delta_{X/Y}$ is a cl. imm, since f sep, and i is a cl. imm. by cor. since g sep. Hence $\Delta_{X/Z}$ is a cl. imm. //

Prop Open and closed immersions are separated.

pf If $X \hookrightarrow S$ is an open or closed imm. then $\Delta_{X/S}: X \xrightarrow{\sim} X \times_S X$ is an iso, hence, a cl. imm. Prove this if $X \hookrightarrow S$ is closed.

$$\begin{tikzcd} X \arrow[r, "\Delta_{X/S}"] \arrow[dr, equal] & X \times_S X \arrow[r, hook, "p_1"] \arrow[d, "p_2"', "\text{cl. imm.}"] \arrow[dr, phantom, "\Leftarrow"] & X \arrow[d, "\text{cl. imm.}"] \\ & X \arrow[r, hook] & S \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{O}_{X,x} & \mathcal{O}_{X \times_S X, y} \arrow[l, "\Delta_{X/S}"'] & \mathcal{O}_{X,x} \arrow[l, two heads, "p_2"'] \arrow[ll, bend left, "\mathrm{Id}"] \end{tikzcd}$$

//

A morphism $X \xrightarrow{f} Y$ is of <u>finite type</u> if for every affine open $U \subset Y$, $f^{-1}(U)$ is quasi-compact and for all $V \subset f^{-1}(U)$ affine open, $\Gamma(V, \mathcal{O}_X)$ is a f.g. $\Gamma(U, \mathcal{O}_Y)$-algebra

<u>Rem</u> Enough to check this for $U = U_i$, $\{U_i\}$ an affine open cover of Y.

<u>Def</u> A morphism $X \xrightarrow{f} Y$ is <u>proper</u> if it is separated, of finite type, and universally closed.

<u>ex</u> Closed immersions are proper. //

Let K be a field and let $A, B \subset K$ be local rings which are not fields. Then B <u>dominates</u> A if $A \subset B$ and $m_A = A \cap m_B$. A valuation ring(*) $V \subset K$ dominates every local ring it contains, and conversely, every local ring $A \subset B$, which is not a field, is contained in some valuation ring $V \subset K$.

<u>Lemma</u> Let V be a valuation ring, let Z be an integral scheme, and let $f: Z \to \operatorname{Spec} V$ be a surjective, separated, and birational morphism. Then f is an isomorphism.

(*) over

Pf Let K be the fraction field of V. Let s, η be the closed and generic point of $\operatorname{Spec} V$. Let $x \in Z$ be a point s.t. $f(x) = s$, and let $y \in Z$ be the generic point. Then we have:

$$\begin{array}{ccc} \mathcal{O}_{Z,x} & \hookrightarrow & \mathcal{O}_{Z,y} \\ \uparrow f_x^* & & \wr\uparrow f_y^* \\ V & \hookrightarrow & K \end{array}$$

I.e. $\mathcal{O}_{Z,x}$ is a local ring in K which dominates V, so f_x^* is an isomorphism.

⌈To see $\mathcal{O}_{Z,x} \hookrightarrow \mathcal{O}_{Z,y}$, let $U = \operatorname{Spec} A$ be an open affine s.t. $x \in U$. Then also $y \in U$ and

$$\mathcal{O}_{Z,x} = A_{\mathfrak{p}_x} \hookrightarrow A_{(0)} = \mathcal{O}_{Z,y}\,.⌋$$

Consider the composite:

$$g : \operatorname{Spec} V \xrightarrow{(f_x^*)^{-1}} \operatorname{Spec} \mathcal{O}_{Z,x} \longrightarrow Z.$$

By def^n, $fg = \mathrm{id}_{\operatorname{Spec} V}$. Consider $gf : Z \to Z$. Since $Z \to \operatorname{Spec} V \to \operatorname{Spec} \mathbb{Z}$ is separated,

$$\{ z \in Z \mid gf(z) = z,\ (gf)^* = \mathrm{id} : k(z) \to k(z) \}$$

is a closed subset of Z. It also contains the generic point, so is all of Z. Now let $a \in \Gamma(U, \mathcal{O}_Z)$ be a section. Then

$$(gf)^* a - a \in \Gamma(U, \mathcal{O}_Z)$$

is a section whose value at every point $z \in U$ is $0 \in k(z)$. Since Z is reduced, $(gf)^* a - a = 0$, and hence, $gf = \mathrm{id}_Z$. //

Prop Let $X \xrightarrow{f} Y$ be a proper morphism. Then in every diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} V \arrow[r] \arrow[ur, dashed, "\exists !"] & Y \end{tikzcd}$$

with V a valuation ring, a unique lift exists.

pf A most one lifting exist since f is separated. Consider

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] & X \times_Y \operatorname{Spec} V \arrow[r, "p_1"] \arrow[d, "p_2"] & X \arrow[d, "f"] \\ & \operatorname{Spec} V \arrow[r] & Y \end{tikzcd}$$

$\eta \in \operatorname{Spec} K \mapsto y \in X \times_Y \operatorname{Spec} V$

Let $Z = \overline{\{y\}}$ with the reduced induced scheme structure. Since f is universally closed, $p_2(Z)$ is a closed subset of $\operatorname{Spec} V$ which contains the generic point. Hence $Z \to \operatorname{Spec} V$ is surj. It is birational since $\operatorname{Spec} K \to Z \to \operatorname{Spec} V$

is the canonical map. It is separated since

$$\begin{tikzcd}
Z \arrow[r, hook, "\text{cl. imm.}"'] \arrow[dr, bend right, "\text{sep.}"'] & X \times_Y \operatorname{Spec} V \arrow[r] \arrow[d, "\text{sep}"] \arrow[dl, phantom, "\Leftarrow"] & X \arrow[d, "f\ \text{sep}"] \arrow[dl, phantom, "\Leftarrow"] \\
 & \operatorname{Spec} V \arrow[r] & Y .
\end{tikzcd}$$

//

Thm $\mathbb{P}^n_{\mathbb{Z}} = \operatorname{Proj}(\mathbb{Z}[x_0, \dots, x_n]) \to \operatorname{Spec} \mathbb{Z}$ is proper.

pf It is finite type since

$$U_i = \operatorname{Spec} \mathbb{Z}\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right], \quad 0 \le i \le n,$$

is a cover. It is separated: use the def$^{\underline{n}}$. To show that $\mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} S \to S$ is closed, we can assume that $S = \operatorname{Spec} R$ is affine. So we must show that

$$\mathbb{P}^n_R \xrightarrow{f} \operatorname{Spec} R$$

is closed. Let $Z \subset \mathbb{P}^n_R$ be a closed subscheme,

$$Z \times_{\mathbb{P}^n_R} U_i = \operatorname{Spec} R\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] / I(Z \cap U_i), \quad 0 \le i \le n.$$

Define $I_d \subset R_d[x_0, \dots, x_n]$ by

$$I_d = \left\{ f \mid f\left(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right) \in I(Z \cap U_i), 0 \le i \le n \right\}.$$

Then $I = \bigoplus_{d \ge 0} I_d$ is a graded ideal, and

<u>Lemma</u> For all $0 \le i \le n$, and all $g \in I(Z \cap U_i)$, there exists $f \in I_d$, some $d \geq 0$, s.t.

$$g = f\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right)$$

Pf For $m \gg 0$, $f' := x_i^m g \in R[x_0, \ldots, x_n]_m$. Consider

$$g_j := \frac{f'}{x_j^m} \in R\left[\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}\right].$$

Clearly, g_j is zero on $U_i \cap U_j$. Hence $\frac{x_i}{x_j} g_j$ is zero on U_j, so $f = x_i f' \in I_{m+1}$ and $g = f\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right)$. //

