This is the cat. of simpl. associated with the simpl. set with q-simpl.:

$$\begin{tikzcd}
B_0 \arrow[r, tail] & B_1 \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & B_q \arrow[r, tail] \arrow[d, two heads] & B' \\
& A_1 \arrow[r, tail] & \cdots \arrow[r, tail] & A_q &
\end{tikzcd}$$

which is equivalent to the cat. of simpl. in the simpl. set $N_{B'}$ with q-simpl.:

$$B_0 \rightarrowtail B_1 \rightarrowtail \cdots \rightarrowtail B_q \rightarrowtail B'$$

s.t. B_q/B_0 (and hence B_i/B_j) is in $\mathcal{A}$.

It suffices to show $|N_{B'}| \simeq *$; for

$$|N.\mathrm{simp}(N_{B'})| \xrightarrow{\sim} |N_{B'}|.$$

Choose a filtration of B' with filtr. quotients in $\mathcal{A}$,

$$0 = C_0 \rightarrowtail C_1 \rightarrowtail \cdots \rightarrowtail C_n = B'.$$

Define

$$N_{B'} \xrightarrow{F_i} N_{B'}$$

$$B_0 \rightarrowtail \cdots \rightarrowtail B_q \rightarrowtail B' \longmapsto B_0 + C_i \rightarrowtail \cdots \rightarrowtail B_q + C_i \rightarrowtail B'$$

$$\begin{tikzcd}
B_j \oplus C_i \arrow[rr] \arrow[dr, two heads] & & B' \\
& B_j + C_i \arrow[ur, tail] &
\end{tikzcd}$$

Since $B_q/B_0 \twoheadrightarrow (B_q+C_i)/(B_0+C_i)$, this map is well-defined; $F_0 = \mathrm{id}$, F_n = constant.

Define simplicial homotopies

$$\begin{tikzcd}
N_{B'} \arrow[d, hook, "i_0"] \arrow[dr, "F_{i-1}"] & \\
N_{B'} \times \Delta[1] \arrow[r] & N_{B'} \\
N_{B'} \arrow[u, hook, "i_1"'] \arrow[ur, "F_i"'] &
\end{tikzcd}$$

$$(B_0 \rightarrowtail \cdots \rightarrowtail B_q \rightarrowtail B', [q] \xrightarrow{\alpha} [1]) \longmapsto$$

$$B_0 + C_{i-1} \rightarrowtail \cdots \rightarrowtail B_t + C_{i-1} \rightarrowtail B_{t+1} + C_i \rightarrowtail \cdots \rightarrowtail B_q + C_i \rightarrowtail B'$$

if $\alpha(0) = \cdots = \alpha(t) = 0$, $\alpha(t+1) = \cdots = \alpha(t+1) = 1$

This is well-defined since

$$B_q/B_0 \oplus C_i/C_{i-1} = B_q \oplus C_i / B_0 \oplus C_{i-1}$$

$$\twoheadrightarrow B_q + C_i / B_0 + C_{i-1} \qquad //$$

3/21/02

$\mathcal{A}$ ab. cat., $\mathcal{W} \subset \mathcal{H} \subset \mathcal{A}$ exact subcat. Let

$$\mathcal{E} = C^b(\mathcal{A}; \mathcal{H}, \mathcal{W})$$

be the cat. w. cof. and w.e. where:

$\mathcal{E}$ = bd. cx. in $\mathcal{A}$ w. homology in $\mathcal{H}$,

co $\mathcal{E}$ = consider as subcat. w. cof. of $C^b(\mathcal{A})$.

w $\mathcal{E}$ = maps whose mapping cone has homology in $\mathcal{W}$.

Let X be a noeth. scheme, let $Y \subset X$ be a closed subscheme, and let $U = X \setminus Y$. Let

$\mathcal{M}_X$ = coh. $\mathcal{O}_X$-modules.

$\mathcal{M}_{X,Y}$ = coh. $\mathcal{O}_X$-modules supported on Y.

We wish to show

$$\begin{tikzcd}
K(C^b(\mathcal{M}_X; \mathcal{M}_{X,Y}, 0)) \arrow[r] & K(C^b(\mathcal{M}_X, \mathcal{M}_X, 0)) \arrow[r] & K(C^b(\mathcal{M}_X, \mathcal{M}_X, \mathcal{M}_{X,Y})) \arrow[d, "\sim"', "j^*"] \\
K(C^b(\mathcal{M}_{X,Y}; \mathcal{M}_{X,Y}, 0)) \arrow[u, "\sim"', "\text{approx. thm.}"] & & K(C^b(\mathcal{M}_U; \mathcal{M}_U, 0)) \\
K(C^b(\mathcal{M}_Y; \mathcal{M}_Y, 0)) \arrow[u, "\sim"', "\text{devissage}"] & &
\end{tikzcd}$$

only need:

