18.917: *Topics in algebraic topology*

The course will give an introduction to higher algebraic K-theory following [1] and [2]. To give an idea of what this is about, suppose first that k is a field. The determinant $\det\colon GL_n(k) \to k^*$ factors through the group homomorphisms

$$GL_n(k) \hookrightarrow GL(k) \twoheadrightarrow GL(k)^{\mathrm{ab}},$$

and the induced map $GL(k)^{\mathrm{ab}} \xrightarrow{\sim} k^*$, it turns out, is an isomorphism. If k is any ring, we still have the group homomorphisms above, and therefore, we can define the "determinant" of $M \in GL_n(k)$ to be the image $[M] \in GL(k)^{\mathrm{ab}}$. The abelian group $K_1(k) = GL(k)^{\mathrm{ab}}$ is the first algebraic K-group or the Whitehead group.

There is no algebraic definition definition of the higher K-groups. Instead, let $BGL(k)$ be the classifying space with fundamental group $GL(k)$ and with all higher homotopy groups trivial. There is a space $K(k)$ which can be thought of as the "abelianization" of $BGL(k)$. The fundamental group of $K(k)$ is the group $K_1(k) = GL(k)^{\mathrm{ab}}$, but the construction also introduces higher homotopy groups. These, by definition, are the higher K-groups, $K_q(k) = \pi_q K(k)$.

Clearly, the groups $K_q(k)$ depend only on the homotopy type of $K(k)$, and it is useful to consider many different homeomorphism types within this homotopy type. One example of a result we will prove is the so-called localization sequence: Let V be discrete valuation ring with field of fractions K and residue field k. Then there is a long-exact sequence of K-groups

$$\cdots \to K_q(k) \to K_q(V) \to K_q(K) \xrightarrow{\partial} K_{q-1}(k) \to \ldots,$$

which for $q = 1$ is isomorphic to the sequence

$$0 \to V^* \to K^* \xrightarrow{v} \mathbb{Z} \to 0.$$

References

[1] D. Quillen, *Higher algebraic K-theory I*, Algebraic K-theory I: Higher K-theories (Battelle Memorial Inst., Seattle, Washington, 1972), Lecture Notes in Mathematics, vol. 341, Springer-Verlag, 1973.

[2] F. Waldhausen, *Algebraic K-theory of spaces*, Algebraic and geometric topology, Lecture Notes in Mathematics, vol. 1126, Springer-Verlag, 1985, pp. 318–419.

$$K_0(R) := \frac{\mathbb{Z}\{P \mid P \text{ f.g. proj. (left) } R\text{-module}\}}{\mathbb{Z}\{P' - P + P'' \mid P' \rightarrowtail P \twoheadrightarrow P'' \text{ exact}\}}$$

$$\left(\xrightarrow{\sim} \frac{\mathbb{Z}\{[P] \mid P \text{ f.g. proj. (left) } R\text{-mod.}\}}{\mathbb{Z}\{[P'] - [P] + [P''] \mid P' \rightarrowtail P \twoheadrightarrow P'' \text{ s.e.s.}\}} \right)$$

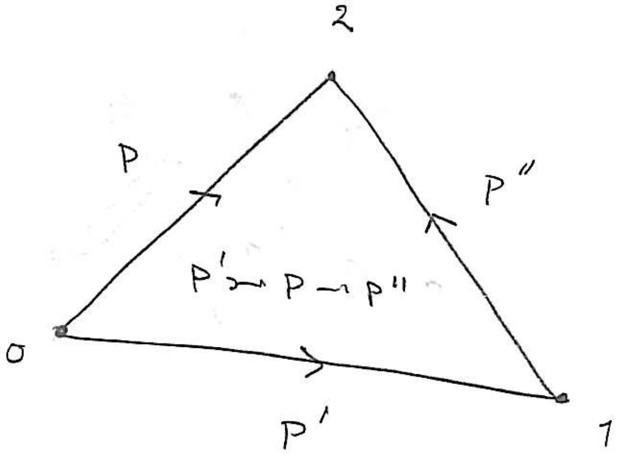
Build 2-dim'l cx X:

vertices : $*$ (chosen null object)

edges : P

faces : $P' \rightarrowtail P \twoheadrightarrow P''$

attach as follows:

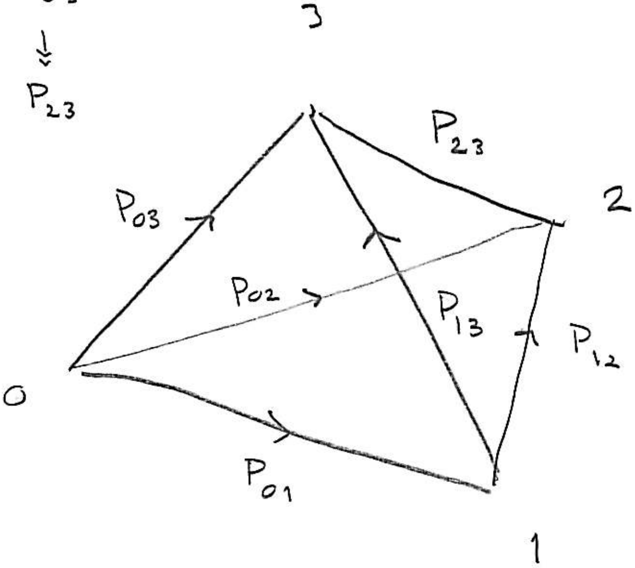


then

$$K_0(R) = H_1(X) \xleftarrow{\sim} \pi_1(X)^{ab}$$

Add k-cells for all k as follows:

$$\begin{tikzcd} P_{02} \arrow[r, tail] & P_{03} \arrow[d, two heads] \\ & P_{23} \end{tikzcd} \qquad \begin{tikzcd} P_{12} \arrow[r, tail] & P_{13} \arrow[d, two heads] \\ & P_{23} \end{tikzcd}$$



$$\begin{tikzcd} P_{01} \arrow[r, tail] & P_{03} \arrow[d, two heads] \\ & P_{13} \end{tikzcd} \qquad \begin{tikzcd} P_{01} \arrow[r, tail] & P_{02} \arrow[d, two heads] \\ & P_{12} \end{tikzcd}$$

index 3-simpl. by

$$\begin{tikzcd} P_{01} \arrow[r, tail] & P_{02} \arrow[r, tail] \arrow[d, two heads] & P_{03} \arrow[d, two heads] \\ & P_{12} \arrow[r, tail] & P_{13} \arrow[d, two heads] \\ & & P_{23} \end{tikzcd} \quad \text{s.t.} \quad \begin{tikzcd} P_{ij} \arrow[r, tail] & P_{ik} \arrow[d, two heads] \\ & P_{jk} \end{tikzcd} \quad \text{s.e.s.}$$

k-simplices indexed by

$$\begin{tikzcd}
P_{01} \arrow[r, tail] & P_{02} \arrow[r, tail] \arrow[d, two heads] & P_{03} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & P_{0n} \arrow[d, two heads] \\
& P_{12} \arrow[r, tail] & P_{13} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & P_{1n} \arrow[d, two heads] \\
& & P_{23} \arrow[r, tail] & \cdots \arrow[r, tail] & P_{2n} \arrow[d, two heads] \\
& & & & \vdots \arrow[d, two heads] \\
& & & & P_{n-1,n}
\end{tikzcd}$$

s.t.
$$\begin{tikzcd}
P_{ij} \arrow[r, tail] & P_{ik} \arrow[d, two heads] \\
& P_{jk}
\end{tikzcd}$$
s.e.s.;

attach as follows: the i'th face of a k-simplex is the (k-1)-simplex obtained by leaving out all P_{jk} with $j = i$ or $k = i$. (Degeneracies.)

Call this space X; then

$$K_i(R) := \pi_{i+1}(X).$$

In more detail:

$\mathcal{C}$ = cat. of f.-g. proj. (left) R-mod.;

$co\,\mathcal{E}$ = subcat. of split monomorphisms (but with same objects);

$*$ = chosen null-object.

$[n] = \{0 \to 1 \to 2 \to \cdots \to n\}$

$Ar[n]$ = cat. of arrows in $[n]$

define

$$S_n\mathcal{E} \subset \mathcal{E}^{Ar[n]}$$

to be full subcat. of functors

$$Ar[n] \xrightarrow{A} \mathcal{E}$$

s.t.

(i) $A_{ii} = *$

(ii) $A_{ij} \rightarrowtail A_{ik} \in co\,\mathcal{E}$

(iii)

$$\begin{array}{ccc} A_{ij} & \rightarrowtail & A_{ik} \\ \downarrow & \text{p.-o.} & \downarrow \\ A_{jj} & \rightarrowtail & A_{jk} \end{array}$$

$$[m] \xrightarrow{\theta} [n] \rightsquigarrow \mathrm{Ar}[m] \xrightarrow{\theta_*} \mathrm{Ar}[n] \rightsquigarrow$$

$$\begin{array}{ccc} \mathcal{E}^{\mathrm{Ar}[n]} & \xrightarrow{\theta^*} & \mathcal{E}^{\mathrm{Ar}[m]} \\ \cup & & \cup \\ S_n\mathcal{E} & \overset{\theta^*}{\dashrightarrow} & S_m\mathcal{E} \end{array}$$

so $S_\cdot \mathcal{E}$ is a simplicial category.

Write out $A \in \mathrm{ob}\, S_n \mathcal{E}$:

$$\begin{array}{ccccccccc}
* = A_{0,0} & \rightarrowtail & A_{0,1} & \rightarrowtail & A_{0,2} & \rightarrowtail & \cdots & \rightarrowtail & A_{0,n} \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
 & & * = A_{1,1} & \rightarrowtail & A_{1,2} & \rightarrowtail & \cdots & \rightarrowtail & A_{1,n} \\
 & & & & \downarrow & & & & \downarrow \\
 & & & & * = A_{2,2} & \rightarrowtail & \cdots & \rightarrowtail & A_{2,n} \\
 & & & & & & & & \downarrow \\
 & & & & & & & & \vdots \\
 & & & & & & & & \downarrow \\
 & & & & & & & & A_{n-1,n}
\end{array}$$

Recall

$$d_i : S_n\mathcal{E} \longrightarrow S_{n-1}\mathcal{E}, \quad 0 \le i \le n,$$

$$d_i = d^{i*}; \quad d^i : [n-1] \longrightarrow [n]$$

injective, misses $i \in [n]$

$$\begin{tikzcd}
Ar[n] \arrow[r, "A"] & \mathcal{C} \\
Ar[n-1] \arrow[u, "d^i_*"] \arrow[ur, "d_i A"'] &
\end{tikzcd}$$

so $d_i A$ obtained from A by removing $A_{j,k}$ with one of j,k equal to i; e.g.

$$\begin{tikzcd}
A_{0,1} \arrow[r, tail] & A_{0,2} \arrow[r, tail] \arrow[d] & A_{0,3} \arrow[d] \\
& A_{1,2} \arrow[r, tail] & A_{1,3} \arrow[d] \\
& & A_{2,3}
\end{tikzcd}$$

$$\overset{d_0}{\longmapsto} \quad \begin{tikzcd}
A_{1,2} \arrow[r, tail] & A_{1,3} \arrow[d] \\
& A_{2,3}
\end{tikzcd}$$

$$\overset{d_1}{\longmapsto} \quad \begin{tikzcd}
A_{0,2} \arrow[r, tail] & A_{0,3} \arrow[d] \\
& A_{2,3}
\end{tikzcd}$$

$$\overset{d_2}{\longmapsto} \quad \begin{tikzcd}
A_{0,1} \arrow[r, tail] & A_{0,3} \arrow[d] \\
& A_{1,3}
\end{tikzcd}$$

$$\overset{d_3}{\longmapsto} \quad \begin{tikzcd}
A_{0,1} \arrow[r, tail] \arrow[d] & A_{0,2} \\
A_{1,2} &
\end{tikzcd}$$

$s_i : S_{n-1}\mathcal{C} \to S_n \mathcal{C}$, $0 \le i \le n$

$s_i = s^{i*}$, $s^i : [n] \to [n-1]$

surjective, repeats $i \in [n]$.

<u>Lemma</u> Let $f, g : \mathcal{E} \to \mathcal{D}$ be a pair of exact functors. A natural isomorphism from f to g induces a simplicial homotopy

$$S.\mathcal{E} \times \Delta[1] \longrightarrow S.\mathcal{D}$$

from $S.f$ to $S.g$.

<u>Pf</u> A natural transformation from f to g determines a functor

$$\mathcal{E} \times [1] \xrightarrow{h} \mathcal{D} \ ;$$

this gives a functor

$$\begin{array}{ccc} \mathcal{E}^{Ar[n]} \times Hom_{\Delta}([n],[1]) & \longrightarrow & \mathcal{D}^{Ar[n]} \\ \cup & & \cup \\ S_n\mathcal{E} \times Hom_{\Delta}([n],[1]) & \dashrightarrow & S_n\mathcal{D} \end{array}$$

if h nat. iso

which takes $(Ar[n] \xrightarrow{A} \mathcal{E},\ [n] \xrightarrow{a} [1])$ to

$$Ar[n] \xrightarrow{(A, a_*)} \mathcal{E} \times Ar[1] \xrightarrow{\mathcal{E} \times p} \mathcal{E} \times [1] \longrightarrow \mathcal{D} \ ;$$

here $p(0 \to 0) = 0$, $p(0 \to 1) = p(1 \to 1) = 1$.

e.g. $n = 3$, $a : [3] \to [1]$, $a(0) = a(1) = a(2) = 0$,
$a(3) = 1$

$$
\begin{tikzcd}
f(A_{0,1}) \arrow[r, tail] & f(A_{0,2}) \arrow[r, tail] \arrow[d] & g(A_{0,3}) \arrow[d] \\
& f(A_{1,2}) \arrow[r, tail] & g(A_{1,3}) \arrow[d] \\
& & g(A_{2,3})
\end{tikzcd}
$$

need h to be nat'l iso for this to be an object of $S_3 \mathcal{D}$. //

Cor An exact equivalence $\mathcal{E} \simeq \mathcal{D}$ induces a homotopy equivalence $S.\mathcal{E} \simeq S.\mathcal{D}$. //

ex Can replace $\mathcal{E}$ by a skeleton cat., i.e. one object for each isomorphism class of objects in $\mathcal{E}$.

$wS.\mathcal{E} \subset S.\mathcal{E}$ subcat. of w-e.

$\Omega|N.wS.\mathcal{E}| = K(\mathcal{E})$

Lemma $|\mathrm{ob}\, S.\mathcal{E}| = |N_0 iS.\mathcal{E}| \xrightarrow{\sim} |N.iS.\mathcal{E}|$

pf $N_k\mathcal{E} = \mathrm{ob}\,\mathcal{E}^{[k]}$; write $\mathbb{N}_k\mathcal{E} = \mathcal{E}^{[k]}$;

let $\mathbb{N}_k^i\mathcal{E} \subset \mathbb{N}_k\mathcal{E}$ be the full subcategory with $\mathrm{ob}\,\mathbb{N}_k^i\mathcal{E} = N_k i\mathcal{E}$.

ob $\mathbb{N}_k^i\mathcal{E}$: $c_0 \xrightarrow{\sim} c_1 \xrightarrow{\sim} \cdots \xrightarrow{\sim} c_k$

mor $\mathbb{N}_k^i\mathcal{E}$:

$$\begin{array}{ccccccc} c_0 & \xrightarrow{\sim} & c_1 & \xrightarrow{\sim} & \cdots & \xrightarrow{\sim} & c_k \\ \downarrow & & \downarrow & & & & \downarrow \\ c_0' & \xrightarrow{\sim} & c_1' & \xrightarrow{\sim} & \cdots & \xrightarrow{\sim} & c_k' \end{array}$$

$\mathbb{N}^i S.\mathcal{E} \cong S.\mathbb{N}^i_{\cdot}\mathcal{E}$

—

$\mathbb{N}_0^i\mathcal{E} \underset{d}{\overset{s}{\rightleftarrows}} \mathbb{N}_k^i\mathcal{E}$ exact eq. of cat.

$c \longmapsto c = c = \cdots = c$

$c_0 \longleftarrow\!\shortmid\; c_0 \xrightarrow{\sim} c_1 \xrightarrow{\sim} \cdots \xrightarrow{\sim} c_k$

so

$$S.N_0^i\mathcal{E} \rightleftarrows S.N_k^i\mathcal{E}$$

simpl. homotopy equivalence. On objects, this is:

$$N_0 i S.\mathcal{E} \overset{\sim}{\rightleftarrows} N_k i S.\mathcal{E}$$

$$|N_0 i S.\mathcal{E}| \xrightarrow{\sim} |N_k i S.\mathcal{E}|$$

Hence

$$|N.iS.\mathcal{E}| \cong |[k] \mapsto |N_k i S.\mathcal{E}||$$

$$\xleftarrow{\sim} |[k] \mapsto |N_0 i S.\mathcal{E}||$$

$$= |N_0 i S.\mathcal{E}|.$$

//

X simplicial set; $|X| = \left(\coprod_{n\geq 0} X_n \times \Delta^n\right) / (\theta^* x, z) \sim (x, \theta_* z)$.

<u>Lemma</u> Let $x \in X_n$. There exists a unique non-deg. $y \in X_m$ and a unique morphism $[n] \xrightarrow{\theta} [m]$ s.t. $x = \theta^* y$.

<u>pf</u> There exists $[n] \xrightarrow{\theta} [m]$ s.t. x is in the image of $X_m \xrightarrow{\theta^*} X_n$ (e.g. $\theta = id$). Let m be minimal with this property and pick $y \in X_m$ with $x = \theta^* y$. Then, necessarily, y is non-deg. and θ surjective. Pick $[m] \xrightarrow{\psi} [n]$ s.t. $\theta\psi = id_{[m]}$. Suppose also $x = \theta'^* y'$, $y' \in X_m$, $[n] \xrightarrow{\theta'} [m]$. The calculation

$$(\theta'\psi)^* y' = \psi^* \theta'^* y' = \psi^* x$$

$$= \psi^* \theta^* y = (\theta\psi)^* y = y$$

shows that $\theta'\psi$ is surjective; or else y would be degenerate. But then $\theta'\psi = id_{[m]}$ and $y' = y$. Finally, we show that $\theta' = \theta$. Given $i \in [n]$, we pick $[m] \xrightarrow{\psi} [n]$ s.t. $\theta\psi = id_{[m]}$ (and hence $\theta'\psi = id_{[m]}$) and $\psi\theta(i) = i$. Then $\theta'(i) = \theta'\psi\theta(i) = \theta(i)$. //

Call $(x,z) \in X_n \times \Delta^n$ non-deg. if $x \in X_n$ is non-deg. and $z \in \mathrm{Int}(\Delta^n)$.

<u>Lemma</u> Every point $\xi \in |X|$ is represented by a unique non-deg. point $(x,z) \in X_n \times \Delta^n$.