Let $y \in \operatorname{Spec} R \setminus p(Z)$. Then $y \in \operatorname{Spec} R_{\mathfrak{p}_y}$ is closed and $\mathfrak{p}_y R_{\mathfrak{p}_y}\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right]$ is the ideal which defines the fiber of $U_i \times_{\operatorname{Spec} R} \operatorname{Spec} R_{\mathfrak{p}_y} \to \operatorname{Spec} R_{\mathfrak{p}_y}$ over y. I claim that this closed subscheme does not meet $(Z \cap U_i) \times_{\operatorname{Spec} R} \operatorname{Spec} R_{\mathfrak{p}_y}$, i.e.,

$$I(Z \cap U_i) + \mathfrak{p}_y R_{\mathfrak{p}_y}\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right] = R_{\mathfrak{p}_y}\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right].$$

⋮

In general, the colimit of a diagram of sets is given by

$$\operatorname*{colim}_{I} X_i = \left(\coprod_{i \in I} X_i \right) / \sim$$

where $\sim$ is the equivalence rel. gen. by the relation $x_i \in X_i \sim x_j \in X_j$ if there exists $i \xrightarrow{\varphi} j$ s.t. $X_i \xrightarrow{\varphi} X_j$ takes x_i to x_j. So in general $x_i \in X_i \sim x_j \in X_j$ if there exists arrows in I

$$\begin{tikzcd}[column sep=small] & j_1 & & j_2 & & \cdots & & j_n & \\ i \arrow[ur] & & i_1 \arrow[ul] \arrow[ur] & & i_2 \arrow[ul] \arrow[ur] & & i_{n-1} \arrow[ul] \arrow[ur] & & j \arrow[ul] \end{tikzcd}$$

s.t. the induced maps

$$\begin{tikzcd}[column sep=small] & X_{j_1} & & X_{j_2} & & \cdots & & X_{j_n} & \\ X_i \arrow[ur] & & X_{i_1} \arrow[ul] \arrow[ur] & & X_{i_2} \arrow[ul] \arrow[ur] & & X_{i_{n-1}} \arrow[ul] \arrow[ur] & & X_j \arrow[ul] \end{tikzcd}$$

map

$$\begin{tikzcd}[column sep=small] & x_{j_1} & & x_{j_2} & & \cdots & & x_{j_n} & \\ x_i \arrow[ur, mapsto] & & x_{i_1} \arrow[ul, mapsto] \arrow[ur, mapsto] & & x_{i_2} \arrow[ul, mapsto] \arrow[ur, mapsto] & & x_{i_{n-1}} \arrow[ul, mapsto] \arrow[ur, mapsto] & & x_j \arrow[ul, mapsto] \end{tikzcd}.$$

The colimit is much more well-behaved if the indexing category I is <u>filtered</u>: $I \neq \emptyset$,

(i) for every pair of objects $i, j \in I$, there exists

$$\begin{tikzcd} i \arrow[dr] & \\ & k \\ j \arrow[ur] & \end{tikzcd}$$

(ii) for every pair of parallel arrows $i \overset{u}{\underset{v}{\rightrightarrows}} j$, there exists $j \xrightarrow{w} k$ s.t. $wu = wv$.

<u>Lemma</u> Suppose that I is filtered. Then

(i) $\operatorname{colim}_I X_i = (\coprod_{i \in I} X_i)/\sim$ where $x_i \in X_i \sim x_j \in X_j$ if and only if there exists $i \to k$ and $j \to k$ s.t. $X_i \to X_k$ and $X_j \to X_k$ map x_i and x_j to the same element.

(ii) any two elements $x_i \in X_i$, $x_j \in X_j$ are equivalent under $\sim$ to two elements of the same X_k.

(iii) two elements $x_i, x_i' \in X_i$ are equivalent under $\sim$ if and only if there exists $i \to j$ s.t. their images under $X_i \to X_j$ are equal.

Pf (i) to see that $\sim$ is an eq. rel. use:

$$\begin{tikzcd}[column sep=small]
 & & & \exists\, n & & & \\
 & & l \arrow[ur] & & m \arrow[ul] & & \\
 & i \arrow[ur] & & j \arrow[ul] \arrow[ur] & & k \arrow[ul] &
\end{tikzcd}$$

(iii) $x_i \sim x_i' \in X_i \iff \exists\, i \overset{u}{\underset{v}{\rightrightarrows}} j$ s.t. $u_*(x_i) = v_*(x_i')$.
Pick $j \xrightarrow{w} k$ s.t. $wu = wv$. Then the images of x_i, x_i' under $X_i \to X_k$ are equal. //

The limit of a J-diagram of sets is the equalizer

$$\lim_J X_j \to \prod_{j \in J} X_j \rightrightarrows \prod_{j \to j'} X_{j'}$$

i.e. the subset of the product consisting of tuples $(x_j)_{j \in J}$ s.t. for all $j \to j' \in J$, the map $X_j \to X_{j'}$ takes x_j to $x_{j'}$. Suppose J is <u>finite</u> i.e. finitely many objects and morphisms. Then we can obtain $\lim_J X_j$ iterated fiber products. For, clearly, we can get the products $\prod_{j \in J} X_j$ and $\prod_{j \to j'} X_{j'}$, and the equalizer of $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is equal to the fiber-prod

$$\begin{tikzcd}
X \times_{X \times Y} X \arrow[r] \arrow[d] & X \arrow[d, "(\mathrm{id}, f)"] \\
X \arrow[r, "(\mathrm{id}, g)"] & X \times Y
\end{tikzcd}$$

<u>Prop</u> Filtered colimits and finite limits of sets commute, i.e. if I is filtered, if J is finite, and if $X_{i,j}$ is an $I \times J$ diagram of sets, the can. map is an iso:

$$\operatorname*{colim}_I \lim_J X_{i,j} \xrightarrow{\sim} \lim_J \operatorname*{colim}_I X_{i,j}.$$

pf Enough to show

$$\operatorname*{colim}_I \, (X_i \times_{S_i} Y_i) \xrightarrow{\sim} \operatorname*{colim}_I X_i \times_{\operatorname*{colim}_I S_i} \operatorname*{colim}_I Y_i .$$

Write down the inverse: let $(\bar{x}, \bar{y})$ be an element on the right. Then $\bar{x}$ is repr. by $x_i \in X_i$, $\bar{y}$ is repr. by $y_j \in Y_j$ and $f(x_i) \in S_i \sim g(y_j) \in S_j$. Pick $\begin{matrix} i \to \\ j \to \end{matrix} k$, let $X_i \to X_k$, $x_i \mapsto x_k$; $Y_j \to Y_k$, $y_j \mapsto y_k$. Then x_k repr. $\hat{x}$, y_k repr. $\bar{y}$ and $f(x_k) \sim g(y_k) \in S_k$. Pick $k \to m$ s.t. $f(x_k)$ and $g(y_k)$ have the same image in S_m. Let $X_k \to X_m$ take $x_k \mapsto x_m$ and $Y_k \to Y_m$ take $y_k \mapsto y_m$. Then x_m repr. $\bar{x}$, y_m repr. $\bar{y}$ and $f(x_m) = g(y_m) \in S_m$. I.e. $(x_m, y_m) \in X_m \times_{S_m} Y_m$. So (x_m, y_m) repr. an element $\bar{z} \in \operatorname*{colim}_I (X_i \times_{S_i} Y_i)$ and the can. map takes $\bar{z}$ to $(\bar{x}, \bar{y})$. //