Lemma Let C be a bd. cx. of coh. $\mathcal{O}_U$-mod. Then there exists a bd. cx. of coh. $\mathcal{O}_X$-mod. C' and a monomorphism $C' \hookrightarrow j_* C$ s.t.

$$j^* C' \overset{\sim}{\hookrightarrow} j^* j_* C \underset{\sim}{\xrightarrow{\varepsilon}} C .$$

pf On a noeth. scheme X, every quasi-coh. $\mathcal{O}_X$-mod. M is the filtered colimit of its coh. submodules:

$$\operatorname{colim}_{\alpha} M_\alpha \xrightarrow{\sim} M .$$

Indeed, on a noeth. scheme, every open subset $V \subset X$ is quasi-compact. It follows that

$$V \longmapsto \operatorname{colim}_{\alpha} M_\alpha(V)$$

is a sheaf; hence it **IS** the colimit of the M_α. If $V \subset X$ is aff. open, then

$$\operatorname{colim}_{\alpha} M_\alpha(V) \xrightarrow{\sim} M(V),$$

since a module is the filtered colimit of its f.g. submodules. In general, $V \subset X$ open is the union of finitely many affine open, say $V_1, \dots, V_n$. Get

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{colim}_{\alpha} M_\alpha(V) \arrow[r] \arrow[d] & \prod_i \operatorname{colim}_{\alpha} M_\alpha(V_i) \arrow[r] \arrow[d, "\sim"] & \prod_{i,j} \operatorname{colim}_{\alpha} M_\alpha(V_i \cap V_j) \arrow[d, hook] \\
0 \arrow[r] & M(V) \arrow[r] & \prod_i M(V_i) \arrow[r] & \prod_{i,j} M(V_i \cap V_j)
\end{tikzcd}$$

which implies

$$\operatorname{colim}_{\alpha} M_\alpha(V) \xrightarrow{\sim} M(V).$$

If M is a coh. $\mathcal{O}_U$-module, then $j_* M$ is a quasi-coh. $\mathcal{O}_X$-module, so

$$\operatorname{colim}_{\alpha} N_\alpha \xrightarrow{\sim} j_* M,$$

and hence,

$$\operatorname{colim}_{\alpha} j^* N_\alpha \xrightarrow{\sim} j^* \operatorname{colim}_{\alpha} N_\alpha \xrightarrow{\sim} j^* j_* M \xrightarrow{\sim} M.$$

$$j_! \dashv j^*$$

Since M is coh., there exists α s.t. $j^* N_\alpha \twoheadrightarrow M$, and $j^* N_\alpha \hookrightarrow M$, for all α.

If C is a bd. cx. of coh. $\mathcal{O}_U$-modules, we proceed by downward induction on the degree. We just need to note that if $C_i' \subset j_* C_i$ is coh. then $d(C_i') \subset j_* C_{i-1}$ is coh., so we can choose $C_{i-1}' \subset j_* C_{i-1}$ with $d(C_i') \subset C_{i-1}'$. //

Let X be a noetherian scheme; let $\mathcal{M}_X^q \subset \mathcal{M}_X$ be the full subcat. of coh. $\mathcal{O}_X$-modules with supp. of codim. $\geq q$.

$$\operatorname{codim} Z = \min_{z \in Z} \dim \mathcal{O}_{X,z}.$$

Finite filtration

$$\mathcal{M}_X = \mathcal{M}_X^0 \supset \mathcal{M}_X^1 \supset \cdots \supset \mathcal{M}_X^n = 0$$

if $n > \dim X$. The fibration thm. gives fiber sequences

$$K(C^b(\mathcal{M}_X; \mathcal{M}_X^{q+1}, 0)) \to K(C^b(\mathcal{M}_X; \mathcal{M}_X^{q}, 0)) \to K(C^b(\mathcal{M}_X; \mathcal{M}_X^{q}, \mathcal{M}_X^{q+1}))\ ;$$

let us write for short

$$K^{q+1}(X) \to K^q(X) \to K^{q/q+1}(X).$$

Taking homotopy groups, we get an exact couple

$$E_{s,t} = \pi_{s+t} K^{-s/-(s-1)}(X)$$

$$D_{s,t} = \pi_{s+t} K^{-s}(X)\ ;$$

hence a (second quadrant) sp. seq.

$$E^1_{s,t} = K^{-s/-(s-1)}_{s+t}(X)$$

$$\Rightarrow K_{s+t}(X).$$

Let $X^{(q)} \subset X$ be the points of codim. q, i.e. pts. s.t. $\dim \mathcal{O}_{X,x} = q$.

