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$$E_r^{p,q} \quad p, q \geq 0.$$

$$E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,\, q-(r-1)}$$

$$E_{r+1} = H(E_r, d_r)$$

$$E_1^{p,q} = H^{p+q}(gr^p)$$

$$E_\infty^{p,q} = gr^p H^{p+q}$$

$$\boxed{H^i(gr^p) = 0, \quad i < p}$$

write this:

$$E_1^{p,q} = H^{p+q}(gr^p C^*)$$

$$\Rightarrow H^{p+q}(C^*)$$

$$\begin{array}{ccccc} F^0 & \supset & F^1 & \supset & F^2 = 0 \\ \downarrow & & \| & & \\ F^0/F^1 & & F^1/F^2 & & \\ \| & & \| & & \\ gr^0 & & gr^1 & & \end{array}$$

$$E_2 = E_\infty$$

$$\begin{array}{ccccc} \vdots & & \vdots & \quad \vdots & \vdots \\ H^2(F^0/F^1) & \xrightarrow{d_1} & H^3(F^1) & \quad E_2^{0,2} & E_2^{1,2} \\ H^1(F^0/F^1) & \xrightarrow{d_1} & H^2(F^1) & \quad E_2^{0,1} & E_2^{1,1} \\ H^0(F^0/F^1) & \xrightarrow{d_1} & H^1(F^1) & \quad E_2^{0,0} & E_2^{1,0} \end{array}$$

$$0 \to E_2^{1,n-1} \to H^n(F^0) \to E_2^{0,n} \to 0$$

$$E_r^{p,q} = \frac{k^{-1}(\mathrm{im}(H^{p+q+1}(F^{p+r}) \to H^{p+q+1}(F^{p+1}))}{j(\ker(H^{p+q}(F^p) \to H^{p+q}(F^{p-(r-1)}))}$$

In particular:

$$E_1^{p,q} = H^{p+q}(gr^p)$$

$$E_\infty^{p,q} = \frac{j(H^{p+q}(F^p))}{j(\ker(H^{p+q}(F^p) \to H^{p+q})} \cong gr^p H^{p+q}$$

(We assume $H^i(gr^p) = 0$, $i < p$.)

$$
\begin{tikzcd}[column sep=small]
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots \arrow[r] H^{p+q-1}(F^p/F^{p+1}) \arrow[r] & H^{p+q}(F^{p+1}) \arrow[r, "j"] \arrow[d, "i"] & H^{p+q}(F^{p+1}/F^{p+2}) \arrow[r, "k"] & H^{p+q+1}(F^{p+2}) \arrow[r, "j"] \arrow[d, "i"] & H^{p+q+1}(F^{p+2}/F^{p+3}) \arrow[r] & \cdots \\
\cdots \arrow[r] H^{p+q-1}(F^{p-1}/F^p) \arrow[r] & H^{p+q}(F^p) \arrow[r, "j"] \arrow[d, "i"] & H^{p+q}(F^p/F^{p+1}) \arrow[r, "k"] & H^{p+q+1}(F^{p+1}) \arrow[r, "j"] \arrow[d, "i"] & H^{p+q+1}(F^{p+1}/F^{p+2}) \arrow[r] & \cdots \\
\cdots \arrow[r] H^{p+q-1}(F^{p-2}/F^{p-1}) \arrow[r] & H^{p+q}(F^{p-1}) \arrow[r, "j"] \arrow[d, "i"] & H^{p+q}(F^{p-1}/F^p) \arrow[r, "k"] & H^{p+q+1}(F^p) \arrow[r, "j"] \arrow[d] & H^{p+q+1}(F^p/F^{p+1}) \arrow[r] & \cdots \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & H^{p+q}(C^*) & & H^{p+q+1}(C^*) &
\end{tikzcd}
$$

$C = \{C^{p,q}\}_{p,q \geq 0}$ double cx:

$$
\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r] & C^{p,q+1} \arrow[u] \arrow[r, "d'"] & C^{p+1,q+1} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{p,q} \arrow[u, "d''"] \arrow[r, "d'"] \arrow[ur, phantom, "(-1)"] & C^{p+1,q} \arrow[u, "d''"] \arrow[r] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}
$$

$$d'd'' + d''d' = 0.$$

Associated <u>total</u> <u>cx</u> $\operatorname{Tot} C$:

$$(\operatorname{Tot} C)^n = \bigoplus_{p+q=n} C^{p,q}$$

with differential $d = d' + d''$. Two filtrations:

$$({}'F^p \operatorname{Tot} C)^n = \bigoplus_{\substack{s+t=n \\ s \geq n}} C^{p,q}$$

$$({}''F^p \operatorname{Tot} C)^n = \bigoplus_{\substack{s+t=n \\ t \geq n}} C^{p,q}$$

Two spectral sequences:

$${}'E_1^{p,q} = H^q(C^{p,*}, d'')$$

$${}'E_2^{p,q} = H^p(H^q(C, d''), d')$$

resp.

$${}''E_2^{p,q} = H^p(H^q(C, d'), d'') .$$

Usually one of these collapses and this calculates $H^* \mathrm{Tot}(C)$. The other gives an interesting spectral sequence.

ex $A \xrightarrow{f} B$ ring-homomorphism, M B-module, N A-module. Let f^*M be M regarded as an A-module. Then

$$\mathrm{Hom}_A(f^*M, N) \cong \mathrm{Hom}_B(M, \mathrm{Hom}_A(B, N)) .$$

Choose:

$M \leftarrow P_\bullet$ proj. res. of B-mod.

$N \longrightarrow I^\bullet$ inj. res. of A-mod.

Then $f^*M \leftarrow f^*P_\bullet$ is a res., but not necessarily by proj. A-modules.

Double cx:

$$\mathrm{Hom}_A(f^*P_\bullet, I^\bullet) \cong \mathrm{Hom}_B(P_\bullet, \mathrm{Hom}_A(B, I^\bullet))$$

Take coh. in P-direction first: since I^p is inj. $\mathrm{Hom}_A(-, I^p)$ is exact, so get

$$'E_1^{p,q} = H^q(\mathrm{Hom}_A(f^*P_\bullet, I^p)) = \begin{cases} \mathrm{Hom}_A(f^*M, I^p), & q=0, \\ 0 & , q>0. \end{cases}$$

so

$$'E_2^{p,q} = \begin{cases} \mathrm{Ext}_A^p(f^*M, N) & , q=0, \\ 0 & , q>0. \end{cases}$$

Hence $'E_\infty = 'E_2$ and this shows:

$$H^*(\mathrm{Tot}\,\mathrm{Hom}_A(f^*P_\bullet, I^\bullet)) = \mathrm{Ext}_A^*(f^*M, N) /$$

Now take coh. first in I-direction. Since P_p is proj. $\mathrm{Hom}_B(P_p, -)$ is exact, so

$$''E_1^{p,q} = H^q(\mathrm{Hom}_B(P_p, \mathrm{Hom}_A(B, I^\bullet))$$

$$= \mathrm{Hom}_B(P_p, \mathrm{Ext}_A^q(B, N))$$

$$''E_2^{p,q} = \mathrm{Ext}_B^p(M, \mathrm{Ext}_A^q(B, N))$$

Hence we have a spectral seq:

$$E_2^{p,q} = \mathrm{Ext}_B^p(M, \mathrm{Ext}_A^q(B, N))$$

$$\Longrightarrow \mathrm{Ext}_A^{p+q}(f^* M, N).$$

Call this change of rings sp. seq.

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$E^\cdot \qquad E^0 \xrightarrow{d} E^1 \xrightarrow{d} E^2 \xrightarrow{d} \cdots$ (cochain) complex

$$H^i(E^\cdot) = \frac{\ker(E^i \xrightarrow{d} E^{i+1})}{\operatorname{im}(E^{i-1} \xrightarrow{d} E^i)}$$

cohomology in degree i, $i \geq 0$.

chain map

$$\begin{array}{cccccccc} E^\cdot & & E^0 & \xrightarrow{d} & E^1 & \xrightarrow{d} & E^2 & \longrightarrow \cdots \\ \downarrow f & & \downarrow f^0 & & \downarrow f^1 & & \downarrow f^2 & \\ E'^\cdot & & E'^0 & \xrightarrow{d'} & E'^1 & \xrightarrow{d'} & E'^2 & \longrightarrow \cdots \end{array}$$

$$H^i(E^\cdot) = \frac{\ker(E^i \xrightarrow{d} E^{i+1})}{\operatorname{im}(E^{i-1} \xrightarrow{d} E^i)}$$

$$\downarrow H^i(f)$$

$$H^i(E'^\cdot) = \frac{\ker(E'^i \xrightarrow{d'} E'^{i+1})}{\operatorname{im}(E'^{i-1} \longrightarrow E'^i)}$$

$f, g : E^\cdot \longrightarrow E'^\cdot$ are (chain) homotopic if there exists $s^n : E^n \longrightarrow E'^{n-1}$, $n \geq 0$, s.t.

$$d' s^n + s^{n+1} d = f^n - g^n .$$

$$\begin{tikzcd} E^0 \arrow[r, "d"] \arrow[d, "f^0-g^0"'] & E^1 \arrow[r, "d"] \arrow[dl, "s^1"] \arrow[d, "f^1-g^1"'] & E^2 \arrow[r, "d"] \arrow[dl, "s^2"] \arrow[d, "f^2-g^2"'] & \cdots \arrow[dl, "s^3"] \\ E'^0 \arrow[r, "d'"] & E'^1 \arrow[r, "d'"] & E'^2 \arrow[r, "d"] & \cdots \end{tikzcd}$$

Lemma If $f \simeq g$ then $H^*(f) = H^*(g)$.

Pf We have $f - g = d's + sd$:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im}(E^{i-1} \xrightarrow{d} E^i) \arrow[r] \arrow[d, "f-g"] & \ker(E^i \xrightarrow{d} E^{i+1}) \arrow[r] \arrow[d, "f-g"] \arrow[dl, "d's"'] & H^i(E^\cdot) \arrow[r] \arrow[d, "H^i(f)-H^i(g)"] & 0 \\
0 \arrow[r] & \operatorname{im}(E'^{i-1} \xrightarrow{d'} E'^i) \arrow[r] & \ker(E'^i \xrightarrow{d'} E'^{i+1}) \arrow[r] & H^i(E'^\cdot) \arrow[r] & 0
\end{tikzcd}$$ //

Note $H^*(f) = H^*(g) \not\Rightarrow f \simeq g$ in general.

A resolution of M is a chain map

$$M \xrightarrow{\eta} E^\cdot$$

s.t. $H^*(\eta)$ is an isomorphism

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & E^0 \arrow[d, "d"] \\
0 \arrow[r] \arrow[d] & E^1 \arrow[d, "d"] \\
0 \arrow[r] \arrow[d] & E^2 \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

$$0 \to M \to E^0 \to E^1 \to E^2 \to \cdots \quad \text{exact.}$$

For all N, $\mathrm{Hom}(-,N)$ is left exact:

$$0 \leftarrow M'' \leftarrow M \leftarrow M' \leftarrow 0 \quad \text{exact} \implies$$

$$0 \to \mathrm{Hom}(M'',N) \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(M',N) \quad \text{exact.}$$

I is injective $\underset{\text{def}}{\iff}$ $\mathrm{Hom}(-,I)$ exact $\iff$

in every diagram, an extension exists:

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[dl, dashed, "\exists"] \\ & I & \end{tikzcd}$$

An <u>injective resolution</u> is a res. $M \to I^\cdot$ s.t. for all $n \geq 0$, I^n is injective.

<u>Lemma</u> Let $M \to E^\cdot$ be a resolution, let $I^\cdot$ be a cx. of injectives, and let

$$M \xrightarrow{\varphi} H^0(I)$$

be a map. Then there exists a chain map $f: E^\cdot \to I^\cdot$ s.t.

$$\begin{tikzcd} H^0(E^\cdot) \arrow[r, "H^0(f)"] & H^0(I^\cdot) \\ M \arrow[u, "\sim"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

and any two such chain maps are homotopic.

pf

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "\varphi"] & E^0 \arrow[r] \arrow[d, dashed, "f^0"] & E^1 \arrow[r] & E^2 \arrow[r] & \cdots \\ 0 \arrow[r] & H^0(I^\cdot) \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & E^n/d(E^{n-1}) \arrow[r, "d"] \arrow[d, "f^n"] & E^{n+1} \arrow[d, dashed, "f^{n+1}"] \\ & I^n/d(I^{n-1}) \arrow[r, "d"] & I^{n+1} \end{tikzcd}$$

//

Cor If both $M \to I^\cdot$ and $M \to J^\cdot$ are injective resolutions then there exists a chain isomorphism $I^\cdot \xrightarrow{f} J^\cdot$ s.t.

$$\begin{tikzcd} H^0(I^\cdot) \arrow[r, "H^0(f)"] & H^0(J^\cdot) \\ M \arrow[u, "\sim"] \arrow[r, equal] & M \arrow[u, "\sim"] \end{tikzcd}$$

and any two such chain isomorphisms are chain homotopic. //

An abelian category has enough injectives if for every object M, there exists

$$0 \to M \to I$$

with I injective.

<u>Lemma</u> If an abelian category has enough inj. then every object has an injective resolution.

<u>pf</u> Build injective res.:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & I^0 \arrow[rr] \arrow[dr] & & I^1 \arrow[rr] \arrow[dr] & & I^2 \arrow[rr] \arrow[dr] & & I^3 \arrow[dr] & \\
& & & C^0 \arrow[ur] \arrow[dr] & & C^1 \arrow[ur] \arrow[dr] & & C^2 \arrow[ur] \arrow[dr] & & \cdot \\
& & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

//

Let F be a (covariant) left exact functor, i.e.

$$0 \to M' \to M \to M'' \to 0 \quad \text{exact} \implies$$

$$0 \to F(M') \to F(M) \to F(M'') \quad \text{exact.}$$

Assume the domain ab. cat. has enough inj. We define the <u>right derived</u> functors of F as follows: given M, choose an injective resolution $M \to I^\cdot$ and define

$$(R^i F)(M) := H^i(F(I^\cdot)).$$

Given $M \xrightarrow{\varphi} N$, and a choice of inj. res. $M \to I^\cdot$, $N \to J^\cdot$, choose a chain map $f : I^\cdot \to J^\cdot$ s.t.

$$\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[d, "\sim"] & N \arrow[d, "\sim"] \\
H^0(I^\cdot) \arrow[r, "H^0(f)"] & H^0(J^\cdot)
\end{tikzcd}$$

and define

$$(R^iF)(\varphi) = H^i(F(f)) .$$

The objects $(R^iF)(M)$ are well-defined up to unique isomorphism. The map $(R^iF)(\varphi)$ does not depend on the choice of f.

Prop (0) there is a natural isomorphism

$$F(M) \xrightarrow{\sim} (R^0F)(M) .$$

(i) for every short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

there is a long exact seq.

$$0 \to R^0F(M') \to R^0F(M) \to R^0F(M'') \xrightarrow{\delta^0} R^1F(M') \to \cdots$$

(ii) for every map of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N'' \arrow[r] & 0 \end{tikzcd}$$

and all $n \geq 0$, the square commutes:

$$\begin{tikzcd} R^nF(M'') \arrow[r, "\delta^n"] \arrow[d] & R^{n+1}F(M') \arrow[d] \\ R^nF(N'') \arrow[r, "\delta^n"] & R^{n+1}F(N') \end{tikzcd}$$

Pf prove (i): it suffices to show that there exists inj. res.