Cor The underlying set of a filtered colimit of rings (resp. modules, resp. ...) is the colimit of the underlying diagram of sets.

pf Let $\{R_i\}_{i \in I}$ be a filtered diagram of rings. Then the colimit of the underlying diagram of sets has a natural ring structure:

$$(\operatorname{colim}_I R_i) \times (\operatorname{colim}_I R_i) \xleftarrow{\sim} \operatorname{colim} (R_i \times R_i)$$

$$\longrightarrow \operatorname{colim} R_i .$$

Therefore, this $\underline{\text{is}}$ the colimit in the category of rings. //

ex This is not true if I is not filtered. For example, if I is $\underset{\circlearrowleft}{1} \quad \underset{\circlearrowleft}{2}$, then in

Sets : $\operatorname{colim}_I R_i = R_1 \amalg R_2$ (disjoint union)

C. Rings : $\operatorname{colim}_I R_i = R_1 \otimes R_2$ (tensor prod.)

Ab. grps : $\operatorname{colim}_I R_i = R_1 \oplus R_2$ (direct sum)

Groups : $\operatorname{colim}_I R_i = R_1 * R_2$ (free prod.) //

Let X be a space; let $X^\wedge$ be the category of presheaves of sets on X; let $X^\sim$ be the full subcategory of sheaves of sets on X. Will show:

Thm The forgetful functor $X^\sim \overset{\upsilon}{\hookrightarrow} X^\wedge$ has a left adjoint $X^\wedge \xrightarrow{a} X^\sim$, and a commutes with finite limits. /

If $\{U_\alpha \to U\}$ is a covering of U, define

$$H^0(\{U_\alpha \to U\}, F) \to \prod_\alpha F(U_\alpha) \rightrightarrows \prod_{\alpha, \alpha'} F(U_\alpha \underset{U}{\times} U_{\alpha'})$$

to be the equalizer. A map

$$\{U_\alpha \to U\}_{\alpha \in A} \xrightarrow{f} \{V_\beta \to U\}_{\beta \in B}$$

is a map $f: A \to B$ s.t. $U_\alpha \to V_{f(\alpha)}$. It induces

$$H^0(\{V_\beta \to U\}, F) \xrightarrow{f^*} H^0(\{U_\alpha \to U\}, F).$$

Lemma If $f, g: \{U_\alpha \to U\} \to \{V_\beta \to V\}$ are any two maps then $f^* = g^*$.

Pf Exercise - or see Godement. //

let I be the category w. ob. $\{U_\alpha \to U\}$ and with one morphism $\{U_\alpha \to U\} \to \{V_\beta \to U\}$ if there exists a map $\{U_\alpha \to U\} \to \{V_\beta \to U\}$. Then

$$\{U_\alpha \to U\} \mapsto H^\circ(\{U_\alpha \to U\}, F)$$

is a contravariant functor from I to sets. Define

$$F^+(U) = \operatorname*{colim}_{I^{op}} H^\circ(\{U_\alpha \to U\}, F).$$

This is a filtered colimit:

$$\begin{tikzcd}
\{U_\alpha \to U\} & \\
& \{U_\alpha \times_U V_\beta \to U\} \arrow[ul] \arrow[dl] \\
\{V_\beta \to U\} &
\end{tikzcd}$$

and no parallel morphisms.

We say that $F \in X^\wedge$ is <u>separated</u> if for all $\{U_\alpha \to U\}$,

$$F(U) \longrightarrow \prod_\alpha F(U_\alpha)$$

is injective.

<u>Lemma</u> F^+ is separated.

pf Let $\{U_\alpha \to V\}$ be a covering and suppose $\bar{\xi}_1, \bar{\xi}_2$ in $F^+(U)$ have the same image in $F^+(U_\alpha)$, for all α. Since I^{op} is filtered, we can repr. $\bar{\xi}_1, \bar{\xi}_2$ by elements

$$\xi_1, \xi_2 \in \ker\left(\prod_\beta F(V_\beta) \rightrightarrows \prod_{\beta,\beta'} F(V_\beta \underset{U}{\times} V_{\beta'})\right)$$

The the images of $\bar{\xi}_i$ in $F^+(U_\alpha)$ are repr. by the images of ξ_i in

$$\ker\left(\prod_\beta F(U_\alpha \underset{U}{\times} V_\beta) \rightrightarrows \prod_{\beta,\beta'} F(U_\alpha \underset{U}{\times} V_\beta \underset{U}{\times} V_{\beta'})\right)$$

Since $\bar{\xi}_1$ and $\bar{\xi}_2$ have the same image in $F^+(U_\alpha)$, there exists $\{W_{\alpha,\nu} \to U_\alpha\} \to \{U_\alpha \underset{U}{\times} V_\beta \to U_\alpha\}$ s.t. the images of ξ_1, ξ_2 in $\prod_\nu F(W_{\alpha,\nu})$ are equal. But then $\{W_{\alpha,\nu} \to U\} \to \{V_\beta \to U\}$ and the images of ξ_1, ξ_2 in $\prod_{\alpha,\nu} F(W_{\alpha,\nu})$ are equal. I.e. $\bar{\xi}_1 = \bar{\xi}_2 \in F^+(U)$. //

<u>Lemma</u> If F is separated, then F^+ is a sheaf.

pf We first note: If $\{V_\beta \to U\} \to \{U_\alpha \to U\}$ then

$$\ker\left(\prod_\alpha F(U_\alpha) \rightrightarrows \prod_{\alpha,\alpha'} F(U_\alpha \underset{U}{\times} U_{\alpha'})\right)$$

$$\hookrightarrow \ker\left(\prod_\beta F(V_\beta) \rightrightarrows \prod_{\beta,\beta'} F(V_\beta \underset{U}{\times} V_{\beta'})\right),$$

For consider

$$\begin{tikzcd}[column sep=small] & \{V_\beta \times_U U_\alpha \to U\} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \{V_\beta \to U\} \arrow[rr, "f"] & & \{U_\alpha \to U\} \end{tikzcd}$$

If we show $p_2^* = p_1^* f^*$ is 1-1 then so is f^*. For fixed α, $\{V_\beta \times_U U_\alpha \to U_\alpha\}$ is a covering, and since F is sep.

$$\prod_\alpha F(U_\alpha) \hookrightarrow \prod_{\alpha,\beta} F(U_\alpha \times_U V_\beta).$$

Hence p_2^* is injective. Now let

$$(\bar{\xi}_\alpha) \in \ker\left(\prod_\alpha F^+(U_\alpha) \rightrightarrows \prod_{\alpha,\alpha'} F(U_\alpha \times_U U_{\alpha'})\right).$$

Wish to find $\bar{\xi} \in F^+(U)$ which restricts to $\bar{\xi}_\alpha \in F^+(U_\alpha)$. For each α, choose $\{V_{\alpha,\nu} \to U_\alpha\}$ s.t. $\bar{\xi}_\alpha$ is repr. by

$$(\xi_{\alpha,\nu}) \in \ker\left(\prod_\nu F(V_{\alpha,\nu}) \rightrightarrows \prod_{\nu,\nu'} F(V_{\alpha,\nu} \times_{U_\alpha} V_{\alpha,\nu'})\right)$$

We show that these match up: consider

$$\begin{tikzcd}
\{V_{\alpha,\nu} \underset{U}{\times} V_{\beta,\mu} \to U\} \arrow[r] \arrow[d] & \{V_{\alpha,\nu} \underset{U}{\times} U_\beta \to U\} \arrow[r] \arrow[d] & \{V_{\alpha,\nu} \to U\} \arrow[d] \\
\{U_\alpha \underset{U}{\times} V_{\beta,\mu} \to U\} \arrow[r] \arrow[d] & \{U_\alpha \underset{U}{\times} U_\beta \to U\} \arrow[r] \arrow[d] & \{U_\alpha \to U\} \arrow[d] \\
\{V_{\beta,\mu} \to U\} \arrow[r] & \{U_\beta \to U\} \arrow[r] & \{U \to U\}
\end{tikzcd}$$