Lemma $K(C^b(m_X^q; m_X^q, m_X^{q+1})) \xrightarrow{\sim} K(C^b(m_X; m_X^q, m_X^{q+1}))$.

pf If $C \in C^b(m_X; m_X^q)$, let $Y \subset X$ be the union of the supp. of the homology sheaves of C. We can find q.is.

$$C' \xrightarrow{\sim} C$$

with C' supported on Y. //

Cave in and assume $X = \operatorname{Spec} A$ affine. Subcat. w. cof. and w.e.

$$\coprod_{\operatorname{ht} \mathfrak{p} = q} \bigcup_{n \geq 1} C^b(\mathcal{M}_{A/\mathfrak{p}^n}; \mathcal{M}_{A/\mathfrak{p}^n}, \mathcal{M}^1_{A/\mathfrak{p}^n})$$

$$\hookrightarrow C^b(\mathcal{M}^q_A; \mathcal{M}^q_A, \mathcal{M}^{q+1}_A).$$

<u>Lemma</u> The inclusion above induces a w.e. on K-theory.

Pf If $M \in \mathcal{M}^q_A$, and $\operatorname{ht} \mathfrak{p} = q$, let

$$M(\mathfrak{p}) = \operatorname{im}(M \to M_{\mathfrak{p}}).$$

Then $M(\mathfrak{p})_{\mathfrak{p}} \xrightarrow{\sim} M_{\mathfrak{p}}$, and $M(\mathfrak{p}) \neq 0$ if and only if $\mathfrak{p}$ is a minimal ass. prime of M. Consider

$$M' \rightarrowtail M \twoheadrightarrow M'' = \bigoplus_{\operatorname{ht} \mathfrak{p} = q} M(\mathfrak{p}).$$

Then M' is supported in codimension $q+1$. Construction is functorial, so applies to complexes. //

Lemma Taking stalks induces a w.e.:

$$K\Big(\bigcup_{n\geq 1} C^b(\mathcal{M}_{A/\mathfrak{p}^n}; \mathcal{M}_{A/\mathfrak{p}^n}, \mathcal{M}^1_{A/\mathfrak{p}^n})$$

$$\xrightarrow{\sim} K\Big(\bigcup_{n\geq 1} C^b(\mathcal{M}_{A_\mathfrak{p}/\mathfrak{p}^n A_\mathfrak{p}}; \mathcal{M}_{A_\mathfrak{p}/\mathfrak{p}^n A_\mathfrak{p}}, 0\Big).$$

pf Can view an $A_\mathfrak{p}/\mathfrak{p}^n A_\mathfrak{p}$-module N as an $A/\mathfrak{p}^n$-module. Write as colim. of f.g. submodules; pick one M s.t. $M_\mathfrak{p} \xrightarrow{\sim} N$. That's the one we need. //

By devissage

$$C^b(\mathcal{M}_{k(\mathfrak{p})}) \hookrightarrow \bigcup_{n\geq 1} C^b(\mathcal{M}_{A_\mathfrak{p}/\mathfrak{p}^n A_\mathfrak{p}})$$

induces a w.e. on K-theory. Putting everything together, we get:

$$K_*^{q/q+1}(X) \cong \bigoplus_{x\in X^{(q)}} K_*(k(x)).$$

4/2/02

S scheme

E loc. free $\mathcal{O}_S$ - mod. of rk. r

$SE = S_{\mathcal{O}_S}(E)$ symm. alg.

$X = \mathrm{Proj}(SE) \xrightarrow{f} S$

$\mathcal{O}_X(1)$ can. line bdl. on X

Recall coh. calc.

Lemma (a) for every q.-coh. $\mathcal{O}_X$ - mod. F, $R^q f_* F$ is a q.-coh. $\mathcal{O}_S$ - mod.; it is zero if $q \geq r$.

(b) for every q.-coh. $\mathcal{O}_X$ - mod. F and loc. free $\mathcal{O}_S$ - mod. E',

$$R^q f_*(F \otimes_{\mathcal{O}_S} E') \xleftarrow{\sim} R^q f_*(F) \otimes_{\mathcal{O}_S} E'$$

(c) for every q.-coh. $\mathcal{O}_S$ - mod. N,

$$R^q f_*(\mathcal{O}_X(n)) \otimes_{\mathcal{O}_S} N \xrightarrow{\sim} R^q f_*(\mathcal{O}_X(n) \otimes_{\mathcal{O}_S} N)$$

(d) $R^q f_*(\mathcal{O}_X(n)) = 0, \quad q \neq 0, r-1.$

(e) $f_* \mathcal{O}_X(n) = S_n E$

(f) $R^{r-1} f_* \mathcal{O}_X(-r) \cong \Lambda^r_{\mathcal{O}_S} E$

(g) there is a perfect pairing

$$f_*\mathcal{O}_X(n) \otimes_{\mathcal{O}_S} R^{r-1} f_* \mathcal{O}_X(-n-r)$$
$$\xrightarrow{\sim} R^{r-1} f_* \mathcal{O}_X(-r) .$$ //