<u>pf</u> Let $(a,u) \in X_k \times \Delta^k$ repr. $\xi \in |X|$. There is a unique injective map $[m] \xhookrightarrow{\psi} [k]$ and $v \in \mathrm{Int}(\Delta^m)$ s.t. $u = \psi_* v$. So

$$(a,u) = (a, \psi_* v) \sim (\psi^* a, v) =: \lambda(a,u)$$

By lemma, there is a unique surjective map $[m] \xrightarrow{\theta} [n]$ and $x \in X_n$ non-deg. s.t. $\psi^* a = \theta^* x$. Now

$$(a,u) \sim (\theta^* x, v) \sim (x, \theta_* v) =: \rho(\theta^* x, v)$$

and since θ is surj., $z = \theta_* v \in \mathrm{Int}(\Delta^n)$. So $(x,z) \in X_n \times \Delta^n$ is non-deg. and repr. ξ.

Uniqueness: suppose also $(b,w) \in X_\ell \times \Delta^\ell$ repr. ξ, i.e. $a = f^* b$, $w = f_* u$, some $[k] \xrightarrow{f} [\ell]$.

$$\begin{tikzcd}
{[k]} \arrow[d, "f"] & {[m]} \arrow[l, hook', "\psi"'] \arrow[r, two heads, "\theta"] \arrow[d, two heads, "\sigma"] & {[n]} \arrow[d, equal] \\
{[\ell]} & {[r]} \arrow[l, hook', "\delta"'] \arrow[r, two heads, "\alpha"] & {[s]}
\end{tikzcd}$$

— by lemma 1

Here $f\psi = \delta\sigma$ is the unique factorization as surj. followed by inj., and

$$\lambda(b,w) = (\delta^* b, \sigma_* v).$$

The map α is with $\delta^* b = \alpha^* c$, $c \in \Delta^s$ non-deg., so by def$^{\underline{n}}$

$$\rho\lambda(b,w) = (c, \alpha_* \sigma_* v).$$

By the uniqueness of lemma 1, $\alpha\sigma = \theta$ and $c = x$. So $\alpha_*\sigma_* v = \theta_* v = z$. //

Cor There is a CW-structure on $|X|$ with one n-cell for each non-deg. $x \in X_n$. //

ex $\Delta[n] = \mathrm{Hom}_{/\!\Delta}(-,[n])$; an m-simpl. $[m] \xrightarrow{\alpha} [n]$ is non-degenerate iff α is injective; claim:

$$\Delta^n \xrightarrow{\sim} |\Delta[n]|, \quad z \mapsto (\mathrm{id}_{[n]}, z).$$

The map is surjective, since

$$([m] \xrightarrow{\theta} [n], w) \sim (\mathrm{id}_{[n]}, \theta_* w).$$

Suppose $z, z' \in \Delta^n$ have the same image. There are (unique) injective maps $[m] \xrightarrow{\alpha} [n]$ and $[m'] \xrightarrow{\alpha'} [n]$ and interior pts. $v \in \Delta^m$ and $v' \in \Delta^{m'}$ s.t. $z = \alpha_*(v)$ and $z' = \alpha'_*(v')$;

$$(\alpha, v) \sim (\mathrm{id}_{[n]}, z) = (\mathrm{id}_{[n]}, z') \sim (\alpha', v').$$

By uniqueness of non-deg. repr., $(\alpha, v) = (\alpha', v') \Rightarrow z = z'$. //

$$X_n = \underset{\text{deg.}}{L_n} \amalg \underset{\text{non-deg.}}{Y_n}$$

$$\begin{tikzcd}
Y_n \times \partial\Delta^{n-1} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\text{p-o.}"] & sk_{n-1}|X| \arrow[d, tail] \\
Y_n \times \Delta^n \arrow[r] & sk_n|X|
\end{tikzcd}$$

alternatively:

$$\begin{tikzcd}
X_n \times \partial\Delta^n \underset{L_n \times \partial\Delta^n}{\cup} L_n \times \Delta^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p-o.}"] & sk_{n-1}|X| \arrow[d] \\
X_n \times \Delta^n \arrow[r] & sk_n|X|
\end{tikzcd}$$

and L_n is the colimit of

$$\begin{tikzcd}
X_n & & X_n & & X_n & & X_n & & X_n \\
& X_{n-1} \arrow[ul, "s_0"] \arrow[ur, "s_0"'] & & X_{n-1} \arrow[ul, "s_1"] \arrow[ur, "s_1"'] & \cdots \arrow[ul, "s_2"] \arrow[ur, "s_{n-2}"'] & & & X_{n-1} \arrow[ul, "s_{n-1}"] \arrow[ur] &
\end{tikzcd}$$

<u>Lemma</u> The canonical map is a homeomorphism:

$$|\Delta[r]\times\Delta[s]| \xrightarrow{\cong} |\Delta[r]|\times|\Delta[s]| .$$

pf Call this map p; construct inverse q. Let $\xi\in|\Delta[r]|\times|\Delta[s]|$ have non-deg. repr. $((a,u),(b,v))$, $[n]\overset{a}{\rightarrowtail}[r]$, $u\in \mathrm{Int}(\Delta^n)$, $[m]\overset{b}{\rightarrowtail}[s]$, $v\in\mathrm{Int}(\Delta^m)$. We need the following topological input (exercise):

Claim: given $\eta\in\Delta^r\times\Delta^s$, there exists a unique (injective) morphism $[\ell]\xrightarrow{\gamma}[r]\times[s]$, and a unique $x\in\mathrm{Int}(\Delta^\ell)$ s.t. $\eta=\gamma_* x$.

Let $\eta=(u,v)$ and define $q(\xi)\in|\Delta[r]\times\Delta[s]|$ to be point with non-deg. repr.

$$((a,b)\circ\gamma\,,\,x).$$

Check this is inverse of $p=(pr_1,pr_2)$:

$$\begin{aligned} pr_1\circ q(\xi) &= pr_1((a,b)\gamma, x)\\ &= (a\circ pr_1\circ\gamma, x) = (a, pr_{1*}\gamma_* x)\\ &= (a, pr_{1*}(u,v)) = (a,u); \end{aligned}$$

let $\eta \in |\Delta[r] \times \Delta[s]|$ have non-deg. repr.
$([\ell] \overset{(f,g)}{\rightarrowtail} [r]\times[s]\,,\ x)$; factor

$$f : [\ell] \overset{\psi_1}{\twoheadrightarrow} [m] \overset{a}{\rightarrowtail} [r]$$

$$g : [\ell] \overset{\psi_2}{\twoheadrightarrow} [n] \overset{b}{\rightarrowtail} [s] \ ;$$

then

$$p(\eta) = \big(([\ell] \xrightarrow{f} [r], x), ([\ell] \xrightarrow{g} [s], x)\big)$$

$$\sim \big(([m] \overset{a}{\rightarrowtail} [r], \psi_{1*}x), ([n] \overset{b}{\rightarrowtail} [s], \psi_{2*}x)\big).$$

But $x \in \mathrm{Int}(\Delta^\ell)$ and $\psi_* x = (\psi_{1*}x, \psi_{2*}x)$, so

$$q(p(\eta)) = ((a,b)\circ\psi, x) = ((f,g), x). \quad //$$

Thm The canonical map is a homeomorphism

$$|X \times Y| \xrightarrow{\cong} |X| \times |Y|$$

provided that the product on the left is given the compactly generated topology.

Pf Write $X \times Y$ as coequalizer:

$$\coprod_{\substack{[r'] \to [r] \\ [s'] \to [s]}} X_r \times Y_s \times \Delta[r'] \times \Delta[s'] \ \rightrightarrows$$

$$\coprod_{[r],[s]} X_r \times Y_s \times \Delta[r] \times \Delta[s] \longrightarrow X \times Y$$

~~Realization commutes with colimits, since it has a right adjoint, so enough to show~~

$$|\Delta[r] \times \Delta[s]| \xrightarrow{\cong} |\Delta[r]| \times |\Delta[s]| . \qquad //$$

Thm Let X be a bisimplicial set. There are canonical and natural homeomorphisms

$$|[k] \mapsto |[l] \mapsto X_{k,l}|| \cong |[n] \mapsto X_{n,n}|$$

$$\cong |[l] \mapsto |[k] \mapsto X_{k,l}|| .$$

pf We have

$$|[k] \mapsto |[l] \mapsto \Delta[r]_k \times \Delta[s]_l||$$

$$\xleftarrow{\cong} |[k] \mapsto \Delta[r]_k \times \Delta^s|$$

$$\xrightarrow{\cong} |[k] \mapsto \Delta[r]_k| \times \Delta^s$$

$$\xrightarrow{\cong} |[k] \mapsto \Delta[r]_k| \times |[l] \mapsto \Delta[s]_l|$$

$$\xleftarrow{\cong} |[n] \mapsto \Delta[r]_n \times \Delta[s]_n|$$

so OK, if $X = \Delta[r] \times \Delta[s]$. In general, write X as coequalizer

$$\coprod_{\substack{[r]\to[r'] \\ [s]\to[s']}} X_{r,s}\times\Delta[r']\times\Delta[s'] \rightrightarrows$$

$$\coprod_{[r],[s]} X_{r,s}\times\Delta[r]\times\Delta[s] \longrightarrow X. \qquad //$$

<u>Lemma</u> (Gluing lemma) Let $X\xrightarrow{f}Y$ be a map of bisimplicial sets and suppose that for all $k\geq 0$, the induced map

$$|[\ell]\mapsto X_{k,\ell}| \xrightarrow{\sim} |[\ell]\mapsto Y_{k,\ell}|$$

is a weak equivalence. Then f induces a weak equivalence

$$|X| \xrightarrow{\sim} |Y|.$$

pf $|X| = |[k]\mapsto|[\ell]\mapsto X_{k,\ell}||$; the map

$$|[\ell]\mapsto X_{k-1,\ell}| \xrightarrow{s_i} |[\ell]\mapsto X_{k,\ell}|$$

is the inclusion of a sub-CW-complex; it follows that the map of latching spaces in the k-direction induced by

f is a w.e. and that the inclusion of the latching space, fco, is the incl. of a sub-CW-cx. //

2/12/02

$\mathcal{E}$ cat. $\rightsquigarrow$ $N.\mathcal{E}$ simpl. set, $N_k\mathcal{E} = \mathcal{E}^{[k]}$ $\rightsquigarrow$

$|\mathcal{E}| := |N.\mathcal{E}|$ space; e.g. $|[n]| = \Delta^n$.

$\mathcal{E} \xrightarrow{f} \mathcal{E}'$ functor $\rightsquigarrow$ $|\mathcal{E}| \xrightarrow{|f|} |\mathcal{E}'|$ map.

$$\begin{tikzcd}
\mathcal{E} \arrow[d, hook, "i_0"] \arrow[dr, bend left, "f"] & \\
\mathcal{E}\times[1] \arrow[r, "h"] & \mathcal{E}' \\
\mathcal{E} \arrow[u, hook, "i_1"'] \arrow[ur, bend right, "g"'] &
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
{|\mathcal{E}|} \arrow[d, hook, "i_0"] \arrow[r, equal] & {|\mathcal{E}|} \arrow[d, hook, "i_0"] \arrow[dr, bend left, "|f|"] & \\
{|\mathcal{E}|\times[0,1]} & {|\mathcal{E}\times[1]|} \arrow[l, "\sim"'] \arrow[r, "|h|"] & {|\mathcal{E}'|} \\
{|\mathcal{E}|} \arrow[u, hook, "i_1"'] \arrow[r, equal] & {|\mathcal{E}|} \arrow[u, hook, "i_1"'] \arrow[ur, bend right, "g"'] &
\end{tikzcd}$$

nat'l transf. $\rightsquigarrow$ homotopy.

ex suppose $\mathcal{E}$ has initial object x_0:

functors: $[0] \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{E}$; $f(0) = x_0$, $g(x) = 0$

nat'l tr:
$$\begin{array}{ccc} f\circ g(x) & \xrightarrow{\exists!} & x \\ \| & & \downarrow \\ f\circ g(x') & \longrightarrow & x' \end{array}$$
unique morphism

so

$$\text{pt.} = |[0]| \underset{|g|}{\overset{|f|}{\rightleftarrows}} |\mathcal{E}|$$

$+$ htpy $|f|\circ|g| = |f\circ g| \simeq id_{|\mathcal{E}|}$

i.e.

$|\mathcal{E}| \simeq *$. //

$X \xrightarrow{f} Y$ cts. map; htpy fiber over $y \in Y$:

$$\begin{tikzcd} F(f,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & X \arrow[d, "f"] \\ P(Y,y) \arrow[r] & Y \end{tikzcd};$$

If $x \in f^{-1}(y)$, long-exact seq.

$$\cdots \to \pi_q(F(f,y),\bar{x}) \to \pi_q(X,x) \xrightarrow{f} \pi_q(Y,y) \to \cdots$$

so

$$F(f,y) \simeq *, \ \forall y \in Y \Rightarrow f \text{ w.e.}$$

$\mathcal{E} \xrightarrow{f} \mathcal{E}'$ functor; f/Y cat. over Y:

ob: $(X, f(X) \xrightarrow{u} Y)$

mor: $\begin{matrix} X \\ \downarrow v \\ X' \end{matrix}$ s.t. $\begin{tikzcd} f(X) \arrow[d, "f(v)"'] \arrow[dr, "u"] & \\ f(X') \arrow[r, "u'"'] & Y \end{tikzcd}$ commutes.

<u>Thm</u> (Quillen) If $|f/Y| \simeq *$, for all ob. Y of $\mathcal{E}'$, then $|f|$ is a w.e.

$$\begin{tikzcd} \Delta^{op} \times \Delta^{op} \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_2"'] \arrow[rr, bend left, "XL"] \arrow[rr, bend right, "XR"'] & \Delta^{op} \arrow[r, "X"] & \text{Sets} \end{tikzcd} \qquad \begin{matrix} |XL| \\ \shortparallel \\ |X| \\ \shortparallel \\ |XR| \end{matrix}$$

$T(f)$ bisimpl. set; elem. of $T(f)_{p,q}$:

$$(X_q \to \cdots \to X_0,\ f(X_0) \to Y_0 \to \cdots \to Y_p)$$

$$(N.\,\mathcal{C}^{op})R \xleftarrow{p_1} T(f) \xrightarrow{p_2} (N.\,\mathcal{C}')L$$

$$(X_0 \leftarrow \cdots \leftarrow X_q) \longleftarrow\!\shortmid (X_q \to \cdots \to X_0,\ f(X_0) \to Y_0 \to \cdots \to Y_p)$$

$$(X_q \to \cdots \to X_0,\ f(X_0) \to Y_0 \to \cdots \to Y_p) \longmapsto (Y_0 \to \cdots \to Y_p).$$

Consider p_1: realize first in p-direction

$$\begin{array}{ccc} |[p] \mapsto T(f)_{p,q}| & \xrightarrow{p_1} & N_q(\mathcal{C}^{op}) \\ \| & & \| \\ \coprod\limits_{X_q \to \cdots \to X_0} |f(X_0)\backslash \mathcal{C}'| & \xrightarrow{\sim} & \coprod\limits_{X_q \to \cdots \to X_0} pt \end{array} \quad \overset{\text{gluing lemma}}{\Longrightarrow}$$

$$|T(f)| \underset{\sim}{\xrightarrow{p_1}} |\mathcal{C}^{op}|.$$

Consider p_2: realize first in q-direction

$$\begin{array}{ccc} |[q] \mapsto T(f)_{p,q}| & \xrightarrow{p_2} & N_p\,\mathcal{C}' \\ \| & & \| \\ \coprod\limits_{Y_0 \to \cdots \to Y_p} |(f/Y_0)^{op}| & \xrightarrow{\sim} & \coprod\limits_{Y_0 \to \cdots \to Y_p} pt. \end{array} \quad \overset{\text{gluing lemma}}{\Longrightarrow}$$

by assumption; so

$$|T(f)| \xrightarrow[\sim]{p_2} |\mathcal{E}'| .$$

Finally, consider

$$\begin{tikzcd}
|\mathcal{E}^{op}| \arrow[d, "f"] & |T(f)| \arrow[l, "p_1"', "\sim"] \arrow[r, "p_2", "\sim"'] \arrow[d] & |\mathcal{E}'| \arrow[d, equal] \\
|\mathcal{E}'^{op}| & |T(id_{\mathcal{E}'})| \arrow[l, "p_1"'] \arrow[r, "p_2", "\sim"'] & |\mathcal{E}'|
\end{tikzcd}$$

//

A category with cof. and w.e. is a pointed category $(\mathcal{E}, *)$ with two subcategories $co\,\mathcal{E}$ and $w\,\mathcal{E}$ s.t.:

cof 1: isomorphisms are in $co\,\mathcal{E}$.

cof 2: $* \to A$ is in $co\,\mathcal{E}$, $\forall A \in ob\,\mathcal{E}$.

cof 3: If $A \rightarrowtail B$ is in $co\,\mathcal{E}$ and if $A \to C$ is any morphism, then the p.o. $B \cup_A C$ exists and $C \rightarrowtail B \cup_A C$ is in $co\,\mathcal{E}$.

weq 1: isomorphisms are in $w\,\mathcal{E}$.

weq 2: given

$$\begin{tikzcd} B \arrow[d, "\sim"] & A \arrow[l, tail] \arrow[r] \arrow[d, "\sim"] & C \arrow[d, "\sim"] \\ B' & A' \arrow[l, tail] \arrow[r] & C' \end{tikzcd}$$

then $B \cup_A C \xrightarrow{\sim} B' \cup_{A'} C'$.

—

$S_n\mathcal{E} \subset \mathcal{E}^{Ar[n]}$ full subcategory of A s.t.

(i) $A_{i,i} = *$

(ii) $A_{i,j} \rightarrowtail A_{i,k}$

(iii)
$$\begin{tikzcd} A_{i,j} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "p.o."] & A_{i,k} \arrow[d] \\ *=A_{j,j} \arrow[r, tail] & A_{j,k} \end{tikzcd}$$

Prop If $\mathcal{E}$ is a category w. cof. and w.e. then so is $S_n\mathcal{E}$, if we define:

$$A \rightarrowtail A' \underset{\text{def}}{\iff} A_{i,j}' \cup_{A_{i,j}} A_{i,k} \rightarrowtail A_{i,k}', \quad 0 \le i \le j \le k \le n$$

$$A \xrightarrow{\sim} A' \underset{\text{def}}{\iff} A_{i,j} \xrightarrow{\sim} A_{i,j}', \quad 0 \le i \le j \le n \quad //$$

So can iterate $S.$ – construction. Define the K-theory spectrum of $\mathcal{E}$ by:

$$K(\mathcal{E})_n = |N.wS.^{(n)}\mathcal{E}|$$

with structure maps

$$|N.wS.^{(n)}\mathcal{E}| \wedge S^1 \xrightarrow{\sigma_n} |N.wS.^{(n+1)}\mathcal{E}|$$

given by the inclusion of the 1-skeleton in the $S.$ – direction ... explain $n=0$:

$$|N.wS.\mathcal{E}| = |[q] \mapsto |N.wS_q\mathcal{E}||$$

$$= (* \times \Delta^0 \amalg |N.w\mathcal{E}| \times \Delta^1 \amalg \cdots)/\sim$$

$$\xleftarrow{\sigma_0} |N.w\mathcal{E}| \wedge \Delta^1/\partial\Delta^1 = |N.w\mathcal{E}| \wedge S^1$$

Rem $$|N.w\mathcal{E}| = \coprod_{[X]\in\pi_0|\mathcal{E}|} B\,\mathrm{End}_{w\mathcal{E}}(X),$$

so the maps

$$|N.w\mathcal{E}| \wedge S^n \xrightarrow{\sigma_{n-1}\circ\cdots\circ\sigma_0} K(\mathcal{E})_n$$

give rise to maps of spectra

$$\Sigma^\infty B\,\mathrm{End}_{w\mathcal{E}}(X)_+ \longrightarrow K(\mathcal{E}).$$

If $\mathcal{E} = \mathcal{P}_R$, then $\mathrm{End}_{i\mathcal{E}}(P) = \mathrm{Aut}(P)$,

so have

$$\Sigma^\infty B\mathrm{Aut}(P)_+ \longrightarrow K(R). \qquad //$$

$\mathbb{N}_k\mathcal{E} = \mathcal{E}^{[k]}$ functor cat.; $N_k\mathcal{E} = \mathrm{ob}\,\mathbb{N}_k\mathcal{E}$.