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I'^{\bullet} \arrow[r] & I^{\bullet} \arrow[r] & I''^{\bullet} \arrow[r] & 0
\end{tikzcd}$$

s.t. the bottom row is a short exact seq. of complexes. First choose

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, hook] & M \arrow[r] \arrow[d, "=: I^0"] & M'' \arrow[r] \arrow[d, hook] & 0 & \\
0 \arrow[r] & I'^0 \arrow[r] \arrow[d] & I'^0 \oplus I''^0 \arrow[r] \arrow[d] & I''^0 \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & C'^0 \arrow[r] \arrow[d, hook] & C^0 \arrow[r] \arrow[d, "=: I^1"] & C''^0 \arrow[r] \arrow[d, hook] & 0 & \text{(snake lemma)} \\
0 \arrow[r] & I'^1 \arrow[r] \arrow[d] & I'^1 \oplus I''^1 \arrow[r] \arrow[d] & I''^1 \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & C'^1 \arrow[r] & C^1 \arrow[r] & C''^1 \arrow[r] & 0 & \text{(--- '' ---)} \\
& \vdots & \vdots & \vdots & &
\end{tikzcd}$$

//

ex The functor $\mathrm{Hom}(M, -)$ is left-exact. Define

$$\mathrm{Ext}^i(M,N) := (R^i\,\mathrm{Hom}(M,-))(N).$$

The group $\mathrm{Ext}^1(M,N)$ is naturally iso.

to equivalence classes of extensions (= short exact sequences)

$$0 \to N \to E \to M \to 0$$

where two such are equivalent if there exists:

$$\begin{array}{ccccccccc} 0 & \to & N & \to & E & \to & M & \to & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \to & N & \to & E' & \to & M & \to & 0 \end{array}$$

The isomorphism is given by:

$$0 \to N \to E \to M \to 0 \quad \rightsquigarrow$$

$$0 \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(M,E) \to \mathrm{Hom}(N,N) \xrightarrow{\delta^0} \mathrm{Ext}^1(M,N)$$

$$\mathrm{id}_N \longmapsto [E]$$

with inverse: choose inj. hull:

$$0 \to N \to I \xrightarrow{g} C \to 0$$

then

since I inj.

$$0 \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(M,I) \to \mathrm{Hom}(M,C) \to \mathrm{Ext}^1(M,N) \to 0$$

choose $f \longmapsto [E]$

get ext. from:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & N \arrow[r] \arrow[d] & I \arrow[r, "g"] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E \arrow[r] \arrow[d] & I \oplus M \arrow[r, "g \oplus f"] \arrow[d] & C \arrow[r] & 0 \\
 & M \arrow[r, equal] \arrow[d] & M \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}
$$

//

<u>Lemma</u> Let $(X, \mathcal{O}_X)$ be a ringed space and let M be an $\mathcal{O}_X$-module. Then

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, M) \xrightarrow{\sim} \Gamma(X, M).$$

pf The isomorphism takes an $\mathcal{O}_X$-linear homomorphism $\varphi : \mathcal{O}_X \to M$ to

$$
\begin{array}{ccc}
\Gamma(X, \mathcal{O}_X) & \longrightarrow & \Gamma(X, M) \\
\cup & & \cup \\
1_X & \longmapsto & \varphi(1_X)
\end{array}
$$

Conversely, $\varphi(1_X)$ determines φ:

$$
\begin{tikzcd}
1_X \in \Gamma(X, \mathcal{O}_X) \arrow[r] \arrow[d, mapsto] & \Gamma(X, M) \arrow[d] & \Gamma(X, \mathcal{O}_X)\text{-lin.} \\
1_U \in \Gamma(U, \mathcal{O}_X) \arrow[r] & \Gamma(U, M) & \Gamma(U, \mathcal{O}_X)\text{-lin.}
\end{tikzcd}
$$

$$\ulcorner \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, M) = \mathrm{Hom}_{X^{\vee}}(e, M) = \Gamma(X, M) \lrcorner$$

//

Define sheaf-cohomology of M by:

$$H^i(X, M) = \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X, M)$$

$$= (R^i\Gamma(X, -))(M). \quad //$$

=

(X, A) ringed space;

$U \xhookrightarrow{j} X$ open immersion

$j_! \dashv j^* \dashv j_*$ adjoint pairs of functors between categories of sheaves of ab. groups.

$A_U := j_! j^* A$

<u>Lemma</u> $\mathrm{Hom}_A(A_U, M) \xrightarrow{\sim} \Gamma(U, M)$.

<u>pf</u> $\mathrm{Hom}_A(A_U, M) = \mathrm{Hom}_A(j_! j^* A, M)$

$\cong \mathrm{Hom}_{j^*A}(j^*A, j^*M) \xrightarrow{\sim} \Gamma(U, j^*M)$

$= \Gamma(U, M)$. //

A *fully injective* res. of a (cochain) cx. $C^\cdot$ is a bicomplex $I^{\cdot,\cdot}$, a map of cx.'s

$$C^\cdot \to I^{\cdot,0}$$

s.t. for all $p \geq 0$, the following are inj. res.:

$$C^p \to I^{p,\cdot}$$

$$B^p \to {}'B^{p,\cdot} \qquad \text{(boundaries w.r.t. } d'\text{)}$$

$$Z^p \to {}'Z^{p,\cdot}$$

$$H^p \to {}'H^{p,\cdot}$$

Lemma In an abelian cat. which has enough injectives, every cx. has a fully injective res.

Pf See Lang's alg. book and/or Cartan-Eilenberg. //

Let F be a left-exact functor s.t. the right derived functors R^qF are defined. An object M is called F-*acyclic* if $(R^qF)(M) = 0$, for $q > 0$.

Thm Let F, G be left-exact functors s.t. if I is injective then $G(I)$ is F-acyclic. Then there is a functorial sp. seq.

$$E_2^{p,q} = (R^pF)(R^qG)(M) \Rightarrow R^{p+q}(FG)(M).$$

Pf First choose inj. res. $M \to J^{\cdot}$; then choose fully injective res. $G(J^{\cdot}) \to I^{\cdot,\cdot}$, and consider the double cx. $F(I^{\cdot,\cdot})$. (We have to introduce signs to get a dbl. cx. from the bi-cx.) We look at the two sp. seq. ass. w. $F(I^{\cdot,\cdot})$.

If we take coh. w.r.t. d'', get:

$$'E_1^{p,q} = H^q F(I^{p,\cdot}) = (R^q F)(G(J^p))$$

$$= \begin{cases} (FG)(J^p) & , q = 0, \\ 0 & , q > 0. \end{cases}$$

Here we use that $G(J^p) \to I^{p,\cdot}$ is an inj. res. and that $G(J^p)$ is F-acyclic. Hence

$$'E_2^{p,0} = H^p(FG(J^{\cdot})) = R^p(FG)(M),$$

and $'E_2^{p,q} = 0$, $q > 0$. So the sp. seq. calc. $R^{\cdot}(FG)(M)$.

To calc. coh. w.r.t. d' of $F(I^{\cdot,\cdot})$ we use that $I^{\cdot,\cdot}$ is a fully inj. res. The seq.

$$0 \to {'Z^{p,q}} \to I^{p,q} \to {'B^{p+1,q}} \to 0$$

$$0 \to {'B^{p,q}} \to {'Z^{p,q}} \to {'H^{p,q}} \to 0$$

split since the terms are injective. Hence, we have exact seq. of cx. (which degreewise split)

$$0 \to F('Z^{\cdot,q}) \to F(I^{\cdot,q}) \to F('B^{\cdot+1,q}) \to 0$$

$$0 \to F('B^{\cdot,q}) \to F('Z^{\cdot,q}) \to F('H^{p,q}) \to 0$$

and all cx. except $F(I^{\cdot,q})$ have differential zero. The ass. long-exact coh. seq. give:

$$F('Z^{p,q})/F('B^{p,q}) \xrightarrow{\sim} H^p(I^{\cdot,q})$$

$$F('Z^{p,q})/F('B^{p,q}) \xrightarrow{\sim} F('H^{p,q})$$

so

$$''E_1^{q,p} = F('H^{p,q}).$$

Since

$$(R^pG)(M) = H^p(G(J^\cdot)) \to 'H^{p,\cdot}$$

is an injective res., we get

$$''E_2^{q,p} = (R^qF)(R^pG)(M). \qquad //$$

ex Let $f: A \to B$ be a ring homomorphism, let $f^*: \mathrm{Mod}_B \to \mathrm{Mod}_A$ be the functor which views a B-module as an A-module via f. This, clearly, is an exact functor and has a left adjoint $\mathrm{Hom}_A(B, -)$, i.e.

$$\mathrm{Hom}_A(f^*M, N) \cong \mathrm{Hom}_B(M, \mathrm{Hom}_A(B, N))$$

or

$$\begin{tikzcd}
\mathrm{Mod}_A \arrow[rr, "\mathrm{Hom}_A(B,-)"] \arrow[dr, "\mathrm{Hom}_A(f^*M,-)"'] & & \mathrm{Mod}_B \arrow[dl, "\mathrm{Hom}_B(M,-)"] \\
& \mathcal{A}b &
\end{tikzcd}$$

The functor $\mathrm{Hom}_A(B,-)$ has an exact left adjoint - namely f^* - so it is left-exact and preserves injectives. So have a Grothendieck sp. seq.

$$E_2^{p,q} = R^p\mathrm{Hom}_B(M,-)\, R^q\mathrm{Hom}_A(B,-)\,(N)$$

$$\Rightarrow R^{p+q}\mathrm{Hom}_A(f^*M,-)\,(N)$$

or

$$E_2^{p,q} = \mathrm{Ext}^p_B(M, \mathrm{Ext}^q_A(B,N))$$

$$\Rightarrow \mathrm{Ext}^{p+q}_A(f^*M, N).$$

(X, A) ringed space

$$C^{\sim} \underset{}{\overset{a}{\leftrightarrows}} C^{\wedge}$$

$$C_A^{\sim} \underset{\underline{H}^0}{\overset{a}{\leftrightarrows}} C_A^{\wedge}$$

a exact.

$\underline{H}^0$ left-exact, takes inj. to inj.

$$\underline{H}^q M := (R^q \underline{H}^0)(M)$$

<u>Lemma</u> $a(\underline{H}^q M) = 0, \quad q > 0.$

<u>pf</u> $a\,\underline{H}^0 = \mathrm{id}$

$$E_2^{p,q} = (R^p a)(R^q \underline{H}^0)(M) = \begin{cases} R^q \underline{H}^0(M), & p = 0, \\ 0, & p > 0, \end{cases}$$

$$\Rightarrow (R^{p+q}\,\mathrm{id})(M) = \begin{cases} M, & p, q = 0, \\ 0, & \text{else.} \end{cases}$$ //

<u>Lemma</u> $\Gamma(U, \underline{H}^q(M)) = H^q(U, M|_U).$

<u>pf</u> $$E_2^{p,q} = R^p \Gamma(U, R^q \underline{H}^0(M)) = \begin{cases} \Gamma(U, \underline{H}^q(M)), & p = 0, \\ 0, & p > 0, \end{cases}$$

$$\Rightarrow (R^{p+q}\Gamma(U, -))(M) =: H^{p+q}(U, M|_U).$$

The identification of the E_2-term uses that for presheaves, global sections is exact. //

$\mathcal{U} = \{U_i \to X\}$ covering, $F \in \hat{C_A}$.

Define Čech complex $C^\cdot(\mathcal{U}, F)$:

$$\prod_{i_0} F(U_{i_0}) \leftleftarrows \prod_{i_0, i_1} F(U_{i_0} \times U_{i_1}) \substack{\leftarrow\\ \leftarrow\\ \leftarrow} \prod_{i_0, i_1, i_2} F(U_{i_0} \times U_{i_1} \times U_{i_2}) \substack{\leftarrow\\ \vdots\\ \leftarrow} \cdots$$

and Čech cohomology of F w.r.t. covering $\mathcal{U}$:

$$\check{H}^q(\mathcal{U}, F) = H^q(C^\cdot(\mathcal{U}, F)).$$

We proceed to show

$$\check{H}^q(\mathcal{U}, F) = (R^q \check{H}^0(\mathcal{U}, -))(F).$$

First, let $U \overset{j}{\hookrightarrow} X$ be open; then $F \mapsto F(U)$ is represented by $A_U \in \hat{C_A}$,

$$A_U := j_! j^* A =: j_! A|_U$$

⌜ i.e.

$$A_U(V) = \begin{cases} A(V), & V \subset U, \\ 0, & \text{else}. \end{cases}$$ ⌟

Indeed,

$$\mathrm{Hom}_{\hat{C_A}}(A_U, F) := \mathrm{Hom}_{\hat{C_A}}(j_! j^* A, F)$$

$$\cong \mathrm{Hom}_{\hat{C}_{j^*A}}(j^*A, j^*F) \cong F(U).$$

Let $A_{\mathcal{U}}$ be the cokernel (in $\hat{C_A}$):

$$0 \leftarrow A_{\mathcal{U}} \leftarrow \bigoplus_{i_0} A_{U_{i_0}} \leftarrow \bigoplus_{i_0, i_1} A_{U_{i_0} \times U_{i_1}}$$

Then, since $\mathrm{Hom}_{\hat{C_A}}(-, F)$ is left-exact:

$$\begin{array}{ccccc} 0 \to \mathrm{Hom}(A_{\mathcal{U}}, F) & \to & \mathrm{Hom}(\bigoplus_{i_0} A_{U_{i_0}}, F) & \to & \mathrm{Hom}(\bigoplus_{i_0, i_1} A_{U_{i_0} \times U_{i_1}}, F) \\ \wr\wr & \Leftarrow & \wr\wr & & \\ 0 \to \check{H}^0(\mathcal{U}, F) & \to & \prod_{i_0} F(U_{i_0}) & \to & \prod_{i_0, i_1} F(U_{i_0} \times U_{i_1}) \end{array}$$

<u>Lemma</u> The complex

$$\bigoplus_{i_0} A_{U_{i_0}} \leftarrow \bigoplus_{i_0, i_1} A_{U_{i_0} \times U_{i_1}} \leftarrow \bigoplus_{i_0, i_1, i_2} A_{U_{i_0} \times U_{i_1} \times U_{i_2}} \leftarrow \cdots$$

is a projective resolution of $A_{\mathcal{U}}$.

<u>Pf</u> First, A_U projective means that

$$\mathrm{Hom}_{\hat{C_A}}(A_U, -) = \Gamma(U, -)$$

is exact, and this is true for presheaves. Write $P_\bullet$ for the cx. in the statement. Must show $0 \leftarrow A_{\mathcal{U}} \leftarrow P_\bullet$ exact, i.e. that for all $V \subset X$ open $0 \to A_{\mathcal{U}}(V) \to P_\bullet(V)$ is exact. Let

$$S = \{ i \mid V \subset U_i \}.$$

then

$$P_n(V) = \Big(\bigoplus_{i_0,\ldots,i_n} A_{U_{i_0} \times \cdots \times U_{i_n}}\Big)(V)$$

$$= A(V)\{\underbrace{S \times \cdots \times S}_{n+1}\}$$

$$= A(V)\{S\} \underset{A(V)}{\otimes} \cdots \underset{A(V)}{\otimes} A(V)\{S\}$$

If $S = \emptyset$, $P_\cdot = 0$, so we are done. If $S \neq \emptyset$, choose $i \in S$, define $s : P_n \to P_{n+1}$ by

$$s(f_0 \otimes \cdots \otimes f_n) = i \otimes f_0 \otimes \cdots \otimes f_n) ;$$

then $ds + sd = \mathrm{id}$:

$$ds(f_0 \otimes \cdots \otimes f_n) = d(i \otimes f_0 \otimes \cdots \otimes f_n)$$

$$= f_0 \otimes \cdots \otimes f_n - i \otimes f_1 \otimes \cdots \otimes f_n + i \otimes f_0 \otimes f_2 \otimes \cdots \otimes f_n - \cdots$$
$$\cdots + (-1)^{n+1} i \otimes f_0 \otimes \cdots \otimes f_{n-1}$$

$$sd = s(f_1 \otimes \cdots \otimes f_n - f_0 \otimes f_2 \otimes \cdots \otimes f_n + \cdots + (-1)^n f_0 \otimes \cdots \otimes f_{n-1})$$

$$- i \otimes f_1 \otimes \cdots \otimes f_n - i \otimes f_0 \otimes f_2 \otimes \cdots \otimes f_n + \cdots$$
$$\cdots + (-1)^n i \otimes f_0 \otimes \cdots \otimes f_{n-1}$$

i.e. on $H_*(0 \leftarrow A_U \leftarrow P_\cdot)$, the identity map and the zero map induce the same map. But then the homology is zero. //

Prop $\check{H}^q(\mathcal{U}, F) = \operatorname{Ext}^q_{C_A^\wedge}(A_{\mathcal{U}}, F) = R^q \check{H}^0(\mathcal{U}, -)(F)$.

pf $\check{H}^q(\mathcal{U}, F) := H^q(\operatorname{Hom}_{C_A^\wedge}(P_\cdot, F))$. //

Prop (Cartan - Leray) Let $\mathcal{U} = \{U_i \to X\}$ be a covering and let $M \in C_A^\sim$ be a sheaf of A-modules. Then

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q M) \Rightarrow H^{p+q}(X, M).$$

pf Since M is a sheaf, $\check{H}^0(\mathcal{U}, \underline{H}^0 M) = M(X)$, i.e.

$$\begin{tikzcd} C_A^\sim \arrow[rr, "\underline{H}^0"] \arrow[dr, "\Gamma(X,-)"'] & & C_A^\wedge \arrow[dl, "\check{H}^0(\mathcal{U},-)"] \\ & \mathcal{A}b & \end{tikzcd}$$

The functor $\underline{H}^0$ has an exact left-adjoint, so is left-exact and preserves injectives. The sp. seq. is Grothendieck's sp. seq. //

$$\begin{tikzcd} U \arrow[rr, "i"] \arrow[dr, "j"'] & & U' \arrow[dl, "j'"] \\ & X & \end{tikzcd} \quad \rightsquigarrow \quad A_U \xrightarrow{A_i} A_{U'} \text{ given by}$$

$$A_U = j_! j^* A = j'_! i_! i^* j'^* A \xrightarrow{\text{can}} j'_! j'^* A = A_{U'}$$

$$\begin{array}{l} \mathfrak{U} = \{U_i \to X\} \\ \downarrow \phi \\ \mathfrak{U}' = \{U'_{i'} \to X\} \end{array} \qquad \begin{tikzcd} U_i \arrow[rr] \arrow[dr] & & U'_{\phi(i)} \arrow[dl] \\ & X & \end{tikzcd}$$

induces

$$\begin{tikzcd} A_{\mathfrak{U}} \arrow[d] & \bigoplus_{i_0} A_{U_{i_0}} \arrow[l] \arrow[d] & \bigoplus_{i_0, i_1} A_{U_{i_0} \times_X U_{i_1}} \arrow[l, shift left] \arrow[l, shift right] \arrow[d] \\ A_{\mathfrak{U}'} & \bigoplus_{i'_0} A_{U'_{i'_0}} \arrow[l] & \bigoplus_{i'_0, i'_1} A_{U'_{i'_0} \times_X U'_{i'_1}} \arrow[l, shift left] \arrow[l, shift right] \end{tikzcd}$$

<u>Note</u> Any two maps $\mathfrak{U} \to \mathfrak{U}'$ induce the <u>same</u> map $A_{\mathfrak{U}} \to A_{\mathfrak{U}'}$.