Cor For every q.-coh. $\mathcal{O}_S$-mod. N,

$$R^q f_*(\mathcal{O}_X(n) \otimes_{\mathcal{O}_S} N) = \begin{cases} S_n E \otimes_{\mathcal{O}_S} N , & q = 0 \\ (S_{-r-n} E)^\vee \otimes_{\mathcal{O}_S} \Lambda^r_{\mathcal{O}_S} E \otimes_{\mathcal{O}_S} N , & q = r-1 \\ 0 , & \text{else.} \end{cases}$$

Call q.-coh. $\mathcal{O}_X$-mod. F <u>regular</u> if

$$R^q f_*(F(-q)) = 0 , \quad q > 0 .$$

<u>ex</u> $F = \mathcal{O}_X(n) \otimes_{\mathcal{O}_S} N$ is reg. if $n \geq 0$:

$$R^{r-1} f_*(F(-r+1)) = R^{r-1} f_*(\mathcal{O}_X(n-r+1) \otimes_{\mathcal{O}_S} N)$$

$$= (S_{-r-n+r-1} E)^\vee \otimes \Lambda^r E \otimes N = 0 .$$ //

Forgot to mention: every q.-coh. $\mathcal{O}_X$-mod. of f. type is a quotient of $\mathcal{O}_X(n)^k$, some n, k.

<u>Rem</u> $\mathrm{rk}_{\mathcal{O}_S} R^{r-1} f_* \mathcal{O}_X(n) = \binom{-n-1}{r-1}$.

<u>Lemma</u> There is a nat. exact seq.

$$0 \to F(-r) \otimes_{\mathcal{O}_S} \Lambda^r E \to \cdots \to F(-1) \otimes_{\mathcal{O}_S} E \to F \to 0,$$

for q.-coh. $\mathcal{O}_X$-mod. F.

<u>pf</u> It suffices to consider $F = \mathcal{O}_X$. The seq. is the complex

$$\bigotimes_{\mathcal{O}_X} \Big(\underset{1}{\mathcal{O}_X(-1) \otimes_{\mathcal{O}_S} E} \longrightarrow \underset{0}{\mathcal{O}_X} \Big).$$ //

<u>Cor</u> F reg. $\Rightarrow$ $F(n)$ reg., $n \geq 0$.

<u>pf</u> Note that $(F(-p) \otimes_{\mathcal{O}_S} \Lambda^p E)(p)$ is reg.:

$$R^q f_* ((F(-p) \otimes_{\mathcal{O}_S} \Lambda^p E)(p-q))$$

$$\xleftarrow{\sim} R^q f_* (F(-q)) \otimes_{\mathcal{O}_S} \Lambda^p E = 0.$$

Split seq. of lemma in s.e.s.:

$$0 \to Z_p \to F(-p) \otimes_{\mathcal{O}_S} \Lambda^p E \to Z_{p-1} \to 0$$

Claim that $Z_p(p+1)$ is reg., $p \geq 0$. OK if $p = r$. Induction step:

$$0 \to Z_p(p) \to \overset{\text{reg.}}{(F(-p)\otimes_{\mathcal{O}_S} \Lambda^p E)(p)} \to Z_{p-1}(p) \to 0 \qquad \rightarrow$$

$$\overset{\overset{0}{\shortparallel}}{R^q f_*((F(-p)\otimes\Lambda^p E)(p-q))} \to R^q f_*(Z_{p-1}(p-q))$$

$$\xrightarrow{\partial} \underset{\underset{0}{\shortparallel \text{ — induction}}}{R^{q+1} f_*(Z_p(p+1-(q+1)))}$$

so $Z_p(p+1)$ reg., $p \geqslant 0$. In particular, $Z_0(1) = F(1)$ is reg. By induction, $F(n)$ is reg., $n \geqslant 0$. //

Lemma F reg. $\Rightarrow$ $\mathcal{O}_X \otimes_{\mathcal{O}_S} f_*(F) \twoheadrightarrow F$.

Pf $0 \to Z_1 \to F(-1)\otimes_{\mathcal{O}_S} E \to F \to 0$

Claim: $R^1 f_*(Z_1(n)) = 0$, $n \geqslant 1$

Indeed, $Z_1(2)$ is reg., hence so is $Z_1(n+1)$, for $n \geqslant 1$. So $R^1 f_*(Z_1(n+1-1)) = 0$.