$\mathbb{N}^w_k\mathcal{E} \subset \mathbb{N}_k\mathcal{E}$ <u>full</u> subcat. $\mathrm{ob}\,\mathbb{N}^w_k\mathcal{E} = N_k w\mathcal{E}$.

$\mathrm{ob}\,\mathbb{N}^w_k\mathcal{E}$: $\quad c_0 \xrightarrow{\sim} c_1 \xrightarrow{\sim} \cdots \xrightarrow{\sim} c_k$

$\mathrm{mor}\,\mathbb{N}^w_k\mathcal{E}$:

$$\begin{array}{ccccccc} c_0 & \xrightarrow{\sim} & c_1 & \xrightarrow{\sim} & \cdots & \xrightarrow{\sim} & c_k \\ \downarrow & & \downarrow & & & & \downarrow \\ c_0' & \xrightarrow{\sim} & c_1' & \xrightarrow{\sim} & \cdots & \xrightarrow{\sim} & c_k \end{array},$$

Note: $\mathbb{N}^w_p S_q \mathcal{E} \cong S_q \mathbb{N}^w_p \mathcal{E}$

Prop The degeneracy maps induce a w.e.

$$|\mathrm{ob}\, S.\mathcal{E}| \xrightarrow{\sim} |\mathbb{N}.iS.\mathcal{E}| .$$

pf Write this as

$$\begin{array}{ccc}
|\mathrm{ob}\,\mathbb{N}^i_0 S.\mathcal{E}| & \longrightarrow & |\mathrm{ob}\,\mathbb{N}^i_. S.\mathcal{E}| \\
\uparrow \cong & & \uparrow \cong \\
|\mathrm{ob}\, S.\mathbb{N}^i_0 \mathcal{E}| & \longrightarrow & |\mathrm{ob}\, S.\mathbb{N}^i_. \mathcal{E}| \\
\shortparallel & & \shortparallel \\
|[k] \mapsto |\mathrm{ob}\, S.\mathbb{N}^i_0 \mathcal{E}|| & \rightarrow & |[k] \mapsto |\mathrm{ob}\, S.\mathbb{N}^i_k \mathcal{E}||
\end{array}$$

so enough to show $\mathbb{N}^i_0 \mathcal{E} \to \mathbb{N}^i_k \mathcal{E}$ is a natural equivalence of categories. The inverse takes $c_0 \xrightarrow{\sim} \cdots \xrightarrow{\sim} c_k$ to c_0; the non-trivial nat'l iso is given by:

$$\begin{array}{ccccccc}
c_0 & \xrightarrow{id} & c_0 & \xrightarrow{id} & \cdots & \xrightarrow{id} & c_0 \\
\downarrow id & & \downarrow \sim & & & & \downarrow \sim \\
c_0 & \xrightarrow{\sim} & c_1 & \xrightarrow{\sim} & \cdots & \xrightarrow{\sim} & c_k
\end{array}$$

//

We wish show:

<u>Thm</u> (Additivity thm.) The following equivalent statements are true:

(1) $|N.wS.E(\mathcal{A},\mathcal{C},\mathcal{B})| \xrightarrow{\sim} |N.wS.\mathcal{A}| \times |N.wS.\mathcal{B}|$

$$A \rightarrowtail C \twoheadrightarrow B \quad \longmapsto \quad (A\ ,\ B)$$

(2) $|N.wS.S_2\mathcal{E}| \underset{\sim}{\xrightarrow{(d_2,d_0)}} |N.wS.\mathcal{E}| \times |N.wS.\mathcal{E}|$

(3) $|N.wS.S_2\mathcal{E}| \xrightarrow{d_1 \simeq d_2 \vee d_0} |N.wS.\mathcal{E}|$

(4) if $F' \rightarrowtail F \twoheadrightarrow F''$ is an exact seq. of functors from $\mathcal{E}$ to $\mathcal{E}'$, then

$$|N.wS.S_2\mathcal{E}| \xrightarrow{F \simeq F' \vee F''} |N.wS.\mathcal{E}'| .$$

Show the statements are equivalent:

(1) ⟹ (2) trivial.

(2) ⟹ (3)
$$\begin{tikzcd}
{|N.wS.\mathcal{E}| \times |N.wS.\mathcal{E}|} \arrow[rr, "\sim", "{(A,B) \,\mapsto\, A \rightarrowtail A\vee B \twoheadrightarrow B}"'] \arrow[dr, "{(A,B) \,\mapsto\, A\vee B}"'] & & {|N.wS.S_2\mathcal{E}|} \arrow[dl, shift right, "d_1"'] \arrow[dl, shift left, "d_2 \vee d_0"] \\
& {|N.wS.\mathcal{E}|} &
\end{tikzcd}$$

$(3) \Rightarrow (4)$ $\qquad \mathcal{E} \xrightarrow{G} S_2\mathcal{E}'$, $\quad c \mapsto F'(c) \rightarrowtail F(c) \twoheadrightarrow F''(c)$

$$\begin{tikzcd}
{|N.wS.\mathcal{E}|} \arrow[r, "G"] \arrow[rr, bend left, "F"] \arrow[rr, bend right, "F' \vee F''"'] & {|N.wS.S_2\mathcal{E}'|} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0 \vee d_2"'] & {|N.wS.\mathcal{E}|}
\end{tikzcd}$$

$(4) \Rightarrow (1)$ $\qquad |N.wS.E(\mathcal{A},\mathcal{E},\mathcal{B})| \underset{s}{\overset{r}{\rightleftarrows}} |N.wS.\mathcal{A}| \times |N.wS.\mathcal{B}|$

$$(A \rightarrowtail C \twoheadrightarrow B) \longmapsto (A, B)$$

$$(A \rightarrowtail A \vee B \twoheadrightarrow B) \longleftarrow\!\shortmid (A, B)$$

$rs = \mathrm{id}$; $sr \simeq \mathrm{id}$: exact seq. of endofunctors

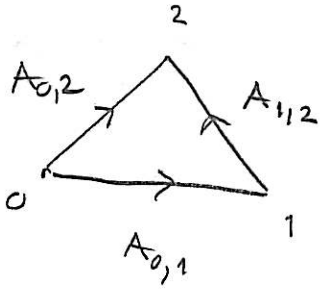
$$\begin{tikzcd}
(A \arrow[r, tail, "id"] & A \arrow[r, two heads] \arrow[d] & *) \\
(A \arrow[r, tail] & C \arrow[r, two heads] \arrow[d] & B) \\
(* \arrow[r, tail] & B \arrow[r, two heads] & B)
\end{tikzcd}$$

so (4) shows id. is homotopic to

$$(A \rightarrowtail C \twoheadrightarrow B) \mapsto (A \rightarrowtail A \twoheadrightarrow *) \vee (* \rightarrowtail B \twoheadrightarrow B)$$

$$=: (A \rightarrowtail A \vee B \twoheadrightarrow B) \, . \qquad //$$

$$|N.wS_2\mathcal{E}| \times \Delta^2 \longrightarrow |N.wS.\mathcal{E}|$$



$$d_1 \simeq d_0 + d_2$$

$\uparrow$

loop sum

i.e.

$$|N.wS_2\mathcal{E}| \underset{d_0+d_2}{\overset{d_1}{\rightrightarrows}} \Omega|N.wS.\mathcal{E}|$$

homotopic; we wanted

$$|N.wS_2\mathcal{E}| \underset{d_0 \vee d_2}{\overset{d_1}{\rightrightarrows}} \Omega|N.wS.\mathcal{E}|$$

homotopic. But $d_0+d_2 \simeq d_0 \vee d_2$ by standard trick, see [Spanier; chap. 1, sec. 6, thm. 8].

So additivity theorem holds in

$$\Omega^\infty|N.wS.^{(\infty)}\mathcal{E}| := \varinjlim_n \Omega^n|N.wS.^{(n)}\mathcal{E}|,$$

i.e. add. thm. follows if

$$\Omega|N.wS.\mathcal{E}| \longrightarrow \Omega^\infty|N.wS.^{(\infty)}\mathcal{E}|$$

is a w.e. Show opposite is true, too.

$$\begin{tikzcd} F(f,g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & X \arrow[d] \\ P(Y,g) \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} F(f,g) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Z \arrow[r, "g"] & Y \end{tikzcd}$$

$$F(f,g) = X \underset{f \searrow Y \swarrow ev_0}{\times} Y^{[0,1]} \underset{ev_1 \searrow Y \swarrow g}{\times} Z$$

$$\begin{tikzcd} W \arrow[dr, dashed] \arrow[drr, bend left, "a"] \arrow[ddr, bend right, "b"'] & & \\ & F(f,g) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & Z \arrow[r, "g"] & Y \end{tikzcd}$$

$W \to Y^{[0,1]} \rightsquigarrow$

$W \times [0,1] \to Y$

$W \to F(f,g) \quad \longleftrightarrow \quad W \xrightarrow{a} X$, $W \xrightarrow{b} Z$, and htpy $W \times [0,1] \to Y$ from fa to gb.

$$\begin{tikzcd} W \arrow[r, "a"] \arrow[d] \arrow[dr, phantom, "G"] & X \arrow[d, "f"] \\ Z \arrow[r, "g"] & Y \end{tikzcd} \quad \underset{\text{def}}{\Longleftrightarrow} \quad W \xrightarrow{\sim} F(f,g)$$

(use const. htpy.)

homotopy cart.

<u>exercise</u> Show

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & X \end{tikzcd}$$

not h.-cart., but

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & PX \arrow[d] \\ * \arrow[r] & X \end{tikzcd}$$

is.

Show that a h.cart. square gives a Mayer-Vietoris l.e.s. on homotopy.

<u>Lemma</u> Let $X \to Y \to Z$ be a sequence of bi-simpl. sets s.t. $X \to Z$ is constant equal to z. Suppose that for all $n \geq 0$,

$$\begin{tikzcd} {|[m] \mapsto X_{m,n}|} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{h. Cart.}"] & {|[m] \mapsto Y_{m,n}|} \arrow[d] \\ pt \arrow[r, "z"] & {|[m] \mapsto Z_{m,n}|} \end{tikzcd}$$

and $|[m] \mapsto Z_{m,n}|$ is connected. Then

$$\begin{tikzcd} {|X|} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{h. Cart.}"] & {|Y|} \arrow[d] \\ pt \arrow[r, "z"] & {|Z|} \end{tikzcd}$$

.

//

$\Delta \xrightarrow{\sigma} \Delta$ shift functor; $\sigma([n]) = [n+1]$,

$\sigma(\theta)(0) = 0$, $\sigma(\theta)(i+1) = \sigma(\theta)(i) + 1$.

$\Delta^{op} \xrightarrow{X}$ Sets simpl. set; define path space:

$$PX : \Delta^{op} \xrightarrow{\sigma} \Delta^{op} \xrightarrow{X} \text{Sets}.$$

The morphisms in Δ,

$$[n+1] \longleftarrow [0] \longleftarrow [n+1] \quad \rightsquigarrow$$

$$0 \longleftarrow\!\shortmid\ 0$$

$$PX \xrightarrow{e} X_0 \xrightarrow{f} PX$$

Lemma There is a natural homotopy

$$PX \times \Delta[1] \longrightarrow PX$$

from

pf The homotopy is given by the map

$$\begin{array}{ccc} PX_n \times \Delta[1]_n & \longrightarrow & PX_n \\ \| & & \| \\ X_{n+1} \times \mathrm{Hom}_{\Delta}([n],[1]) & \longrightarrow & X_{n+1} \end{array}$$

adjoint to

$$\mathrm{Hom}_{\Delta}([n],[1]) \longrightarrow \mathrm{Map}(X_{n+1}, X_{n+1})$$

$$([n] \xrightarrow{a} [1]) \longmapsto \varphi_a^*$$

$$\varphi_a(0) = 0, \quad \varphi_a(j+1) = \begin{cases} j+1, & a(j) = 1, \\ 0, & a(j) = 0. \end{cases}$$

Must check that for $[m] \xrightarrow{\Theta} [n]$,

$$\begin{tikzcd} PX_n \times \Delta[1]_n \arrow[r] \arrow[d, "\Theta^* \times \Theta^*"] & PX_n \arrow[d, "\Theta^*"] \\ PX_m \times \Delta[1]_m \arrow[r] & PX_m \end{tikzcd}$$

or

$$\begin{tikzcd} X_{n+1} \times \Delta[1]_n \arrow[r] \arrow[d, "\sigma(\Theta)^* \times \Theta^*"] & X_{n+1} \arrow[d, "\sigma(\Theta)^*"] \\ X_{m+1} \times \Delta[1]_m \arrow[r] & X_{m+1} \end{tikzcd}$$

--- //

We nat'l maps of simpl. sets

$$X_1 \longrightarrow PX \longrightarrow X$$

incl. of 0-skeleton $\qquad$ $X_{n+1} \xrightarrow{d_0} X_n$

with composite $X_1 \xrightarrow{d_0} X_0 \hookrightarrow X$.

In particular, we have

$$\begin{tikzcd}
N.w\mathcal{E} = N.wS_1\mathcal{E} \arrow[r] \arrow[d, "d_0"] & P(N.wS.\mathcal{E}) \arrow[d] \\
pt = N.wS_0\mathcal{E} \arrow[r] & N.wS.\mathcal{E}
\end{tikzcd}$$

Prop The following square is homotopy-cart.:

$$\begin{tikzcd}
|N.wS.\mathcal{E}| \arrow[r] \arrow[d] & |P(N.wS.S.\mathcal{E})| \arrow[d] \\
pt. \arrow[r] & |N.wS.S.\mathcal{E}|
\end{tikzcd}$$

and the upper right hand term is canonically contractible.

pf Since $|N.wS.S_n\mathcal{E}|$ is connected, $n \geq 0$, it will suffice to show that for all $n \geq 0$, the back square is htpy-cart:

$$\begin{tikzcd}
|N.wS.\mathcal{E}| \arrow[rr] \arrow[dr, equal] \arrow[dd] & & |N.wS.S_{n+1}\mathcal{E}| \arrow[dr, "\simeq"] \arrow[dd] & \\
& |N.wS.\mathcal{E}| \arrow[rr] \arrow[dd] & & |N.wS.\mathcal{E}| \times |N.wS.S_n\mathcal{E}| \arrow[dd, "pr_2"] \\
pt. \arrow[rr] \arrow[dr, equal] & & |N.wS.S_n\mathcal{E}| \arrow[dr, equal] & \\
& pt. \arrow[rr] & & |N.wS.S_n|
\end{tikzcd}$$

The two maps are:

$$|N.wS.S_{n+1}\mathcal{E}| \underset{g}{\overset{f}{\rightleftarrows}} |N.wS.\mathcal{E}| \times |N.wS.S_n\mathcal{E}|$$

$$(A_0 \rightarrowtail \cdots \rightarrowtail A_n) \longmapsto (A_0, A_1/A_0 \rightarrowtail \cdots \rightarrowtail A_n/A_0)$$

$$(A \vee A_1 \rightarrowtail \cdots \rightarrowtail A \vee A_n) \longleftarrow\!\shortmid (A, A_1 \rightarrowtail \cdots \rightarrowtail A_n)$$

$f \circ g = \mathrm{id}$, $g \circ f \simeq \mathrm{id}$, by add. thm. //

Cor $|N.wS.\mathcal{E}| \xrightarrow{\sim} \Omega|N.wS.S.\mathcal{E}|$. //

2/21/02

$$\begin{tikzcd}
\Delta \times \Delta \arrow[rr, "\amalg"] \arrow[rr, bend left=60, "pr_1"] \arrow[rr, bend right=60, "pr_2"'] & \arrow[u, phantom, "\Uparrow \varepsilon_L"] \arrow[d, phantom, "\Downarrow \varepsilon_R"] & \Delta
\end{tikzcd}$$

$$\begin{tikzcd}
pr_1([m],[n]) \arrow[r, "\varepsilon_L"] \arrow[d, phantom, "\shortparallel"] & {[m] \amalg [n]} \arrow[d, phantom, "\shortparallel"] & pr_2([m],[n]) \arrow[l, "\varepsilon_R"'] \arrow[d, phantom, "\shortparallel"] \\
{[m]} \arrow[r, hook, "\text{first}"'] & {[m+n+1]} & {[n]} \arrow[l, hook', "\text{last}"']
\end{tikzcd}$$

$$\begin{tikzcd}
{[m]} \arrow[r, hook, "\varepsilon_L"] \arrow[d, "\Theta"] & {[m] \amalg [n]} \arrow[d, "\Theta \amalg \eta"] & {[n]} \arrow[l, hook', "\varepsilon_R"'] \arrow[d, "\eta"] \\
{[m']} \arrow[r, hook, "\varepsilon_L"] & {[m'] \amalg [n']} & {[n']} \arrow[l, hook', "\varepsilon_R"']
\end{tikzcd}$$

$$\begin{tikzcd}
{[m]} \arrow[r, hook, shift left, "\varepsilon_L"] \arrow[d, equal] & {[m] \amalg [n]} \arrow[l, shift left, "\iota_L"] \arrow[d, "id \amalg \eta"] \\
{[m]} \arrow[r, hook, shift left, "\varepsilon_L"] & {[m] \amalg [n']} \arrow[l, shift left, "\iota_L"]
\end{tikzcd}$$

$\iota_L([n]) = m \in [m]$

<u>Lemma</u> The composites are naturally simplicially homotopic to the identity:

$$PX_{m,\cdot} \xrightarrow{\varepsilon_L} X_m \xrightarrow{\iota_L} P_{m,\cdot}$$

$$PX_{\cdot,n} \xrightarrow{\varepsilon_R} X_n \xrightarrow{\iota_R} P_{\cdot,n} .$$

