**18.727: Problem Set 3**

*Due*: 3/7/01

**1.** Show that open immersions (resp. closed immersions, resp. morphisms of finite type) are preserved under base change.

**2.** Let $f\colon X \to Y$ be a morphism of schemes. Show that the following conditions are equivalent:

(i) there exists an affine open cover $\{V_i\}_{i\in I}$ of $Y$ such that for all $i \in I$, $f^{-1}(V_i) \subset X$ is quasi-compact.

(ii) for every affine open $V \subset Y$, $f^{-1}(V) \subset X$ is quasi-compact.

Such a morphism is called *quasi-compact*.

**3.** Let $S = \bigoplus_{d\geq 0} S_d$ be a graded ring and let $S_+$ be the homogeneous ideal of elements of positive degree. Let $f \in S_+$ be a homogeneous element of degree $d$. The ring of fractions $S_f = S[\frac{1}{f}]$ is naturally $\mathbb{Z}$-graded, and we let $S_{(f)}$ be the subring of elements of degree zero, i.e.

$$S_{(f)} = \{\frac{a}{f^n} \in S_f \mid a \in S_{nd}\}.$$

Let $X_f = \operatorname{Spec} S_{(f)}$.

(i) Show that the morphism

$$\varphi_{fg}\colon X_{fg} \to X_f, \qquad \frac{ag^n}{(fg)^n} \leftarrow \frac{a}{f^n},$$

is an open immersion. (*Hint*: The image is a distringuished open.)

(ii) Conclude that the schemes $X_f$ glue to give a scheme $X$.

(iii) Let $U_f \subset X$ denote the image of $X_f$ in $X$. Show that the $U_f$ form a basis for the topology on $X$.

(iv) Show that $\{U_{f_i}\}_{i\in I}$ cover $X$ if and only if for all $x \in S_+$, some power of $x$ is contained in the ideal generated by the elements $f_i$, $i \in I$. (*Hint*: Note that $\{U_{f_i}\}_{i\in I}$ cover $X$ if and only if each $U_f$ is covered by $\{U_f \cap U_{f_i}\}_{i\in I}$.)

(v) Descibe the scheme $X$ if $S = \mathbb{Z}[x_0, \ldots, x_n]$, where each $x_i$ has degree one.