It follows that

$$f_*(F(n-1))\otimes_{\mathcal{O}_S} E \to f_*(F(n)) \xrightarrow{\partial} 0$$

if $n \geqslant 1$. Inductively, get

$$f_*(F)\otimes_{\mathcal{O}_S} S_n E \longrightarrow f_*(F(n)) \to 0$$

and hence

$$f_*(F) \otimes_{\mathcal{O}_S} SE \twoheadrightarrow \bigoplus_{n\geq 0} f_*(F(n)).$$

Thus

$$\begin{tikzcd}
(f_*(F) \otimes_{\mathcal{O}_S} SE)^{\sim} \arrow[r, two heads] \arrow[d, equal] & (\bigoplus_{n\geq 0} f_* F(n))^{\sim} \arrow[d, "\sim"] \\
f_*(F) \otimes_{\mathcal{O}_S} \mathcal{O}_X \arrow[r, two heads] & F
\end{tikzcd}$$ //

Prop For reg. $\mathcal{O}_X$-mod. F, there is a natural exact seq.

$$0 \to \mathcal{O}_X(-r+1) \otimes_{\mathcal{O}_S} T_{r-1}(F) \to \cdots \to \mathcal{O}_X \otimes_{\mathcal{O}_S} T_0(F) \to F \to 0$$

where $F \mapsto T_i(F)$, $0 \leq i < r$, are exact functors from reg. $\mathcal{O}_X$-mod. to q.-coh. $\mathcal{O}_S$-mod.

To begin the proof, suppose F admits a res.

$$0 \to \mathcal{O}_X(-r+1) \otimes_{\mathcal{O}_S} T_{r-1} \to \cdots \to \mathcal{O}_X \otimes_{\mathcal{O}_S} T_0 \to F \to 0$$

with T_i q.-coh. $\mathcal{O}_S$-mod. Then F is necessarily reg. For example, if $r=2$:

$$0 \to \mathcal{O}_X(-1) \otimes_{\mathcal{O}_S} T_1 \to \mathcal{O}_X \otimes_{\mathcal{O}_S} T_0 \to F \to 0 \quad \Rightarrow$$

$$\begin{tikzcd}
R^q f_*(\mathcal{O}_X \otimes_{\mathcal{O}_S} T_0(-q)) \arrow[r] \arrow[d, equal] & R^q f_*(F(-q)) \arrow[r, "\partial"] & {} \\
0 & R^{q+1}((\mathcal{O}_X(-1) \otimes_{\mathcal{O}_S} T_1)(-q)) \arrow[d, equal] & \\
 & R^{q+1}((\mathcal{O}_X \otimes_{\mathcal{O}_S} T_1)(-(q+1))) = 0 &
\end{tikzcd}$$

.

__Lemma__ The $\mathcal{O}_S$-mod. T_i are determined up to can. iso. by F.

Pf For all $n \geq 0$, the exact seq.

$$0 \to \mathcal{O}_X(n-r+1) \otimes_{\mathcal{O}_S} T_{r-1} \to \cdots \to \mathcal{O}_X(n) \otimes_{\mathcal{O}_S} T_0 \to F(n) \to 0$$

consists of $\mathcal{O}_X$-mod. which are acyclic for $R^q f_*$, $q \geq 1$. So the seq. remains exact if we apply f_*. Get:

$$0 \to S_{n-r+1}E \otimes T_{r-1} \to \cdots \to S_n E \otimes T_0 \to f_*(F(n)) \to 0$$

If $0 \leq n < r$, this reads:

$$0 \to T_n \to E \otimes_{\mathcal{O}_S} T_{n-1} \to \cdots \to S_n E \otimes T_0 \to f_*(F(n)) \to 0$$

The lemma follows.

Define

$T_n = T_n(F)$ q-coh. $\mathcal{O}_S$-mod

$Z_n = Z_n(F)$ q-coh. $\mathcal{O}_X$-mod

as follows: let

$Z_{-1} = F$,

and, inductively,

$T_n = f_*(Z_{n-1}(n))$

$$0 \to Z_n \xrightarrow[\text{def}]{} \mathcal{O}_X(-n) \otimes_{\mathcal{O}_S} T_n \xrightarrow{\text{can}} Z_{n-1}$$

Claim F reg. $\Rightarrow$ $Z_n(n+1)$ reg.

Pf $Z_{-1}(0) = F$ reg. Induction:

$$0 \to Z_n(n) \to \mathcal{O}_X \otimes_{\mathcal{O}_S} T_n \to Z_{n-1}(n) \to 0$$

| by induction

$$\begin{tikzcd} R^q f_*(Z_{n-1}(n-q)) \arrow[r, "\partial"] \arrow[d, phantom, "\shortparallel"] & R^{q+1}(Z_n(n+1-(q+1))) \arrow[r] & {} \\ 0 & R^{q+1}(\mathcal{O}_X \otimes_{\mathcal{O}_S} T_n(-q)) \arrow[d, phantom, "\shortparallel"] & \\ & R^{q+1}((\mathcal{O}_X \otimes_{\mathcal{O}_S} T_n)(1)(-(q+1))) \arrow[d, phantom, "\shortparallel\ \text{lemma}"] & \\ & 0 & \end{tikzcd}$$

/

Note: $f_* Z_n(n) = 0$, by def'n.

<u>Claim</u> f_* exact on reg. $\mathcal{O}_X$-mod.

