18.727: Problem Set 6

Due: 4/11/01

1. In an abelian category which both has enough projectives and injectives, show that there is a canonical isomorphism

$$(R^i \operatorname{Hom}(M,-))(N) \cong (R^i \operatorname{Hom}(-,N))(M).$$

(*Hint*: let $M \leftarrow P.$ (resp. $N \to I^{\cdot}$) be a projective (resp. injective) resolution and consider the spectral sequences associated with the double complex $\operatorname{Hom}(P., I^{\cdot})$.)

2. Show that if an additive functor has an exact left adjoint then it preserves injectives.

3. Let $C^{\cdot}$ be a (cochain) complex with a descending filtration

$$C^{\cdot} = \operatorname{Fil}^0 C^{\cdot} \supset \operatorname{Fil}^1 C^{\cdot} \supset \cdots \supset \operatorname{Fil}^p C^{\cdot} \supset \dots$$

and assume that $H^s(\operatorname{Fil}^p C^{\cdot})$ vanishes, if $s < p$. For every $p \geq 0$, we have the long-exact homology sequence

$$\cdots \to H^s(\operatorname{Fil}^{p+1} C^{\cdot}) \xrightarrow{i} H^s(\operatorname{Fil}^p C^{\cdot}) \xrightarrow{j} H^s(\operatorname{gr}^s C^{\cdot}) \xrightarrow{k} H^{s+1}(\operatorname{Fil}^{p+1} C^{\cdot}) \to \dots$$

We let

$$E_r^{p,q} = \frac{k^{-1}(\operatorname{im}(H^{p+q+1}(\operatorname{Fil}^{p+r} C^{\cdot}) \xrightarrow{i^{r-1}} H^{p+q}(\operatorname{Fil}^{p+1} C^{\cdot})))}{j(\ker(H^{p+q}(\operatorname{Fil}^p C^{\cdot}) \xrightarrow{i^{r-1}} H^{p+q}(\operatorname{Fil}^{p-(r-1)} C^{\cdot})))}.$$

and define the rth differential

$$d_r \colon E_r^{p,q} \to E_r^{p+r,q-(r-1)}$$

as follows: Given $x \in E_r^{p,q}$, we first choose $\tilde{x} \in H^{p+q}(\operatorname{gr}^p C^{\cdot})$ which represents x. Then $k(\tilde{x}) \in H^{p+q+1}(\operatorname{Fil}^{p+1} C^{\cdot})$ only depends on x, and by the definition of $E_r^{p,q}$, we can choose $\tilde{y} \in H^{p+q+1}(\operatorname{Fil}^{p+r} C^{\cdot})$ such that $i^{r-1}(\tilde{y}) = k(\tilde{x})$. Then, by definition, $d_r x \in E_r^{p+r,q-(r-1)}$ is the class represented by $j(\tilde{y}) \in H^{p+q+1}(\operatorname{gr}^{p+r} C^{\cdot})$.

(i) Show that $E_{r+1}^{p,q}$ is isomorphic to the cohomology of

$$E_r^{p-r,q+(r-1)} \xrightarrow{d_r} E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-(r-1)}.$$

(ii) Show that $E_\infty^{p,q} \cong \operatorname{gr}^p H^{p+q}(C^{\cdot})$.

4. In an abelian category which has enough injectives, let $P.$ be a chain complex (i.e. maps go $P_0 \leftarrow P_1 \leftarrow \dots$) of projective objects. If M is any object then $\operatorname{Hom}(P., M)$ is a cochain complex. Show that there is a spectral sequence

$$E_2^{s,t} = \operatorname{Ext}^s(H_t(P.), M) \Rightarrow H^{s+t}(\operatorname{Hom}(P., M)).$$

This is called the universal coefficient spectral sequence.