18.727: Problem Set 9

Due: 5/9/01

1. Let X be an irreducible projective scheme over a field k. Show that $H^0(X, \mathcal{O}_X)$ is a finite extension field of k.

2. Let A be a noetherian ring, let $X = \operatorname{Spec} A$, and let M and N be finitely generated A-modules.

(i) Show that for all $q \geq 0$, the groups $\operatorname{Ext}^q_{\mathcal{O}_X}(\tilde{M}, \tilde{N})$ and $\operatorname{Ext}^q_A(M, N)$ are naturally isomorphic.

(ii) Show that for all $q \geq 0$, the sheaves $\underline{\operatorname{Ext}}^q_{\mathcal{O}_X}(\tilde{M}, \tilde{N})$ and $\operatorname{Ext}^q_A(M, N)^{\tilde{}}$ are naturally isomorphic.

3. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module M is of *finite type* if there exists a covering $\{U_i \to X\}_{i \in I}$, and for all $i \in I$, a surjection from a finite direct sum of copies of $\mathcal{O}_X|_{U_i}$ onto $M|_U$. It is *coherent* if it is of finite type and if for every open subset $U \subset X$ and every homomorphism $f\colon (\mathcal{O}_X|_U)^n \twoheadrightarrow M|_U$, the kernel of f is of finite type on $(U, \mathcal{O}_X|_U)$. Show:

(i) If M is a coherent $\mathcal{O}_X$-module and $N \subset M$ is a sub-$\mathcal{O}_X$-module of finite type, then N is coherent.

(ii) If two out of three $\mathcal{O}_X$-modules in an exact sequence of $\mathcal{O}_X$-modules

$$0 \to M' \to M \to M'' \to 0$$

are coherent, then so is the third.

(iii) If M and N are coherent $\mathcal{O}_X$-modules, then so are $M \otimes_{\mathcal{O}_X} N$ and $\underline{\operatorname{Hom}}_{\mathcal{O}_X}(M, N)$.