Define for $F \in X^\wedge_A$:

$$\check{H}^q(U, F) = \operatorname*{colim}_{\{U_i \to U\}} \check{H}^q(\{U_i \to U\}, F)$$

as the filtered colimit over coverings $\mathfrak{U}$ with one morphism $\mathfrak{U} \to \mathfrak{U}'$ if there exists a map $\mathfrak{U} \to \mathfrak{U}'$.

These form a presheaf of A-modules,

$$\underline{\check{H}}^q(F)(U) = \check{H}^q(U, F).$$

$\ulcorner$ $\underline{\check{H}}^0(F) = F^+$, $\underline{\check{H}}^0\underline{\check{H}}^0(F) = aF$. $\lrcorner$

<u>Lemma</u> (i) $\check{H}^q(U,F) = R^q\check{H}^0(U,-)(F)$,

(ii) $\underline{\check{H}}^q(F) = (R^q\underline{\check{H}}^0)(F)$.

<u>pf</u> (i) Let $F \to I^\cdot$ be an inj. res. Since filt. colimits are exact and $\check{H}^q(\mathcal{U},-) = R^q\check{H}^0(\mathcal{U},-)$,

$$\begin{aligned} R^q\check{H}^0(U,-)(F) &:= H^q(\check{H}^0(U,I^\cdot)) \\ &= H^q(\operatorname*{colim}_{\mathcal{U}} \check{H}^0(\mathcal{U},I^\cdot)) \overset{\cdot\cdot}{=} \operatorname*{colim}_{\mathcal{U}} H^q(\check{H}^0(\mathcal{U},I^\cdot)) \\ &= \operatorname*{colim}_{\mathcal{U}} \check{H}^q(\mathcal{U},F) = \check{H}^q(U,F). \end{aligned}$$

(ii) Let $F \to I^\cdot$ be an inj. res. Since $\Gamma(U,-)$ is an exact functor from $X^\wedge_A$ to $\mathrm{Mod}_{A(U)}$,

$$\begin{aligned} \Gamma(U, \underline{\check{H}}^q(F)) &:= \check{H}^q(U,F) \overset{(i)}{=} H^q(\check{H}^0(U,I^\cdot)) \\ &= H^q(\Gamma(U, \underline{\check{H}}^0(I^\cdot))) = \Gamma(U, H^q(\underline{\check{H}}^0(I^\cdot))) \\ &= \Gamma(U, (R^q\underline{\check{H}}^0)(F)). \end{aligned}$$

//

Thm There are sp. seq.

(i) $E_2^{p,q} = \check{H}^p(U, \underline{H}^q(M)) \Rightarrow H^{p+q}(U, M)$

(ii) $E_2^{p,q} = \check{\underline{H}}^p(\underline{H}^q(M)) \Rightarrow \underline{H}^{p+q}(M)$.

Moreover, $E_2^{0,q} = 0$, for $q > 0$.

pf These are the Grothendieck sp. seq. The first sp. seq. is obtained from the latter by taking $\Gamma(U, -)$. Show that for $q > 0$,

$$\check{\underline{H}}^0(\underline{H}^q(M)) = 0.$$

We proved last time:

$$\check{\underline{H}}^0(\check{\underline{H}}^0(\underline{H}^q(M))) =: a\,\underline{H}^q(M) = 0.$$

But $\check{\underline{H}}^0(\underline{H}^q(M))$ is a separated presheaf, and for every separated presheaf F, $F \to \check{\underline{H}}^0(F)$ is injective. //

Cor $\check{H}^1(X, \underline{H}^0(M)) \xrightarrow{\sim} H^1(X, M)$.

pf The spectral seq. gives an exact seq.

$$0 \to \check{H}^1(X, \underline{H}^0(M)) \to H^1(X, M) \to \check{H}^0(X, \underline{H}^1(M)) \to 0$$

and the right hand term is zero. //

Cor Let (X, A) be a ringed space. The forgetful functor from sheaves of A-modules to sheaves of abelian groups takes injective objects to $\Gamma(X, -)$-acyclic objects.

Pf Let M be a sheaf of A-modules and let $\bar{M}$ be the underlying sheaf of ab. groups. Then, $M(U) \xrightarrow{\sim} \bar{M}(U)$, and hence for every covering $\mathcal{U} = \{U_i \to U\}$,

$$\check{H}^q(\mathcal{U}, M) \xrightarrow{\sim} \check{H}^q(\mathcal{U}, \bar{M}).$$

Taking colimits, we get for all $U \subset X$,

$$\check{H}^q(U, M) \xrightarrow{\sim} \check{H}^q(U, \bar{M}),$$

i.e.,

$$\check{\underline{H}}^q(M) \xrightarrow{\sim} \check{\underline{H}}^q(\bar{M}),$$

Suppose $M = I$ is injective. Then $\underline{H}^q(I) = 0$, $q > 0$, and the spectral sequence

$$E_2^{p,q} = \check{\underline{H}}^p(\underline{H}^q(I)) \Rightarrow \underline{H}^{p+q}(I)$$

shows that also $\check{\underline{H}}^q(I) = 0$, $q > 0$. Hence, $\check{\underline{H}}^q(\bar{I}) = 0$, $q > 0$. Since in the sp. seq.

$$E_2^{p,q} = \check{\underline{H}}^p(\underline{H}^q(\bar{I})) \Rightarrow \underline{H}^{p+q}(\bar{I})$$

we have $E_2^{0,q} = 0$, $q > 0$, we find that

$\underline{H}^n(\bar{I}) = 0$, $n > 0$, as desired. Let us write out the induction argument. For $n=1$,

$$\underline{H}^1(\bar{I}) \xleftarrow{\sim} \check{\underline{H}}^1(\bar{I}) = 0.$$

Suppose $\underline{H}^i(\bar{I}) = 0$, $0 < i < n$. Then $E_2^{p,i} = 0$, $0 < i < n$. We also know $E_2^{0,q} = 0$ and $E_2^{q,0} = \check{\underline{H}}^q(\bar{I}) = 0$. So for all $p+q = n$, $E_2^{p,q} = 0$. Hence $E_\infty^{p,q} = 0$, if $p+q = n$. But the sp. seq. gives a filtration

$$\underline{H}^n(\bar{I}) = \mathrm{Fil}^0 \underline{H}^n(\bar{I}) \supset \cdots \supset \mathrm{Fil}^{n+1} \underline{H}^n(\bar{I}) = 0$$

with filtration quotients

$$gr^p \underline{H}^n(\bar{I}) = E_\infty^{p,q}.$$

Since these are all zero, so is $\underline{H}^n(\bar{I})$. //

Cor Let (X, A) be a ringed space, let M be a sheaf of A-modules, and let $\bar{M}$ be the underlying sheaf of ab. grps. Then for all $q \geq 0$,

$$H^q(X, M) \xrightarrow{\sim} H^q(X, \bar{M}).$$

pf The functor $M \mapsto \bar{M}$ is exact (trivial) and takes injectives to $H^0(X, -)$-acyclics. Use a Grothendieck sp. seq. //

Let $(X, \mathcal{O}_X)$ be a scheme, and let $\underline{M}$ be a quasi-coherent $\mathcal{O}_X$-module. We shall prove that if $U \subset X$ is affine and $\mathcal{U} = \{U_i \to X\}$ a finite aff. open cover, then for all $q > 0$,

$$\check{H}^q(\mathcal{U}, \underline{M}) = 0.$$

This is the cohomology of the complex $C^\cdot(\mathcal{U}, \underline{M})$

$$\prod_{i_0} \underline{M}(U_{i_0}) \rightrightarrows \prod_{i_0, i_1} \underline{M}(U_{i_0} \times_X U_{i_1}) \overrightarrow{\equiv} \cdots$$

Let $A = \Gamma(U, \mathcal{O}_X)$, $A_i = \Gamma(U_i, \mathcal{O}_X)$, $B = \prod_i A_i$, and $M = \Gamma(U, \underline{M})$. Then $C^\cdot(\mathcal{U}, \underline{M})$ is iso. to:

$$M \otimes_A B \rightrightarrows M \otimes_A B \otimes_A B \overrightarrow{\equiv} \cdots$$

Moreover:

<u>Lemma</u> The map $A \xrightarrow{f} B$ is faithfully flat, i.e.

$$M' \to M \to M'' \quad \text{exact} \iff$$

$$M' \otimes_A B \to M \otimes_A B \to M'' \otimes_A B \quad \text{exact}$$

Pf Flatness can be checked locally: $A \xrightarrow{f} B$ is flat if and only if for every prime ideal $\mathfrak{p} \subset B$, $A_{f^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$ is flat. But $\operatorname{Spec} B = \coprod_i U_i \to U = \operatorname{Spec} A$, and

$A_{f^{-1}(\beta)} \to B_\beta$ is an isomorphism.

So $M \mapsto M \otimes_A B$ is an exact functor. It is also faithful, i.e. $M \otimes_A B = 0 \Rightarrow M = 0$. //

Prop Let $A \xrightarrow{f} B$ be a faithfully flat map. Then the complex

$$0 \to A \xrightarrow{f} B \rightrightarrows B \otimes_A B \rightrightarrows B \otimes_A B \otimes_A B \rightrightarrows \cdots$$

is exact

pf Suppose that f has a section $B \xrightarrow{g} A$. Then a contracting homotopy is given by:

$$s(b_0 \otimes \cdots \otimes b_n) = g(b_0) b_1 \otimes b_2 \otimes \cdots \otimes b_n.$$

Indeed,

$$ds(b_0 \otimes \cdots \otimes b_n) = d(g(b_0) b_1 \otimes b_2 \otimes \cdots \otimes b_n)$$

$$= 1 \otimes g(b_0) b_1 \otimes b_2 \otimes \cdots \otimes b_n - g(b_0) b_1 \otimes 1 \otimes b_2 \otimes \cdots \otimes b_n + \cdots$$

$$\cdots + (-1)^n g(b_0) b_1 \otimes b_2 \otimes \cdots \otimes b_n \otimes 1$$

$$sd(b_0 \otimes \cdots \otimes b_n) = b_0 \otimes b_1 \otimes \cdots \otimes b_n - g(b_0) \otimes b_1 \otimes \cdots \otimes b_n +$$

$$g(b_0) b_1 \otimes 1 \otimes b_2 \otimes \cdots \otimes b_n - \cdots + (-1)^{n+1} g(b_0) b_1 \otimes \cdots \otimes b_n \otimes 1$$

so $ds + sd = \mathrm{id}$. In general, it suffices, since $A \to B$ is faithfully flat, to show that the cx. is exact after $B \otimes_A -$;

$$0 \to B \to B \otimes_A B \rightrightarrows B \otimes_A B \otimes_A B \rightrightarrows \cdots$$

$$b \mapsto b \otimes 1$$

A section is given by $g(b_{-1} \otimes b_0) = b_{-1} b_0$. //

Thm Let X be an affine scheme and M a quasi-coherent $\mathcal{O}_X$-module. Then

$$H^i(X, M) = \begin{cases} \Gamma(X, M), & i = 0, \\ 0, & i > 0. \end{cases}$$

Pf If $\mathcal{U} = \{U_i \to U\}$ is a finite affine open cover, then in the notation above, the cx. $\check{C}^\bullet(\mathcal{U}, M)$ is

$$M \otimes_A B \rightrightarrows M \otimes_A B \otimes_A B \rightrightarrows M \otimes_A B \otimes_A B \otimes_A B \rightrightarrows \cdots$$

whose coh., by the prop., is M concentrated in degree zero, i.e.,

$$\check{H}^i(\mathcal{U}, M) = \begin{cases} \Gamma(U, M), & i = 0, \\ 0, & i > 0. \end{cases}$$

Since U is quasi-cpt.

$$\check{H}^i(U, M) = \operatorname*{colim}_{\substack{\mathcal{U} \\ \text{finite, aff.}}} \check{H}^i(\mathcal{U}, M)$$

(every cover has fin. aff. refinement)

Consider the spectral sequence

$$E_2^{p,q} = \check{H}^p(U, \underline{H}^q(M)) \Rightarrow H^{p+q}(U, M),$$

where $E_2^{0,q} = 0$. We have showed that for all $U \subset X$ aff. and all $p > 0$, $\check{H}^p(U, \underline{H}^0(M)) = 0$. This shows inductively that for all $U \subset X$ aff. and all $p > 0$, $H^p(U, M) = 0$. //

Cor Let X be a separated scheme and let M be a q-coh. $\mathcal{O}_X$-module. Then for every affine open cover $\mathcal{U}$,

$$\check{H}^*(\mathcal{U}, M) \xrightarrow{\sim} H^*(X, M).$$

pf Let $\mathcal{U} = \{U_i \to X\}$. Since X is separated, $U_{i_0} \times_X \cdots \times_X U_{i_n}$ is again affine. Hence

$$\underline{H}^q(M)(U_{i_0} \times_X \cdots \times_X U_{i_n}) = \begin{cases} M(U_{i_0} \times_X \cdots \times_X U_{i_n}), & q = 0, \\ 0, & q > 0. \end{cases}$$

Thus the Cartan-Leray sp. seq.

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(M)) \Rightarrow H^{p+q}(X, M)$$

has $E_2^{p,q} = 0$, $q > 0$. //

4/23/01

S graded ring, $S_+ = \bigoplus_{d>0} S_d$ ideal.

$S_{(f)} = (S_f)_0$, $f \in S_d$, $d > 0$.

$S_{(f)} \longrightarrow S_{(fg)} = (S_{(f)})_{g^d/f^e}$, $f \in S_d, g \in S_e$

$\frac{a}{f^n} \longmapsto \frac{ag^n}{(fg)^n}$

The schemes $\operatorname{Spec} S_{(f)}$, $f \in S_d$, $d > 0$, glue to give

$$X = \operatorname{Proj}(S);$$

let $D_+(f) \subset X$ be the image of $\operatorname{Spec} S_{(f)}$. The family $\{D_+(f_i)\}_{i \in I}$ is a covering if and only if for every $x \in S_+$, there exists $n \geq 1$ s.t. $x^n \in (f_i \mid i \in I)$. X is separated.

⌜As a topological space, $\operatorname{Proj}(S)$ is naturally homeomorphic to the set of graded prime ideals in S which do not contain S_+ with the Zariski topology given by graded ideals $I \subset S$.⌟

M graded S-module.

$M_{(f)} = (M_f)_0$, $f \in S_d$, $d > 0$ $\rightsquigarrow$

$\tilde{M}_{(f)}$ q.coh. $\mathcal{O}_{\operatorname{Spec} S_{(f)}}$-module $\underset{\text{glue}}{\rightsquigarrow}$

$\tilde{M}$ q-coh. $\mathcal{O}_X$ - module. // $(S^\sim = \mathcal{O}_X)$

M graded S-module $\rightsquigarrow$ $M(n)$, $M(n)_d = M_{n+d}$.

<u>Lemma</u> Let $f \in S_d$, $d > 0$. Then

$$S(nd)^\sim|_{D_+(f)} \xleftarrow{\sim} \mathcal{O}_X|_{D_+(f)}$$

Pf This corresponds to the iso. of $S_{(f)}$-mod.:

$$S(nd)_{(f)} \underset{\sim}{\xleftarrow{f^n \cdot}} S_{(f)}.$$ //

<u>Cor</u> If S_1 generates S_+, then

$$\mathcal{O}_X(n) := S(n)^\sim$$

is an invertible $\mathcal{O}_X$-module. //

Define Serre twist of $\mathcal{O}_X$-module F:

$$F(n) := F \otimes_{\mathcal{O}_X} \mathcal{O}_X(n).$$

<u>Lemma</u> Suppose S_1 generates S_+. Then for every graded S-module M and every $n \in \mathbb{Z}$,

$$M(n)^\sim \xrightarrow{\sim} \tilde{M}(n).$$

Pf
$$\begin{array}{ccc} M(n)_d & \xrightarrow{\sim} & (M \otimes_S S(n))_d \\ \| & & \uparrow \\ M_{n+d} & \longrightarrow & M_{n+d} \otimes S(n)_{-n} \\ z & \longmapsto & z \otimes 1 \end{array}$$ //

Suppose S_1 gen. S_+. Then also $\mathcal{O}_X(n) = \mathcal{O}_X(1)^{\otimes n}$.
If F is an $\mathcal{O}_X$-module, define

$$\Gamma_*(F) := \bigoplus_{n \in \mathbb{Z}} \Gamma(X, F(n)).$$

<u>Lemma</u> Suppose S_1 generates S_+. Then $((-)^\sim, \Gamma_*(-))$ form an adjoint pair, i.e.

$$\mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, F) \overset{\sim}{\rightleftarrows} \mathrm{Hom}_S(M, \Gamma_*(F)).$$

Pf We define natural maps

$$M \xrightarrow{\alpha} \Gamma_*(\tilde{M})$$

$$\Gamma_*(F)^\sim \xrightarrow{\beta} F .$$

Then $\Gamma_*(F)$ is regarded as a graded S-mod. via the map $S \xrightarrow{\alpha} \Gamma_*(\mathcal{O}_X)$, and the maps of the statement are given by

$$\tilde{M} \xrightarrow{f} F \longmapsto M \xrightarrow{\alpha} \Gamma_*(\tilde{M}) \xrightarrow{\Gamma_*(f)} \Gamma_*(F)$$

$$\tilde{M} \xrightarrow{\tilde{g}} \Gamma_*(F)^\sim \xrightarrow{\beta} F \longleftarrow M \xrightarrow{g} \Gamma_*(F).$$

To define α, note that the maps are comp.:

$$M_n \to \Gamma(D_+(f), M(n))$$

$$M_n \to (M_f)_n = (M(n)_f)_0 .$$

Define β on $D_+(f)$, $f \in S_d$, $d \geq 0$, by

$$\begin{array}{ccc} \left(\bigoplus_{n\in\mathbb{Z}} \Gamma(X, F(n))\right)_{(f)} & \longrightarrow & \Gamma(D_+(f), F) \\ \psi & & \psi \\ \dfrac{z}{f^m} & \longmapsto & \dfrac{z \,|\, D_+(f)}{\alpha(f^m) \,|\, D_+(f)} \end{array}$$

$z \in \Gamma(X, F(md))$.