<u>pf</u> The second map is induced from

$$[m] \amalg [n] \longrightarrow [m] \amalg [n]$$

which is the identity on $[n]$ and maps $[m]$ to $0 \in [n]$. The htpy to the identity is given by

$$([m] \xrightarrow{a} [1]) \longmapsto (PX_{m,n} \xrightarrow{\varphi_a^*} PX_{m,n})$$

where

$$[m] \amalg [n] \xrightarrow{\varphi_a} [m] \amalg [n]$$

is the identity on $[n]$, and

$$\varphi_a(j) = \begin{cases} j & , \text{ if } a(j) = 0 \\ m+1 & , \text{ if } a(j) = 1 \end{cases}$$

$$(= 0 \in [n])$$

Let $[m] \xrightarrow{a} [1]$, $[m'] \xrightarrow{\theta} [m]$, $a' = a \circ \theta : [m'] \to [1]$.

$$\begin{array}{ccc} [m'] \amalg [n] & \xrightarrow{\varphi_{a'}} & [m'] \amalg [n] \\ \downarrow \theta \amalg id & & \downarrow \theta \amalg id \\ [m] \amalg [n] & \xrightarrow{\varphi_a} & [m] \amalg [n] \end{array} .$$

//

$\mathcal{E} \xrightarrow{f} \mathcal{D}$; bi-simplicial categories:

$$\begin{tikzcd} S_\cdot f \mid \mathcal{D} \arrow[r] \arrow[d, "\varepsilon_L"] \arrow[dr, phantom, "\text{p.b.}"] & PS_\cdot \mathcal{D} \arrow[d, "\varepsilon_L"] \\ (S_\cdot \mathcal{E})L \arrow[r, "f"] & (S_\cdot \mathcal{D})L \end{tikzcd}$$

Ignoring choices of quotients, the (m,n)-simpl. of $S_\cdot f \mid \mathcal{D}$ consists of:

$$C_1 \rightarrowtail \cdots \rightarrowtail C_m$$

$$D_1 \rightarrowtail \cdots \rightarrowtail D_m \rightarrowtail S_0 \rightarrowtail \cdots \rightarrowtail S_n$$

s.t. $f(C_i) = D_i$, $1 \le i \le m$. The map ε_L forgets second row; the section ι_L induces a section — for fixed m — of ε_L; it maps

$$C_1 \rightarrowtail \cdots \rightarrowtail C_m$$

to

$$C_1 \rightarrowtail \cdots \rightarrowtail C_m$$

$$D_1 \rightarrowtail \cdots \rightarrowtail D_m = D_m = \cdots = D_m,$$

where $D_i = f(C_i)$. The htpy of the

lemma induces one from the composite

$$(S.f|\mathcal{D})_{m,\cdot} \xrightarrow{\varepsilon_L} S_m\mathcal{E} \xrightarrow{\iota_L} (S.f|\mathcal{D})_{m,\cdot}$$

to the identity. There are natural maps of pull-back diagrams of simpl. cat.:

$$\begin{tikzcd}
 & (PS.\mathcal{D})_{\cdot,n} \arrow[dr, "\varepsilon_R"] \arrow[dd, "\varepsilon_L"] & & \\
 & & S_n\mathcal{D} \arrow[dr, "\iota_R"] \arrow[dd] & \\
S.\mathcal{E} \arrow[r] \arrow[dr] & S.\mathcal{D} \arrow[dr] & & (PS.\mathcal{D})_{\cdot,n} \arrow[dd, "\varepsilon_L"] \\
 & * \arrow[r] \arrow[dr] & * \arrow[dr] & \\
 & & S.\mathcal{E} \arrow[r] & S.\mathcal{D}
\end{tikzcd}$$

which induces

$$(S.f|\mathcal{D})_{\cdot,n} \xrightarrow{\varepsilon_R} S_n\mathcal{D} \xrightarrow{\iota_R} (PS.f|\mathcal{D})_{\cdot,n} ;$$

the composite takes

$$C_1 \rightarrowtail \cdots \rightarrowtail C_m$$

$$D_1 \rightarrowtail \cdots \rightarrowtail D_m \rightarrowtail S_0 \rightarrowtail \cdots \rightarrowtail S_n$$

to

$$* = \cdots = *$$

$$* = \cdots = * = S_0/S_0 \rightarrowtail \cdots \rightarrowtail S_n/S_0$$

Prop Suppose that for all $n \geq 0$, the composite

$$(S.f|D)_{\cdot,n} \xrightarrow{\varepsilon_R} S_n D \xrightarrow{L_R} (S.f|D)_{\cdot,n}$$

is simplicially homotopic to the identity. Then $|obS.\mathcal{E}| \underset{\sim}{\xrightarrow{|f|}} |obS.D|$ is a w.e.

Pf This follows from the diagram

$$\begin{tikzcd}
|ob(S.\mathcal{E})L| \arrow[d, "f"] & |obS.f|D| \arrow[l, "\varepsilon_L"', "\sim \text{ always}"] \arrow[r, "\varepsilon_R", "\sim \text{ assump.}"'] \arrow[d, "f"] & |ob(S.D)R| \arrow[d, equal] \\
|ob(S.D)L| & |obPS.D| \arrow[l, "\varepsilon_L"', "\sim \text{ always}"] \arrow[r, "\varepsilon_R", "\sim \text{ always}"'] & |ob(S.D)R|
\end{tikzcd}$$

Upper right hand horizontal map is

$$|[n] \mapsto |(S.f|D)_{\cdot,n}|| \rightarrow |[n] \mapsto S_n D|$$

which is a w.e. by the gluing lemma and assumption. ▲

$$S_2\mathcal{E} \longrightarrow \mathcal{E}\times\mathcal{E} \quad ; \quad A \rightarrowtail C \twoheadrightarrow B \longmapsto (A, B)$$

$$\begin{tikzcd}
A_1 \arrow[r, tail] \arrow[d, tail] & A_2 \arrow[r, tail] \arrow[d, tail] & \cdots \arrow[r, tail] & A_m \arrow[d, tail] \\
C_1 \arrow[r, tail] \arrow[d, two heads] & C_2 \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & C_m \arrow[d, two heads] \\
B_1 \arrow[r, tail] & B_2 \arrow[r, tail] & \cdots \arrow[r, tail] & B_m
\end{tikzcd} \longmapsto$$

$$A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_m \rightarrowtail S_0 \rightarrowtail S_1 \rightarrowtail \cdots \rightarrowtail S_n$$

$$B_1 \rightarrowtail B_2 \rightarrowtail \cdots \rightarrowtail B_m \rightarrowtail T_0 \rightarrowtail T_1 \rightarrowtail \cdots \rightarrowtail T_n$$

$$\begin{tikzcd}
* \arrow[r, equal] \arrow[d, tail] & * \arrow[r, equal] \arrow[d, tail] & \cdots \arrow[r, equal] & * \arrow[d, tail] \\
B_1 \arrow[r, tail] \arrow[d, equal] & B_2 \arrow[r, tail] \arrow[d, equal] & \cdots \arrow[r, tail] & B_m \arrow[d, equal] \\
B_1 \arrow[r, tail] & B_2 \arrow[r, tail] & \cdots \arrow[r, tail] & B_m
\end{tikzcd}$$

$$* = * = \cdots = * = S_0/S_0 \rightarrowtail S_1/S_0 \rightarrowtail \cdots \rightarrowtail S_n/S_0$$

$$B_1 \rightarrowtail B_2 \rightarrowtail \cdots \rightarrowtail B_m \rightarrowtail T_0 \rightarrowtail T_1 \rightarrowtail \cdots \rightarrowtail T_n$$

We give a simplicial homotopy from the identity to this map. Let $X_i = C_i \cup_{A_i} S_0$ s.t.

$$\begin{tikzcd}
A_i \arrow[r, tail] \arrow[d, tail] & S_0 \arrow[d, tail] \\
C_i \arrow[r, tail] \arrow[d, two heads] & X_i \arrow[d, two heads] \\
B_i \arrow[r, equal] & B_i
\end{tikzcd}$$

for $a = 1 : [m] \to [1]$ we use the map above;

If $a(0) = \dots = a(i) = 0$, $a(i+1) = \dots = a(m) = 1$, $0 \le i \le m$

we map the given element to:

$$\begin{array}{ccccccccc}
A_1 & \rightarrowtail \cdots \rightarrowtail & A_i & \rightarrowtail & S_0 & = & \cdots & = & S_0 \\
\wr & & \wr & & \wr & & & & \wr \\
C_1 & \rightarrowtail \cdots \rightarrowtail & C_i & \rightarrowtail & X_{i+1} & \rightarrowtail & \cdots & \rightarrowtail & X_m \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
B_1 & \rightarrowtail \cdots \rightarrowtail & B_i & \rightarrowtail & B_{i+1} & \rightarrowtail & \cdots & \rightarrowtail & B_m
\end{array}$$

$$\begin{array}{l}
A_1 \rightarrowtail \cdots \rightarrowtail A_i \rightarrowtail S_0 = \cdots = S_0 = S_0 \rightarrowtail \cdots \rightarrowtail S_n \\
B_1 \rightarrowtail \cdots \rightarrowtail B_i \rightarrowtail B_{i+1} \rightarrowtail \cdots \rightarrowtail B_m \rightarrowtail T_0 \rightarrowtail \cdots \rightarrowtail T_n
\end{array}$$

One must check that this is indeed a simplicial map

$$(S_\cdot(d_2, d_0) | \mathcal{E}^2)_{\cdot, n} \times \Delta[1] \to (S_\cdot(d_2, d_0) | \mathcal{E}^2)_{\cdot, n}.$$

We next define a simpl. homotopy from the map above to the map which to the given element assigns

$$\begin{array}{ccccccc}
* & = & * & = & \cdots & = & * \\
\wr & & \wr & & & & \wr \\
* & = & * & = & \cdots & = & * \\
\downarrow & & \downarrow & & & & \downarrow \\
* & = & * & = & \cdots & = & *
\end{array}$$

$$\begin{array}{l}
* = * = \cdots = * = S_0/S_0 \rightarrowtail S_1/S_0 \rightarrowtail \cdots \rightarrowtail S_n/S_0 \\
* = * = \cdots = * = T_0/T_0 \rightarrowtail T_1/T_0 \rightarrowtail \cdots \rightarrowtail T_n/T_0
\end{array}$$

For $a = 1 : [m] \to [1]$ we use the map just described; for $a(0) = \cdots = a(i) = 0$, $a(i+1) = \cdots = a(m) = 1$, $0 \leq i \leq m$, we map the given element to

$$
\begin{tikzcd}
* \arrow[r, equal] \arrow[d, rightarrowtail] & \cdots \arrow[r, equal] & * \arrow[r, equal] \arrow[d, rightarrowtail] & * \arrow[r, equal] \arrow[d, rightarrowtail] & \cdots \arrow[r, equal] & * \arrow[d, rightarrowtail] \\
B_1 \arrow[r, rightarrowtail] \arrow[d, equal] & \cdots \arrow[r, rightarrowtail] & B_i' \arrow[r, rightarrowtail] \arrow[d, equal] & T_0 \arrow[r, equal] \arrow[d, equal] & \cdots \arrow[r, equal] & T_0 \arrow[d, equal] \\
B_1 \arrow[r, rightarrowtail] & \cdots \arrow[r, rightarrowtail] & B_i \arrow[r, rightarrowtail] & T_0 \arrow[r, equal] & \cdots \arrow[r, equal] & T_0
\end{tikzcd}
$$

$$
\begin{array}{l}
* = \cdots = * = * = \cdots = * = S_0/S_0 \rightarrowtail S_1/S_0 \rightarrowtail \cdots \rightarrowtail S_n/S_0 \\
B_1 \rightarrowtail \cdots \rightarrowtail B_i' \rightarrowtail T_0 = \cdots = T_0 = T_0 \rightarrowtail T_1 \rightarrowtail \cdots \rightarrowtail T_n
\end{array}
$$

2/26/02

$$\begin{tikzcd} A \arrow[r, "j_1"] \arrow[dr, "f"'] & T(f) \arrow[d, "p"] & B \arrow[l, "j_2"'] \arrow[dl, equal] \\ & B & \end{tikzcd} \quad \longleftarrow\!\!| \quad A \xrightarrow{f} B$$

Cyl 1: $(A \xrightarrow{f} B) \longmapsto (A \vee B \overset{j_1 \vee j_2}{\rightarrowtail} T(f))$ exact, i.e.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A' \arrow[r, "f'"] & B' \end{tikzcd} \longmapsto \begin{tikzcd} A \vee B \arrow[r, tail] \arrow[d] & T(f) \arrow[d] \arrow[dl] \\ A' \vee B' \arrow[r, tail] \arrow[ur, phantom, "\text{p.o.}"] & T(f') \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\sim"] & B \arrow[d, "\sim"] \\ A' \arrow[r, "f'"] & B' \end{tikzcd} \longmapsto \begin{tikzcd} A \vee B \arrow[r, tail] \arrow[d, "\sim"] & T(f) \arrow[d, "\sim"] \\ A' \vee B' \arrow[r, tail] & T(f') \end{tikzcd}$$

Cyl 2: $T(* \to A) = A$

$$\begin{tikzcd} * \arrow[r] \arrow[dr] & T(* \to A) \arrow[d, "p = id"] & A \arrow[l, "j_2 = id"'] \arrow[dl, equal] \\ & A & \end{tikzcd}$$

ex pointed "spaces":

$$\begin{tikzcd}
A \arrow[r, tail, "j_1"] \arrow[dr, "f"'] & A \wedge [0,1]_+ \underset{A \times \{1\}}{\cup} B \arrow[d, "p"] & B \arrow[l, tail] \arrow[dl, equal] \\
& B &
\end{tikzcd}$$

chain complexes:

$$T(A \xrightarrow{f} B)_n = A_n \oplus A_{n-1} \oplus A_n \underset{A_n}{\oplus} B_n$$

$$\mathrm{id} \nwarrow A_n \nearrow f$$

$$[a, a', a'', b] \overset{\sim}{\longmapsto} (a, a', f(a'') + b)$$

$$\downarrow d$$

$$[da + (-1)^{n-1}(-a'), da', (-1)^{n-1} a' + da'', db]$$

$$\downarrow \sim$$

$$(da + (-1)^n a', da', db + (-1)^{n-1} f(a') + f(da''))$$

so identifying

$$T(A \to B)_n = A_n \oplus A_{n-1} \oplus B_n,$$

$$d(a, a', b) = (da + (-1)^n a', da', db + (-1)^{n-1} f(a')).$$

modules: no cylinder functor.

not true in top. spaces

Axioms on $w\mathcal{E} \subset \mathcal{E}$:

Cylinder axiom : $T(f) \xrightarrow{p} A$ in $w\mathcal{E}$.

Saturation axiom : If $f \circ g$ and f (resp. g) are in $w\mathcal{E}$, so is g (resp. f).

Lemma A cylinder functor on $\mathcal{E}$ defines one on $S_n\mathcal{E}$.

pf
$$\begin{array}{ccc}
Ar(S_n\mathcal{E}) & \xrightarrow{T} & \text{diagr.}(S_n\mathcal{E}) \\
\downarrow \cong & & \downarrow \cong \\
S_n(Ar\,\mathcal{E}) & \xrightarrow{S_nT} & S_n(\text{diagr.}\,\mathcal{E})
\end{array}$$

$$\begin{array}{ccc}
Ar(S_n\mathcal{E}) & \longrightarrow & F_1 S_n \mathcal{E} \\
\downarrow \cong & & \downarrow \cong \\
S_n(Ar\,\mathcal{E}) & \underset{\text{exact}}{\xrightarrow{S_n(\cdot)}} & S_n F_1 \mathcal{E}
\end{array}$$
//

Cone : $cA := T(A \longrightarrow *)$

Suspension : $\Sigma A := (cA)/A$.

<u>lemma</u> If $\mathcal{E}$ has a cylinder functor and $w\mathcal{E}$ satisfies the cylinder axiom, then

$$id \vee \Sigma \simeq * : |N.wS.\mathcal{E}| \longrightarrow |N.wS.\mathcal{E}|.$$

pf Exact seq. of endo-functors on $\mathcal{E}$:

$$id \rightarrowtail c \twoheadrightarrow \Sigma ;$$

by additivity theorem

$$id \vee \Sigma \simeq c : |N.wS.\mathcal{E}| \to |N.wS.\mathcal{E}|.$$

But $cA = T(A \to *) \xrightarrow[\sim]{p} *$ is a nat'l transformation though w.e. (since $w\mathcal{E}$ satisfies cylinder axiom), so $c \simeq *$. //

We need a technical lemma: let $\bar{w}\mathcal{E} = w\mathcal{E} \cap co\mathcal{E} \xhookrightarrow{i} w\mathcal{E}$.

<u>lemma</u> If $\mathcal{E}$ has a cylinder functor, and if $w\mathcal{E}$ satisfies the cylinder and saturation axioms, then

$$|N.\bar{w}\mathcal{E}| \xhookrightarrow{\sim} |N.w\mathcal{E}|.$$

pf We use Quillen's theorem A: it suffices to show that for all B in $ob(w\mathcal{E}) = ob\,\mathcal{E}$, $|N.\,i/B| \simeq *$.

ob i/B : $A \xrightarrow{\sim} B$

mor i/B :

$$\begin{tikzcd} A \arrow[r, "\sim"] \arrow[d, tail, "\sim"] & B \arrow[d, equal] \\ A' \arrow[r, "\sim"] & B \end{tikzcd}$$

Define $i/B \xrightarrow{t} i/B$ by

$$t\,(A \underset{\sim}{\xrightarrow{f}} B) = (T(f) \underset{\sim}{\xrightarrow{p}} B)\ ;$$

natural transf.:

$$\begin{tikzcd} A \arrow[r, tail, "j_1", "\sim"'] \arrow[d, "f"] & T(f) \arrow[d, "p", "\sim"'] & B \arrow[l, tail, "j_2"', "\sim"] \arrow[d, equal] \\ B \arrow[r, equal] & B \arrow[r, equal] & B \end{tikzcd}$$

So $|\mathrm{id}_{i/B}| \simeq |t| \simeq |\text{const.}|$, i.e.

