18.727: Problem Set 8

Due: 4/25/01

1. Let (X, A) be a ringed space, and let $\mathfrak{U} = \{U_i \to X\}_{i \in I}$ be a family of open subsets. We define $A_{\mathfrak{U}}$ to be the presheaf of A-modules given by

$$A_{\mathfrak{U}}(V) = \begin{cases} A(V), & \text{if } V \subset U_i, \text{ some } i \in I, \\ 0, & \text{else.} \end{cases}$$

(If the family $\mathfrak{U}$ has only one element $U \to X$, we write A_U instead of $A_{\mathfrak{U}}$.) We choose a well-ordering of the index set I and consider the complex

$$\bigoplus_{i_0} A_{U_{i_0}} \leftarrow \bigoplus_{i_0 < i_1} A_{U_{i_0} \cap U_{i_1}} \leftarrow \bigoplus_{i_0 < i_1 < i_2} A_{U_{i_0} \cap U_{i_1} \cap U_{i_2}} \leftarrow \cdots .$$

(i) Show that this is a resolution of the presheaf of A-modules $A_{\mathfrak{U}}$ by projective presheaves of A-modules.

(*Hint*: A sequence of presheaves of A-modules, by definition, is exact if and only if for all $V \subset X$ open, the sequence of $A(V)$-modules obtained by taking sections over V is exact.)

(ii) Conclude that if $\mathfrak{U}$ is a covering, the Čech cohomology $\check{H}^*(\mathfrak{U}, M)$ is canonically isomorphic to the cohomology of the complex

$$\prod_{i_0} M(U_{i_0}) \to \prod_{i_0 < i_1} M(U_{i_0} \cap U_{i_1}) \to \prod_{i_0 < i_1 < i_2} M(U_{i_0} \cap U_{i_1} \cap U_{i_2}) \to \cdots .$$

This is called the complex of *alternating* Čech cochains of M with respect to the covering $\mathfrak{U}$.

(*Warning:* The above is not true, in general, for a ringed topos.)

2. Prove that every non-empty quasi-compact scheme has a closed point.

3. Let X be a quasi-compact scheme and suppose that for all quasi-coherent $\mathcal{O}_X$-modules M and for all $q > 0$, $H^q(X, M)$ is zero.

(i) Let $x \in X$ is a closed point, let $x \in U \subset X$ be an affine open neighborhood, and let $Y = X \setminus U$ be the complement considered as a closed subscheme of X with the reduced induced scheme structure. Show that there exists $f \in \Gamma(X, \mathcal{J}_Y)$ such that $f(x) = 1 \in k(x)$.

(*Hint*: Use the exact sequence $0 \to \mathcal{J}_{Y \cup \{x\}} \to \mathcal{J}_Y \to i_{x*}k(x) \to 0$.)

(ii) Conclude that for every closed point $x \in X$, there exists $x \in \Gamma(X, \mathcal{O}_X)$ such that $X_f = \{x \in X \mid f(x) \neq 0\}$ is an affine open neighborhood of x.

(iii) Show that X is affine.

(*Hint*: It suffices to find $f_1, \dots, f_n \in \Gamma(X, \mathcal{O}_X)$ such that all X_{f_i} are affine and such that $(f_1, \dots, f_n) = \Gamma(X, \mathcal{O}_X)$.)