It remains to prove that the two composites

$$\tilde{M} \xrightarrow{\tilde{\alpha}} \Gamma_*(\tilde{M})^\sim \xrightarrow[\sim]{\beta} \tilde{M}$$

$$\Gamma_*(F) \xrightarrow{\alpha} \Gamma_*(\Gamma_*(F)^\sim) \xrightarrow{\Gamma_*(\beta)} \Gamma_*(F)$$

are the identity maps. //

Thm Suppose that S_+ is generated by finitely many $f_1, \ldots, f_r \in S_1$. Then for every quasi-coh $\mathcal{O}_X$-module F,

$$\Gamma_*(F)^\sim \xrightarrow[\sim]{\beta} F.$$

pf See EGA II, 2.7.5. The essential part is to prove that

$$\Gamma_*(F)_{(\alpha(f))} \xrightarrow{\sim} \Gamma(D_+(f), \Gamma_*(F)).$$

This uses (EGA I, 9.3.6) that X is quasi-cpt. and this follows from $X = \bigcup_{i=1}^{r} D_+(f_i)$. //

Cor For all $n \geq 0$, there exists an epimorphism

$$\mathcal{O}_X^k \twoheadrightarrow \mathcal{O}_X(n)$$

for some $k \geq 1$.

Pf Since $S_n = S_1^n$ and since S_1 is a f-g. S_0-module, S_n is a f-g. S_0-module. Choose a set of gen. $a_1, \dots, a_k$ and consider

$$\bigoplus_{j=1}^k S \cdot e_j \longrightarrow S(n) \quad , \quad e_j \longmapsto a_j .$$

The cokernel is killed by $(-)_f$, $f \in S_d$, $d > 0$. //

Cor Let F be a quasi-coh. $\mathcal{O}_X$-module of finite type. Then there exists $n_0 \in \mathbb{Z}$ s.t. for all $n \geq n_0$, there exists $k \geq 1$ and an epi.

$$\mathcal{O}_X^k \twoheadrightarrow F(n) .$$

Pf By the thm, $F = \tilde{M}$ for a graded S-module M. Let a_i, $i \in I$, be a set of homogeneous generators of M. For every fin. subset $H \subset I$, let $M_H \subset M$ be the submod. gen. by a_i, $i \in H$. Then

$$\operatorname{colim}_H M_H \xrightarrow{\sim} M .$$

Since M is of finite type and since X

is quasi-cpt, it follows (exercise) that for some $H \subset I$ finite, $M_H \xrightarrow{\sim} M$. Hence M is finitely generated as an S-module, say, $M = S \cdot a_1 + \dots + S \cdot a_r$, $\deg(a_i) = m_i$. Then we have epi.

$$\bigoplus_{i=1}^{r} S(m_i) \twoheadrightarrow M$$

and

$$\bigoplus_{i=1}^{r} S(m_i+n) \twoheadrightarrow M(n).$$

Choose $n_0 \geq \max\{-m_i\}_{1 \leq i \leq r}$. Use previous corollary. //

Thm Let $X = \mathbb{P}^r_A = \mathrm{Proj}(S)$, $S = A[x_0, \dots, x_r]$.

(i) $S \xrightarrow{\alpha} \Gamma_*(\mathcal{O}_X)$ is an iso.

(ii) $H^i(X, \mathcal{O}_X(n)) = 0$ if $i \neq 0, r$.

(iii) $H^r(X, \mathcal{O}_X(-r-1)) \cong A$.

(iv) there is a perfect pairing of f.g. free A-modules

$$H^0(X, \mathcal{O}_X(n)) \otimes_A H^r(X, \mathcal{O}_X(-n-r-1)) \to H^r(X, \mathcal{O}_X(-r-1)).$$

Pf We can calculate $H^*(X, \mathcal{O}_X(n))$ as Čech coh. $\check{H}^*(\mathcal{U}, \mathcal{O}_X(n))$ w.r.t. the affine cover $\mathcal{U}$.

Take $\mathcal{U} = \{D_+(x_i)\}_{0 \le i \le r}$. Since this is finite, $\check{C}^\bullet(\mathcal{U}, \mathcal{O}_X(n))$ is the degree n part of the graded cx. $\check{C}^\bullet(\mathcal{U}, F)$, $F = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n)$. The cx.

$$0 \to S \to \prod_{0 \le i_0 \le r} F(U_{i_0}) \to \prod_{0 \le i_0 < i_1 \le r} F(U_{i_0} \times_X U_{i_1}) \to \cdots$$

is can. iso. to

$$\bigotimes_{i=0}^{r}{}_S \, (S \longrightarrow S_{x_i}) .$$

This is a tensor product of flat S-mod,

$$H^q(\mathcal{U}, F) \cong \left(\bigotimes_{i=0}^{r}{}_S \, H^*(S \to S_{x_i}) \right)^{q+1}$$

Since $H^*(S \to S_{x_i}) = S_{x_i}/S$ conc. in deg. 1,

$$H^q(\mathcal{U}, F) = \begin{cases} S & , q = 0 \\ \bigotimes_{i=0}^{r}{}_S \, S_{x_i}/S & , q = r \\ 0 & , \text{else} \end{cases}$$

The degree n part is $H^*(\mathcal{U}, \mathcal{O}_X(n))$. Note:

$$\bigotimes_{i=0}^{r}{}_S \, S_{x_i}/S \leftarrow A\{x_0^{l_0} \cdots x_r^{l_r} \mid l_i < 0\},$$

so $H^r(X, \mathcal{O}_X(-r-1)) = A \cdot x_0^{-1} \cdots x_r^{-1}$ //

4/25/01

Thm Let A be a noetherian ring and let F be a coherent sheaf on $\mathbb{P}^r_A$. Then

(i) $H^i(X,F)$ is a f.g. A-module, $i \geq 0$,

(ii) there exists $n_0 \in \mathbb{Z}$ s.t. for all $n \geq n_0$ and all $i > 0$, $H^i(X, F(n)) = 0$.

pf Since F is quasi-coh. of finite type, we can find

$$E := \mathcal{O}_X(-n)^k \twoheadrightarrow F \,;$$

since F is coh., the kernel K of $E \to F$ is again coherent. Prove (i) by descending induction on i. If $i > r$, $H^i(X,F) = 0$. This is true for any quasi-coh. sheaf on X: X has a covering by $r+1$ aff. open, so the alternating Čech cx. is non-zero only in degrees $0 \leq i \leq r$. Suppose (i) holds for $i+1$, and for all coh. sheaves on X. Then:

$$\cdots \to \underset{\substack{| \\ \text{f.g. by calc.}}}{H^i(X,E)} \to H^i(X,F) \to \underset{\substack{| \\ \text{f.g. by induction}}}{H^{i+1}(X,K)} \to \cdots$$

Since A noeth., $H^i(X,F)$ f.g. To prove (ii) use:

$$\cdots \to \underset{\substack{\| \\ 0 \\ \text{if } n \gg 0 \text{ by calc.}}}{H^i(X,E(n))} \to H^i(X,F(n)) \to \underset{\substack{\| \\ 0 \\ \text{if } n \gg 0 \text{ by induction}}}{H^{i+1}(X,K(n))} \to \cdots$$

//

Let $X = \mathbb{P}^r_A$, $\omega_X = \Omega^r_{X/A} \cong \mathcal{O}_X(-r-1)$.

$$\mathrm{Ext}^r_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X) = H^r(X, \omega_X) \xleftarrow{\sim} A$$

$$\check{C}^r(\mathfrak{U}, \omega_X) \ni \frac{x_0}{x_1 \cdots x_r}\, d\left(\frac{x_1}{x_0}\right) \cdots d\left(\frac{x_r}{x_0}\right) \longleftarrow\!\!| \ 1$$

This is a natural isomorphism.

Thm Let k be a field, $X = \mathbb{P}^r_k$. Then for every coh. $\mathcal{O}_X$-module F, the composition

$$\mathrm{Ext}^{r-i}_{\mathcal{O}_X}(\mathcal{O}_X, F) \otimes_k \mathrm{Ext}^i_{\mathcal{O}_X}(F, \omega_X) \to \mathrm{Ext}^r_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X)$$

is a perfect pairing of f.d. k-vector spaces

Pf Identifying $\mathrm{Ext}^r_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X) \xleftarrow{\sim} k$, the statement is a nat'l iso:

$$\mathrm{Ext}^i_{\mathcal{O}_X}(F, \omega_X) \xrightarrow{\sim} \mathrm{Hom}_k(H^{r-i}(X, F), k).$$

For $i=0$, this is

$$\mathrm{Hom}_{\mathcal{O}_X}(F, \omega_X) \xrightarrow{\sim} \mathrm{Hom}_k(H^r(X, F), k).$$

If $F = \mathcal{O}_X(q)$,

$$\mathrm{Hom}_{\mathcal{O}_X}(F, \omega_X) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(q), \mathcal{O}_X(-r-1))$$

$$\cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X(-q-r-1))$$

$$= H^0(X, \mathcal{O}_X(-q-r-1))$$

$$\cong \mathrm{Hom}_k(H^r(X, \mathcal{O}(q)), k)$$

In general, write

$$E_1 \to E_0 \to F \to 0$$

with E_i a finite sum of $\mathcal{O}_X(q_s)$. Since $\mathrm{Hom}_{\mathcal{O}_X}(-, \omega)$ and $\mathrm{Hom}_k(H^r(X, -), k)$ are both left-exact, we get the statement for $i=0$, for any F.

To prove the statement for $i \geq 0$, we note that we have a map of δ-functors

$$\mathrm{Ext}^i_{\mathcal{O}_X}(-, \omega_X) \longrightarrow \mathrm{Hom}_k(H^{r-i}(X, -), k)$$

which is an isomorphism for $i=0$. So it will suffice to prove that both are effaceable.

Lemma $\mathrm{Ext}^p_{\mathcal{O}_X}(F(-n), G) \cong \mathrm{Ext}^p_{\mathcal{O}_X}(F, G(n))$.

pf $F \mapsto F(n)$ is exact and takes inj. to inj. so Grothendieck sp. seq.

$$E_2^{p,q} = \begin{cases} \mathrm{Ext}^p_{\mathcal{O}_X}(F, G(n)), & q=0, \\ 0 & , q>0, \end{cases}$$

$$\Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_X}(F(-n), G).$$

//

Write F as a quotient of $E = \mathcal{O}_X(-n)^N$. We can choose n as large as we wish. Then

$$\mathcal{E}xt^i_{\mathcal{O}_X}(E, \omega_X) = \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_X(-n)^N, \omega_X)$$

$$\xleftarrow{\sim} \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_X(-n), \omega_X)^N$$

$$\cong \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X(n))^N$$

which is zero, for $n \gg 0$. So $\mathcal{E}xt^i_{\mathcal{O}_X}(-, \omega_X)$ is effaceable, for $i > 0$. Similarly,

$$H^{r-i}(X, E) \cong H^{r-i}(X, \mathcal{O}_X(-n))^N$$

is zero for $i > 0$, if $n > 0$. Hence also $\mathrm{Hom}_k(H^r(X, -), k)$ is effaceable. //

A δ- <u>functor</u> between two ab. categories is:

(i) a family of add. functors $\{T^i\}_{i\geq 0}$;

(ii) for every short-exact sequence

$$0 \to M' \to M \to M'' \to 0,$$

a map

$$T^i(M'') \xrightarrow{\delta^i} T^{i+1}(M)$$

s.t.

(1) for every short-exact seq.

$$0 \to M' \to M \to M'' \to 0$$

the sequence

$$\begin{aligned} 0 \to T^0(M') &\to T^0(M) \to T^0(M'') \\ &\xrightarrow{\delta^0} T^1(M') \to T^1(M) \to T^1(M'') \\ &\xrightarrow{\delta^1} T^2(M') \to \cdots \end{aligned}$$

is exact, and

(2) for every map of s.e.s.

$$\begin{array}{ccccccccc} 0 & \to & M' & \to & M & \to & M'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & N' & \to & N & \to & N'' & \to & 0 \end{array}$$

and all $i \geq 0$, the diagr. commutes:

$$\begin{array}{ccc} T^i(M'') & \xrightarrow{\delta^i} & T^{i+1}(M') \\ \downarrow & & \downarrow \\ T^i(N'') & \xrightarrow{\delta^i} & T^{i+1}(N') \end{array}$$

A δ-functor $\{T^i, \delta^i\}_{i \geq 0}$ is <u>universal</u> if for every δ-functor $\{T'^i, \delta'^i\}_{i \geq 0}$ and every map $f^0 : T^0 \to T'^0$ there exists a unique family of natural maps $f^i : T^i \to T'^i$ s.t. $\delta'^i f^i = f^i \delta^i$.

An additive functor F is <u>effaceable</u> if for every M, there exists a monomorphism $M \to A$ s.t. $F(M) \to F(A)$ is zero.

<u>Prop</u> A δ-functor $\{T^i, \delta^i\}$ is universal if and only if for all $i > 0$, T^i is effaceable. //

Let $(X, \mathcal{O}_X)$ be a ringed space, and let F, G be two $\mathcal{O}_X$-modules. Define

$$(F \otimes_{\mathcal{O}_X} G)(U) = a\left(V \mapsto F(V) \otimes_{\mathcal{O}_X(V)} G(V)\right)(U)$$

$$\underline{Hom}_{\mathcal{O}_X}(F, G)(U) = Hom_{\mathcal{O}_X|_U}(F|_U, G|_U).$$

These are adjoint functors:

$$Hom_{\mathcal{O}_X}(F \otimes_{\mathcal{O}_X} G, H) \xrightarrow{\sim} Hom_{\mathcal{O}_X}(F, \underline{Hom}_{\mathcal{O}_X}(G, H)).$$

In particular, $\underline{Hom}_{\mathcal{O}_X}(F, -)$ is left-exact; define

$$\underline{Ext}^q_{\mathcal{O}_X}(F, G) = R^q \underline{Hom}_{\mathcal{O}_X}(F, -)(G).$$

<u>Lemma</u> There is a spectral sequence

$$E_2^{p,q} = H^p(X, \underline{Ext}^q_{\mathcal{O}_X}(F, G)) \Rightarrow Ext^{p+q}_{\mathcal{O}_X}(F, G).$$

<u>pf</u> This is the Grothendieck sp. seq. ass. with:

$$\begin{tikzcd} X^{\sim}_{\mathcal{O}_X} \arrow[rr, "\underline{Hom}_{\mathcal{O}_X}(F, -)"] \arrow[dr, "Hom_{\mathcal{O}_X}(F, -)"'] & & X^{\sim}_{\mathcal{O}_X} \arrow[dl, "\Gamma(X, -)"] \\ & Mod_{\Gamma(X, \mathcal{O}_X)} & \end{tikzcd}$$

We need to show that for I inj. $\mathcal{O}_X$-mod,

$$H^q(X, \underline{Hom}_{\mathcal{O}_X}(F, I)) = 0, \quad q > 0.$$

— —

<u>Lemma</u> Let $(X, \mathcal{O}_X)$ be a ringed space and suppose $\mathcal{O}_X$ is coherent. Then for every pair F, G of coherent $\mathcal{O}_X$-modules, $\underline{\mathrm{Ext}}^q_{\mathcal{O}_X}(F, G)$ is again coherent.

<u>pf</u> Since $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X, G) = G$, $\underline{\mathrm{Ext}}^q_{\mathcal{O}_X}(\mathcal{O}_X, G) = 0$, if $q > 0$. Hence, $\underline{\mathrm{Ext}}^q_{\mathcal{O}_X}(L, G) = 0$, if $q > 0$, for L locally free of finite type. Since F is coh., we can find, locally, an exact seq.