$|i/B| \simeq *$. //

let $\mathcal{E}$ be a cat. with cof. $co\,\mathcal{E}$ and two subcat. of w.e. $v\mathcal{E} \subset w\mathcal{E}$; let $\mathcal{E}^w \subset \mathcal{E}$ be the subcat. w. cofibrations with object c s.t. $c \to *$ is in $w\mathcal{E}$; let

$$v\mathcal{E}^w = v\mathcal{E} \cap \mathcal{E}^w, \quad w\mathcal{E}^w = w\mathcal{E} \cap \mathcal{E}^w.$$

$$\mathbb{N}^{v,w}_{\cdot,\cdot}\mathcal{E} := \mathbb{N}^v_\cdot(\mathbb{N}^w\mathcal{E}) \cong \mathbb{N}^w_\cdot(\mathbb{N}^v_\cdot\mathcal{E})$$

<u>Lemma</u> (Swallowing lemma) The inclusion of the 0-skeleton induces a w.e.

$$|\mathrm{ob}\,\mathbb{N}^w_\cdot\mathcal{E}| \xrightarrow{\sim} |\mathrm{ob}\,\mathbb{N}^{v,w}_{\cdot,\cdot}\mathcal{E}| .$$

pf write this

$$|\mathrm{ob}\,\mathbb{N}^w_\cdot\mathcal{E}| \longrightarrow |[n] \mapsto |\mathrm{ob}.\mathbb{N}^w_\cdot(\mathbb{N}^v_n\mathcal{E})||$$

so enough to show

$$|N_\cdot w\mathcal{E}| \rightleftarrows |N_\cdot w(\mathbb{N}^v_n\mathcal{E})|$$

$$[0] \rightleftarrows [n] \qquad 0 \mapsto 0$$

$\supset = \mathrm{id}$; $\subset$ nat'l transf. in $w\mathcal{E}$:

$$\begin{tikzcd}
c_0 \arrow[r, equal] \arrow[d] & c_0 \arrow[r, equal] \arrow[d, "f_1"] & \cdots \arrow[r, equal] & c_0 \arrow[d, "f_n \circ \cdots \circ f_1"] \\
c_0 \arrow[r, "f_1", "\sim"'] & c_1 \arrow[r, "f_2", "\sim"'] & \cdots \arrow[r, "f_n"] & c_n
\end{tikzcd}$$

//

Thm (Fibration thm) If $\mathcal{E}$ has a cylinder functor, and if $w\mathcal{E}$ satisfies the cylinder, saturation, and extension axioms, then the square

$$\begin{tikzcd}
{|N.vS.\mathcal{E}^w|} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{h. cart.}"] & {|N.wS.\mathcal{E}^w|} \arrow[d] \\
{|N.vS.\mathcal{E}|} \arrow[r] & {|N.wS.\mathcal{E}|}
\end{tikzcd}$$

is homotopy cartesian, and the upper right hand term is contractible by a canonical null-htpy.

Pf Extend to a diagram:

$$\begin{tikzcd}
{|\mathrm{ob}\,N^v_\cdot S.\mathcal{E}^w|} \arrow[r] \arrow[d] & {|\mathrm{ob}\,N^{v,\bar w}_\cdot S.\mathcal{E}^w|} \arrow[r, "\sim"] \arrow[d] & {|\mathrm{ob}\,N^{v,w}_\cdot S.\mathcal{E}^w|} \arrow[r, "\sim"] \arrow[d] & {|\mathrm{ob}\,N^{w}_\cdot S.\mathcal{E}^w|} \arrow[d] \\
{|\mathrm{ob}\,N^v_\cdot S.\mathcal{E}|} \arrow[r] & {|\mathrm{ob}\,N^{v,\bar w}_\cdot S.\mathcal{E}|} \arrow[r, "\sim"] & {|\mathrm{ob}\,N^{v,w}_\cdot S.\mathcal{E}|} \arrow[r, "\sim"] & {|\mathrm{ob}\,N^{w}_\cdot S.\mathcal{E}|}
\end{tikzcd}$$

cylinder lemma — swallowing lemma

Show left hand square is h.-cart.

Define $S.(\mathcal{E} \to \mathcal{D})$ as the pull-back

$$\begin{tikzcd} S.(\mathcal{E} \to \mathcal{D}) \arrow[r] \arrow[d] & PS.\mathcal{D} \arrow[r, phantom, "="] \arrow[d] & (\mathbb{P}S.\mathcal{D})_{0,\cdot} \arrow[d, "\varepsilon_R"] \\ S.\mathcal{E} \arrow[r] & S.\mathcal{D} \arrow[r, phantom, "="] & S.\mathcal{D} \end{tikzcd}$$

so ob $S_n(\mathcal{E} \to \mathcal{D})$ are (neglecting choices):

$$c_1 \rightarrowtail c_2 \rightarrowtail \cdots \rightarrowtail c_n$$
$$d_0 \rightarrowtail d_1 \rightarrowtail d_2 \rightarrowtail \cdots \rightarrowtail d_n \qquad d_i/d_0 = f(c_i).$$

Same proof as for $\mathcal{E} = \mathcal{E}$ shows:

Thm The square is homotopy-cartesian:

$$\begin{tikzcd} |N.wS.\mathcal{E}| \arrow[r] \arrow[d] & |N.wS.S.(\mathcal{E} \to \mathcal{E})| \arrow[d] \\ |N.wS.\mathcal{D}| \arrow[r] & |N.wS.S.(\mathcal{E} \to \mathcal{D})| \end{tikzcd},$$

where the horizontal maps are the incl. of the appropriate 0-skeleton.

pf Exercise — do it! //

Return to proof of fibration theorem:
I claim that

$$\begin{tikzcd}
N^v_\cdot S_\cdot \mathcal{E}^w \arrow[r] \arrow[d] & N^{v,\bar{w}}_\cdot S_\cdot \mathcal{E}^w \arrow[d] & = \\
N^v_\cdot S_\cdot \mathcal{E} \arrow[r] & N^{v,\bar{w}}_\cdot S_\cdot \mathcal{E} &
\end{tikzcd}$$

$$\begin{tikzcd}
N^v_\cdot S_\cdot \mathcal{E}^w \arrow[r] \arrow[d] & N^v_\cdot S_\cdot S_\cdot (\mathcal{E}^w = \mathcal{E}^w) \arrow[d] \\
N^v_\cdot S_\cdot \mathcal{E} \arrow[r] & N^v_\cdot S_\cdot S_\cdot (\mathcal{E}^w \to \mathcal{E})
\end{tikzcd} ,$$

enough to see

$$\begin{tikzcd}
\mathcal{E}^w \arrow[r] \arrow[d] & N^{\bar{w}}_n \mathcal{E}^w \arrow[d] & = \\
\mathcal{E} \arrow[r] & N^{\bar{w}}_n \mathcal{E} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{E}^w \arrow[r] \arrow[d] & S_n (\mathcal{E}^w = \mathcal{E}^w) \arrow[d] \\
\mathcal{E} \arrow[r] & S_n (\mathcal{E}^w \to \mathcal{E})
\end{tikzcd}$$

look at lower right hand term:

An object (neglecting choices) is:

$$c_1 \rightarrowtail c_2 \rightarrowtail \cdots \rightarrowtail c_n \qquad c_i \xrightarrow{\sim} *$$

$$d_0 \rightarrowtail d_1 \rightarrowtail d_2 \rightarrowtail \cdots \rightarrowtail d_n \ , \quad d_i / d_0 = c_i$$

By saturation axiom:

$$\underbrace{* \rightarrowtail c_i \xrightarrow{\sim} *}_{\mathrm{id}} \quad \Longrightarrow \quad * \overset{\sim}{\rightarrowtail} c_i$$

$$\begin{tikzcd} & * \arrow[dl, tail, "\sim"'] \arrow[dr, tail, "\sim"] & \\ c_i \arrow[rr, tail] & & c_{i+1} \end{tikzcd} \quad \Longrightarrow \quad c_i \overset{\sim}{\rightarrowtail} c_{i+1}$$

by extension and saturation axiom:

$$\begin{tikzcd} d_0 \arrow[r, equal] \arrow[d, equal] & d_0 \arrow[r, two heads] \arrow[d, tail] & * \arrow[d, tail, "\sim"] \\ d_0 \arrow[r, tail] & d_i \arrow[r, two heads] & c_i \end{tikzcd} \quad \Longrightarrow \quad d_0 \overset{\sim}{\rightarrowtail} d_i$$

$$\begin{tikzcd} & d_0 \arrow[dl, tail, "\sim"'] \arrow[dr, tail, "\sim"] & \\ d_i \arrow[rr, tail] & & d_{i+1} \end{tikzcd} \quad \Longrightarrow \quad d_i \overset{\sim}{\rightarrowtail} d_{i+1}$$

so data "same" as

$$d_0 \overset{\sim}{\rightarrowtail} d_1 \overset{\sim}{\rightarrowtail} \cdots \overset{\sim}{\rightarrowtail} d_n \ . \qquad //$$

 2/28/02

$\mathcal{A}$ abelian category; choose null-object 0; let
$co\,\mathcal{A}$ = monomorphisms; $w\mathcal{A} = i\mathcal{A}$ = isomorphisms.

$\mathcal{E} \subset \mathcal{A}$ full subcategory s.t. if

$$0 \to A' \to A \to A'' \to 0$$

is an exact sequence in $\mathcal{A}$ then

(i) A', A'' in $\mathcal{E} \Rightarrow A$ in $\mathcal{E}$,
(ii) A, A'' in $\mathcal{E} \Rightarrow A'$ in $\mathcal{E}$.

View $\mathcal{E} \subset \mathcal{A}$ as a subcategory with cof. and w.e.

$C^b(\mathcal{A})$ = bounded cx. in $\mathcal{A}$

$\ulcorner$ $$\cdots \to A_n \to A_{n-1} \to \cdots \to A_0 \to A_{-1} \to \cdots$$ $\lrcorner$

$co\, C^b(\mathcal{A})$ = degree-wise monomorphisms
$w\, C^b(\mathcal{A})$ = quasi-isomorphisms, i.e.

$$A_\cdot \xrightarrow{\sim} A'_\cdot \underset{def}{\Longleftrightarrow} H_*(A_\cdot) \xrightarrow{\cong} H_*(A'_\cdot).$$

$C^b(\mathcal{E}) \subset C^b(\mathcal{A})$ subcat. w. cof. and w.e.

$\ulcorner$ $E_\cdot$ in $C^b(\mathcal{E}) \not\Rightarrow H_n(E_\cdot)$ in $\mathcal{E}$. $\lrcorner$

$\mathcal{E} \hookrightarrow C^b(\mathcal{E})$ exact.

$E \longmapsto E$ conc. in degree zero.

<u>Thm</u> $|N.iS.\mathcal{E}| \xrightarrow{\sim} |N.wS.C^b(\mathcal{E})|$.

<u>pf</u> $C^b(\mathcal{E})$ has cylinder functor and $wC^b(\mathcal{E})$ satisfies cylinder axiom, saturation axiom, and extension axiom. So

$$
\begin{tikzcd}
{*\ \cong\ |N.iS.\mathcal{E}^i|} \arrow[r] \arrow[d] & {|N.iS.C^b(\mathcal{E})^w|} \arrow[r, "\simeq *"] \arrow[d] \arrow[dr, phantom, "h.\text{-cart.}"] & {|N.wS.C^b(\mathcal{E})^w|} \arrow[d] \\
{|N.iS.\mathcal{E}|} \arrow[r] & {|N.iS.C^b(\mathcal{E})|} \arrow[r] & {|N.wS.C^b(\mathcal{E})|}
\end{tikzcd}
$$

Enough to show left square h.-cart. Let $\mathcal{E}_a^b \subset C^b(\mathcal{E})$ be the full subcat. of E_* s.t. $E_i = 0$ if $i \notin \{a, a+1, \dots, b\}$. Consider $\mathcal{E}_a^b$ a subcat. w. cof. First show that for all $a \leq 0 \leq b$:

$$
\begin{tikzcd}
{|N.iS.\mathcal{E}^i|} \arrow[r] \arrow[d] \arrow[dr, phantom, "h.\text{-cart.}"] & {|N.iS.\mathcal{E}_a^{bw}|} \arrow[d] \\
{|N.iS.\mathcal{E}|} \arrow[r] & {|N.iS.\mathcal{E}_a^b|}
\end{tikzcd}
$$

<u>Claim</u> $|N.iS.\mathcal{E}_a^b| \xrightarrow{\sim} \prod_{a \leq s \leq b} |N.iS.\mathcal{E}|$

$$E_* \longmapsto (E_b, E_{b-1}, \dots, E_a)$$

$a = b$: map is iso.

$a<b$: exact equivalence of cat.:

$$\mathcal{E}_a^b \longrightarrow E(\mathcal{E}_a^a, \mathcal{E}_a^b, \mathcal{E}_{a+1}^b)$$

$$E_* \longmapsto (\sigma_{\le a} E_* \rightarrowtail E_* \twoheadrightarrow \sigma_{>a} E_*)$$

$$E_* \longleftarrow\!\shortmid \quad E'_* \rightarrowtail E_* \twoheadrightarrow E''_*$$

so

$$|N.iS.\mathcal{E}_a^b| \xrightarrow{\sim} |N.iS.E(\mathcal{E}_a^a, \mathcal{E}_a^b, \mathcal{E}_{a+1}^b)|$$

$$\underset{\text{add. thm.}}{\xrightarrow{\sim}} |N.iS.\mathcal{E}| \times |N.iS.\mathcal{E}_{a+1}^b| .$$

Claim follows by induction. /

Claim: $|N.iS.\mathcal{E}_a^{bw}| \xrightarrow{\sim} \prod_{a\le s<b} |N.iS.\mathcal{E}|$

$$E_* \longmapsto (B_{b-1}, B_{b-2}, \ldots, B_a) \underset{\text{acyclic}}{=} (Z_{b-1}, Z_{b-2}, \ldots, Z_a)$$

This is an exact functor:

$$0 \to E'_* \to E_* \to E''_* \to 0 \quad \Rightarrow$$

$$\begin{array}{ccccccc} & B'_i & \to & B_i & \to & B''_i & \to 0 \\ & \| & & \| & & \| & \\ 0 \to & Z'_i & \to & Z_i & \to & Z''_i & \end{array}$$

$a = b-1$: exact equivalence of cat.:

$$\mathcal{E}_a^{bw} \xrightarrow{\sim} \mathcal{E} \qquad \Rightarrow$$

$$(E_{a+1} \underset{\cong}{\xrightarrow{d}} E_a) \longmapsto E_a$$

$$(E = E) \longleftarrow E$$

$$|N.iS.\mathcal{E}_a^{bw}| \xrightarrow{\sim} |N.iS.\mathcal{E}|$$

$a < b-1$: exact equivalence of categories:

$$\mathcal{E}_a^{bw} \xrightarrow{\sim} E(\mathcal{E}_{b-1}^{bw}, \mathcal{E}_a^{bw}, \mathcal{E}_a^{(b-1)w})$$

$$E_* \longmapsto (\tau_{\geq b-1} E_* \rightarrowtail E_* \twoheadrightarrow \tau_{<b-1} E_*)$$

$$E_* \longleftarrow E_*' \rightarrowtail E_* \twoheadrightarrow E_*''$$

This is exact since cx. acyclic. So

$$|N.iS.\mathcal{E}_a^{bw}| \xrightarrow{\sim} |N.iS.E(\mathcal{E}_{b-1}^{bw}, \mathcal{E}_a^{bw}, \mathcal{E}_a^{(b-1)w})|$$

$$\underset{\text{add. thm.}}{\xrightarrow{\sim}} |N.iS.\mathcal{E}_{b-1}^{bw}| \times |N.iS.\mathcal{E}_a^{(b-1)w}|$$

$$\xrightarrow{\sim} |N.iS.\mathcal{E}| \times |N.iS.\mathcal{E}_a^{(b-1)w}|$$

composite induced from

$$E_* \longmapsto (B_{b-1}, \tau_{<b-1} E_*).$$

Claim follows by induction. /

Consider the diagram:

$$
\begin{array}{ccccc}
E_* & \longmapsto & (B_{b-1}, B_{b-2}, \ldots, B_a) & & \\
|N.iS.\mathcal{E}_a^{bw}| & \xrightarrow{\sim} & \prod_{a\le s<b} |N.iS.\mathcal{E}| & & (E_{b-1}, \ldots, E_a) \\
\downarrow & & \downarrow & & \downarrow \\
|N.iS.\mathcal{E}_a^{b}| & \xrightarrow{\sim} & \prod_{a\le s\le b} |N.iS.\mathcal{E}| & & (E_{b-1}, E_{b-1}\vee E_{b-2}, \ldots, E_{a+1}\vee E_a, E_a) \\
E_* & \longmapsto & (E_b, E_{b-1}, \ldots, E_a) & &
\end{array}
$$

It is homotopy-commutes by the add. thm: if E_* is in $\mathcal{E}_a^{bw}$, then

$$B_s = Z_s \rightarrowtail E_s \twoheadrightarrow B_{s-1} \quad \Longrightarrow$$

$$(E_* \mapsto E_s) \simeq (E_* \mapsto B_s \vee B_{s-1}).$$

Extend to

$$
\begin{array}{ccccc}
|N.iS.\mathcal{E}^i| & \to & |N.iS.\mathcal{E}_a^{bw}| & \xrightarrow{\sim} & \prod_{a\le s<b} |N.iS.\mathcal{E}| \\
\downarrow & & \downarrow & & \downarrow \\
|N.iS.\mathcal{E}| & \to & |N.iS.\mathcal{E}_a^{b}| & \xrightarrow{\sim} & \prod_{a\le s\le b} |N.iS.\mathcal{E}|
\end{array}
$$

The outer diagram is h.-cartesian: look at horizontal htpy fibers; hence so is the left hand square.

$$\operatorname*{colim}_{a \leq b} \mathcal{E}_a^b \xrightarrow{\cong} C^b(\mathcal{E}) \quad \Longrightarrow$$

$$\operatorname*{colim}_{a \leq b} N.iS.\mathcal{E}_a^b \xrightarrow{\cong} N.iS.C^b(\mathcal{E}) \quad \Longrightarrow$$

$$\operatorname*{colim}_{a \leq b} |N.iS.\mathcal{E}_a^b| \xrightarrow{\cong} |N.iS.C^b(\mathcal{E})| \quad \Longrightarrow$$

$$\begin{tikzcd}
{|N.iS.\mathcal{E}^{i}|} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{h. cart.}"] & \operatorname*{colim}_{a \leq b} |N.iS.\mathcal{E}_a^{bw}| \arrow[d] \\
{|N.iS.\mathcal{E}|} \arrow[r] & \operatorname*{colim}_{a \leq b} |N.iS.\mathcal{E}_a^b|
\end{tikzcd}$$

. //

$$\begin{tikzcd}
\sigma_{\leq a}E_* \arrow[r, tail] & E_* \arrow[r, two heads] & \sigma_{>a}E_* \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
0 \arrow[r, tail] \arrow[d] & E_{a+1} \arrow[r, equal] \arrow[d] & E_{a+1} \arrow[d] \\
E_a \arrow[r, equal] \arrow[d] & E_a \arrow[r] \arrow[d] & 0 \arrow[d] \\
E_{a-1} \arrow[r, equal] \arrow[d] & E_{a-1} \arrow[r] \arrow[d] & 0 \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

$$\begin{tikzcd}
\tau_{\geq a}E_* \arrow[r, tail] & E_* \arrow[r, two heads] & \tau_{<a}E_* \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
E_{a+1} \arrow[r, equal] \arrow[d] & E_{a+1} \arrow[r] \arrow[d] & 0 \arrow[d] \\
Z_a \arrow[r, equal] \arrow[d] & E_a \arrow[r, "d"] \arrow[d] & B_{a-1} \arrow[d, hook] \\
0 \arrow[r, tail] \arrow[d] & E_{a-1} \arrow[r, equal] \arrow[d] & E_{a-1} \arrow[d] \\
0 \arrow[r, tail] \arrow[d] & E_{a-2} \arrow[r, equal] \arrow[d] & E_{a-2} \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