Then ξ_α defines an element

$$\xi^1_{\alpha\beta} \in \ker\Big(\prod_\nu F(V_{\alpha,\nu} \underset{U}{\times} U_\beta) \rightrightarrows \prod_{\nu,\nu'} F(V_{\alpha,\nu} \underset{U}{\times} U_\beta \underset{U_\alpha \underset{U}{\times} U_\beta}{\times} V_{\alpha,\nu'} \underset{U}{\times} U_\beta)\Big)$$

and ξ_β an element

$$\xi^2_{\alpha\beta} \in \ker\Big(\prod_\mu F(U_\alpha \underset{U}{\times} V_{\beta,\mu}) \rightrightarrows \prod_{\mu,\mu'} F(U_\alpha \underset{U}{\times} V_{\beta,\mu} \underset{U_\alpha \underset{U}{\times} U_\beta}{\times} U_\alpha \underset{U}{\times} V_{\beta,\mu'})\Big).$$

And by the assumption on (ξ_α), $\xi^1_{\alpha\beta}$ and $\xi^2_{\alpha\beta}$ define the same element in $F^+(U_\alpha \underset{U}{\times} U_\beta)$. Since I is filtered, we can find common ref.

$$\{V_{\alpha,\nu} \underset{U}{\times} U_\beta \to U_\alpha \underset{U}{\times} U_\beta\} \leftarrow \{W_\gamma \to U_\alpha \underset{U}{\times} U_\beta\} \to \{U_\alpha \underset{U}{\times} V_{\beta,\mu} \to U_\alpha \underset{U}{\times} U_\beta\}$$

s.t.

$$\xi^1_{\alpha\beta} \in \ker\Big(\prod_{\nu} F(V_{\alpha,\nu} \times_U U_\beta) \rightrightarrows \prod_{\nu,\nu'} F(V_{\alpha,\nu} \times_U U_\beta \times_{U_\alpha \times_U U_\beta} V_{\alpha,\nu'} \times_U U_\beta)\Big)$$

$$\begin{array}{ccc} \downarrow & \qquad\qquad\qquad & \hookdownarrow \end{array}$$

$$\text{same} \in \ker\Big(\prod_{\gamma} F(W_\gamma) \rightrightarrows \prod_{\gamma,\gamma'} F(W_\gamma \times_{U_\alpha \times_U U_\beta} W_{\gamma'})\Big)$$

$$\begin{array}{ccc} \uparrow & \qquad\qquad\qquad & \hookuparrow \end{array}$$

$$\xi^2_{\alpha\beta} \in \ker\Big(\prod_{\mu} F(U_\alpha \times_U V_{\beta,\mu}) \rightrightarrows \prod_{\mu,\mu'} F(U_\alpha \times_U V_{\beta,\mu} \times_{U_\alpha \times_U U_\beta} U_\alpha \times V_{\beta,\mu'})\Big)$$

But then the same is also true for the common refinement $\{V_{\alpha,\nu} \times_U V_{\beta,\mu} \to U_\alpha \times_U U_\beta\}$, i.e. $\xi^1_{\alpha\beta}$ and $\xi^2_{\alpha\beta}$ have the same image in $\prod_{\nu,\mu} F(V_{\alpha,\nu} \times_U V_{\beta,\mu})$.
Hence

$$(\xi_{\alpha,\nu}) \in \ker\Big(\prod_{\alpha,\nu} F(V_{\alpha,\nu}) \rightrightarrows \prod_{\alpha,\nu,\beta,\mu} F(V_{\alpha,\nu} \times_U V_{\beta,\mu})\Big)$$

defines $\bar{\xi} \in F^+(U)$ which restricts to $\bar{\xi}_\alpha$, $\forall \alpha$. //

Pf Define $a(F) = F^{++}$. Clearly, a is left adjoint to $\cup$ and commutes with finite limits. //

<u>Lemma</u> Suppose (u, v) is an adjoint pair. Then

$$\operatorname*{colim}_I u(X_i) \xrightarrow{\sim} u(\operatorname*{colim}_I X_i), \quad v(\lim_I Y_i) \xrightarrow{\sim} \lim_I v(Y_i).$$

<u>pf</u> Write down inverse: by adjointness

$$u(\operatorname*{colim}_I X_i) \to \operatorname*{colim}_I u(X_i) \quad \leftrightsquigarrow$$

$$\operatorname*{colim}_I X_i \to v(\operatorname*{colim}_I u(X_i)) \quad \leftrightsquigarrow$$

$$X_i \to v(\operatorname*{colim}_I u(X_i)), \quad \forall i \quad \leftrightsquigarrow$$

$$u(X_i) \to \operatorname*{colim}_I u(X_i), \quad \forall i$$

and these are the canonical maps. //

<u>Prop</u> (Kan extension) Let C be a category which have all limits and colimits, and let $u : I \to J$ be a functor. Then the induced functor of diagram categories

$$C^I \xleftarrow{u^*} C^J$$

has both a left adjoint $u_!$ and a right adjoint u_* given by

$$(u_! F)(j) = \operatorname*{colim}_{(i,\, u(i) \to j)} F(i)$$

$$(u_* F)(j) = \lim_{(i,\, j \to u(i))} F(i).$$

pf Must show

$$\mathrm{Hom}_{C^J}(u_! F, G) \cong \mathrm{Hom}_{C^I}(F, u^* G).$$

Given $u_! F \xrightarrow{\varphi} G$, get $F \xrightarrow{\tilde{\varphi}} u^* G$ by

$$\begin{tikzcd}
(u_! F)(u(i)) \arrow[r, "\varphi"] & G(u(i)) = (u^* G)(i) \\
\operatorname*{colim}_{(i', u(i') \to u(i))} F(i') \arrow[u, equal] & \\
F(i) \atop (i, u(i) \xrightarrow{id} u(i)) \arrow[u] \arrow[uur, "\tilde{\varphi}"'] &
\end{tikzcd}$$

Conversely, given $F \xrightarrow{\psi} u^* G$, get $u_! F \xrightarrow{\psi^+} G$ by

$$\operatorname*{colim}_{(i, u(i) \to j)} F(i) \xrightarrow{\psi} \operatorname*{colim}_{(i, u(i) \to j)} G(u(i)) \longrightarrow G(j).$$

//

Let X be a space and let $O(X)$ be the category:

$\mathrm{ob}\, O(X) = \{ U \subset X \text{ open} \}$

$\mathrm{mor}\, O(X)$: one arrow $U \to U'$ if $U \subset U'$.

The cat. $X^\wedge$ of presheaves of sets on X is the diagram category

$$X^\wedge = \mathrm{Sets}^{O(X)^{op}}.$$

A cts. map $f : X \to Y$ defines a functor

$$O(X) \xleftarrow{u} O(Y), \qquad f^{-1}(V) \leftarrow\!\shortmid V,$$

and hence

$$\begin{tikzcd} X^\wedge \arrow[r, "u^*"] & Y^\wedge \arrow[l, bend right, "u_!"'] \arrow[l, bend left, "u_*"] \end{tikzcd}$$

given by

$$(f_p G)(V) := (u^* G)(V) = G(f^{-1}(V))$$

$$(f^p F)(U) := (u_! F)(U) = \operatorname*{colim}_{(V,\ f(U) \subset V)} F(V).$$

The <u>direct</u> <u>image</u> functor f_p preserves sheaves, but f^p does not. Recall

$$f_p U = U f_*$$

$$\begin{tikzcd} X^\sim \arrow[r, hook, "U"'] & X^\wedge \arrow[l, bend right, "a"'] \end{tikzcd},$$

Define <u>inverse</u> <u>image</u> $f^* F$ by:

$$f^* F = a(f^p(UF)),$$

the sheaf associated with the presheaf $f^p UF$.