<u>pf</u> $0 \to F' \to F \to F'' \to 0 \Rightarrow$

$$0 \to f_*F' \to f_*F \to f_*F'' \xrightarrow{\partial} R^1 f_* F'$$

$$\| $$

$$R^1 f_*(F'(1-1)) = 0. \quad //$$

<u>Cor</u> $T_n = f_*(Z_{n-1}(n))$ is an exact functor from reg. $\mathcal{O}_X$-mod. to q.-coh. $\mathcal{O}_S$-mod.

<u>pf</u> Inductively, $Z_{n-1}(n)$ is an exact endo-functor on reg. $\mathcal{O}_X$-mod. //

<u>Claim</u> $Z_{r-1} = 0$.

<u>pf</u> $0 \to Z_{n+q}(n) \to \mathcal{O}_X(-q) \otimes_{\mathcal{O}_S} T_{n+q} \to Z_{n+q-1}(n) \to 0 \Rightarrow$

$$R^{q-1} f_*(Z_{n+q-1}(n)) \to R^q(Z_{n+q}(n)) \to R^q f_*(\mathcal{O}_X \otimes_{\mathcal{O}_S} T_{n+q}(-q)) = 0$$

Since $f_*(Z_n(n)) = 0$, get inductively that $R^q f_*(Z_{n+q}(n)) = 0$, all $n, q \geq 0$. Hence $Z_{r-1}(r-1)$ is regular:

$$R^q f_*(Z_{r-1}(r-1-q)) = 0, \quad 1 \leq q \leq r-1,$$

by the above, and $R^q f_* F = 0$, $q \geq r$, for all F. Hence

$$\begin{tikzcd}
\mathcal{O}_X \otimes_{\mathcal{O}_S} f_*(Z_{r-1}(r-1)) \arrow[r, two heads] \arrow[d, equal] & Z_{r-1}(r-1) \\
0 &
\end{tikzcd}$$

so $Z_{r-1}(r-1) = 0$, hence $Z_{r-1} = 0$. //

This finishes the proof of prop.

4/4/02

S noeth. scheme

E loc. free $\mathcal{O}_S$-mod., $\mathrm{rk}_{\mathcal{O}_S} E = r$

$X = \mathrm{Proj}(SE) \xrightarrow{f} S$

Lemma (Serre) If F is a coh. $\mathcal{O}_X$-module, then $R^q f_* F$ is a coh. $\mathcal{O}_S$-mod., $q \geq 0$.

Pf Since F is a q.-coh. $\mathcal{O}_X$-mod. of f. type, there exists an epi. $\mathcal{O}_X(n)^k \twoheadrightarrow F$, some k, n. Since F is coh., the kernel K is again coh. Prove lemma by downward induction on $q \geq 0$. Ok for $q \geq r$, since $R^q f_* = 0$. Induction step: consider

$$\cdots \to \underset{\substack{| \\ \text{coh. by calc.}}}{R^q f_*(\mathcal{O}_X(n)^k)} \to R^q f_*(F) \xrightarrow{\partial} \underset{\substack{| \\ \text{coh. by ind.}}}{R^{q+1} f_*(K)} \to \cdots$$

Since S is noeth., every submodule of a coh. $\mathcal{O}_S$-mod. is coh. And if two out of three mod. in a s.e.s. are coh., so is the third. //

Cor For reg. coh. $\mathcal{O}_X$-mod. F, there is a functorial resolution

$$0 \to \mathcal{O}_X(-r+1) \otimes_{\mathcal{O}_S} T_{r-1}(F) \to \cdots \to \mathcal{O}_X \otimes_{\mathcal{O}_S} T_0(F) \to F \to 0$$

where the T_i are exact functors from reg. coh. $\mathcal{O}_X$-mod. to coh. $\mathcal{O}_S$-mod.

pf Recall : $Z_{-1} = F$, and inductively,

$$T_n = f_*(Z_{n-1}(n))$$

$$0 \to Z_n \to \mathcal{O}_X(-n) \otimes_{\mathcal{O}_S} T_n \to Z_{n-1} \to 0 ,$$

use lemma. //

Beilinson has shown :

$$T_i(F) = f_*(\Omega^i_{X/S}(i) \otimes_{\mathcal{O}_X} F) .$$

$$\mathcal{O}_X(-1) \otimes_{\mathcal{O}_S} E \xrightarrow{d} \mathcal{O}_X \quad \Rightarrow$$

$$\Lambda^n_{\mathcal{O}_X}(\mathcal{O}_X(-1) \otimes_{\mathcal{O}_S} E) \xrightarrow{d} \Lambda^{n-1}_{\mathcal{O}_X}(\mathcal{O}_X(-1) \otimes_{\mathcal{O}_S} E)$$

$$x_1 \wedge \dots \wedge x_n \longmapsto \sum_{i=1}^{n} (-1)^i d(x_i)\, x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge x_n$$

<u>Lemma</u> This is exact.