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & R \arrow[r] & L_{q-1} \arrow[rr] \arrow[dr] & & L_{q-2} \arrow[r] & \cdots \arrow[r] & L_1 \arrow[rr] \arrow[dr] & & L_0 \arrow[r] & F|_U \arrow[r] & 0 \\
& & & K_{q-2} \arrow[ur] \arrow[dr] & & & & K_0 \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

with L_i free of finite type and R coh. We find

$$\underline{\mathrm{Ext}}^q_{\mathcal{O}_X|_U}(F|_U, G|_U) \xleftarrow{\sim} \underline{\mathrm{Ext}}^{q-1}_{\mathcal{O}_X|_U}(K_0, G|_U) \xleftarrow{\sim} \cdots$$

$$\cdots \xleftarrow{\sim} \underline{\mathrm{Ext}}^1_{\mathcal{O}_X|_U}(K_{q-2}, G|_U)$$

and

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X|_U}(L_{q-1}, G|_U) \to \underline{\mathrm{Hom}}_{\mathcal{O}_X|_U}(R, G|_U) \to \underline{\mathrm{Ext}}^1_{\mathcal{O}_X|_U}(K_{q-2}, G|_U) \to 0$$

Thus we are reduced to the case $q = 0$. See FAC. //

<u>Lemma</u> If L is locally free of finite type, then

$$\underline{Ext}^q_{\mathcal{O}_X}(F,G)\otimes_{\mathcal{O}_X} L \xrightarrow{\sim} \underline{Ext}^q_{\mathcal{O}_X}(F, G\otimes_{\mathcal{O}_X} L).$$

<u>pf</u> This is true, if $q=0$. Moreover, $-\otimes_{\mathcal{O}_X} L$ preserves injectives:

$$Hom_{\mathcal{O}_X}(-, I\otimes_{\mathcal{O}_X} L) \cong Hom_{\mathcal{O}_X}(-\otimes_{\mathcal{O}_X} L^{\vee}, I)$$

is exact. Consider the two spectral seq.:

$$\begin{tikzcd}
\tilde{X}_{\mathcal{O}_X} \arrow[dr, "\underline{Hom}_{\mathcal{O}_X}(F,-)"'] & \tilde{X}_{\mathcal{O}_X} \arrow[l, "-\otimes_{\mathcal{O}_X} L"'] \arrow[r, "\underline{Hom}_{\mathcal{O}_X}(F,-)"] \arrow[d] & \tilde{X}_{\mathcal{O}_X} \arrow[dl, "-\otimes_{\mathcal{O}_X} L"] \\
& \tilde{X}_{\mathcal{O}_X} &
\end{tikzcd}$$

//

<u>Thm</u> Let X be a noeth. space of dim. n, and let F be a sheaf of ab. groups on X. Then $H^i(X, F) = 0$, if $i > n$.

<u>pf</u> Later. //

<u>Lemma</u> Let k be a field, let $P = \mathbb{P}^N_k$, and let $X \xhookrightarrow{i} P$ be a closed immersion of codim. r. Then for every $0 \le q < r$,

$$\underline{Ext}^q_{\mathcal{O}_P}(i_*\mathcal{O}_X, \omega_P) = 0.$$

pf Since $\mathcal{O}_P(n)$ is invertible, it suffices to show that for $n \gg 0$ and $0 \le q < r$, the sheaf

$$\underline{Ext}^q_{\mathcal{O}_P}(i_*\mathcal{O}_X, \omega_P)(n) = \underline{Ext}^q_{\mathcal{O}_P}(i_*\mathcal{O}_X, \omega_P(n))$$

is zero. But for $n \gg 0$, this is generated by global sections, so it suffices to show that for $n \gg 0$ and $0 \le q < r$,

$$H^0(P, \underline{Ext}^q_{\mathcal{O}_P}(i_*\mathcal{O}_X, \omega_P(n))) = 0.$$

Consider the sp. seq.

$$E_2^{p,q} = H^p(P, \underline{Ext}^q_{\mathcal{O}_P}(i_*\mathcal{O}_X, \omega_P(n)))$$

$$\Rightarrow Ext^{p+q}_{\mathcal{O}_P}(i_*\mathcal{O}_X, \omega_P(n)).$$

Since $\underline{Ext}^q_{\mathcal{O}_P}(i_*\mathcal{O}_X, \omega_P)$ is coh., we know that for $n \gg 0$, $E_2^{p,q} = 0$, if $p > 0$. So we may instead show that for $n \gg 0$ and $0 \le q < r$,

$$Ext^q_{\mathcal{O}_P}(i_*\mathcal{O}_X, \omega_P(n)) = 0.$$

But by duality for P,

$$Ext^q_{\mathcal{O}_P}(i_*\mathcal{O}_X, \omega_P(n)) = Ext^q_{\mathcal{O}_P}(i_*\mathcal{O}_X(-n), \omega_P)$$

$$\xrightarrow{\sim} Hom_k(H^{N-q}(P, i_*\mathcal{O}_X(-n)), k).$$

And, since i_* is exact and takes injectives to injectives ($i^* \dashv i_* \dashv i^!$),

$$H^{N-q}(P, i_*\mathcal{O}_X(-n)) \xleftarrow{\sim} H^{N-q}(X, \mathcal{O}_X(-n))$$

which is zero, if $N-q > N-r$, or equivalently, if $q < r$. //

The functor $F \mapsto i_* F$ defines an equivalence between the cat. $X^{\sim}_{\mathcal{O}_X}$ and $P^{\sim}_{i_*\mathcal{O}_X}$. Let ω_X° be the coh. $\mathcal{O}_X$-mod. s.t.

$$i_*\omega_X^{\circ} = \underline{\mathrm{Ext}}^r_{\mathcal{O}_P}(i_*\mathcal{O}_X, \omega_P).$$

Cor There is a natural iso. for coherent $\mathcal{O}_X$-modules F:

$$\mathrm{Hom}_{\mathcal{O}_X}(F, \omega_X^{\circ}) \cong \mathrm{Ext}^r_{\mathcal{O}_P}(i_*F, \omega_P).$$

Pf We identify $X^{\sim}_{\mathcal{O}_X}$ with $P^{\sim}_{i_*\mathcal{O}_X}$. The forgetful functor

$$P^{\sim}_{i_*\mathcal{O}_X} \xrightarrow{U} P^{\sim}_{\mathcal{O}_P}$$

is exact and has a left adjoint:

$$\underline{\mathrm{Hom}}_{\mathcal{O}_P}(i_*\mathcal{O}_X, G) \leftarrow\!\shortmid G,$$

i.e.

$$\mathrm{Hom}_{\mathcal{O}_P}(UF, G) \cong \mathrm{Hom}_{i_*\mathcal{O}_X}(F, \underline{\mathrm{Hom}}_{\mathcal{O}_P}(i_*\mathcal{O}_X, G)).$$

Hence, we have a Grothendieck sp. sequence

$$E_2^{p,q} = \mathrm{Ext}^p_{i_*\mathcal{O}_X}(F, \underline{\mathrm{Ext}}^q_{\mathcal{O}_P}(i_*\mathcal{O}_X, \omega_P))$$

$$\Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_P}(UF, \omega_P),$$

and the lemma shows that $E_2^{p,q} = 0$, if $0 \le q < r$. In particular, the edge hom. is an iso:

$$\mathrm{Hom}_{i_*\mathcal{O}_X}(F, \underline{\mathrm{Ext}}^r_{\mathcal{O}_P}(i_*\mathcal{O}_X, \omega_P)) \xrightarrow{\sim} \mathrm{Ext}_{\mathcal{O}_P}(UF, \omega_P). \quad //$$

Prop There is a natural map

$$H^d(X, \omega_X^\circ) \xrightarrow{t} k, \qquad d = \dim X,$$

and the pairing

$$\mathrm{Hom}_{\mathcal{O}_X}(F, \omega_X^\circ) \otimes H^d(X, F) \to H^d(X, \omega_X^\circ) \xrightarrow{t} k$$

is perfect.

Pf $\mathrm{Hom}_{\mathcal{O}_X}(F, \omega_X^\circ) \cong \mathrm{Ext}^r_{\mathcal{O}_P}(i_*F, \omega_P)$

$$\cong \mathrm{Hom}_k(H^d(P, i_*F), k) \cong \mathrm{Hom}_k(H^d(X, F), k).$$

Let t correspond to $\omega_X^\circ \xrightarrow{\mathrm{id}} \omega_X^\circ$. //

Let X be a scheme over a field k. By a <u>dualizing sheaf</u> on X we mean a pair (ω_X°, t), where ω_X° is a coh. $\mathcal{O}_X$-module and t a k-linear map $H^n(X, \omega_X^\circ) \xrightarrow{t} k$ s.t. for every coh. $\mathcal{O}_X$-module F, the pairing is perfect:

$$\mathrm{Hom}_{\mathcal{O}_X}(F, \omega_X^\circ) \otimes_k H^n(X, F) \xrightarrow{\circ} H^n(X, \omega_X^\circ) \xrightarrow{t} k$$

<u>Lemma</u> A dualizing sheaf (ω_X°, t) is unique up to canonical isomorphism.

<u>pf</u> The data (ω_X°, t) gives a natural iso.

$$\mathrm{Hom}_k(H^n(X, F), k) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}(F, \omega_X^\circ),$$

i.e. ω_X° represents the functor $\mathrm{Hom}_k(H^n(X, -), k)$. And representing objects are unique up to can. isomorphism. //

Suppose that $X \xrightarrow{f} \mathrm{Spec}\, k$ is <u>projective</u>, i.e. can be factored

$$X \xhookrightarrow{i} \mathbb{P}^N_k \xrightarrow{pr} \mathrm{Spec}\, k$$

with i a closed immersion. Then we proved last time that X has a dualizing sheaf (ω_X°, t) s.t.

$$i_* \omega_X^\circ = \underline{\mathcal{E}xt}^r_{\mathcal{O}_P}(i_* \mathcal{O}_X, \omega_P),$$

where $r = N - n$ is the codimension of i.

The lemma shows that up to canonical iso. (ω_X°, t) does not depend on the choice of factorization of f. Let $n = \dim X$. We wish to show that for all $0 \le q \le n$, the pairing

$$\operatorname{Ext}^q_{\mathcal{O}_X}(F, \omega_X^\circ) \otimes_k \operatorname{Ext}^{n-q}_{\mathcal{O}_X}(\mathcal{O}_X, F) \xrightarrow{\circ} \operatorname{Ext}^n_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X^\circ) \xrightarrow{t} k$$

is perfect, or equivalently, that the adjoint map

$$\operatorname{Ext}^q_{\mathcal{O}_X}(F, \omega_X^\circ) \longrightarrow \operatorname{Hom}_k(H^n(X, F), k)$$

is an isomorphism. Try to repeat the proof for $X = P$. The two sides are δ-functors in F, so it suffices to show that they are effaceable. Choose $X \xrightarrow{i} P = \mathbb{P}^N_k$, let $\mathcal{O}_X(n) = i^*\mathcal{O}_P(n)$

<u>Lemma</u> Let F be a coherent $\mathcal{O}_X$-module. Then for $n \gg 0$, $F(n)$ is generated by global sections.

<u>Pf</u> Since i_*F is a coh. $\mathcal{O}_P$-module, $(i_*F)(n)$ $= i_*(F(n))$ is gen. by global sections if $n \gg 0$, i.e. there exists

$$\mathcal{O}_P^m \longrightarrow\!\!\!\!\rightarrow i_*(F(n)).$$

But then

$$\mathcal{O}_X^m = i^*\mathcal{O}_P^m \longrightarrow\!\!\!\!\rightarrow i^*i_*(F(n)) \xrightarrow{\sim} F(n). \qquad //$$

instead try to show that for $0 \le j < n$,

$$\underline{Ext}^{N-j}_{\mathcal{O}_P}(i_* \mathcal{O}_X, \omega_P(s)) = 0,$$

i.e. for $q > r$,

$$\underline{Ext}^{q}_{\mathcal{O}_P}(i_* \mathcal{O}_X, \omega_P(s)) = 0.$$

This holds if and only all the stalk at every (closed) point $x \in P$ is zero. Will show later that

$$\underline{Ext}^{q}_{\mathcal{O}_P}(i_* \mathcal{O}_X, \omega_P(s))_x \xleftarrow{\sim} Ext^{q}_{\mathcal{O}_{P,x}}((i_* \mathcal{O}_X)_x, \omega_P(s)_x)$$

The right hand side is zero unless $x \in X$. So let $x \in X$ be a closed point. We wish to show that for $q > r$,

$$Ext^{q}_{\mathcal{O}_{P,x}}(\mathcal{O}_{X,x}, \omega_{P,x}) = 0.$$

This, of course, is true, if as an $\mathcal{O}_{P,x}$-module, $\mathcal{O}_{X,x}$ has projective dimension $\le r$.

⌈ Some commutative algebra: $\mathcal{O}_{P,x}$ is a regular local ring. So by [M, 19.1, 19.2]

$$\mathrm{pd}_{\mathcal{O}_{P,x}} \mathcal{O}_{X,x} + \mathrm{depth}_{\mathcal{O}_{P,x}} \mathcal{O}_{X,x} = \dim \mathcal{O}_{P,x}$$

But

$$\mathrm{depth}_{\mathcal{O}_{P,x}} \mathcal{O}_{X,x} = \mathrm{depth}_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}$$

<u>Lemma</u> Let F be a coherent $\mathcal{O}_X$-module. Then for $n >> 0$ and $q > 0$, $H^q(X, F(n)) = 0$.

<u>Pf</u> The Leray sp. seq. collapses:

$$H^q(X, F(n)) = H^q(P, i_* F(n)) = 0,$$

if $n >> 0$ and $q > 0$. //

<u>Cor</u> $Ext^q_{\mathcal{O}_X}(-, \omega_X^\circ)$ is effaceable, $q > 0$.

<u>Pf</u> Choose $E = \mathcal{O}_X(-s)^m \twoheadrightarrow F$. Then

$$Ext^q_{\mathcal{O}_X}(E, \omega_X^\circ) \cong Ext^q_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X^\circ(s))$$

is zero, if $s >> 0$ and $q > 0$. //

Again consider $E = \mathcal{O}_X(-s)^m \twoheadrightarrow F$. We wish to show that for $s >> 0$ and $0 \le j < n$,

$$H^j(X, \mathcal{O}_X(-s)) = 0.$$

By Leray,

$$H^j(X, \mathcal{O}_X(-s)) \cong H^j(P, i_* \mathcal{O}_X(-s))$$

and by duality for P, this is the dual of

$$Ext^{N-j}_{\mathcal{O}_P}(i_* \mathcal{O}_X(-s), \omega_P) \cong Ext^{N-j}_{\mathcal{O}_P}(i_* \mathcal{O}_X, \omega_P(s)).$$

We have the sp. seq.

$$E_2^{p,q} = H^p(P, \underline{Ext}^q_{\mathcal{O}_P}(i_* \mathcal{O}_X, \omega_P(s))) \Rightarrow Ext^{p+q}_{\mathcal{O}_P}(i_* \mathcal{O}_X, \omega_P(s))$$

and for $s >> 0$, $E_2^{p,q} = 0$, if $p > 0$. So can so we need depth $\mathcal{O}_{X,x} = n$. Suppose all irreducible components of X have the same dim. n. Then for every cl. pt. $x \in X$, $\mathcal{O}_{X,x}$ has dimension n. Hence, we need:

$$\operatorname{depth} \mathcal{O}_{X,x} = \dim \mathcal{O}_{X,x}.$$

This is <u>Cohen - Macauley</u>. ⌋

<u>Thm</u> Let $X \to \operatorname{Spec} k$ be projective and let (ω_X°, t) be a dualizing sheaf on X. Suppose that X is Cohen-Macauley and that all irreducible components of X have the same dim. n. Then for every coherent $\mathcal{O}_X$-module F and every $0 \le q \le n$, the pairing

$$\operatorname{Ext}^q_{\mathcal{O}_X}(F, \omega_X^\circ) \otimes_k \operatorname{Ext}^{n-q}_{\mathcal{O}_X}(\mathcal{O}_X, F) \to \operatorname{Ext}^n_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X^\circ) \xrightarrow{t} k$$

is perfect. //

We have used (among other things):

<u>Lemma</u> Let $(X, \mathcal{O}_X)$ be a ringed space s.t. $\mathcal{O}_X$ is a coherent $\mathcal{O}_X$-module. Then for every coherent $\mathcal{O}_X$-module F, every $\mathcal{O}_X$-module G and every $q \ge 0$,

$$\operatorname{Ext}^q_{\mathcal{O}_{X,x}}(F_x, G_x) \xrightarrow{\sim} \underline{\operatorname{Ext}}^q_{\mathcal{O}_X}(F, G)_x .$$

Pf Since $\mathcal{O}_X$ is coh., we can find a resolution $L. \xrightarrow{\sim} F$ of F by locally free $\mathcal{O}_X$-modules of finite type. Let $G \to I^{\cdot}$ be an injective resolution. The two spectral sequences ass. w. the dbl. cx. $\underline{Hom}_{\mathcal{O}_X}(L., I^{\cdot})$ shows that

$$\underline{Ext}^q_{\mathcal{O}_X}(F, G) \cong \underline{H}^q \underline{Hom}_{\mathcal{O}_X}(L., G).$$

So it suffices to prove the statement for $q=0$ and F loc. free of finite type. But this is obvious. //

A closed imm. $X \xhookrightarrow{i} Y$ is a <u>regular immersion</u> of codim. r if for every $x \in X$, there exists $i(x) \in U \subset Y$ open affine s.t. the ideal $\Gamma(U, J) \subset \Gamma(U, \mathcal{O}_X)$ is generated by a regular sequence of length r.