<u>Lemma</u> The functors (f^*, f_*) form an adjoint pair, and f^* preserves finite limits.

<u>Pf</u> Since colimit which defines f^p is

filtered, so f^p preserves finite limits; so does a, and U preserves all limits. Hence f^* preserves finite limits.

$$\mathrm{Hom}_{X^\sim}(f^*F, G) = \mathrm{Hom}_{X^\sim}(a f^p U F, G)$$

$$\cong \mathrm{Hom}_{X^\wedge}(f^p UF, UG) \cong \mathrm{Hom}_{Y^\wedge}(UF, f_p UG)$$

$$\cong \mathrm{Hom}_{Y^\wedge}(UF, U f_* G) = \mathrm{Hom}_{Y^\sim}(F, f_* G)$$

since f_* preserves sheaves — since sheaves is a full subcat. of presheaves. //

Suppose $j : U \hookrightarrow X$ is the inclusion of an open set. Get functor

$$O(U) \xrightarrow{v} O(X), \qquad V \subset U \longmapsto V \subset X.$$

$$\begin{tikzcd} U^\wedge \arrow[r, bend left, "v_!"] & X^\wedge \arrow[l, "v^*"'] \end{tikzcd} \quad \begin{tikzcd} U^\wedge \arrow[r, bend right, "v_*"'] & X^\wedge \end{tikzcd}$$

$$(v^*F)(V) = F(V) \xrightarrow{\sim} \operatorname*{colim}_{\substack{(W,\, j(V) \subset W) \\ \text{open in } X}} F(W) = (j^p F)(V)$$

$$(v_* G)(W) = \lim_{\substack{(V,\, W \supset j(V)) \\ \text{open in } U}} F(W) \xrightarrow{\sim} F(W \cap V)$$

$$= (j_p F)(W)$$

$$(v_! F)(W) = \operatorname*{colim}_{(V,\, W \subset j(V))} F(W) = \begin{cases} F(V), & \text{if } V \subset U, \\ \emptyset, & \text{if } V \not\subset U. \end{cases}$$

The functors j^p, j_p both preserve sheaves, so define (j^*, j_*). Define $j_! F = a\, v_! U F$.

<u>Lemma</u> The functors $(j_!, j^*)$ and (j^*, j_*) form adjoint pairs of functors. //

<u>Rem</u> The functor $v_!$ does <u>not</u> preserve finite limits. So for a (pre)sheaves of ab. groups, must instead use

$$(v_! F)(W) = \operatorname*{colim}_{(V, W \subset j(V))} F(W) = \begin{cases} F(V), & V \subset U, \\ \{0\}, & V \not\subset U. \end{cases}$$

So $j_! F$ means different things for sheaves of sets and sheaves of abelian groups. //

<u>Lemma</u> Let $U \xhookrightarrow{j} X$ be the inclusion of an open subset and let $Z \xhookrightarrow{i} X$ be the incl. of the closed complement. Then the sq.:

$$\begin{array}{ccc} F & \longrightarrow & j_* j^* F \\ \downarrow & & \downarrow \\ i_* i^* F & \longrightarrow & i_* i^* j_* j^* F \end{array}$$

is cartesian, for every F in $X^\sim$.

pf Look at stalks. If $x \in U$, the diagr. is

$$\begin{array}{ccc} F_x & \longrightarrow & F_x \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \end{array}.$$

Indeed,

$$(i_* i^* F)_x = \operatorname*{colim}_{x \in V \subset X} (i_* i^* F)(V)$$

$$\xleftarrow{\sim} \operatorname*{colim}_{x \in V \subset U} (i^* F)(V \cap U) = \operatorname*{colim}_{x \in V \subset U} (i^* F)(\emptyset)$$

$$= \operatorname*{colim}_{x \in V \subset U} \{*\} = * \quad \text{(since filt. colimit).}$$

Similarly, if $x \in Z$ we get

$$\begin{array}{ccc} F_x & \longrightarrow & (j_* j^* F)_x \\ \downarrow & & \downarrow \\ F_x & \longrightarrow & (j_* j^* F)_x \end{array}.$$

Here:

$$(i_* i^* F)_x = \operatorname*{colim}_{x \in V \subset X} (i_* i^* F)(V)$$

$$= \operatorname*{colim}_{x \in V \subset X} i^* F(V \cap Z) = (i^* F)_x$$

$$= F_{i(x)} = F_x. \qquad //$$

Cor The category $X^\sim$ is naturally equivalent to the category with objects the triples of $F_1 \in Z^\sim$, $F_2 \in U^\sim$, and $F_1 \xrightarrow{\varphi} i^* j_* F_2 \in Z^\sim$, and with morphisms pairs $F_1 \xrightarrow{\psi_1} F_1' \in Z^\sim$, $F_2 \xrightarrow{\psi_2} F_2' \in U^\sim$ s.t.

$$\begin{array}{ccc} F_1 & \xrightarrow{\varphi} & i^* j_* F_2 \\ \downarrow \psi_1 & & \downarrow i^* j_* \psi_2 \\ F_1' & \xrightarrow{\varphi'} & i^* j_* F_2' \end{array}$$

commutes. The equivalence takes $F \in X^\sim$ to the triple $(i^*F,\ j^*F,\ i^*F \to i^* j_* j^* F)$. //

Describe $j_!$, j^*, j_* and i^*, i_* in terms of triples:

$$i^*(F_1, F_2, \varphi) = F_1$$

$$i_*(F_1) = (F_1,\ *,\ F_1 \to *)$$

$$j_!(F_2) = (\emptyset,\ F_2,\ \emptyset \to F_2)$$

$$j^*(F_1, F_2, \varphi) = F_2$$

$$j_*(F_2) = (i^* j_* F_2,\ F_2,\ \mathrm{id}).$$

For sheaves of ab. groups, we have instead

$$j_!(F_2) = (0,\ F_2,\ 0 \to F_2).$$

$$i^!(F_1, F_2, \varphi) := \ker \varphi$$

exercise Show that $(i_*, i^!)$ is an adjoint pair. //

The category $X^\wedge$ has all (small) limits and colimits:

$$(\lim_I F_i)(U) = \lim_I F_i(U)$$

$$(\operatorname{colim}_I F_i)(U) = \operatorname{colim}_I F_i(U).$$

The category $X^\sim$ also has all (small) limits and colimits:

$$(\lim_I F_i)(U) = \lim_I F_i(U)$$

$$(\operatorname{colim}_I F_i)(U) = a\,(V \mapsto \operatorname{colim}_I F_i(V))\,(U).$$

The category of ab. groups in $X^\sim$ (resp. $X^\wedge$) is abelian:

$$\ker(A \xrightarrow{f} B) = \lim\left(A \underset{0}{\overset{f}{\rightrightarrows}} B \right)$$

$$\operatorname{coker}(A \xrightarrow{f} B) = \operatorname{colim}\left(\begin{tikzcd} A\times B \arrow[r, shift left, "pr_2"] \arrow[r, shift right] \arrow[dr, "f\times id"'] & B \\ & B\times B \arrow[u, "+"'] \end{tikzcd}\right)$$

Rem Note that i^*, i_*, $j_!$, and j^* are exact, and that j_* and $i^!$ are left exact. //

Let $(X, \mathcal{O}_X)$ be a ringed space. This is the same as a space X and a ring $\mathcal{O}_X \in X^\sim$. A map of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is the same as a cts. map $X \xrightarrow{f} Y$ and a ring homomorphism $\mathcal{O}_Y \to f_* \mathcal{O}_X$, or equivalently, $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. Define

$$\{\mathcal{O}_X\text{-mod}\} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \{\mathcal{O}_Y\text{-mod}\}$$

by

$$f_* N = f_* N$$

$$f^* M = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1} N.$$

Then (f^*, f_*) is an adjoint pair of functors.