<u>pf</u> The question is local, so we can assume $S = \operatorname{Spec} A$, $E = A\{t_1, \dots, t_r\}$, $SE = A[t_1, \dots, t_r]$. The cx. is:

$$\bigotimes_{i=1}^{r}{}_{SE} \left(SE \xrightarrow{t_i} SE \right)$$

which has homology conc. in degree 0:

$$H_0 = \bigotimes_{i=1}^{r}{}_{A[t_1,\dots,t_r]} A[t_1, \dots, t_r]/(t_i)$$

$$= A[t_1, \dots, t_r]/(t_1, \dots, t_r) = A.$$

But $A^\sim = 0$, so the cx. of v.b. on $\operatorname{Proj}(SE)$ is exact. //

$$0 \to \mathcal{O}_X(-r)\underset{\mathcal{O}_S}{\otimes}\Lambda^r E \to \cdots \to \mathcal{O}_X(-1)\underset{\mathcal{O}_S}{\otimes} E \to \mathcal{O}_X \to 0 \quad \xRightarrow{\underline{Hom}_{\mathcal{O}_X}(-,\mathcal{O}_X)}$$

$$0 \leftarrow \mathcal{O}_X(r)\underset{\mathcal{O}_S}{\otimes}\Lambda^r E^\vee \leftarrow \cdots \leftarrow \mathcal{O}_X(1)\underset{\mathcal{O}_S}{\otimes} E^\vee \leftarrow \mathcal{O}_X \leftarrow 0 \quad \xRightarrow{F\underset{\mathcal{O}_X}{\otimes} -}$$

$$0 \to F \to F(1)\underset{\mathcal{O}_S}{\otimes} E^\vee \to \cdots \to F(r)\underset{\mathcal{O}_S}{\otimes}\Lambda^r E^\vee \to 0$$

$\mathcal{M}_n \subset \mathcal{M}_X$: $R^q f_*(F(n)) = 0$, $n \geq k$ and $q \geq 0$.

Serre: $\bigcup_{n\geq 0} \mathcal{M}_n = \mathcal{M}_X$.

$p>0$, then $F \mapsto F(p)\underset{\mathcal{O}_S}{\otimes}\Lambda^p E$ is an exact functor from $\mathcal{M}_n$ to $\mathcal{M}_{n-p} \hookrightarrow \mathcal{M}_{n-1}$, so defines

$$K_*(\mathcal{M}_n) \xrightarrow{u_p} K_*(\mathcal{M}_{n-1});$$

by add. thm,

$$K_*(\mathcal{M}_n) \xrightarrow{\sum_{p>0}(-1)^p u_p} K_*(\mathcal{M}_{n-1})$$

is a htpy inv. of the map ind. by the inclusion $\mathcal{M}_{n-1} \hookrightarrow \mathcal{M}_n$. So

$$K_*(\mathcal{M}_0) \xrightarrow{\sim} K_*(\mathcal{M}_1) \xrightarrow{\sim} \cdots \xrightarrow{\sim} K_*(\mathcal{M}_X).$$

$R_n \subset \mathcal{M}_X$: F s.t. $F(k)$ reg. , $n \geq k$.

<u>Claim</u> $\bigcup_{n \geq 0} R_n = \mathcal{M}_X$.

<u>Pf</u> Serre: given F, there exists $n_0 \in \mathbb{Z}$ s.t. for $n \geq n_0$, $R^q f_*(F(n)) = 0$, $q > 0$. Let $k = n_0 + r - 1$, then for $0 < q < r$ and $n \geq k$,

$$R^q f_*(F(n-q)) = 0$$

since $n - q \geq n - r + 1 \geq n_0$. So $F \in R_n$. //

<u>Cor</u> $K(R_0) \xhookrightarrow{\sim} K(\mathcal{M}_X) = K'(X)$. //

—

$\mathcal{M}_S \xrightarrow{U_n} \mathcal{M}_0$ exact $\leadsto$ $K'(S) \xrightarrow{u_n} K(\mathcal{M}_0)$

$N \longmapsto \mathcal{O}_X(-n) \otimes_{\mathcal{O}_S} N$

$\mathcal{M}_0 \xrightarrow{v_n} \mathcal{M}_S$ exact $\Rightarrow$ $K(\mathcal{M}_0) \xrightarrow{v_n} K'(S)$

$F \longmapsto f_*(F(n))$ \ if $n \geq 0$ by def$^{\underline{n}}$ of $\mathcal{M}_0$.

consider

$$K'_q(S)^r \underset{v}{\overset{u}{\rightleftarrows}} K_q(\mathcal{M}_0)$$

$$(a_n)_{0 \leq n < r} \longmapsto u_0(a_0) + \cdots + u_{r-1}(a_{r-1})$$

$$(v_n(x))_{0 \leq n < r} \longmapsfrom x$$

$$v_n u_m(N) = f_*(\mathcal{O}_X(n-m) \otimes_{\mathcal{O}_S} N) = S_{n-m}E \otimes_{\mathcal{O}_S} N \Rightarrow$$

$K'_q(S)^r \xrightarrow{vu} K'_q(S)^r$ given by triangular matrix w. 1 on diag. $\Rightarrow$

$K'_q(S)^r \xhookrightarrow{u} K_q(\mathcal{M}_0)$ injective.

