18.727: Problem Set 5

Due: 3/21/01

1. Let X be a scheme and let $x, x' \in X$. If $x' \in \{x\}^-$, we say that x' is a *specialization* of x. Let $f\colon X \to Y$ be a quasi-compact morphism. Show that the following are equivalent:

(i) The morphism f is closed.

(ii) For every $x \in X$ and every specialization y' of $y = f(x)$, there exists a specialization x' of x such that $f(x') = y'$.

2. Let $f\colon X \to S$ be a morphism, let $x \in X$, let $y = f(x) \in Y$, and let $y' \neq y$ be a specialization of y. Show that there exists a commuatative diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{g'} & X \\ \big\downarrow & & \big\downarrow f \\ \operatorname{Spec} V & \xrightarrow{g} & S, \end{array}$$

where V is a valuation ring with quotient field K, such that if s (resp η) denotes the closed point of $\operatorname{Spec} V$ (resp. the unique point of $\operatorname{Spec} K$), then $g(s) = y'$ and $g'(\eta) = x$.

3. Suppose that the diagonal morphism $\Delta_{X/S}\colon X \to X \times_S X$ is quasi-compact. Show that if in every diagram of the form

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \operatorname{Spec} V & \longrightarrow & S, \end{array}$$

where V is a valuation ring with quotient field K, at most one lifting exists, then f is separated.

4. Let $f\colon X \to S$ be a quasi-compact and separated morphism, and suppose that in every diagram of the form

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \operatorname{Spec} V & \longrightarrow & S, \end{array}$$

where V is a valuation ring with quotient field K, a (necessarily unique) lift exists. Show that f is universally closed. (*Hint*: First show that f is closed. Then show that the hypothesis is satisfied for every base change of the morphism f.)

(*over*)

5. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a map of schemes. Let M (resp. N) be an $\mathcal{O}_X$-module (resp. $\mathcal{O}_Y$-module). Show that there is an isomorphism

$$f_*M \otimes N \xrightarrow{\sim} f_*(M \otimes f^*N).$$

(*Hint*: Use universal properties to construct the map. Consider stalks to show that it is an isomorphism.)