Def Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module M is <u>quasi</u>-<u>coherent</u> if it is locally presented, i.e. there exists an open covering $\{j_\alpha : U_\alpha \hookrightarrow X\}$ s.t. the $j_\alpha^{-1}\mathcal{O}_X$-module $j_\alpha^{-1} M$ has a presentation

$$\bigoplus_{i \in I_1} j_\alpha^{-1}\mathcal{O}_X \longrightarrow \bigoplus_{i \in I_0} j_\alpha^{-1}\mathcal{O}_X \longrightarrow j_\alpha^{-1} M \longrightarrow 0.$$

An $\mathcal{O}_X$-module is <u>coherent</u> if it is locally finitely presented, i.e. I_0, I_1 can be chosen finite. //

In general, limits (e.g. kernel, product) of a diagram $\{F_i\}_{i\in I}$ of sheaves on X is given by:

$$(\lim_i F_i)(U) = \lim_i F_i(U).$$

The colimit (e.g. coker, direct sum) is given by

$$(\operatorname{colim}_i)(U) = a(V \longmapsto \operatorname{colim}_i F_i(V))(U).$$

Def Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module M is quasi-coherent if it is locally presented, i.e. for all $x\in X$, there exists $x\in U \overset{j}{\hookrightarrow} X$ open and an exact seq.

$$\bigoplus_{i\in I_1} j^*\mathcal{O}_X \longrightarrow \bigoplus_{i\in I_0} j^*\mathcal{O}_X \longrightarrow j^*M \longrightarrow 0.$$

If I_0, I_1 can be chosen finite, we say that M is coherent. //

Let $(X, \mathcal{O}_X) = (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$, let M be an R-module. Define $\mathcal{O}_X$-module $\tilde{M}$ by

$$\tilde{M} = a\left(U \longmapsto \begin{cases} M_f & \text{if } U = D(f) \\ 0 & \text{else} \end{cases}\right)$$

Then $\Gamma(D(f), \tilde{M}) \xleftarrow{\sim} M_f$, $\tilde{M}_x \xrightarrow{\sim} M_{\mathfrak{p}_x}$, and

<u>Lemma</u> $\mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, N) \xrightarrow{\sim} \mathrm{Hom}_R(M, \Gamma(X, N))$

<u>pf</u> The map takes a map of sheaves to the induced map of global sections. Conversely, if $M \to \Gamma(X, N)$ is R-linear, there is a unique R_f-linear map $M_f \to \Gamma(D(f), N)$ s.t.

$$\begin{array}{ccc} M & \longrightarrow & \Gamma(X, N) \\ \downarrow & & \downarrow \\ M_f & \overset{\exists !}{\dashrightarrow} & \Gamma(D(f), N) \end{array}$$

commutes. //

<u>Lemma</u> A seq. of R-modules $M' \to M \to M''$ is exact if and only if the seq. of $\mathcal{O}_X$-modules $\tilde{M}' \to \tilde{M} \to \tilde{M}''$ is exact.

<u>pf</u> Both statements are equivalent to:

$$M'_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \longrightarrow M''_{\mathfrak{p}}$$

is exact, for all $\mathfrak{p} \subset R$ prime. //

<u>Cor</u> If $\tilde{M} \to \tilde{N}$ is $\mathcal{O}_X$-linear, then its kernel and cokernel are $\mathcal{O}_X$-modules of the form $\tilde{K}$, for some R-module K. //

<u>Thm</u> Let $(X, \mathcal{O}_X)$ be a scheme and let M be an $\mathcal{O}_X$-module. The following are equivalent:

(i) M is quasi-coherent.

(ii) for all $U \subset V \subset X$ open aff., the can. map is an isomorphism:

$$\Gamma(U, \mathcal{O}_X) \otimes_{\Gamma(V, \mathcal{O}_X)} \Gamma(V, M) \xrightarrow{\sim} \Gamma(U, M).$$

(iii) for all $U \subset X$ open aff., $M|_U \cong \tilde{N}$, for some $\Gamma(U, \mathcal{O}_X)$-module N.

(iv) there exists an aff. open cover $\{U_\alpha\}$ of X s.t. for all α, $M|_{U_\alpha} \cong \tilde{N}_\alpha$, for some $\Gamma(U_\alpha, \mathcal{O}_X)$-module N_α.

Pf (i) $\Rightarrow$ (iv) we can cover X by affine open $U_\alpha = \operatorname{Spec} R_\alpha$ s.t. there are exact seq.

$$\bigoplus_{i \in I_1} \mathcal{O}_X|_{U_\alpha} \to \bigoplus_{i \in I_0} \mathcal{O}_X|_{U_\alpha} \to M|_{U_\alpha} \to 0.$$

Since $M \mapsto M^\sim$ has a right adjoint, it preserves colimits. In particular, it preserves sums, so

$$\bigoplus_i \mathcal{O}_X|_{U_\alpha} = \bigoplus_i \tilde{R}_\alpha \xrightarrow{\sim} \Big(\bigoplus_i R_\alpha\Big)^\sim.$$

By the cor., $M|_{U_\alpha}$ has the form $\tilde{N}_\alpha$.

(iv) $\Rightarrow$ (iii) let $U = \operatorname{Spec} R$, then by (iv), U has an open aff. cover U_α s.t. $M|_{U_\alpha} \cong \tilde{N}_\alpha$. We can assume $U_\alpha = D(g_\alpha)$ is distinguished. Since U is quasi-cpt, we can assume $\{U_\alpha\}$ finite.

Let $U_\alpha \xhookrightarrow{j_\alpha} U$ and $U_\alpha \cap U_\beta \xrightarrow{j_{\alpha\beta}} U$ be the incl.
Then the seq. of $\mathcal{O}_X|_U$ - modules is exact:

$$(\dagger) \quad 0 \to M|_U \to \prod_\alpha j_{\alpha *}(M|_{U_\alpha}) \to \prod_{\alpha,\beta} j_{\alpha,\beta *}(M|_{U_\alpha \cap U_\beta})$$

Here $M|_{U_\alpha} \cong \tilde{N}_\alpha$, $M|_{U_\alpha \cap U_\beta} \cong \widetilde{(N_\alpha)_{g_\beta}} \cong \widetilde{(N_\beta)_{g_\alpha}}$.

Claim $j_{\alpha *}(\tilde{N}_\alpha) = \tilde{N}_\alpha$

N_α R_{g_α}-mod $\qquad\qquad$ N_α viewed as R-mod.

pf $\Gamma(D(g), j_{\alpha *}(\tilde{N}_\alpha)) = \Gamma(D(g) \cap U_\alpha, \tilde{N}_\alpha)$
$= (N_\alpha)_g = \Gamma(D(g), \tilde{N}_\alpha)$. /

Since the covering $\{U_\alpha\}$ is finite

$$\prod_\alpha j_{\alpha *}(M|_{U_\alpha}) \xleftarrow{\sim} \bigoplus_\alpha j_{\alpha *}(M|_{U_\alpha})$$

So the two products in $(\dagger)$ are both $\mathcal{O}_U$-mod. of the form $\tilde{N}$, some R-module N. Hence, so is $M|_U$ by cor.