3/05/02

Define $S.(\mathcal{E} \xrightarrow{f} \mathcal{D})$ as pull-back

$$\begin{tikzcd} S.(\mathcal{E}\to\mathcal{D}) \arrow[r] \arrow[d, "\varepsilon"] & PS.\mathcal{D} \arrow[d] & = & (\mathbb{P}S.\mathcal{D})_{0,\cdot} \arrow[d, "\varepsilon_R"] \\ S.\mathcal{E} \arrow[r, "f"] & S.\mathcal{D} & = & S.\mathcal{D} \end{tikzcd}$$

So, neglecting choices of quotients, an object of $S_n(\mathcal{E}\to\mathcal{D})$ is:

$$c_1 \rightarrowtail c_2 \rightarrowtail \cdots \rightarrowtail c_n$$

s.t.

$$d_0 \rightarrowtail d_1 \rightarrowtail d_2 \rightarrowtail \cdots \rightarrowtail d_n$$

$$d_1/d_0 \rightarrowtail \cdots \rightarrowtail d_n/d_0 = f(c_1) \rightarrowtail \cdots \rightarrowtail f(c_n).$$

In particular, inclusion of the 0-skeleton defines a map

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[rr, bend right, "*"'] & S.(\mathcal{E}\to\mathcal{D}) \arrow[r, "\varepsilon"] & S.\mathcal{E} \end{tikzcd}$$

Prop The square is homotopy-cartesian:

$$\begin{tikzcd} {|N.wS.\mathcal{D}|} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{h. cart.}"] & {|N.wS.S.(\mathcal{E}\to\mathcal{D})|} \arrow[d, "\varepsilon"] \\ * \arrow[r] & {|N.wS.S.\mathcal{E}|} \end{tikzcd}$$

pf Square of tri-simplicial sets; show

that for all $n \geq 0$,

$$\begin{array}{ccc} |N.wS.\mathcal{D}| & \xrightarrow{i} & |N.wS.S_n(\mathcal{E}\to\mathcal{D})| \\ \downarrow & \text{h. cart.} & \downarrow \varepsilon \\ * & \longrightarrow & |N.wS.S_n\mathcal{E}| \end{array}$$

Since the base is connected, the prop. will follow. Define section of ε:

$$\sigma(c_1 \rightarrowtail c_2 \rightarrowtail \cdots \rightarrowtail c_n) =$$

$$\begin{array}{c} c_1 \rightarrowtail c_2 \rightarrowtail \cdots \rightarrowtail c_n \\ * \rightarrowtail f(c_1) \rightarrowtail f(c_2) \rightarrowtail \cdots \rightarrowtail f(c_n) \end{array};$$

define retraction of i:

$$\rho\begin{pmatrix} & c_1 \rightarrowtail \cdots \rightarrowtail c_n \\ d_0 \rightarrowtail & d_1 \rightarrowtail \cdots \rightarrowtail d_n \end{pmatrix} = d_0 .$$

Consider

$$\begin{array}{c} |N.wS.\mathcal{D}| \times |N.wS.S_n\mathcal{E}| \\ i\times\sigma \downarrow\uparrow \rho\times\varepsilon \\ |N.wS.S_n(\mathcal{E}\to\mathcal{D})| \end{array}$$

$\circlearrowright$ = id ; $\curvearrowright$ $\simeq$ id by add. thm. ?

$$i \circ p \rightarrowtail \mathrm{Id} \twoheadrightarrow \sigma \circ \varepsilon$$

is an exact sequence of endo-functors of $S_n(\mathcal{C} \to \mathcal{D})$:

$$\begin{array}{c}
* = * = \cdots = * \\
d_0 = d_0 = d_0 = \cdots = d_0 \\
\downarrow \\
c_1 \rightarrowtail c_2 \rightarrowtail \cdots \rightarrowtail c_n \\
d_0 \rightarrowtail d_1 \rightarrowtail d_2 \rightarrowtail \cdots \rightarrowtail d_n \\
\downarrow \\
c_1 \rightarrowtail c_2 \rightarrowtail \cdots \rightarrowtail c_n \\
* \rightarrowtail f(c_1) \rightarrowtail f(c_2) \rightarrowtail \cdots \rightarrowtail f(c_n)\,.
\end{array}$$

//

<u>Cor</u> Given $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ exact functors,

$$\begin{array}{ccc}
|N.wS.\mathcal{B}| & \longrightarrow & |N.wS.S.(\mathcal{A} \to \mathcal{B})| \\
\downarrow & & \downarrow \\
|N.wS.\mathcal{C}| & \longrightarrow & |N.wS.S.(\mathcal{A} \to \mathcal{C})|
\end{array}$$

is homotopy-cartesian.

pf This follows from:

$$\begin{tikzcd}
{|N.wS.\mathcal{B}|} \arrow[rr] \arrow[dr] \arrow[dd] & & {|N.wS.S.(\mathcal{A} \to \mathcal{B})|} \arrow[dr] \arrow[dd] & \\
& {|N.wS.\mathcal{E}|} \arrow[rr] \arrow[dd] & & {|N.wS.S.(\mathcal{A} \to \mathcal{E})|} \arrow[dd] \\
{*} \arrow[rr] \arrow[dr, equal] & & {|N.wS.S.\mathcal{A}|} \arrow[dr, equal] & \\
& {*} \arrow[rr] & & {|N.wS.S.\mathcal{A}|}
\end{tikzcd}$$

. //

Thm If $\mathcal{E} \to \mathcal{D}$ is exact, then

$$\begin{tikzcd}
{|N.wS.\mathcal{E}|} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{h. cart.}"] & {|N.wS.S.(\mathcal{E} = \mathcal{E})|} \simeq * \arrow[d] \\
{|N.wS.\mathcal{D}|} \arrow[r] & {|N.wS.S.(\mathcal{E} \to \mathcal{D})|}
\end{tikzcd}$$

where $S.(\mathcal{E} = \mathcal{E}) = PS.\mathcal{E}$ is contractible. //

let us say that

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is a fiber sequence if $gf = z$ is constant, and the canonical map

$$X \xrightarrow{\sim} F(g, z)$$

a weak equivalence. Then for every $x \in X$, there is a long-exact sequence

$$\cdots \to \pi_n(X,x) \to \pi_n(Y,y) \to \pi_n(Z,z) \xrightarrow{\partial} \pi_{n-1}(X,x) \to \cdots$$

$$y = f(x) \qquad z = gf(x)$$

It is the same to say that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ \{z\} & \longrightarrow & Z \end{array}$$

is a homotopy - cartesian square.

ex (1) a fiber - bundle is a fiber - seq:

$$F = p^{-1}(b) \to E \xrightarrow{p} B$$

(2) a Serre - fibration gives a fiber - seq:

$$F = p^{-1}(b) \to E \xrightarrow{p} B.$$

Thm Let $X_{\cdot,\cdot} \xrightarrow{f} Y_{\cdot,\cdot} \xrightarrow{g} Z_{\cdot,\cdot}$ be a sequence of bi-simplicial sets s.t. gf is constant. Assume that for all $n \geq 0$,

$$|X_{\cdot,n}| \to |Y_{\cdot,n}| \to |Z_{\cdot,n}|$$

is a fiber-sequence and $|Z_{\cdot,n}|$ is connected. Then

$$|X_{\cdot,\cdot}| \to |Y_{\cdot,\cdot}| \to |Z_{\cdot,\cdot}|$$

is a fiber-sequence.

pf Will only consider the case where $Z_{\cdot,n}$ is reduced, i.e. $Z_{0,n} = \{z\}$. (This is the case in our application.)

⌈ $G_\cdot$ simplicial group ; universal principal bundle

$$G_\cdot \longrightarrow EG_\cdot \longrightarrow BG_\cdot \quad ;$$

in degree n :

$$G_n \xrightarrow{\Delta} (G_n)^{n+1} \longrightarrow (G_n)^n$$

$$(g_0, \dots, g_n) \longmapsto (g_0^{-1} g_1, g_1^{-1} g_2, \dots, g_{n-1}^{-1} g_n)$$

$F_\cdot$ right $G_\cdot$-action $\rightsquigarrow$ fiber bundle

$$F_\cdot \longrightarrow F_\cdot \times_{G_\cdot} EG_\cdot \longrightarrow BG_\cdot$$

$$\begin{array}{ccccc} |F_\bullet| & \longrightarrow & |F_\bullet \times_{G_\bullet} EG_\bullet| & \longrightarrow & |BG_\bullet| \\ \| & & \downarrow \cong & & \\ |F_\bullet| & \longrightarrow & |F_\bullet| \times_{|G_\bullet|} |EG_\bullet| & \longrightarrow & |BG_\bullet| \end{array}$$

so realization is a fiber-bundle, hence a fiber sequence. ⌋

<u>Claim</u> The functor

$$\{\text{simpl. grps}\} \xrightarrow{B} \{\text{reduced simpl. sets}\}$$

has a left-adjoint G, and the unit of the adjunction is a w.e.

$$Z_\bullet \xrightarrow{\sim} BGZ_\bullet \qquad //$$

let $G_{\bullet,\bullet} = GZ_{\bullet,\bullet}$; consider

$$\begin{array}{ccccc} X_{\bullet,n} & = & X_{\bullet,n} & \xrightarrow{\sim} & F_{\bullet,n} \\ \downarrow & & \downarrow & & \downarrow \\ Y_{\bullet,n} & \xleftarrow{\sim} & F_{\bullet,n} \times_{G_{\bullet,n}} EG_{\bullet,n} & = & F_{\bullet,n} \times_{G_{\bullet,n}} EG_{\bullet,n} \\ \downarrow & \text{p.b.} & \downarrow & & \downarrow \\ BG_{\bullet,n} & \xleftarrow{\sim} & EG_{\bullet,n} \times_{G_{\bullet,n}} EG_{\bullet,n} & \xrightarrow{\sim} & BG_{\bullet,n} \end{array}$$

$$Y_{\bullet,n} \longrightarrow Z_{\bullet,n} \xrightarrow{\sim} BG_{\bullet,n}$$

Here $X_{\cdot,n} \xrightarrow{\sim} F_{\cdot,n}$ by assumption. Consider (the diagonal of)

$$\begin{tikzcd}
 & X_{\cdot,\cdot} \arrow[r, equal] \arrow[d] & X_{\cdot,\cdot} \arrow[r, "\sim"] \arrow[d] & F_{\cdot,\cdot} \arrow[d] \\
 & Y_{\cdot,\cdot} \arrow[d] \arrow[dl] & F_{\cdot,\cdot} \times_{G_{\cdot,\cdot}} EG_{\cdot,\cdot} \arrow[l, "\sim"'] \arrow[r, equal] \arrow[d] \arrow[dl, phantom, "\text{p.b.}"] & F_{\cdot,\cdot} \times_{G_{\cdot,\cdot}} EG_{\cdot,\cdot} \arrow[d] \\
Z_{\cdot,\cdot} \arrow[r, "\sim"] & BG_{\cdot,\cdot} & EG_{\cdot,\cdot} \times_{G_{\cdot,\cdot}} EG_{\cdot,\cdot} \arrow[l, "\sim"'] \arrow[r, "\sim"] & BG_{\cdot,\cdot}
\end{tikzcd}$$

where

$$\begin{tikzcd}
|X_{\cdot,\cdot}| \arrow[r, phantom, "="] \arrow[d] & \big|[n] \mapsto |X_{\cdot,n}|\big| \arrow[d, "\sim"] \\
|F_{\cdot,\cdot}| \arrow[r, phantom, "="] & \big|[n] \mapsto |F_{\cdot,n}|\big|
\end{tikzcd}$$

//

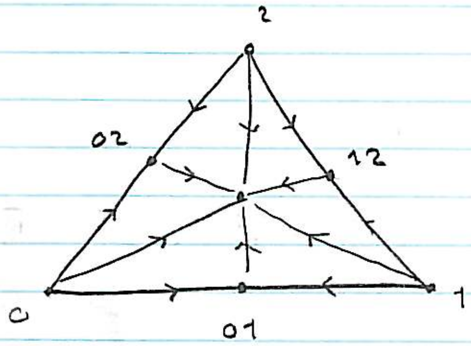
Can think of diagr. above as:

$$\begin{tikzcd}
X_{\cdot,\cdot} \arrow[r, equal] \arrow[d] & X_{\cdot,\cdot} \arrow[r, "\sim"] \arrow[d] & F_{\cdot,\cdot} \arrow[d] \\
Y_{\cdot,\cdot} \arrow[d] & F_{\cdot,\cdot} \times_{\Omega Z_{\cdot,\cdot}} E\Omega Z_{\cdot,\cdot} \arrow[l, "\sim"'] \arrow[r, equal] \arrow[d] & F_{\cdot,\cdot} \times_{\Omega Z_{\cdot,\cdot}} E\Omega Z_{\cdot,\cdot} \arrow[d] \\
Z_{\cdot,\cdot} & PZ_{\cdot,\cdot} \times_{\Omega Z_{\cdot,\cdot}} E\Omega Z_{\cdot,\cdot} \arrow[l, "\sim"'] \arrow[r, "\sim"] & B\Omega Z_{\cdot,\cdot}
\end{tikzcd}$$

$\mathrm{simp}^{n.d.}(\Delta[n]) = \{ [k] \overset{\theta}{\hookrightarrow} [n] \text{ injective} \}$ p.o. set:

$$\begin{tikzcd} {[k]} \arrow[r, hook] \arrow[d] & {[n]} \\ {[l]} \arrow[r, hook] & {[n]} \end{tikzcd} \iff \begin{tikzcd} {[k]} \arrow[r, hook] \arrow[d, hook, "\exists"'] & {[n]} \arrow[d, equal] \\ {[l]} \arrow[r, hook] & {[n]} \end{tikzcd}$$

Define $\mathrm{sd}\,\Delta[n] = |N.\,\mathrm{simp}^{n.d.}(\Delta[n])|$



For any $[n] \xrightarrow{\psi} [m]$ let

$$\mathrm{simp}^{n.d.}(\Delta[m]) \xrightarrow{\mathrm{sd}\,\psi} \mathrm{simp}^{n.d.}(\Delta[n])$$

be the map

$$[k] \overset{\theta}{\hookrightarrow} [m] \longmapsto [l] \overset{(\mathrm{sd}\,\psi)(\theta)}{\hookrightarrow} [n]$$

given by the unique factorization

$$\begin{tikzcd} {[k]} \arrow[r, hook, "\theta"] \arrow[d, two heads] & {[m]} \arrow[d, "\psi"] \\ {[l]} \arrow[r, hook, "(\mathrm{sd}\,\psi)(\theta)"] & {[n]} \end{tikzcd}$$

Induced map

$$\mathrm{sd}\,\Delta[m] \xrightarrow{\mathrm{sd}\,\psi} \mathrm{sd}\,\Delta[n],$$

If X is any simpl. set, we have the coequalizer

$$\coprod_{[m]\to[n]} X_m \times \Delta[m] \rightrightarrows \coprod_{[n]} X_n \times \Delta[n] \to X$$

and defined $\operatorname{sd} X$ as the coequalizer

$$\coprod_{[m]\to[n]} X_m \times \operatorname{sd}\Delta[n] \rightrightarrows \coprod_{[n]} X_n \times \operatorname{sd}\Delta[n] \to \operatorname{sd} X.$$

There are compatible non-simplicial homeo.

$$|\operatorname{sd}\Delta[n]| \xrightarrow{\cong} |\Delta[n]|$$

$$[k] \overset{\Theta}{\hookrightarrow} [n] \longmapsto \text{bary-center of } \Theta_*\Delta^k \subset \Delta^n$$

Since realization commutes w. colimits, we get an induced natural homeo.

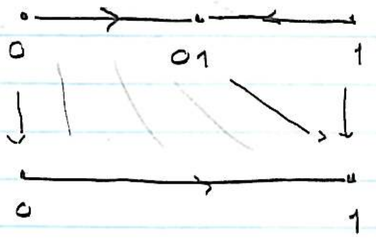
$$|\operatorname{sd} X| \xrightarrow{\cong} |X|.$$

This homeomorphism is homotopic to a simplicial map

$$|\operatorname{sd} X| \xrightarrow[\sim]{\delta} |X|.$$

induced from

$$\begin{array}{ccc} \operatorname{simp}^{n.d.}(\Delta[n]) & \xrightarrow{\delta} & [n] \\ \cup & & \cup \\ [k] \overset{\Theta}{\hookrightarrow} [n] & \longmapsto & \Theta(k) \end{array}$$



Call X non-singular if for every non-deg. simpl. $x \in X_n$, some n, the map

$$\Delta[n] \longrightarrow X$$

$$[m] \xrightarrow{\theta} [n] \longmapsto \theta^* x$$

is injective. Then $\mathrm{simp}^{n.d.}(X)$ is a p.o. set: if $x \in X_n$ and $x' \in X_m$ are non-deg. and $n \geq m$, then either $x' = \theta^* x$ for a unique $[m] \xhookrightarrow{\theta} [n]$ or not.

<u>Lemma</u> If X is non-sing. then

$$\coprod_{[m] \overset{\theta}{\hookrightarrow} [n]} X_m^{n.d.} \times \Delta[n] \rightrightarrows \coprod_{[n]} X_n^{n.d.} \times \Delta[n] \longrightarrow X$$

is a coequalizer.

<u>pf</u> Need non-sing. to have maps defined. //

<u>Prop</u> If X is non-sing., there is a canonical isomorphism

$$\mathrm{sd}\, X \xrightarrow{\cong} N.\,\mathrm{simp}^{n.d.}(X)$$

<u>pf</u> It suffices to show that the diagr. is a coequalizer

$$\coprod_{[m] \hookrightarrow [n]} X_m^{n.d.} \times \mathrm{simp}^{n.d.} \Delta[n] \rightrightarrows \coprod_{[n]} X_n^{n.d.} \times \mathrm{simp}^{n.d.} \Delta[n] \to \mathrm{simp}^{n.d.} X$$

Can check a degree – say k – at a time. Since $\Delta[n]_k^{n.d.} = \{[k] \hookrightarrow [n]\}$ this becomes

$$\coprod_{[k] \hookrightarrow [m] \hookrightarrow [n]} X_m^{n.d.} \rightrightarrows \coprod_{[k] \hookrightarrow [n]} X_n^{n.d.} \longrightarrow X_k^{n.d.}$$

The right hand map is clearly onto and the necessary relations is obtained from $[k] = [k] \hookrightarrow [n]$. //

Cor For non-sing. X, there is a natural non-simpl. homeo.

$$|N.\mathrm{simp}^{n.d.}(X)| \xrightarrow{\cong} |X|$$

which is homotopic to a simpl. map. //

Thm (Simplicial approx.) Let K and X be simpl. sets with K finite. Then for every cts. map $|K| \xrightarrow{f} |X|$ there exists $n \geq 0$ and $sd^n K \xrightarrow{g} X$ s.t.

$$\begin{tikzcd} {|K|} \arrow[rr, "f"] & & {|X|} \\ & {|sd^n K|} \arrow[ul, "\cong"] \arrow[ur, "|g|"'] & \end{tikzcd}$$

homotopy-commutes. //

An exact functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ has the approx. prop. if

App 1: $F^{-1}(w\mathcal{D}) = w\mathcal{C}$

App 2: given $F(c) \xrightarrow{v} d$ there exists $c \rightarrowtail{u} c'$ and $F(c') \underset{\sim}{\xrightarrow{v'}} d$ s.t. $v = v' \circ F(u)$.