⌈ If A is a ring and M an A-module, a sequence $x_1, \dots, x_r \in X$ is a regular sequence for M if for all $1 \leq i \leq r$, x_i is a non-zero-divisor in $M/(x_1, \dots, x_{i-1})M$. ⌋

In this case, J/J^2 is a loc. free $\mathcal{O}_X$-module of rank d — the conormal bdl. — and the canonical map is an iso:

$$S^{\cdot}_{\mathcal{O}_X}(J/J^2) \xrightarrow{\sim} gr^{\cdot}_J(\mathcal{O}_Y).$$

If Y is Cohen-Macauley and $X \xhookrightarrow{i} Y$ a regular imm. then X is Cohen-Macauley.

<u>Lemma</u> Let $X \xhookrightarrow{i} P = \mathbb{P}^N_k$ be a regular imm. of codim. r. Then there is a natural iso:

$$\omega_X^{\circ} \cong i^*\omega_P \otimes_{\mathcal{O}_X} \Lambda^r_{\mathcal{O}_X} \underline{Hom}_{\mathcal{O}_X}(J/J^2, \mathcal{O}_X)$$

In particular, ω_X° is an inv. $\mathcal{O}_X$-module.

Pf let $U \subset P$ be aff. open s.t. $\Gamma(U, J)$ is gen. by a reg. seq. $f_1, \ldots, f_r \in \Gamma(U, \mathcal{O}_P)$. Then we have a res. of $i_* \mathcal{O}_X|_U$ by loc. free $\mathcal{O}_P$-mod of finite type,

$$\begin{array}{ccc} i_*\mathcal{O}_X|_U & \xleftarrow{\sim} & K(f_1, \ldots, f_r; \mathcal{O}_P|_U), \\ & & \| \\ & & \bigotimes_{\mathcal{O}_P|_U} (\mathcal{O}_P|_U \xleftarrow{f_i} \mathcal{O}_P|_U). \end{array}$$

So

$$\begin{aligned} \underline{Ext}^r_{\mathcal{O}_P}(i_*\mathcal{O}_X, \omega_P)|_U &\cong \underline{Ext}^r_{\mathcal{O}_P|_U}(i_*\mathcal{O}_X|_U, \omega_P|_U) \\ &= \underline{H}^r(\underline{Hom}_{\mathcal{O}_P}(K(f_1, \ldots, f_r; \mathcal{O}_P|_U), \omega_P|_U)) \\ &\cong \underline{H}^r(\bigotimes_{\mathcal{O}_P|_U}(\omega_P|_U \xrightarrow{f_i} \omega_P|_U)) \\ &\cong \omega_P|_U / (f_1, \ldots, f_r)\omega_P|_U \\ &\cong (\omega_P \otimes_{\mathcal{O}_P} i_*\mathcal{O}_X)|_U . \end{aligned}$$

This isomorphism, however, depends on our choice of gen. $f_1, \ldots, f_r$. The change of basis $f_i \longmapsto g_i = \sum a_{ij} f_j$ induces mult. by $\det(a_{ij})$ on $(\omega_P \otimes_{\mathcal{O}_P} i_*\mathcal{O}_X)|_U$. Throw in $\underline{Hom}_{\mathcal{O}_X}(J/J^2, \mathcal{O}_X) =: N_X P$ to change this. //

The morphism

$$\begin{array}{ccc} X & = & \operatorname{Spec} A[x_1,\dots,x_{n+d}]/(f_1,\dots,f_n) \\ \downarrow f & & \downarrow \\ Y & = & \operatorname{Spec} A \end{array}$$

is <u>smooth</u> <u>at</u> $x \in X$ (of relative dim. d) if

$$\operatorname{rk}_{k(x)} \left(\frac{\partial f_i}{\partial x_j}(x) \right) = n$$

A morphism $X \xrightarrow{f} Y$ is <u>smooth</u> (of rel. dim. d), if for all $x \in X$, there exists $x \in U \subset X$ and $f(x) \subset V \subset Y$ with $f(U) \subset V$ and a commutative diagr.

$$\begin{array}{ccc} U & \hookrightarrow & \operatorname{Spec} A[x_1,\dots,x_{n+d}]/(f_1,\dots,f_n) \\ \downarrow f & & \downarrow \\ V & \hookrightarrow & \operatorname{Spec} A \end{array}$$

s.t. the horizontal maps are open imm. and s.t. the right hand vertical map is smooth at x.

Let k be a perfect field, and let

$$X \xrightarrow{f} \operatorname{Spec} k$$

be a morphism of finite type. Then f is smooth if and only if for all $x \in X$, $\mathcal{O}_{X,x}$ is a regular local ring.

If $X \xrightarrow{f} Y$ is a morphism of S-schemes and if $X \to S$ is smooth, then

$$0 \to \mathcal{J}/\mathcal{J}^2 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to 0 .$$

Cor Let $X \xrightarrow{f} \operatorname{Spec} k$ be a smooth projective morphism. Then $\omega_X^\circ \cong \omega_X$.

pf If we embed $X \overset{i}{\hookrightarrow} \mathbb{P}^N_k$, then i is automatically a regular immersion. So

$$\omega_X^\circ \cong i^*\omega_P \underset{\mathcal{O}_X}{\otimes} \Lambda^r_{\mathcal{O}_X}(\mathcal{J}/\mathcal{J}^2)^\vee \cong \omega_X . \qquad //$$

$(X, \mathcal{O}_X)$ ringed space

$\mathcal{S} \subset \mathcal{O}_X$ sheaf of non-zero divisors.

$\mathcal{M}_X = \mathcal{S}^{-1}\mathcal{O}_X$ sheaf of meromorphic functions.

Consider exact cohomology sequence:

$$
\begin{array}{ccccccccc}
0 \to H^0(X, \mathcal{O}_X^*) & \to & H^0(X, \mathcal{M}_X^*) & \to & H^0(X, \mathcal{M}_X^*/\mathcal{O}_X^*) & \to & H^1(X, \mathcal{O}_X^*) & \to & \cdots \\
 & & & & \| & & \downarrow \wr & & \\
 & & & & \mathrm{Div}(X) & & \mathrm{Pic}(X) & &
\end{array}
$$

Let $D \in \mathrm{Div}(X)$. Then there exists a covering $\{U_i \to X\}_{i \in I}$ and for all $i \in I$, $f_i \in \Gamma(U_i, \mathcal{M}_X^*)$ s.t. for all $i, j \in I$, $f_i/f_j \in \Gamma(U_i \times_X U_j, \mathcal{O}_X^*)$. We say that the f_i are <u>local equations</u> for D. Let $\mathcal{O}_X(D) \subset \mathcal{M}_X$ be the sub-$\mathcal{O}_X$-module gen. by $f_i^{-1} \in \Gamma(U_i, \mathcal{M}_X^*)$. Then $\mathcal{O}_X(D)$ is an invertible $\mathcal{O}_X$-module. The boundary map

$$\mathrm{Div}(X) \longrightarrow \mathrm{Pic}(X)$$

takes D to the element corresponding to $cl(\mathcal{O}(D))$ under the canonical iso:

$$\mathrm{Pic}(X) = \check{H}^1(X, \mathcal{O}_X^*) \xleftarrow{\sim} H^1(X, \mathcal{O}_X^*).$$

Say that D is <u>principal</u> if D is in the image of

$$H^0(X, \mathcal{M}_X^*) \xrightarrow{div} H^0(X, \mathcal{M}_X^*/\mathcal{O}_X^*)$$

Say that D and D' are <u>linearly equivalent</u>, and write $D \sim D'$, if $D - D' = \operatorname{div}(f)$. Note

$$\mathcal{S} = \mathcal{S}(\mathcal{O}_X) = \mathcal{O}_X \cap \mathcal{M}_X^* .$$

Say that D is <u>effective</u>, and write $D \geq 0$, if D is in the image of

$$\operatorname{Div}^+(X) := H^0(X, \mathcal{S}(\mathcal{O}_X)/\mathcal{O}_X^*) \hookrightarrow \operatorname{Div}(X) .$$

For every $D \in \operatorname{Div}(X)$, the set of effective divisors lin. eq. to D is the hom. space:

$$|D| := H^0(X, \mathcal{O}_X(D) \cap \mathcal{M}_X^*) / H^0(X, \mathcal{O}_X^*) .$$

Let X be a scheme. The group $Z^r(X)$ of codim. r cycles on X is the free abelian group gen. by the closed integral subschemes of X of codim. r. Equivalently, $Z^r(X)$ is the free abelian group gen. by the codim. r points of X. (The codim. of x is $\dim \mathcal{O}_{X,x}$.) Indeed, if $Y \hookrightarrow X$ is a closed integral subscheme of X of codim. r, then the generic pt. of Y is a point of codim. r. And if x is a point of codim. r, then $\overline{\{x\}} \hookrightarrow X$ with the reduced induced scheme structure is a closed integral subscheme of codim. r. The group $Z^1(X)$ is called the group of Weil divisors.

Prop Let X be an integral, separated, noeth. scheme s.t. for all $x \in X$, $\mathcal{O}_{X,x}$ is a UFD. Then there is can. ise.

$$\mathrm{Div}(X) \xrightarrow{\sim} Z^1(X).$$

pf Let $D \in \mathrm{Div}(X)$ and let $\{(U_i, f_i)\}$ be local equations. If $x \in X$ is codim. 1, then $\mathcal{O}_{X,x}$ is a UFD of dim. 1, i.e. a DVR; let v_x be the valuation. If $x \in U_i$, let $v_x(D) := v_x(f_i)$. We map

$$D \longmapsto \sum_{x \in X^{(1)}} v_x(D) \cdot x.$$

Lemma This sum is finite

pf Let $f = f_i \in \Gamma(U_i, \mathcal{M}_X^*)$. There exists an aff. open subset $U = \operatorname{Spec} A \subset U_i$ s.t. f is regular on U. Then $Z = X \setminus U$ is a proper closed subset, and since X is noetherian, $Z \cap X^{(1)}$ is finite. Let $x \in U \cap X^{(1)}$. Then $v_x(f) \geq 0$ and $v_x(f) \neq 0$ if and only if $x \in V(f)$. Since U is noeth. and $V(f) \subset U$ a proper closed subset, there are only fin. many such x. //

Conversely, let $\sum n_x \cdot x \in Z^1(X)$. We pick an aff. open cover $\{U_i\}$ s.t. each U_i contains at most one x with $n_x \neq 0$. Let $U_i \ni x$ with $n_x \neq 0$, and recall that $\mathcal{O}_{X,x}$ is a DVR. Restricting U_i further, we can find $t_i \in \Gamma(U_i, \mathcal{O}_X)$ s.t. $v_x(t_i) = 1$, i.e. t_i is a uniformizer. Now let $f_i = t^{n_i}$. //

Cor Every $D \in \operatorname{Div}(X)$ can be written

$$D = D^+ - D^-$$

with $D^+, D^- \in \operatorname{Div}^+(X)$.

pf Indeed, $D \in \operatorname{Div}^+(X)$ if and only if $v_x(D) \geq 0$, $\forall x \in X^{(1)}$. //

If $D \geq 0$, we can view D as a closed subscheme of X given by the ideal $\mathcal{O}_X(-D) \subset \mathcal{O}_X$, i.e.

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to i_* \mathcal{O}_D \to 0 .$$

Then

$$i_* \mathcal{O}_D = \sum_{x \in X^{(1)}} i_{x*} \left(\mathcal{O}_{X,x} / m_x^{n_x} \right) . \qquad //$$

Prop For any integral scheme X, the map

$$\mathrm{Div}(X) \to \mathrm{Pic}(X), \quad D \mapsto \mathcal{O}_X(D),$$

is surjective.

pf The statement is that every invertible $\mathcal{O}_X$-module $\mathcal{L}$ is isomorphic to a sub-module of $\mathcal{M}_X$. Since X is integral, $\mathcal{M}_X$ is the constant sheaf of $k(X)$. Locally, $\mathcal{L} \cong \mathcal{O}_X$, so $\mathcal{L} \otimes \mathcal{M}_X \cong \mathcal{M}_X$. So $\mathcal{L} \otimes \mathcal{M}_X$ is locally constant, and since X is irr, $\mathcal{L} \otimes \mathcal{M}_X$ is constant. So $\mathcal{L} \otimes \mathcal{M}_X \cong \mathcal{M}_X$, and $\mathcal{L} \hookrightarrow \mathcal{L} \otimes \mathcal{M}_X$. //

By a smooth curve over a field k, we mean a smooth projective geometrically conn. morphism

$$X \xrightarrow{f} \operatorname{Spec} k$$

of relative dimension 1. If F is a coh. $\mathcal{O}_X$-module, we define the Euler-Poincaré characteristic:

$$\chi(F) = \sum_{i \geq 0} (-1)^i \dim_k H^i(X, F).$$

Define the degree map:

$$\begin{array}{ccc} \operatorname{Pic}(X) & \xrightarrow{\deg} & \mathbb{Z} \\ \Psi & & \Psi \\ [\mathcal{L}] & \longmapsto & \chi(\mathcal{L}) - \chi(\mathcal{O}_X) \end{array}$$

Thm (Riemann-Roch) The degree map is a homomorphism. In effect,

$$\deg(\mathcal{O}_X(D)) = \sum_{x \in X^{(1)}} v_x(D) \cdot [k(x) : k].$$

Pf If $D \geq 0$, then

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to i_* \mathcal{O}_D \to 0,$$

and since $\operatorname{supp}(i_* \mathcal{O}_D)$ is finite,

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to i_* \mathcal{O}_D \to 0.$$

Now Euler char. is additive so

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \chi(i_* \mathcal{O}_D) \rightsquigarrow$$

$$\deg(\mathcal{O}_X(D)) = \dim_k H^0(X, i_* \mathcal{O}_D)$$

$$= \sum_{x \in X^{(1)}} v_x(D) \cdot [k(x) : k].$$

In general $D = D^+ - D^-$ with $D^+, D^- \geq 0$. From

$$0 \to \mathcal{O}_X(-D^-) \to \mathcal{O}_X \to i_* \mathcal{O}_{D^-} \to 0 \rightsquigarrow$$

$$0 \to \mathcal{O}_X(D^+ - D^-) \to \mathcal{O}_X(D^+) \to i_* \mathcal{O}_{D^-} \to 0$$

we get

$$\chi(\mathcal{O}_X(D^+)) = \chi(\mathcal{O}_X(\underbrace{D^+ - D^-}_{D})) + \chi(i_* \mathcal{O}_{D^-})$$

or

$$\deg(\mathcal{O}_X(D^+)) + \cancel{\chi(\mathcal{O}_X)} = \deg(\mathcal{O}_X(D)) + \cancel{\chi(\mathcal{O}_X)} + \deg(\mathcal{O}_X(D^-))$$

The theorem follows. //

<u>Thm</u> $\chi(\mathcal{O}_X) = -\frac{1}{2} \deg(\omega_X)$.

<u>Pf</u> By Serre duality, $\chi(\omega_X) = -\chi(\mathcal{O}_X)$.
And $\chi(\omega_X) = \deg(\omega_X) + \chi(\mathcal{O}_X)$. //

<u>Def</u> $\chi(\mathcal{O}_X) = 1 - g$.

Def An elliptic curve over a field k is a pair (E, o) of a smooth curve of genus 1 and a k-rational point. //

Let $\mathrm{Pic}^\circ(E) \subset \mathrm{Pic}(E)$ be the subgroup of elements of degree o.