(iii) $\Rightarrow$ (ii) $M|_V \cong \tilde{N}$, some $\Gamma(V, \mathcal{O}_X)$-mod. N.
Choose a presentation of N:

$$\bigoplus_{i \in I_1} \Gamma(V, \mathcal{O}_X) \to \bigoplus_{i \in I_0} \Gamma(V, \mathcal{O}_X) \to \Gamma(V, M) \to 0$$

By lemma 2, this gives an exact seq. of $\mathcal{O}_X|_V$-modules

$$\bigoplus_{i\in I_1} \mathcal{O}_X|_V \to \bigoplus_{i\in I_0} \mathcal{O}_X|_V \to M|_V \to 0 .$$

Since $j^* = (-)|_U$ is exact, we get

$$\bigoplus_{i\in I_1} \mathcal{O}_X|_U \to \bigoplus_{i\in I_0} \mathcal{O}_X|_U \to M|_U \to 0 .$$

And since $M|_U$ is of the form $\tilde{K}$, lemma 2 show that the seq. is exact:

$$\bigoplus_{i\in I_1} \Gamma(U,\mathcal{O}_X) \to \bigoplus_{i\in I_0} \Gamma(U,\mathcal{O}_X) \to \Gamma(U,M) \to 0.$$

Now use that $\Gamma(U,\mathcal{O}_X)\underset{\Gamma(V,\mathcal{O}_X)}{\otimes} -$ is right exact:

$$\begin{tikzcd}
\Gamma(U,\mathcal{O}_X)\underset{\Gamma(V,\mathcal{O}_X)}{\otimes}\left(\bigoplus_{I_1}\Gamma(V,\mathcal{O}_X)\right) \arrow[r] \arrow[d, "\sim"] & \Gamma(U,\mathcal{O}_X)\underset{\Gamma(V,\mathcal{O}_X)}{\otimes}\left(\bigoplus_{I_0}\Gamma(V,\mathcal{O}_X)\right) \arrow[r] \arrow[d, "\sim"] & \Gamma(U,\mathcal{O}_X)\underset{\Gamma(V,\mathcal{O}_X)}{\otimes}\Gamma(V,M) \arrow[r] \arrow[d] & 0 \\
\bigoplus_{i\in I_1}\Gamma(U,\mathcal{O}_X) \arrow[r] & \bigoplus_{i\in I_0}\Gamma(U,\mathcal{O}_X) \arrow[r] & \Gamma(U,M) \arrow[r] & 0
\end{tikzcd}$$

(ii) $\Rightarrow$ (i) Pick any aff. open $x\in U\subset X$. We show that the can. map

$$\widetilde{\Gamma(U,M)} \longrightarrow M|_U$$

is an isomorphism. For all $f \in \Gamma(U, \mathcal{O}_X)$:

$$\begin{tikzcd}
\Gamma(U_f, \widetilde{\Gamma(U,M)}) \arrow[r] & \Gamma(U_f, M) \\
\Gamma(U_f, \mathcal{O}_X) \otimes_{\Gamma(U,\mathcal{O}_X)} \Gamma(U, M) \arrow[u, "\underline{\text{def}}\ \sim"] \arrow[ur, "\sim \text{ by (ii)}"'] &
\end{tikzcd}$$

So enough to show that $\widetilde{\Gamma(U,M)}$ has a presentation. Then $\Gamma(U, \mathcal{O}_X)$-mod. $\Gamma(U, M)$ has a presentation

$$\bigoplus_{i \in I_1} \Gamma(U, \mathcal{O}_X) \to \bigoplus_{i \in I_0} \Gamma(U, \mathcal{O}_X) \to \Gamma(U, M) \to 0$$

and by lemma 2, the induced seq.

$$\bigoplus_{i \in I_1} \mathcal{O}_X|_U \to \bigoplus_{i \in I_0} \mathcal{O}_X|_U \to \widetilde{\Gamma(U,M)} \to 0$$

is exact. //

<u>ex</u> (1) Let V be a DVR with quotient field K. A sheaf M on Spec V consists of

$$\begin{array}{ccc}
M_0 & = & \Gamma(X, M) \\
\downarrow & & \downarrow \\
M_1 & = & \Gamma(U, M)
\end{array}$$

If M is an $\mathcal{O}_X$-mod. then M_0 is a V-mod.

M_1 is a K-vector space, and $M_0 \to M_1$ is V-linear; M is quasi-coh. iff

$$K \otimes_V M_0 \xrightarrow{\sim} M_1 .$$

(2) Let $Y \xhookrightarrow{i} X$ be a closed immersion; let $\mathcal{I}$ be the corresponding sheaf of ideals:

$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_* \mathcal{O}_Y \to 0$$

Then $i_* \mathcal{O}_Y$, and hence $\mathcal{I}$, is a q.-coh. sheaf of $\mathcal{O}_X$-modules: For let $\{U_\alpha\}$ be an aff. open cover of X. Then we showed that $i^{-1}(U_\alpha)$ is affine and

$$0 \to \Gamma(U_\alpha, \mathcal{I}) \to \Gamma(U_\alpha, \mathcal{O}_X) \to \Gamma(U_\alpha, i_* \mathcal{O}_Y) \to 0$$

so $(i_* \mathcal{O}_Y)|_{U_\alpha} \cong \widetilde{\Gamma(U_\alpha, i_* \mathcal{O}_Y)}$.

Conversely, if $\mathcal{I} \subset \mathcal{O}_X$ is a q.-coh. ideal, let $Y \subset X$ be the support of $\mathcal{O}_X / \mathcal{I}$, i.e.

$$Y = \{ x \in X \mid (\mathcal{O}_X / \mathcal{I})_x \neq 0 \} .$$

Then $Y \xhookrightarrow{i} X$ is closed and $(Y, i^{-1}(\mathcal{O}_X / \mathcal{I}))$ is a scheme. Indeed, we can assume X aff, say $X = \operatorname{Spec} R$. Then $\mathcal{O}_X = \tilde{R}$, $\mathcal{I} = \tilde{I}$, so $Y = \operatorname{Spec} R/I$. But we also have

$$i^{-1}(\mathcal{O}_X / \mathcal{I}) \cong \widetilde{(R/I)} \qquad \text{(stalks equal)}.$$

- For Lars Hesselholt's class.

21 Mar. 2001

QUASICOHERENT SHEAVES AND SHEAF OF RELATIVE DIFFERENTIALS

Defn of $\mathcal{O}_X$-module $\mathcal{F}$ (on $(X, \mathcal{O}_X)$ scheme).

- Sheaf
- $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module
- in a way that agrees with restriction maps.

$$\begin{tikzcd} \mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[d] \\ \mathcal{O}_X(V) \times \mathcal{F}(V) \arrow[r] & \mathcal{F}(V) \end{tikzcd}$$

commutes for all $V \hookrightarrow U$.

Defn $\mathcal{F}$ is qcoh if

$\forall\ U \subset X$ affine open $\mathcal{F}|_U \cong \tilde{M}$ for some $\mathcal{O}_X(U)$-module M

Prop

$\forall\ U \in \{U_i\}$ U_i affine cover.