Lemma If $\mathcal{C} \xrightarrow{F} \mathcal{D}$ has the approximation property, so does $S_n\mathcal{C} \xrightarrow{S_nF} S_n\mathcal{D}$.

Thm (Approx. thm.) Let $\mathcal{E} \xrightarrow{F} \mathcal{D}$ be an exact functor, let $\mathcal{E}$ have a cylinder functor and suppose that $w\mathcal{E}$ (resp $w\mathcal{D}$) satisfies the saturation and cylinder axioms (resp. the saturation axiom). If F has the approximation property, then

$$|N.w\mathcal{E}| \xrightarrow{\sim} |N.w\mathcal{D}|$$

$\ulcorner$

$$\begin{array}{ccc} |N.wS.\mathcal{E}| & \longrightarrow & |N.wS.\mathcal{D}| \\ \uparrow \cong & & \uparrow \cong \\ |[n] \mapsto |N.wS_n\mathcal{E}|| & \xrightarrow{\sim} & |[n] \mapsto |N.wS_n\mathcal{D}|| \end{array}$$

$\lrcorner$

pf Fix notation:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{F} & \mathcal{D} \\ \uparrow & & \uparrow \\ w\mathcal{E} & \xrightarrow{wF} & w\mathcal{D} \end{array}$$

By Quillen's thm A, it suffices to show that for all $d \in \mathrm{ob}\, wD = \mathrm{ob}\, \mathcal{D}$,

$$|wF/d| \simeq *$$

Recall:

$\mathrm{ob}(wF/d)$: $(c, F(c) \xrightarrow{\sim} d)$

$\mathrm{mor}(wF/d)$:
$$\begin{tikzcd} c \arrow[d, "\sim"] & & wF(c) \arrow[d, "\sim"] \arrow[dr, "\sim"] & \\ c' & \text{s.t.} & wF(c') \arrow[r, "\sim"'] & d \end{tikzcd}$$

$|wF/d| \neq \emptyset$: $(*, F(*) \to d) \in \mathrm{ob}(F/d)$;
apply approx. prop.: there exists

$$\begin{tikzcd} * \arrow[d, tail] & F(*) \arrow[d, tail] \arrow[dr] & \\ c & F(c) \arrow[r, "\sim"'] & d \end{tikzcd}$$

so $(c, F(c) \xrightarrow{\sim} d) \in \mathrm{ob}(wF/d)$.

$|wF/d|$ conn. : use approx. prop. to sum:

$$\begin{tikzcd} c \arrow[d, tail] & F(c) \arrow[d, tail] \arrow[dr, "\sim"] & \\ c \vee c' & F(c \vee c') \arrow[r] & d \\ c' \arrow[u, tail] & F(c') \arrow[u, tail] \arrow[ur, "\sim"'] & \end{tikzcd}$$

to get

$$\begin{tikzcd} & c \arrow[dl] \arrow[d, "\sim"] \\ c \vee c' \arrow[r] & c'' \\ & c' \arrow[ul] \arrow[u, "\sim"'] \end{tikzcd} \qquad \begin{tikzcd} & F(c) \arrow[dl] \arrow[d, "\sim"] \arrow[dr, "\sim"] & \\ F(c \vee c') \arrow[r] & F(c'') \arrow[r, "\sim"] & d \\ & F(c') \arrow[ul] \arrow[u, "\sim"'] \arrow[ur, "\sim"'] & \end{tikzcd}$$

use App 1. use wD saturated /

If I is an (index) cat., cone(I) is the category obtained by adding a terminal object ∞ to I.

$$|\mathrm{cone}(I)| \xleftarrow{\cong} \mathrm{cone}(|I|).$$

<u>Lemma</u> Let $I \xrightarrow{\mathfrak{X}} wF/d$ be a diagram and suppose an extension exists:

$$\begin{tikzcd} I \arrow[r, "\mathfrak{X}"] \arrow[dr, hook] & wF/d \arrow[r, hook] & F/d \\ & \mathrm{cone}(I) \arrow[ur, dashed] & \end{tikzcd}$$

Then there is an extension

$$\begin{tikzcd} I \arrow[rr, "\mathfrak{X}"] \arrow[dr, hook] & & wF/d \\ & \mathrm{cone}(I) \arrow[ur, dashed] & \end{tikzcd}$$

<u>pf</u> let $\mathfrak{X}_i = (c_i, F(c_i) \xrightarrow{\sim} d)$, and let

$$\begin{tikzcd} c_i \arrow[d] & F(c_i) \arrow[d] \arrow[dr, "\sim"] & \\ c_\infty & F(c_\infty) \arrow[r] & d \end{tikzcd}$$

be the given extension in F/d; get

$$\begin{tikzcd} & c_i \arrow[dl] \arrow[dd, "\sim"] \\ c_\infty \arrow[dr, tail] & \\ & c'_\infty \end{tikzcd} \qquad \begin{tikzcd} & F(c_i) \arrow[dl] \arrow[dd, "\sim"] \arrow[dr, "\sim"] & \\ F(c_\infty) \arrow[dr, tail] & & d \\ & F(c'_\infty) \arrow[ur, "\sim"] & \end{tikzcd} \quad . \qquad //$$

$$\begin{tikzcd} X \arrow[rr, "q"] & & N.(wF/d) \\ & N.\,\mathrm{simp}^{n.d.}(X) \arrow[ul, "\sim"] \arrow[ur] & \end{tikzcd}$$

— induced by functor

$$\mathrm{simp}^{n.d.}(X) \xrightarrow{q_*} wF/d.$$

If $x \in X_n$ and

$$q(x) = \begin{tikzcd} c_0(x) \arrow[r, "\sim"] & c_1(x) \arrow[r, "\sim"] & \cdots \arrow[r, "\sim"] & c_n(x) \\ F(c_0(x)) \arrow[r, "\sim"] \arrow[d, "\sim"] & F(c_1(x)) \arrow[r, "\sim"] \arrow[d, "\sim"] & \cdots \arrow[r, "\sim"] & F(c_n(x)) \arrow[d, "\sim"] \\ d \arrow[r, equal] & d \arrow[r, equal] & \cdots \arrow[r, equal] & d \end{tikzcd}$$

then

$$q_*(x) = (c_n(x)\,,\ F(c_n(x)) \xrightarrow{\sim} d).$$

Let $s(X)$ = p.o. set of simpl. subsets of X; then

$$\mathrm{simp}^{n.d.}(X) \hookrightarrow s(X)$$

$$\Delta[n] \xrightarrow{x} X \longmapsto x(\Delta[n]).$$

Propose to extend

$$\begin{tikzcd} \mathrm{simp}^{n.d.}(X) \arrow[r, "q_*"] \arrow[d, hook] & wF/d \arrow[d, hook] \\ s(X) \arrow[r, dashed] & F/d \end{tikzcd}$$

by letting t be the unique ext. s.t.

$$t(X_1 \cup_{X_0} X_2) = t(X_1) \cup_{t(X_0)} t(X_2).$$

In more detail: suppose t has been defined on all $Y \in s(X)$ with $Y \subset sk_{n-1} X$, and let $Y \in sk(X)$ with $Y \subset sk_n X$.

Then Y is obtained from $sk_{n-1} Y$ by attaching n-simpl. $y_1, \ldots, y_m$. Let $Y^{(r)}$ be the result of attaching $y_1, \ldots, y_r$ to $sk_{n-1} Y$; then for $1 \le r \le m$:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, tail, "y_r"] \arrow[d, tail] & Y^{(r-1)} \arrow[d, tail] \\ \Delta[n] \arrow[r, tail, "y_r"] & Y^{(r)} \end{tikzcd}$$

Inductively, $t(Y^{(r-1)})$ and $t(y_r(\partial\Delta[n]))$ has been defined, and

$$t(y_r(\Delta[n])) := q_*(y_r).$$

We wish to define

$$t(Y^{(r)}) = t(Y^{(r-1)}) \cup_{t(y_r(\partial\Delta[n]))} t(y_r(\Delta[n])).$$

<u>Problem</u> Only have push-out along cofibrations.

Change q_* by natural transformation to new functor

$$\mathrm{simp}^{n.d.}(X) \xrightarrow{T_q} wF/d$$

s.t. the extension

$$\begin{tikzcd}
\mathrm{simp}^{n.d.}(X) \arrow[r, "T_q"] \arrow[d, hook] & wF/d \arrow[d, hook] \\
S(X) \arrow[r, dashed, "t"] & F/d
\end{tikzcd}$$

can be defined. If

$$q(x) = \begin{array}{ccccc}
c_0(x) & \xrightarrow{\sim} & \cdots & \xrightarrow{\sim} & c_n(x) \\
F(c_0(x)) & \xrightarrow{\sim} & \cdots & \xrightarrow{\sim} & F(c_n(x)) \\
\downarrow\sim & & & & \downarrow\sim \\
d & = & \cdots & = & d
\end{array}$$

define

$$T_q(x) = \begin{array}{c}
T(c_0(x) \xrightarrow{\sim} \cdots \xrightarrow{\sim} c_n(x)) \\
F(T(c_0(x) \xrightarrow{\sim} \cdots \xrightarrow{\sim} c_n(x))) \\
\sim\downarrow p \\
F(c_n(x)) \\
\downarrow\sim \\
d
\end{array}$$

Here p w.e. by cylinder axiom.

Use maps d^i to get functor

$$\mathrm{simp}^{\mathrm{n.d.}}(X) \xrightarrow{T_q} wF/d$$

and the map p to get nat. trans. from T_q to q_*:

$$T(c_0(x) \to \cdots \to c_n(x)) \underset{\sim}{\xrightarrow{p}} c_n(x)$$

$$\begin{tikzcd}
F(T(c_0(x) \to \cdots \to c_n(x))) \arrow[r, "F(p)", "\sim"'] \arrow[d, "\sim"', "F(p)"] & F(c_n(x)) \arrow[dd, "\sim"] \\
F(c_n(x)) \arrow[d, "\sim"] & \\
d \arrow[r, equal] & d
\end{tikzcd}$$

Claim that we can define ext.

$$\begin{tikzcd}
\mathrm{simp.}^{\mathrm{n.d.}}(X) \arrow[r, "T_q"] \arrow[d, hook] & wF/d \arrow[d, hook] \\
s(X) \arrow[r, dashed, "t"] & F/d
\end{tikzcd}$$

by procedure outlined earlier.

Recall situation

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, hook, "y"] \arrow[d, hook] & Y^{(r-1)} \arrow[d, hook] \\ \Delta[n] \arrow[r, hook, "y"] & Y^{(r)} \end{tikzcd}$$

wish to define

$$\begin{tikzcd} t(y(\partial\Delta[n])) \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\text{p.o.}"] & t(Y^{(r-1)}) \arrow[d, tail] \\ t(y(\Delta[n])) \arrow[r] \arrow[d, phantom, "\shortparallel"] & t(Y^{(r)}) \\ T_q(y) & \end{tikzcd}$$

claim —

Write

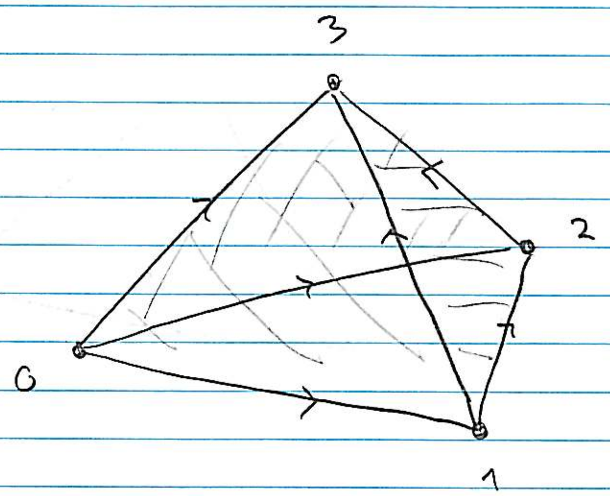
$$\partial\Delta[n] = \Lambda^n[n] \cup \Delta[n-1]$$

$$\partial^n \hookleftarrow \partial\Delta[n-1]$$

The same formula that expresses $\Lambda^n[n]$ as a union of n copies of $\Delta[n-1]$ holds for the cylinder functor, so

$$t(y(\Lambda^n[n])) \cong T(t(y(\partial^n(\partial\Delta[n-1]))) \to t(y(\Delta[0])))$$

last vertex



horn = cone on ∂(missing face)

So $t(y(\partial\Delta[n])) \to t(y(\Delta[n]))$ is identified with

$$\begin{array}{c} t(d_n y(\Delta[n-1])) \cup T(t(d_n y(\partial\Delta[n-1])) \to c_n(y)) \\ t(d_n y(\partial\Delta[n-1])) \end{array}$$

last vertex

$$\downarrow$$

$$T_q(y) := T(t(d_n y(\Delta[n-1])) \to c_n(y))$$

Define iterated mapping cylinder

$$c_0 \to \cdots \to c_n \longmapsto T(c_0 \to \cdots \to c_n)$$

together with maps

$$T(c_0 \to \cdots \to \hat{c}_i \to \cdots \to c_n) \xrightarrow{d^i} T(c_0 \to \cdots \to c_n), \quad 0 \le i \le n,$$

$$T(c_0 \to \cdots \to c_n) \xrightarrow{p} c_n,$$

inductively, as follows:

$$\begin{tikzcd}[column sep=small]
T(c_0 \to \cdots \to c_n) := T(T(c_0 \to \cdots \to c_{n-1}) \arrow[dr, "p"] & & \to c_n) \\
& c_{n-1} \arrow[ur] &
\end{tikzcd}$$

$$d^n = j_1 : T(c_0 \to \cdots \to c_{n-1}) \overset{j_1}{\rightarrowtail} T(T(c_0 \to \cdots \to c_{n-1}) \to c_n)$$

$$d^0 = j_2 : c_1 \rightarrowtail T(c_0 \to c_1), \quad \text{if } n = 1,$$

and for $0 \le i < n$ and $n > 1$,

$$d^i = T(d^{i'}),$$

where

$$\begin{array}{ccc}
T(c_0 \to \cdots \to \hat{c}_i \to \cdots \to c_{n-1}) & \to & c_n \\
\downarrow d^{i} & & \| \\
T(c_0 \to \cdots \to c_i \to \cdots \to c_{n-1}) & \to & c_n
\end{array}.$$

This gives a functor

$$\mathrm{simp}^{n.d.}(\Delta[n]) \longrightarrow \mathcal{E}$$

$$c_0 \to \cdots \to c_n \longmapsto T(c_0 \to \cdots \to c_n).$$

App 2$'$: given $F(A) \to B$ there exists

$$\begin{tikzcd} A \arrow[d] & F(A) \arrow[d] \arrow[dr] & \\ A' & F(A') \arrow[r, "\sim"'] & B \end{tikzcd}$$

Lemma App 2$'$ $\Rightarrow$ App 2.

pf We show that given $F(A) \to B$, we can find

$$\begin{tikzcd} A \arrow[d, tail] & F(A) \arrow[d, tail] \arrow[dr] & \\ A'' & F(A'') \arrow[r, "\sim"'] & B \end{tikzcd}.$$

Apply cylinder functor to $A \to A'$:

$$\begin{tikzcd} A \arrow[r, tail] \arrow[dr] & T(A \to A') \arrow[d, "\sim"] & A' \arrow[l] \arrow[dl, equal] \\ & A' & \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[d, tail] & F(A) \arrow[d, tail] \arrow[dr] & \\ T(A \to A') \arrow[d, "\sim"] & F(T(A \to A')) \arrow[d, "\sim"] & B \\ A' & F(A') \arrow[ur, "\sim"'] & \end{tikzcd}$$

//

$\mathcal{P}_R$ = f.g. proj. R-mod;

$\mathcal{M}_R$ = f.g. R-mod.

Let $F: \mathcal{C} \to \mathcal{D}$ be the inclusion

$$C^b(\mathcal{P}_R) \hookrightarrow C^b(\mathcal{M}_R).$$

Suppose that every M in $\mathcal{M}_R$ has a finite resolution in $\mathcal{P}_R$:

$$M \leftarrow P_0 \leftarrow \cdots \leftarrow P_n \leftarrow 0;$$

then for every d in $C^b(\mathcal{M}_R)$, there exists c in $C^b(\mathcal{P}_R)$ and a q.i.

$$F(c) \xrightarrow{\sim} d.$$

Moreover, using projectivity, we can lift maps to resolutions:

$$\begin{array}{ccccc} c & & F(c) & \longrightarrow & d \\ \downarrow \exists & & \downarrow \exists & & \| \\ c' & & F(c') & \xrightarrow{\sim} & d \end{array}.$$

So F satisfies App 1 and App 2'; hence

$$|N.wS.C^b(\mathcal{P}_R)| \xrightarrow{\sim} |N.wS.C^b(\mathcal{M}_R)|,$$

if R is <u>regular</u>.

X non-singular ; then

$$N_k(\mathrm{simp}^{n.d.}(X)) = \coprod_{[i_0]\hookrightarrow\cdots\hookrightarrow[i_k]} X^{n.d.}_{i_k}$$

$$\coprod_{[n]} X_n^{n.d.} \times N_k(\mathrm{simp}^{n.d.}\Delta[n]) = \coprod_{[i_0]\hookrightarrow\cdots\hookrightarrow[i_k]\hookrightarrow[n]} X_n^{n.d.}$$

$$\uparrow\uparrow$$

$$\coprod_{[m]\hookrightarrow[n]} X_n^{n.d.} \times N_k(\mathrm{simp}^{n.d.}\Delta[m]) = \coprod_{[i_0]\hookrightarrow\cdots\hookrightarrow[i_k]\hookrightarrow[m]\hookrightarrow[n]} X_n^{n.d.}$$

look at $[i_0]\hookrightarrow\cdots\hookrightarrow[i_k] = [i_0]\hookrightarrow[n]$

$$\operatorname*{colim}_\alpha N_k \mathcal{E}_\alpha = \operatorname*{colim}_\alpha \mathcal{E}_\alpha^{[k]} = \left(\operatorname*{colim}_\alpha \mathcal{E}_\alpha\right)^{[k]}$$

V = discrete valuation ring

= P.I.D. with unique non-zero prime ideal m,

k = V/m residue field

K = quotient field.

ex (1) $V = \mathbb{Z}_{(p)}$, $k = \mathbb{F}_p$, $K = \mathbb{Q}$

(2) $V = k[[t]]$, $k = k$, $K = k((t))$.

(3)
$$\begin{array}{ccccc} k & - & V & - & K \\ | & & | & & |\ \text{fin.} \\ \mathbb{F}_p & - & \mathbb{Z}_p & - & \mathbb{Q}_p \end{array}$$

$V \subset K$ integral closure.