Thm The map is a bijection:

$$E(k) \xrightarrow{\sim} \mathrm{Pic}^\circ(X), \quad x \longmapsto \mathcal{O}_E(x-o).$$

In particular, $E(k)$ is an abelian group and $\mathrm{Spec}\, k \xrightarrow{o} E$ is the additive unit.

pf We define the inverse. Let $\mathcal{L} \in \mathrm{Pic}^\circ(X)$. Then $\mathcal{L} \otimes \mathcal{O}_X((o))$ has degree 1, so by Riemann-Roch:

$$h^0(X, \mathcal{L} \otimes \mathcal{O}_X((o))) - h^1(X, \mathcal{L} \otimes \mathcal{O}_X((o))) = 1.$$

By Serre duality

$$h^1(X, \mathcal{L} \otimes \mathcal{O}_X((o))) = h^0(X, \omega_X \otimes \mathcal{L}^{-1} \otimes \mathcal{O}_X(-(o))).$$

Since ω_X has degree zero, $\omega_X \otimes \mathcal{L}^{-1} \otimes \mathcal{O}_X(-(o))$ has degree -1. But then it has no non-zero sections. So

$$h^0(X, \mathcal{L} \otimes \mathcal{O}_X((o))) = 1.$$

Hence, up to a unit, $\mathcal{L} \otimes \mathcal{O}_X((o))$ has a unique section. This section has a single zero $x \in E$ of order 1 and $[k(x):k] = 1$, i.e. x is a rational point. //

Let $X \to \operatorname{Spec} k$ be a smooth, projective, geo. conn. morphism of rel. dim. 2. Let C, D be effective divisors on X. Then

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to i_{D*}\mathcal{O}_D \to 0$$

and since $\mathcal{O}_X(-D)$ locally free:

$$0 \to \underbrace{\mathcal{O}_X(-D) \otimes_{\mathcal{O}_X} i_{C*}\mathcal{O}_C}_{= \; i_{C*} i_C^* \mathcal{O}_X(-D)} \to i_{C*}\mathcal{O}_C \to \underbrace{i_{C*}\mathcal{O}_C \otimes_{\mathcal{O}_X} i_{D*}\mathcal{O}_D}_{= \; i_{C \times_X D *}\mathcal{O}_{C \times_X D}} \to 0$$

Suppose C is a smooth curve and that none of the generic points of D lie on C. Then by RR for C,

$$C \cdot D = h^0(C \times_X D, \mathcal{O}_{C \times_X D}) = \deg(i_C^* \mathcal{O}_X(-D)).$$

This only depends on the linear equivalence class of such D. Define

$$\operatorname{Pic} X \otimes \operatorname{Pic} X \longrightarrow \mathbb{Z}$$
$$C \otimes D \longmapsto C \cdot D$$

to be the unique bilinear pairing extending the above.

ex 1) Let $C, D \subset X$ be two smooth curves s.t. none of the generic points of C are on D (and vice versa). For $x \in C \cap D$, let f_x (resp. g_x) be a local eq. for C (resp. D) near x. Then

$$C.D = \sum_{x \in C \cap D} \text{length}\left(\mathcal{O}_{X,x}/(f, g)\right) [k(x) : k] .$$

2) Let $C \subset X$ be a sm. curve. Then

$$C.C = \deg\left(i_C^* \mathcal{O}_X(C)\right) .$$

Now

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to i_{C*}\mathcal{O}_C \to 0 \quad \rightsquigarrow$$

$$0 \to \mathcal{O}_X(-C) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-C) \to \mathcal{O}_X(-C) \to \mathcal{O}_X(-C) \otimes_{\mathcal{O}_X} i_{C*}\mathcal{O}_C \to 0$$

$$\mathcal{O}_X(-C) \otimes_{\mathcal{O}_X} i_{C*}\mathcal{O}_C = i_{C*} i_C^* \mathcal{O}_X(-C)$$

Since $\mathcal{O}_X(-C)$ is the ideal which defines C in X, we get

$$i_C^* \mathcal{O}_X(C) \cong N_C X$$

so

$$C \cdot C = \deg(N_C X) .$$

<u>Thm</u> Let $X \to \operatorname{Spec} k$ be a smooth projective surface and let D be a divisor on X. Then

$$\chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X) = \tfrac{1}{2} D.(D-K),$$

where K is a canonical divisor.

<u>pf</u> Use without proof that every divisor D is lin. eq. to $C - E$, where $C, E \subset X$ are sm. curves s.t. none of the generic points of C lie on E (and vice versa). Since $\mathcal{O}_X(-E)$ is loc. free hence flat, the seq.

$$0 \to \mathcal{O}_X(-E) \to \mathcal{O}_X \to i_{E*}\mathcal{O}_E \to 0$$

gives

$$0 \to \mathcal{O}_X(C-E) \to \mathcal{O}_X(C) \to \mathcal{O}_X(C) \otimes_{\mathcal{O}_X} i_{E*}\mathcal{O}_E \to 0$$

Similarly, since $\mathcal{O}_X(-C)$ flat:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_X(C) \otimes_{\mathcal{O}_X} i_{C*}\mathcal{O}_C \to 0$$

So

$$\chi(\mathcal{O}_X(C-E)) + \chi(i_E^*\mathcal{O}_X(C))$$

$$= \chi(\mathcal{O}_X) + \chi(i_C^*\mathcal{O}_X(C)).$$

By RR for C, D:

$$\chi(i_E^* \mathcal{O}_X(C)) = \chi(\mathcal{O}_E) + \deg(i_E^* \mathcal{O}_X(C))$$

$$\chi(i_C^* \mathcal{O}_X(C)) = \chi(\mathcal{O}_C) + \deg(i_C^* \mathcal{O}_X(C))$$

Since $E \hookrightarrow X$ is a reg. immersion with ideal $\mathcal{O}_X(-E)$,

$$\omega_E \cong i_E^*(\omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(E))$$

$$\omega_C \cong i_C^*(\omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(C))$$

and hence

$$\chi(\mathcal{O}_E) = -\tfrac{1}{2}\deg(\omega_E) = -\tfrac{1}{2}\deg(i_E^*(\omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(E)))$$

$$\chi(\mathcal{O}_C) = -\tfrac{1}{2}\deg(i_C^*(\omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(C))).$$

Putting everything together:

$$\chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X) = \chi(i_C^* \mathcal{O}_X) - \chi(i_E^* \mathcal{O}_X)$$

$$= -\tfrac{1}{2}\deg(i_C^* \mathcal{O}_X(K+C)) + \deg(i_C^* \mathcal{O}_X(C))$$

$$+ \tfrac{1}{2}\deg(i_E^* \mathcal{O}_X(K+E)) - \deg(i_E^* \mathcal{O}_X(C))$$

$$= -\tfrac{1}{2}\, C.(C+K) + C.C$$

$$+ \tfrac{1}{2}\, E.(E+K) - E.C$$

$$= -\tfrac{1}{2} C.C - \tfrac{1}{2} C.K + C.C + \tfrac{1}{2} E.E + \tfrac{1}{2} E.K - E.C$$

$$= \tfrac{1}{2} (C.C - C.K + E.E + E.K - 2E.C)$$

$$= \tfrac{1}{2} (C-E).(C-E-K)$$

$$= \tfrac{1}{2} D.(D-K).$$ //

<u>Fact</u> If H is an ample divisor on X and if D is an effective divisor, then $H.D > 0$.

<u>Lemma</u> Let H be an ample divisor on X. Then there exists $n_0 \in \mathbb{Z}$ s.t. for every divisor D with $D.H \geq n_0$, $H^2(X, \mathcal{O}_X(D)) = 0$.

<u>pf</u> By Serre duality,

$$h^2(X, \mathcal{O}_X(D)) = h^0(X, \mathcal{O}_X(K-D)).$$

If $h^0(X, \mathcal{O}_X(K-D)) > 0$, then $\mathcal{O}_X(K-D)$ has a regular section, so $K-D$ is equivalent to an effective divisor. But then

$$(K-D).H > 0$$

and hence $K.H > K.D$. Let $n_0 = K.H$. //

Cor Let H be an ample divisor on X, and let D be a divisor s.t. $D.H > 0$ and $D.D > 0$. Then for $n \gg 0$, nD is lin. eq. to an effective divisor.

pf By R.R.

$$\chi(\mathcal{O}_X(nD)) = \tfrac{1}{2} nD\,(nD - K) + \chi(\mathcal{O}_X)$$

$$= \tfrac{1}{2} n^2 D.D - \tfrac{1}{2} n D.K + \chi(\mathcal{O}_X) \to \infty$$

as $n \to \infty$ since $D.D > 0$. Since $D.H > 0$, $nD.H > K.H$, if $n \gg 0$. So $h^2(X, \mathcal{O}_X(nD)) = 0$, if $n \gg 0$. I.e.

$$\chi(\mathcal{O}_X(nD)) = h^0(X, \mathcal{O}_X(nD)) - h^1(X, \mathcal{O}_X(nD)).$$

This is positive, if $n \gg 0$, so $h^0(X, \mathcal{O}_X(nD))$ is non-zero, if $n \gg 0$. So $\mathcal{O}_X(nD)$ has a regular section, hence is nD is eq. to an effective divisor. //

Thm (Hodge index formula) Let H be an ample divisor, and let D be a divisor s.t. $D.H = 0$ and s.t. there exists a divisor E with $D.E \neq 0$. Then $D.D < 0$.

Pf Suppose $D.D > 0$. Let $H' = D + nH$. Then for $n \gg 0$, H' is ample. Since $D.H' = D.D > 0$, the cor. shows that for $m \gg 0$, mD is eq. to eff. divisor. So $mD.H > 0$, hence $D.H > 0$ ↯.

If $D.D = 0$, let E be s.t. $D.E \neq 0$. Replacing E by $E' = (H.H)E - (E.H)H$, can assume $E.H = 0$. Let $D' = nD - E$. Then $D'.H = 0$, and

$$D'.D' = 2n D.E + E.E.$$

So can find n s.t. $D'.D' > 0$. Repeat argument above. //

Let $X \to \operatorname{Spec} k$ be a sm. proj. geo. conn. surf; say that $C, D \in \operatorname{Div}(X)$ are <u>numerically equivalent</u> if for all $E \in \operatorname{Div}(X)$, $(C-D) \cdot E = 0$. Define

$$\operatorname{Num}(X) = \operatorname{Div}(X) / \text{num. eq.}$$

It can be shown that this is a f.g. free abelian group. Can restate Hodge Index Thm:

<u>Thm</u> Let $h, \ell \in \operatorname{Num}(X)$. Suppose h is ample, $h \cdot \ell = 0$ and $\ell \neq 0$. Then $\ell \cdot \ell < 0$. //

Let C_1, C_2 be sm., proj., geo. conn. curves $/ k$; let $X = C_1 \times C_2$. If $p_i : \operatorname{Spec} k \to C_i$ are pts, let $\ell_1 = cl(C_1 \times p_2)$; $\ell_2 = cl(p_1 \times C_2)$.

<u>Lemma</u> (i) $h = \ell_1 + \ell_2$ is ample.

(ii) $\ell_i \in \operatorname{Num}(X)$ is independent of p_i.

(iii) $\ell_1 \cdot \ell_2 = 1$, $\ell_i \cdot \ell_i = 0$, $h \cdot h = 2$, $h \cdot \ell_i = 1$.

<u>Pf</u> (i) don't know.

(ii) let $p_2, p_2' : \operatorname{Spec} k \to C_2$ be two pts; let $D_1 = C_1 \times p_2$, $D_1' = C_1 \times p_2'$; let $\ell_1 = cl(D_1)$, $\ell_1' = cl(D_1')$. Let

$$\ell = \ell_1 - \ell_1' = cl(D_1 - D_1')$$

Then $\ell \cdot \ell_1 = \ell \cdot \ell_2 = 0$, so $\ell \cdot h = 0$. And

$$\ell \cdot \ell = \ell_1^2 - 2\ell_1 \cdot \ell_1' + \ell_1'^2 = 0,$$

since $D_1 \cdot D_1' = 0$. Since h ample, the Hodge

Index Thm. shows that $\ell = 0 \in \mathrm{Num}(X)$.

(iii) This is clear. //

<u>Lemma</u> For all $\ell \in \mathrm{Num}(X)$, $\ell.\ell \leq 2(\ell.\ell_1)(\ell.\ell_2)$.

Pf Let $\ell' = \ell - (\ell.\ell_2)\ell_1 - (\ell.\ell_1)\ell_2$. Then $h.\ell' = 0$.
If $\ell' = 0$, the statement holds with equality.
If $\ell' \neq 0$, the Hodge index thm. shows $\ell'.\ell' < 0$, i.e.

$$(\ell - (\ell.\ell_2)\ell_1 - (\ell.\ell_1)\ell_2).(\ell - (\ell.\ell_2)\ell_1 - (\ell.\ell_1)\ell_2) < 0$$

$$\begin{aligned} \ell.\ell &- (\ell.\ell_2)(\ell.\ell_1) - (\ell.\ell_1)(\ell.\ell_2) \\ &- (\ell.\ell_2)(\ell.\ell_1) + (\ell.\ell_2)(\ell.\ell_1) \\ &- (\ell.\ell_1)(\ell.\ell_2) + (\ell.\ell_1)(\ell.\ell_2) \quad < 0 \end{aligned}$$

Hence $\ell.\ell < 2(\ell.\ell_1)(\ell.\ell_2)$. //

⌜If Y is a separated S-scheme and $X \xrightarrow{f} Y$ a morphism of S-schemes, then the graph morphism $(id, f)_S : X \to X \times_S Y$ is a closed immersion; the graph Γ_f is the corresponding closed subscheme of $X \times_S Y$.⌟

Let $\Gamma \hookrightarrow C_1 \times C_2$ be the graph of a morph. $f : C_1 \to C_2$ of degree d. Then

$$\Gamma.\ell_1 = d, \quad \Gamma.\ell_2 = 1.$$

<u>Lemma</u> (Adjunction formula) Let $C \subset X$ be sm. curve of genus g, and let K be a can. divisor on X. Then

$$2g-2 = C.(C+K).$$

<u>pf</u> We proved this last time:

$$\omega_C \cong i_C^*(\omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(C))$$

gives

$$\deg(\omega_C) = \deg(i_C^*(\omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(C)))$$

$$2g-2 = C.(C+K). \qquad //$$

With $X = C_1 \times C_2$,

$$\left.\begin{aligned} \omega_X &= \pi_1^* \omega_{C_1} \otimes \pi_{C_2}^* \omega_{C_2} \\ cl(\pi_1^* \omega_{C_1}) &= (2g_1-2)\, \ell_2 \in \mathrm{Num}(X) \\ cl(\pi_2^* \omega_{C_2}) &= (2g_2-2)\, \ell_1 \in \mathrm{Num}(X). \end{aligned}\right\} \rightsquigarrow$$

$$cl(\omega_X) = (2g_2-2)\ell_1 + (2g_1-2)\ell_2 .$$

Apply lemma to $\Gamma \cong C_1$:

$$\begin{aligned} \cancel{2g_1-2} &= \Gamma.(\Gamma+K) = \Gamma.\Gamma + \Gamma.K \\ &= \Gamma.\Gamma + (2g_2-2)\Gamma.\ell_1 + (2g_1-2)\Gamma.\ell_2 \\ &= \Gamma.\Gamma + (2g_2-2)d + \cancel{(2g_1-2)} \qquad \Longrightarrow \end{aligned}$$

$$\Gamma.\Gamma = (2-2g_2)d$$

If $c_1 = c_2 = c$, get

$$\Delta . \Delta = 2 - 2g.$$

What about $\Gamma . \Delta$?

Prop If C is a sm., proj., geo. curve of genus g and Γ the graph of a morphism of degree d, then

$$| \Gamma . \Delta - (1+q) | \leq 2g\sqrt{d} .$$

pf Let $\ell = cl(r\Gamma + s\Delta)$, $r, s \in \mathbb{Z}$. By lemma:

$$\ell . \ell \leq 2(\ell . \ell_1)(\ell . \ell_2) .$$

Calculate:

$$\ell . \ell = r^2 \Gamma . \Gamma + 2rs\, \Gamma . \Delta + s^2 \Delta . \Delta$$

$$= r^2 (\cancel{2} - 2g) d + 2rs\, \Gamma . \Delta + s^2 (\cancel{2} - 2g) \leq$$

$$2(\ell . \ell_1)(\ell . \ell_2) = 2(r+s)(rd+s)$$

$$= \cancel{2r^2d} + 2rs + 2drs + \cancel{2s^2}$$

or

$$gdr^2 + (d+1 - \Gamma . \Delta) rs + g s^2 \geq 0 \quad , \forall r, s .$$

Hence $(d+1 - \Gamma . \Delta)^2 - 4g^2 d \leq 0$. //

$$F_X = q\text{'th frob.}$$

$$\begin{tikzcd}
X \arrow[dr, "F_{X/k}"] \arrow[drr, bend left, "F_X"] \arrow[ddr, bend right, "f"'] & & \\
& X^{(q)} \arrow[r, "w"] \arrow[d] \arrow[dr, phantom, "\text{cart.}"] & X \arrow[d, "f"] \\
& \operatorname{Spec} k \arrow[r, "F"] & \operatorname{Spec} k
\end{tikzcd}$$

$$\begin{tikzcd}
\bar{X} \arrow[dr, "F_{\bar X/\bar{\mathbb{F}}_q}"] \arrow[drrr, bend left, "F_{\bar X}"] \arrow[ddddr, bend right] & & & & \\
& \bar{X} \arrow[rr, "w"] \arrow[dr] \arrow[ddd] & & \bar{X} \arrow[dr] \arrow[ddd] & \\
& & X \arrow[rr, equal] \arrow[ddd] & & X \arrow[ddd] \\
& & & & \\
& \operatorname{Spec} \bar{\mathbb{F}}_q \arrow[rr, "F"] \arrow[dr] & & \operatorname{Spec} \bar{\mathbb{F}}_q \arrow[dr] & \\
& & \operatorname{Spec} \bar{\mathbb{F}}_q \arrow[rr, equal] & & \operatorname{Spec} \bar{\mathbb{F}}_q
\end{tikzcd}$$

The relative Frob. $F_{\bar X/\bar{\mathbb{F}}_q}$ is a morphism of $\bar{\mathbb{F}}_q$-schemes of deg. q.