Prop $\forall\ x \in X$, $\exists$ U nbhd of x and exact seq. of $\mathcal{O}_X|_U$-modules.

$$\mathcal{O}_U^{(I)} \to \mathcal{O}_U^{(J)} \to \mathcal{F}|_U \to 0.$$

Example If in Prop above, $\tilde{M} = \mathcal{O}_X(U)^{I_i}$ for some I_i's then $\mathcal{F}$ is locally free.

If all I_i are same, $= r$, $\mathcal{F}$ is l.f. of rank r.

If $r=1$, invertible sheaf.

Idea. Rank r vector bundles $\longleftrightarrow$ l.f. qcs of rank r.

Examples Copies of structure sheaf.

Examples $\rightarrow$ $\mathcal{O}_{\mathbb{P}^n}(k)$ are inv. sheaves on $\mathbb{P}^n$. $(k \in \mathbb{Z})$.
They are all of them, and they are all different.

$\rightarrow$ " X smooth sheaf of section of tgt bundle."

Sheaf of Rel diff's

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd}$$

Spec'd on $\operatorname{Spec} B$

$$\begin{tikzcd} \operatorname{Spec} B \arrow[d] & B \\ \operatorname{Spec} A & A \arrow[u] \end{tikzcd}$$

Defn $\Omega_{B/A}$ to be a B-module

— Generated by $\{d\beta \mid \beta \in B\}$

— Relations. $d(\ \ ) = d(\ \ ) - d(\ \ )$

$$\alpha, \beta \in B \qquad \alpha\, d\beta + \beta\, d\alpha = d(\alpha\beta)$$

$$d\beta = 0 \quad \text{if } \beta \in f(A)$$

<u>Example</u>

$$\begin{tikzcd} B = k[x,y] \\ A = k[x] \arrow[u] \end{tikzcd} \qquad \Omega_{B/A} = k[x,y]\,dy$$

Why?

<u>Prop</u> $\hom_B(\Omega_{B/A}, C) \cong \left\{ \begin{matrix} \text{module of } A\text{-derivations} \\ D : B \to C \end{matrix} \right\}$ a B-module

<u>Reminder</u> A derivation $D : B \to C$

__Examples__ $A = k \quad B = k[x_1, \ldots, x_n]$.

Do it. what if relation $f(x_1, \ldots, x_n)$?

Add df.

eg. $k[x,y]/_{xy}$ $\quad xdy + ydx = 0$

$A = \mathbb{R} \quad B = \mathbb{C} = \mathbb{R}[z]/_{z^2+1}$

0-modules.

Char p $\quad A = \mathbb{F}_p[t] \quad B = \mathbb{F}_p[u]$

$t \longrightarrow u^p$.

$B = A[u]/_{u^p - t} \qquad Adu/_{pu^{p-1}}$.

Adu.

$$B \otimes_A B \xrightarrow{\delta} B \qquad I = \ker \delta$$

$$\beta_1 \otimes \beta_2 \longrightarrow \beta_1 \beta_2$$

I is a $B \otimes_A B$ -module.

I/I^2 is actually a B-module:
$\beta \otimes 1$ acts the same as $1 \otimes \beta$

<u>Thm</u> $I/I^2 \cong \Omega_{B/A}$

$1 \otimes \beta - \beta \otimes 1 \longleftarrow d\beta$

<u>Pf</u> 1) First, check that image lies in I. ✓

2) Next check that it is well-defined map from $\Omega_{B/A}$. a) + c) ok.

b). $d(\alpha\beta) = \alpha d\beta + \beta d\alpha$

$$1 \otimes \alpha\beta - \alpha\beta \otimes 1 \overset{?}{=} \alpha (1 \otimes \beta - \beta \otimes 1) + \beta (1 \otimes \alpha - \alpha \otimes 1)$$

OK.

3) Inverse map $I/I^2 \longrightarrow \Omega_{B/A}$

$E := B \oplus \Omega_{B/A}$ ring $\qquad R + Rd\cancel{x}$

$B \times B \to E$

$(\beta_1, \beta_2) \to (\beta_1\beta_2, \beta_1 d\beta_2)$ — bi-additive $(x+y, z) = (x, z) + (y, z)$
— A-bilinear $(\alpha\beta_1, \beta_2) \mapsto \ldots$
$(\beta_1, \alpha\beta_2) \mapsto \ldots$

$B \otimes_A B \to E$ Ring hom in fact.

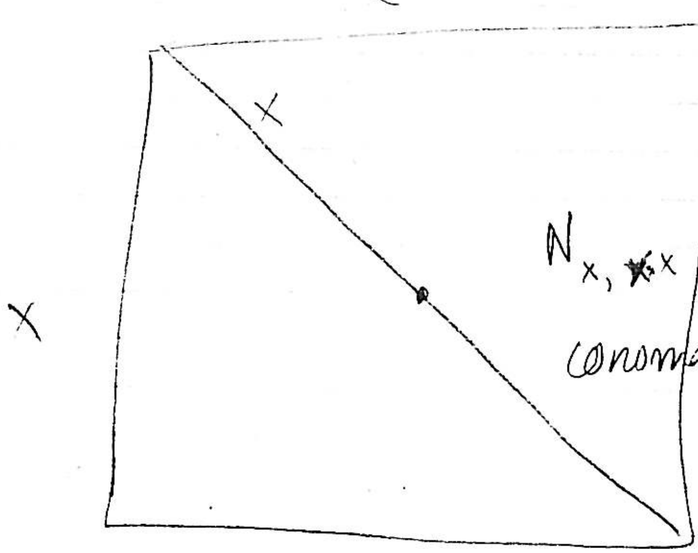
$I \to \Omega_{B/A}$

$I/I^2 \to \Omega_{B/A}$. check that two maps are inverses! //

In general $f: X \to Y$. separated

$X \xhookrightarrow{\Delta} X \times_Y X$ $\quad \mathcal{I}/\mathcal{I}^2$ sheaf on diagonal.
hence on X.

Idea $\quad X$ smooth



$N_{X, X \times X} \cong T_X$.

conormal bundle $\cong \Omega_X$.

Conormal bundle $- I/I^2$.

Point in $\mathbb{A}^n = \operatorname{Spec} k[x_1, \dots, x_n]$. $\quad$ Normal to pt $=$ tgt to $\mathbb{A}^n$ @ pt

$I = (x_1, \dots, x_n)$.

$I^2 = (x_1^2, x_1x_2, \dots x_n^2)$.

$I/I^2 = \{\alpha_1 x_1 + \dots + \alpha_n x_n\}$.

canonically cotangent.

<u>Example</u> If $f: X \to Y$, $Y = \operatorname{Spec} k$,

and $\Omega_{X/Y}$ loc free of rank dim k,

then we say X is smooth (over Y).

Tangent sheaf $= \Omega^*_{X/Y}$.

(Better defn later.)

Example $\mathbb{P}^1$.

$[x;y]$.

U = Patch 1

$y \neq 0$.

Coord. u. $[u;1]=[x,y]$

Spec $k[u]$

$u = x/y$

V = Patch 2.

$x \neq 0$

coord v. $[1;v] = [x,y]$

$v = y/x$

Spec $k[v]$.

$u = \frac{1}{v}$ on $U \cap V$.

Inv. sheaf

$f \in k[u]$ as $k[u]$-module

$g \in k[v]$.

$f = u^{l} g$

$g = v^{l} f$

e.g. Global sections, $l = 0$

$f(u) = g(v)$ Constants 1-dim

$l = 1$ $f(u) = u\, g(\frac{1}{u})$ 2-dim.

This sheaf is called $\mathcal{O}_{\mathbb{P}^1}(l)$. In fact can define $\mathcal{O}_{\mathbb{P}^n}(l)$.

These are all the inv. sheaves on $\mathbb{P}^n$.