—

$$\begin{array}{ccccc}
K\begin{pmatrix}\text{bd. cx. of f.g.} \\ V\text{-mod. w. tors} \\ \text{homology} \\ \\ \text{qis.}\end{pmatrix} & \longrightarrow & K\begin{pmatrix}\text{bd. cx. of f.g.} \\ V\text{-mod.} \\ \\ \text{qis.}\end{pmatrix} & \longrightarrow & K\begin{pmatrix}\text{bd. cx. of} \\ \text{f.g. } V\text{-mod.} \\ \\ K\text{-qis}\end{pmatrix} \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
K\begin{pmatrix}\text{bd. cx. of} \\ \text{f.g. } V\text{-mod} \\ \text{tors. hom.} \\ \\ \text{qis}\end{pmatrix} & \longrightarrow & K\begin{pmatrix}\text{bd. cx. of} \\ \text{f.g. } V\text{-mod} \\ \\ \text{qis.}\end{pmatrix} & \longrightarrow & K\begin{pmatrix}\text{bd. cx. of} \\ \text{f.g. } V\text{-mod} \\ \\ K\text{-qis}\end{pmatrix} \\
\uparrow & & & & \downarrow
\end{array}$$

$$\begin{array}{ccc} \uparrow & & \uparrow \\ K\begin{pmatrix}\text{bd. cx. of f.g.} \\ \text{tor. } V\text{-mod.} \\ \text{qis}\end{pmatrix} & & K\begin{pmatrix}\text{bd. cx. of} \\ \text{f.d. } K\text{-v.s.} \\ \text{qis}\end{pmatrix} \\ \uparrow & & \\ K\begin{pmatrix}\text{bd. cx. of} \\ \text{f.d. } k\text{-v.s.} \\ \text{qis}\end{pmatrix} & & \end{array}$$

"Duality" property of the cat. :

$$A' \rightarrowtail A \twoheadrightarrow A''$$

└ call this a quotient map

Then the quotient maps form a category $\text{quot}(\mathcal{A}) \subset \mathcal{A}$ and

$$(\mathcal{A}^{op}, \text{quot}(\mathcal{A})^{op}, w(\mathcal{A})^{op})$$

is a category w. cof. and w.e. Moreover

$A' \longrightarrow A \longrightarrow A''$ exact in $\mathcal{A}$ $\iff$

$A' \longleftarrow A \longleftarrow A''$ exact in $\mathcal{A}^{op}$.

so

$$S.(\mathcal{A}^{op}) = (S.\mathcal{A})^{op}.$$

$\mathcal{A} \xrightarrow{F} \mathcal{B}$; define new cat. $\mathcal{C}$

ob $\mathcal{C}$: $(A,\ F(A) \xrightarrow{\sim} B)$

mor $\mathcal{C}$:
$$\begin{array}{cccc} A & F(A) & \xrightarrow{\sim} & B \\ \downarrow & \downarrow & & \downarrow \\ A' & F(A') & \xrightarrow{\sim} & B' \end{array}$$

co $\mathcal{C}$:
$$\begin{array}{cccc} A & F(A) & \xrightarrow{\sim} & B \\ \rightarrowtail & \rightarrowtail & & \rightarrowtail \\ A' & F(A') & \xrightarrow{\sim} & B' \end{array}$$

w $\mathcal{C}$:
$$\begin{array}{cccc} A & F(A) & \xrightarrow{\sim} & B \\ \downarrow\sim & \downarrow\sim & & \downarrow\sim \\ A' & F(A') & \xrightarrow{\sim} & B \end{array}$$

functors

$$\mathcal{A} \rightleftarrows \mathcal{C}$$

$A \longmapsto (A, F(A) = F(A))$

$A \longleftarrow (A, F(A) \xrightarrow{\sim} B)$

$\leftarrow = \mathrm{Id}$, $\rightarrow \simeq \mathrm{Id}$:
$$\begin{array}{cccc} A & F(A) & = & F(A) \\ \| & \| & & \downarrow\sim \\ A & F(A) & \xrightarrow{\sim} & B \end{array}$$

so

$$|N.wS.\mathcal{A}| \rightleftarrows |N.wS.\mathcal{C}|$$

homotopy – equivalence.

$$w^{-1}\mathcal{A} \xrightarrow{w^{-1}F} w^{-1}\mathcal{B}$$ eq. of cat. iff :

(0) $w\mathcal{A} = F^{-1}(w\mathcal{B})$

(1) $\forall B$, $\exists\ F(A) \xrightarrow{\sim} B$.

(2) $\forall\ F(A') \xrightarrow{b} F(A'')$, $\exists\ A' \underset{\sim}{\xleftarrow{a'}} A \xrightarrow{a''} A''$ s.t.

$$\begin{tikzcd}
F(A') \arrow[rr, "b"] & & F(A'') \\
& F(A) \arrow[ul, "F(a')", "\sim"'] \arrow[ur, "F(a'')"'] &
\end{tikzcd}$$

chain-homotopy commutes.

(3) $\forall\ A' \xrightarrow{a'} A''$ s.t. $F(A') \xrightarrow{F(a')} F(A'')$ is

chain-null-homotopic, $\exists\ A \xrightarrow{\sim} A'$ s.t.

$A \xrightarrow{\sim} A' \xrightarrow{a'} A''$ is chain-null-homotopic.

<u>Note</u> (2) + (3) automatically satisfied for fully faithful F, e.g.

$$C^b(\mathcal{P}_R) \hookrightarrow C^b(\mathcal{M}_R).$$

$$\begin{tikzcd} FA \arrow[r, "\sim"] & B \\ FA' \arrow[r, "\sim"] & B' \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & & FA_2 \arrow[dl, "\sim"', "(1)"] \arrow[ddr] & FA'' \arrow[l, "\sim", "(2)"'] \arrow[dd, "F(a'')"] \\ & FA' \overset{h}{\underset{B}{\times}} FA \arrow[dl, "\sim"'] \arrow[dr] & & \\ FA' \arrow[d, "\sim"'] \arrow[drr] & & FA \arrow[d, "\sim"] & FA \\ B' \arrow[rr] & & B & \end{tikzcd}$$

Here $FA' \overset{h}{\underset{B}{\times}} FA \xrightarrow{\sim} FA'$ by "Mayer – Vietoris"; our w.e. are some kind of homology eq.

$$\begin{tikzcd} FA \arrow[r, "\sim"] & B \\ FA'' \arrow[u] \arrow[r, "\sim"] \arrow[ur, phantom, "\simeq"] & B' \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} & FA \\ FA''' \arrow[ur, "F(a')"] \arrow[dr, "\sim", "F(a'')"'] & \\ & FA'' \arrow[uu, "b"'] \end{tikzcd}$$

chain-htpy commutes

$$\begin{tikzcd} A & FA \arrow[r, "\sim"] & B \\ A''' \arrow[u, "a'"] & FA''' \arrow[u] \arrow[r, "\sim"] \arrow[ur, phantom, "\simeq"] & B' \arrow[u] \end{tikzcd}$$

$B'' = B \overset{h}{\underset{B}{\times}} B'$; pick chain htpy to get:

$$
\begin{tikzcd}
& FA'' \arrow[rr, "\sim"] \arrow[dl] \arrow[dd] \arrow[dr, "\sim"] & & B' \arrow[dd] \\
FA \arrow[dr, "\sim"'] & & B \overset{h}{\underset{B}{\times}} B' \arrow[ur, "\sim"'] \arrow[dl] & \\
& B \arrow[rr, equal] & & B
\end{tikzcd}
$$

$$\Downarrow$$

$$
\begin{tikzcd}
FA \arrow[r, "\sim"] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[dr, phantom, "\circlearrowright"] & \\
FA'' \arrow[u] \arrow[r, "\sim"] & B \overset{h}{\underset{B}{\times}} B' \arrow[u] & B' \arrow[l, "\sim"'] \arrow[ul, bend right, "\text{orig. map}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
B' \arrow[rr, equal] \arrow[dd] \arrow[dr, "\sim"] & & B' \arrow[dd] \\
& B \overset{h}{\underset{B}{\times}} B' \arrow[ur, "\sim"'] \arrow[dl] & \\
B \arrow[rr, equal] & & B
\end{tikzcd}
$$

$$X \overset{h}{\underset{Z}{\cup}} Y := X \underset{Z}{\cup} [0,1]\times Z \underset{Z}{\cup} Y$$

$$\begin{tikzcd}[column sep=small]
(A \overset{h}{\underset{f^{\nwarrow} C \nearrow_g}{\cup}} B)_n \arrow[d, "d"] & = & A_n \underset{C_n}{\oplus} C_n \oplus C_{n-1} \oplus C_n \underset{C_n}{\oplus} B_n \arrow[d, "d"] & \leftarrow & A_n \oplus C_{n-1} \oplus B_n \arrow[d, "d"] \\
(A \overset{h}{\underset{C}{\cup}} B)_{n-1} & = & A_{n-1} \underset{C_{n-1}}{\oplus} C_{n-1} \oplus C_{n-2} \oplus C_{n-1} \underset{C_{n-1}}{\oplus} B_{n-1} & \rightarrow & A_{n-1} \oplus C_{n-2} \oplus B_n
\end{tikzcd}$$

$$\begin{tikzcd}
A_n \arrow[d, "d"] & C_{n-1} \arrow[dl, "-f"'] \arrow[d, "-d"] \arrow[dr, "g"] & B_n \arrow[d, "d"] \\
A_{n-1} & C_{n-2} & B_{n-1}
\end{tikzcd}$$

$$d(a,c,b) = (da - fc, -dc, db + gc)$$

$A \overset{h}{\underset{C}{\cup}} B \longrightarrow D \quad \leftrightsquigarrow \quad A \to D,\ B \to D$ and

chain htpy between

$C \to A \to D$ and

$C \to B \to D$.

$$X \overset{h}{\underset{Z}{\times}} Y := X \times \underset{Z \xleftarrow{ev_0} \quad \xrightarrow{ev_1} Z}{Z^{[0,1]}} \times Y$$

$$W \longrightarrow X \overset{h}{\underset{Z}{\times}} Y \quad \leftrightsquigarrow \quad \begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

$$(A \overset{h}{\underset{C}{\times}} B)_n = A_n \times C_{n+1} \times B_n$$

$$d(a,c,b) = (da, -fa - dc + gb, db).$$

3/19/02

<u>Lemma</u> Let A be a noetherian ring, let C_* be a bd. cx. of f.g. A-modules, and suppose the homology of C_* is annihilated by some power of an ideal $I \subset A$. Then there exists a quasi-iso.

$$C_* \xrightarrow{\sim} D_*$$

with D_* a bd. cx. of f.g. A-mod. annihilated by some power of I.

<u>pf</u> Suppose that for all $i \geq n$, C_i is annihilated by a power of I; construct qis. $C_* \to C''_*$ of bd. cx. of f.g. A-mod s.t. for all $i \geq n-1$, C''_i is annihilated by a power of I.

The exact sequences

$$0 \to Z_n \to C_n \xrightarrow{d} B_{n-1} \to 0$$

$$0 \to B_{n-1} \to Z_{n-1} \to H_{n-1} \to 0$$

show that Z_{n-1} is annihilated by a power of I, say, by I^r. Since C_{n-1} is f.g. and A noeth., Z_{n-1} is f.g. Hence, by the Artin-Rees lemma, there exists $s \geq 0$ s.t.

$$Z_{n-1} \cap I^s C_{n-1} \subset I^r Z_{n-1} = 0.$$

Define

$$C''_i = C_i \quad , \quad i \neq n-1, n-2$$

$$C''_{n-1} = C_{n-1} / I^s C_{n-1}$$

$$C''_{n-2} = \text{p.o.} :$$

$$\begin{tikzcd}
 & Z_{n-1} \arrow[r, "\sim"] \arrow[d, hook] & Z_{n-1}+I^s C_{n-1} / I^s C_{n-1} \cong Z_{n-1} / Z_{n-1} \cap I^s C_{n-1} \arrow[d, hook] \\
I^s C_{n-1} \arrow[r, tail] \arrow[d, "\cong"] & C_{n-1} \arrow[r, two heads] \arrow[d, "d"] \arrow[dr, phantom, "\text{p.o.}"] & C''_{n-1} \arrow[d, "d''"] \\
I^s C_{n-1} \arrow[r, tail] & C_{n-2} \arrow[r, two heads] & C''_{n-2} = C''_{n-1} \oplus_{C_{n-1}} C_{n-2}
\end{tikzcd}$$

so

$$C' \rightarrowtail C \xrightarrow{\sim} C''$$

and $H_* C' \simeq *$. //

$\mathcal{B}$ ab. cat., $\mathcal{A}$ non-empty full subcategory closed under taking subobjects, quotient-objects, and finite products in $\mathcal{B}$. Then $\mathcal{B}$ is an abelian cat. and $\mathcal{A} \hookrightarrow \mathcal{B}$ exact.

Thm (Devisage) Suppose every object M of $\mathcal{B}$ has a finite filtration

$$0 = M_{-1} \subset M_0 \subset M_1 \subset \cdots \subset M_n = M$$

s.t. M_j / M_{j-1} is in $\mathcal{A}$, $0 \le j \le n$. Then

$$|N.iS.\mathcal{A}| \xrightarrow{\sim} |N.iS.\mathcal{B}|$$

Pf Recall the fiber sequence

$$|N.iS.\mathcal{B}| \to |N.iS.S.(\mathcal{A} \hookrightarrow \mathcal{B})| \to |N.iS.S.\mathcal{A}|;$$

enough to show that for all $n \geq 0$,

$$|N.iS_nS.(\mathcal{A} \hookrightarrow \mathcal{B})| \simeq *$$

Since

$$S_nS.(\mathcal{A} \hookrightarrow \mathcal{B}) \cong S.(S_n\mathcal{A} \hookrightarrow S_n\mathcal{B}),$$

and since $S_n\mathcal{A} \hookrightarrow S_n\mathcal{B}$ satisfies assump. of the theorem (exercise), it is enough to show

$$|N.iS.(\mathcal{A} \hookrightarrow \mathcal{B})| \xleftarrow{\sim} |\mathrm{ob}\, S.(\mathcal{A} \hookrightarrow \mathcal{B})| \simeq *.$$

Recall that objects of $S_n(\mathcal{A} \to \mathcal{B})$ are diagrams:

$$\begin{tikzcd}
B_0 \arrow[r, tail] & B_1 \arrow[r, tail] \arrow[d] & B_2 \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] & B_n \arrow[d] \\
& A_{0,1} \arrow[r, tail] & A_{0,2} \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] & A_{0,n} \arrow[d] \\
& & A_{1,2} \arrow[r, tail] & \cdots \arrow[r, tail] & A_{1,n} \arrow[d] \\
& & & & \vdots \arrow[d] \\
& & & & A_{n-1,n}
\end{tikzcd}$$

in $S_{n+1}\mathcal{B}$ s.t. $B_j / B_i \xrightarrow{\sim} A_{i,j}$ is in $\mathcal{A}$.

For every simpl. set X, there is a w.e.:

$$\operatorname{hocolim}_{\Delta^{op}} X \xrightarrow{\sim} X \,,$$

the map is:

$$\begin{array}{ccccc}
\coprod_{[i_0]} X_{i_0} & \longrightarrow & X_0 & \quad & [i_0] \longleftarrow [0] \\
& & & & i_0 \longleftarrow\!\shortmid\ 0 \\
\uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow & & \\
\coprod_{[i_0]\xrightarrow{\theta_1}[i_1]} X_{i_1} & \longrightarrow & X_1 & & [i_1] \longleftarrow [1] \\
& & & & \theta_1(i_0) \longleftarrow\!\shortmid\ 0 \\
\uparrow\downarrow\uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow\downarrow\uparrow & & i_1 \longleftarrow\!\shortmid\ 1 \\
\coprod_{[i_0]\xrightarrow{\theta_1}[i_1]\xrightarrow{\theta_2}[i_2]} X_{i_2} & \longrightarrow & X_2 & & [i_2] \longleftarrow [2] \\
& & & & \theta_2\theta_1(i_0) \longleftarrow\!\shortmid\ 0 \\
\uparrow \cdots \uparrow & & \uparrow - \uparrow & & \theta_2(i_1) \longleftarrow\!\shortmid\ 1 \\
& & & & i_2 \longleftarrow\!\shortmid\ 2
\end{array}$$

$$\begin{array}{ccc} B_0 \rightarrowtail B_1 \rightarrowtail \cdots \rightarrowtail B_{i_0} & \longmapsto & B_{i_0} \\ A_1 \rightarrowtail \quad \rightarrowtail A_{i_0} & & \end{array}$$

$$\begin{array}{ccc} \coprod\limits_{[i_0]} \operatorname{ob} S_{i_0}(\mathcal{A} \to \mathcal{B}) & \longrightarrow & \operatorname{ob}(\operatorname{co}\mathcal{B}) \\ \uparrow \downarrow \uparrow & & \uparrow \downarrow \uparrow \\ \coprod\limits_{[i_0] \xrightarrow{\theta} [i_1]} \operatorname{ob} S_{i_1}(\mathcal{A} \to \mathcal{B}) & \longrightarrow & \operatorname{mor}(\operatorname{co}\mathcal{B}) \end{array}$$

$$\begin{array}{ccc} B_0 \rightarrowtail B_1 \rightarrowtail \cdots \rightarrowtail B_{i_1} & \longmapsto & B_{\theta(i_0)} \rightarrowtail B_{i_1} \\ A_1 \rightarrowtail \cdots \rightarrowtail A_{i_1} & & \end{array}$$

$$\operatorname{ob} S.(\mathcal{A} \to \mathcal{B}) \to \operatorname{ob} PS.\mathcal{B} \longrightarrow N.\operatorname{co}\mathcal{B}$$

$$\begin{array}{ccccc} \operatorname*{hocolim}\limits_{\Delta^{op}} \operatorname{ob} S.(\mathcal{A} \to \mathcal{B}) & \to & \operatorname*{hocolim}\limits_{\Delta^{op}} N.\operatorname{co}\mathcal{B} & \xrightarrow{\sim} & N.\operatorname{co}\mathcal{B} \simeq * \\ \| & & & & \\ N \operatorname{simp}(\operatorname{ob} S.(\mathcal{A} \to \mathcal{B})) & & \longrightarrow & & N.\operatorname{co}\mathcal{B} \end{array}$$

induced by functor L

use Quillen's Thm A.

Let $B' \in \operatorname{ob} \operatorname{co}\mathcal{B} = \operatorname{ob} \mathcal{B}$,

ob L/B':
$$\begin{array}{l} B_0 \rightarrowtail B_1 \rightarrowtail \cdots \rightarrowtail B_{i_0} \rightarrowtail B' \\ \qquad\quad \downarrow \qquad\qquad\quad \downarrow \\ \qquad\quad A_1 \rightarrowtail \cdots \rightarrowtail A_{i_0} \end{array}$$

mor L/B': $[i_0] \xrightarrow{\theta} [i_1]$
$$\begin{array}{l} B_0 \rightarrowtail B_1 \rightarrowtail \cdots \rightarrowtail B_{i_1} \rightarrowtail B' \\ \qquad\quad \downarrow \qquad\qquad\quad \downarrow \\ \qquad\quad A_1 \rightarrowtail \cdots \rightarrowtail A_{i_1} \end{array}$$