Thm Let X be a sm. proj. geometrically conn. curve over $\mathbb{F}_q$. Then

$$\left| \# X(\mathbb{F}_{q^n}) - (1+q^n) \right| \leq 2g \cdot q^{n/2} .$$

Pf As a set

$$X(\mathbb{F}_{q^n}) = \Gamma_{F^n_{X/\mathbb{F}_q}} \cap \Delta \subset \bar{X}(\bar{\mathbb{F}}_q) \times \bar{X}(\bar{\mathbb{F}}_q) ,$$

Can show (not hard) that $\Gamma_{F_{X/\bar{\mathbb{F}}_q}}$ and Δ intersect transversely. Hence

$$\# X(\mathbb{F}_{q^n}) = \Gamma_{F^n_{\bar{X}/\bar{\mathbb{F}}_q}} \cdot \Delta . \qquad //$$

For $X \to \operatorname{Spec} \mathbb{F}_q$ sm., proj., geo. conn. of dim. N, define zeta function

$$Z(X/\mathbb{F}_q ; T) = \exp\left(\sum_{n=1}^{\infty} \# X(\mathbb{F}_{q^n}) \cdot \frac{T^n}{n} \right) .$$

Then

$$\# X(\mathbb{F}_{q^n}) = \frac{1}{(n-1)!} \frac{d^n}{dT^n} \log Z(X/\mathbb{F}_q ; T) .$$

ex $X = \mathbb{P}^N$. Then

$$\# X(\mathbb{F}_{q^n}) = \frac{q^{n(N+1)} - 1}{q^n - 1}$$

$$\log Z(\mathbb{P}^N/\mathbb{F}_q; T) = \sum_{n=1}^{\infty} \left(\sum_{i=0}^{N} q^{ni} \right) \frac{T^n}{n}$$

$$= \sum_{i=0}^{N} -\log(1 - q^i T)$$

so

$$Z(\mathbb{P}^N/\mathbb{F}_q; T) = \frac{1}{(1-T)(1-qT) \cdots (1-q^N T)} \in \mathbb{Q}(T).$$

The Weil conjectures state:

(i) (Rationality) $Z(X/\mathbb{F}_q; T) \in \mathbb{Q}(T)$.

(ii) (Functional equation) If $\varepsilon = \chi(X)$,

$$Z(X/\mathbb{F}_q; 1/q^N T) = \pm q^{N\varepsilon/2} T^{\varepsilon} Z(X/\mathbb{F}_q; T)$$

(iii) (Riemann Hypothesis)

$$Z(X/\mathbb{F}_q; T) = \frac{P_1(T) \cdots P_{2N-1}(T)}{P_0(T) \cdots P_{2N}(T)}$$

with $P_i(T) \in \mathbb{Z}[T]$, $P_0(T) = 1 - T$, $P_{2N} = 1 - q^N T$, and for $0 < i < 2N$, $P_i(T)$ factors over $\mathbb{C}$

$$P_i(T) = \prod_j (1 - \alpha_{ij} T)$$

with $|\alpha_{ij}| = q^{i/2}$.

Cor: If H is ample on X then $(H|H) > 0$.

Rmk: ① An inv. sheaf on a curve is ample iff $\deg > 0$
② Surface: L ample iff $L \cdot C > 0$ $\forall C \in \mathrm{Div}^+ - \{0\}$ and $L \cdot L > 0$.

Key #2: Suppose H is ample on X. If $(L|L) > 0$ and $(L|H) > 0$ then $h^0(L^{\otimes n}) \neq 0$ for $n \gg 0$.

$\chi(L^{\otimes n}) = \frac{n^2}{2}(L|L) - \frac{n}{2}(L|\omega) + \chi(\mathcal{O}_X)$, so $\chi(L^{\otimes n}) > 0$ for $n \gg 0$ if $(L|L)$:
$(\omega \otimes L^{-n}|H) = (\omega|H) - n(L|H)$ which is negative for $n \gg 0$

Key #1.5: If $(L|\omega) > (L|H)$ then $h^2(L) = 0$.
Pf: $h^2(L) = h^0(\omega \otimes L^{-1})$ and $(\omega \otimes L^{-1}|H) = (\omega|H) - (L|H) < 0$.

Hodge Index Thm: Let h be the class of an ample inv. sheaf ($h \in \mathrm{Num}$). Then $(\ )$ is negative definite on $h^\perp$.

Recall: If L is inv. and H is ample then $L \otimes H^n$ is g.b.g.s for $n \gg 0$. H^m is very ample for some m. $\Rightarrow L \otimes H^{n+m}$ is very ample.

Pf (Thm)

Suppose $L \in \ell \in h^\perp$ and $L^2 > 0$. Consider $L \otimes H^n =: H'$ which is very ample for $n \gg 0$ (see above "recall")
$(L|H') = (L|L) + n(H|L) = (L|L) > 0$ since $L \in \ell \in h^\perp$
$\Rightarrow \exists D \in |L^{\otimes m}|$ for $m \gg 0$ but $(D|H) > 0$ by key #1
$\Rightarrow (L|H) > 0 \Rightarrow \Leftarrow$.
Now say $\ell^2 = 0$ and $\ell \cdot h = 0$.
Claim: $\ell = 0$ in Num.
If not $\exists E \in \mathrm{Pic}$ s.t. $(L|E) > 0$.
Let $E' = (H|H)E - (E|H)H \in h^\perp$
$\Rightarrow (L|E') = (H|H)(L|E) > 0$
Let $\ell' = k\ell + e'$ (for $k \gg 0$). $\ell' \in h^\perp$ but $(\ell')^2 = k^2\overset{0}{\ell^2} + 2k\ell e' + (e')^2 > 0$ for $k \gg 0$

4/27 $X \hookrightarrow \mathbb{P}^3$ smooth degree m

$H := \mathcal{O}_X(1)$ is very ample

$$\begin{tikzcd} X \arrow[r, hook, "\deg m"] & \mathbb{P}^3 \\ C \arrow[u, hook] \arrow[r, hook, "\deg m"] & \mathbb{P}^2 \arrow[u, hook] \end{tikzcd}$$

$C = X \cap \mathbb{P}^2$

$\deg H|_C = m \implies C\cdot C = m$, $H\cdot H = m$

$\omega_{\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3}(-4)$, $\omega_X = \omega_{\mathbb{P}^3}|_X \otimes \left(I_X/I_X^2\right)^\vee = \omega_{\mathbb{P}^3}|_X(m) = \mathcal{O}_X(m-4)$

$H\cdot\omega_X = m\cdot(m-4)$, $\omega_X\cdot\omega_X = \left((m-4)H \mid (m-4)H\right) = (m-4)^2\, H\cdot H = m(m-4)^2$

If $m=1$: $\omega\cdot\omega = 9$, $X = \mathbb{P}^2$, $\omega_X = \mathcal{O}(-3)$

$m=2$: $X \cong \mathbb{P}^1\times\mathbb{P}^1$, $\omega\cdot\omega = 8$

$m=4$: $\omega \cong \mathcal{O}_X$. K3 surface

$m>4$: ω is ample "general type"

Let $X = C_1 \times C_2$. Take $p_i \in C_i$ and let $\pi_i : X \to C_i$ and $L_i = \pi_i^*\mathcal{O}_{C_i}(p_i)$

Let $D_i = \pi_i^{-1}(p_i)$. Then $D_1 = p_1 \times C_2$ and $D_2 = p_2 \times C_1$

$\implies L_1\cdot L_2 = 1$

Write ℓ_i for the class of L_i in Num X.

$(L_i \mid L_i) = \deg L_i|_{D_i}$

$$\begin{tikzcd} X \arrow[r] & C_i \\ D_i \arrow[u, hook] \arrow[r, "\pi_i"] & p_i \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

$\Rightarrow L_i|_{D_i} = \pi_i^*\left(\mathcal{O}_{C_i}(p_i)|_{p_i}\right)$

$\implies L_i|_{D_i} \cong \mathcal{O}_{D_i} \Rightarrow \deg L_i = 0$. Thus

<u>Lemma</u>: $\ell_1\cdot\ell_2 = 1$, $\ell_1\cdot\ell_1 = \ell_2\cdot\ell_2 = 0$, $h = \ell_1+\ell_2$ is ample, $h\cdot h = 2$, and $h\cdot\ell_i = 1$

<u>Rmk</u>: ℓ_i is independent of the choice of p_i:

Let $\ell = $ class $\mathcal{O}_X(D_1 - D_1')$, $D_1\cdot D_1' = 0$, $\ell_1'\cdot\ell_2 = 1$

$\ell\cdot\ell_1 = \ell\cdot\ell_2 = 0 \Rightarrow h\cdot\ell = 0$. What is $\ell\cdot\ell$?

It is $\ell_1^2 - 2\ell_1\cdot\ell_1' + \ell_1'^2 = 0+0+0 = 0 \implies \ell_1 = 0$ by Hodge Index Th

Since ℓ_1 and ℓ_2 are indep. in Num, Num has rank ≥ 2.

$\omega_X \cong \pi_1^* \omega_{C_1} \otimes \pi_2^* \omega_{C_2}$ $\qquad \pi_1^*\omega_{C_1} \sim (2g_1-2)\ell_1$, ...

$\Rightarrow \omega_X$ -mod Num $= (2g_1-2)\ell_1 + (2g_2-2)\ell_2$

$\Rightarrow \omega_X \cdot \omega_X = 8(g_1-1)(g_2-1)$ and $\omega \cdot h = 2g_1 - 2 + 2g_2 - 2$

<u>Lemma:</u> Suppose $\ell \in \text{Num}(X)$. Let $a_i = \ell \cdot \ell_i$. Then $\ell \cdot \ell \leq 2a_1a_2$ with equality iff $\ell = a_2\ell_1 + a_1\ell_2$

<u>Pf:</u>

If is clear.

Only if: Let $\ell' = \ell - a_2\ell_1 - a_1\ell_2$

$\Rightarrow \ell' \cdot h = a_1 + a_2 - a_2 - a_1 = 0$

By Hodge, $(\ell')^2 \leq 0$ and $=0$ iff $\ell' = 0$

So $(\ell')^2 = \ell^2 - 2a_2\ell\ell_1 - 2a_1\ell \cdot \ell_2 + 2a_1a_2\ell_1\ell_2 = \ell^2 - 2a_1a_2$ ✓

<u>Eg:</u> ① $\Gamma : C_1 \longrightarrow C_2$ is the graph of a morphism of deg d.

$\Rightarrow \Gamma \cdot \ell_1 = 1$ and $\Gamma \cdot \ell_2 = d$

Now $\Gamma \cong C_1$ so by Adjunction formula, $\deg(\omega_\Gamma) = \Gamma^2 + \omega_X \cdot \Gamma$

$\Rightarrow 2g_1 - 2 = \Gamma^2 + [(2g_1-2)\ell_1 + (2g_2-2)\ell_2] \cdot \Gamma = \Gamma^2 + (2g_1-2) + (2g_2-2) \cdot d$

$\Rightarrow \Gamma \cdot \Gamma = (2-2g_2)d$

Checking above lemma, $2(1-g_2)d \leq 2d \Rightarrow$ If $g_2 > 0$ then Γ is independent of ℓ_1 and ℓ_2.

② $C_1 = C_2 = C$

$\Delta^2 = 2 - 2g \leq 2$ (diagonal) with $<$ if $g > 0$.

③ What about $\Delta \cdot \Gamma$?

<u>Prop:</u> If C is a curve of genus g and Δ is graph of id and Γ is the graph of a morphism of deg d, then $|\Gamma \cdot \Delta - (d+1)| \leq 2g\sqrt{d}$.

<u>Pf:</u>

Let $D = r\Gamma + s\Delta \in \text{Num}$.

$D^2 = r^2\Gamma^2 + 2rs\,\Gamma \cdot \Delta + s^2\Delta^2 = r^2(2-2g)d + 2rs\,\Gamma \cdot \Delta + s^2(2-2g)$

Now $D^2 \leq 2(D \cdot \ell_1)(D \cdot \ell_2) = 2(r+s)(dr+s) = 2r^2d + 2rsd + 2rs + 2s^2$

So $(\cancel{2}-2g)r^2d + 2rs\,\Gamma \cdot \Delta + s^2(\cancel{2}-2g) \leq \cancel{2r^2d} + 2rsd + 2rs + \cancel{2s^2}$

$\Rightarrow gdr^2 + rs(d - \Gamma\cdot\Delta + 1) + gs^2 \geq 0$. Let $a = d - \Gamma\cdot\Delta + 1$

$\Rightarrow gdr^2 + ars + gs^2 \geq 0 \quad \forall r, s.$

$\Rightarrow a^2 - 4g^2 d \leq 0 \Rightarrow |a| \leq 2g\sqrt{d}.$

4/29 Thm: Let $X/\mathbb{F}_q$ be a smooth, proper, g. connected curve. Then

$$Z(X/\mathbb{F}_q, T) = \frac{\prod_{i=1}^{2g}(1-\alpha_i T)}{(1-T)(1-qT)} \quad \text{and} \quad |\alpha_i| = \sqrt{q} \quad \forall i$$

$$a_n = 1 + q^n - \sum \alpha_i^n \quad \text{and} \quad |a_n - (1+q^n)| \leq 2g\sqrt{q^n}$$

Recall: Let $\bar{X} = X \times_{\mathbb{F}_q} \bar{\mathbb{F}}$. We get Frobenius $F_q : \bar{X} \longrightarrow \bar{X}$ of deg q.

$a_n(X/\mathbb{F}_q) = \#\{x \in \bar{X} : F_q^n x = x\}$

Claim: Γ = graph of F_q^n meets Δ transversely. Thus $\Gamma\cdot\Delta = a_n(X/\mathbb{F}_q)$

Pf:

WLOG, assume $n=1$. Want to show that if $x \in X(\mathbb{F}_q)$ then I_Γ and I_Δ generate $m_{(x,x)}$ at (x,x).

We want to show $I_\Gamma + I_\Delta \longrightarrow m$ is surj. By Nakayama's lemma it suffices to replace m by m/m^2.

Now $m/m^2 \cong \Omega_{X\times X/\mathbb{F}}(x,x) = \Omega_X(x) \oplus \Omega_X(x)$

$\Rightarrow \ldots \Rightarrow$ The graph of any $f: X \to X$ in $\Omega \oplus \Omega$ is just the graph of df in $\Omega' \oplus \Omega'$.

It suffices to prove I_Γ and I_Δ are different at (x,x) is v.s. is 2-dim.

Use: $dF_q : \bar{X} \longrightarrow \bar{X}$ is just the base change of $dF_q : X \longrightarrow X$

$(F_X)^e : X \longrightarrow X$ is id on topo, q^{th} power on fcns.

$(dF_X)(df) = d(F_X^* f) = df^q = qf^{q-1}df = 0$

$\Rightarrow$ They can't be equal.

By above stuff, this proves the Riemann Hypothesis!!

Eg: $\tilde{X}$ = compactification of $y^2 = x^3 - x$. Let $X = \mathbb{P}^1$

$\tilde{X} \longrightarrow X$ by $(x,y) \longmapsto x$

$$Z(\tilde{X};T) = \frac{(1-\alpha T)(1-\bar{\alpha}T)}{(1-T)(1-qT)}, \quad |\alpha| = \sqrt{q}$$

Cor: Let $A_n = \sum_{x \in \mathbb{P}^1(\mathbb{F}_{q^n})} \left(\frac{x}{\mathbb{F}_{q^n}}\right)$. Then $\overline{\lim_{n\to\infty}} \frac{A_n}{1+q^n} = 0$

Let $a_n(+1) = \#\{x \in \mathbb{P}^1(\mathbb{F}_{q^n}) : y \text{ is a square}\}$, $a_n(-1) = \dots$, $a_n(0) = 4$.

$\Rightarrow A_n = a_n(+1) - a_n(-1)$ and $1+q^n = a_n(+1) + a_n(-1) + 4$.

Thm: $\lim_{n\to\infty} \frac{a_n(\pm 1)}{a_n(\mathbb{P}^1/\mathbb{F}_q)} = 1/2$

Pf:

$$a_n(+1) = \frac{1+q^n}{2} - \frac{A_n}{2} - 2$$

$$\Rightarrow \frac{a_n(+1)}{a_n} = \frac{1}{2} - \frac{A_n}{2(1+q^n)} - \frac{2}{1+q^n} \longrightarrow 1/2.$$

5/2 X/Y is a finite extension of smooth, g. connected, complete curves over $\mathbb{F}_q$.

Assume that $K(X)/K(Y)$ is Galois with group G.

Let $V \subseteq Y$ be an affine open set. Then $U = \pi^{-1}(V)$ is G-stable.

Write $U = \operatorname{Spec} A$, $V = \operatorname{Spec} B$. and $B = A^G$

$$\begin{array}{ccc} A & \hookrightarrow & K(X) \\ | & & | \\ B & \hookrightarrow & K(Y) \end{array}$$

Prop: G acts transitively on the fibers of $\pi: \bar{X} \longrightarrow \bar{Y}$

Pf:

$\bar{B} = \bar{A}^G$. Suppose m and m' are maximal ideals of A which are in distinct orbits. Claim: $m \cap \bar{B} \neq m' \cap \bar{B}$

Let $\{m_1, \dots, m_n\}$ be the G-orbit of m. By Ch. Rem. Th, $\exists a \in A$ s.t. $a \notin m_i \ \forall i$ and $a \in m'$.

Let $b = N_G(a)$. $b \in B$ and $b \in m'$ but $b \notin m$ since distinct orbits.