18.727: Problem Set 1

Due: No, but you should do it anyway. Really!

1. (The following is proved in Mumford, but try to do it yourself.) Let X be the prime spectrum of a ring. Show that the closure $\{x\}^-$ of the one-point set $\{x\} \subset X$ is an irreducible closed subset, i.e that it cannot be written as the union of two proper closed subsets. Show that x is a generic point of $\{x\}^-$, i.e. that the only closed subset of $\{x\}^-$ which contains x is the whole set. Show that every irreducible closed subset of X is of the form $\{x\}^-$ and that x is its unique generic point. Conclude that the assignment $x \mapsto \{x\}^-$ defines a one-to-one correspondance between the points of X and the irreducible closed subsets of X.

2. Let X be a space, let $x \in X$ be a point, and let E be a set. We define the *skyscraber sheaf* $i_{x*}E$ on X by assigning to $U \subset X$ the set E, if $x \in U$, and a one-point set, if $x \notin U$. Show that $i_{x*}E$ is indeed a sheaf. Then show that the functor $E \mapsto i_{x*}E$ is right adjoint to the functor $F \mapsto F_x$, which to a sheaf of sets F on X assigns the stalk in the point x. (This means that there is natural one-to-one correspondance between maps of sets $F_x \to E$ and maps of sheaves $F \to i_{x*}E$.) Finally, show that the stalk of $i_{x*}E$ at $x' \in X$ is E, if $x' \in \{x\}^-$, and a one-point set, otherwise.

3. Show that $\operatorname{Spec} R$ is connected if and only if R does not contain non-trivial idempotents. (A non-trivial idempotent is an element $e \neq 1$ such that $e^2 = e$.)

4. Let $(X, \mathcal{O}_X)$ be a scheme, let $f \in \Gamma(X, \mathcal{O}_X)$, and define X_f to be the set of points $x \in X$ such that $f(x) \neq 0 \in k(x)$.

(a) If U is an affine open of X, and if $f|_U$ is the image of f in $\Gamma(U, \mathcal{O}_X)$, show that $U \cap X_f = D(f|_U)$. Conclude that $X_f \subset X$ is open.

(b) Assume that X is quasi-compact. Suppose that the restriction of $a \in \Gamma(X, \mathcal{O}_X)$ to $\Gamma(X_f, \mathcal{O}_X)$ is zero. Show that for some $n \geq 0$, $f^n a = 0$.

(c) Assume in addition that X is quasi-separated, i.e. that the intersection of two affine open subsets $U, V \subset X$ is quasi-compact. Let $b \in \Gamma(X_f, \mathcal{O}_X)$. Show that for some $n \geq 0$, $f^n b$ is the restriction of an element of $\Gamma(X, \mathcal{O}_X)$.

(c) Conclude that if X is quasi-compact and quasi-separated, then the restriction induces an isomorphism

$$\Gamma(X, \mathcal{O}_X)_f \xrightarrow{\sim} \Gamma(X_f, \mathcal{O}_X).$$