$\mathcal{R}_0 \xrightarrow{T_n} \mathcal{M}_S$ exact $\Rightarrow$ $K(\mathcal{R}_0) \xrightarrow{t_n} K'(S)$

$$K_q(\mathcal{R}_0) \xrightarrow{t} K'_q(S)^r \xrightarrow{u} K_q(\mathcal{M}_0)$$

$$x \longmapsto ((-1)^n t_n(x))_{0 \le n < r}$$

comp. equal to map induced by $\mathcal{R}_0 \hookrightarrow \mathcal{M}_0$; since this is an iso., u is onto. Hence:

<u>Thm</u> Let S be a noeth. scheme, let E be a loc. free $\mathcal{O}_S$-mod. of rk. r, and let

$$X = \mathrm{Proj}(S_{\mathcal{O}_S}(E)) \xrightarrow{f} S ;$$

then

$$K'_q(S)^r \xrightarrow[\sim]{u} K'_q(X)$$

$$(a_n)_{0 \le n < r} \longmapsto u_0(a_0) + \dots + u_{r-1}(a_{r-1})$$

is an iso. //

<u>Lemma</u> Suppose S is quasi-cpt. Then for every loc. free $\mathcal{O}_X$-mod. F, there exists an integer n_0 s.t. for all $n \geq n_0$ and all q.-coh. $\mathcal{O}_S$-mod. N,

(a) $R^q f_*(F(n) \otimes_{\mathcal{O}_S} N) = 0$, $q > 0$.

(b) $f_*(F(n) \otimes_{\mathcal{O}_S} N) \xleftarrow{\sim} f_*(F(n)) \otimes_{\mathcal{O}_S} N$

(c) $f_*(F(n))$ is a loc. free $\mathcal{O}_S$-mod.

pf Can assume S affine. Then F being coh. is a quotient for $L = \mathcal{O}_X(-n)^k$, some $n, k \geq 0$. Consider

$$0 \to F' \to L \to F \to 0 .$$

Then F' is a loc. free $\mathcal{O}_X$-mod. and the lemma is true for L by Serre's calc. To prove (a) use

$$0 \to F'(n) \otimes_{\mathcal{O}_S} N \to L(n) \otimes_{\mathcal{O}_S} N \to F(n) \otimes_{\mathcal{O}_S} N \to 0 \quad \Rightarrow$$

$$R^q f_*(L(n) \otimes_{\mathcal{O}_S} N) \to R^q f_*(F(n) \otimes_{\mathcal{O}_S} N) \to R^{q+1} f_*(F'(n) \otimes_{\mathcal{O}_S} N)$$

and downward induction on $q > 0$.

It follows from (a) that for $n \geq n_0$, we have a diagr. w. exact rows

$$\begin{tikzcd}
 & f_*(F'(n)) \otimes_{\mathcal{O}_S} N \arrow[r] \arrow[d, "u'"] & f_*(L(n)) \otimes_{\mathcal{O}_S} N \arrow[r] \arrow[d, "\sim"] & f_*(F(n)) \otimes_{\mathcal{O}_S} N \arrow[r] \arrow[d, "u"] & 0 \\
0 \arrow[r] & f_*(F'(n) \otimes_{\mathcal{O}_S} N) \arrow[r] & f_*(L(n) \otimes_{\mathcal{O}_S} N) \arrow[r] & f_*(F(n) \otimes_{\mathcal{O}_S} N) \arrow[r] & 0
\end{tikzcd}$$

It follows that u is surj. But then u' is surj (apply statement to F'), hence u is iso.

By (a), $N \mapsto f_*(F(n) \otimes_{\mathcal{O}_S} N)$ is exact, for $n \geq n_0$; by (b), $N \mapsto f_*(F(n)) \otimes_{\mathcal{O}_S} N$ is exact, for $n \geq n_0$; hence $f_*(F(n))$ is a flat $\mathcal{O}_S$-mod. Moreover, $f_*(F(n))$, being a quotient of the coh. $\mathcal{O}_S$-mod. $f_*(L(n))$, is of f-type. The same arg. shows that $f_*(F'(n))$ is of fin. type. So $f_*(F(n))$ is of fin. presentation. But a fin. presented flat module is loc. free. //

<u>Lemma</u> If F is a loc. free $\mathcal{O}_X$-mod. and if $R^q f_*(F(n)) = 0$, for all $q > 0$ and $n \geq 0$, then $f_*(F(n))$ is loc. free.

pf Can assume S is affine; by previous lemma, $f_*(F(n))$ is loc. free for $n \gg 0$. Use

$$0 \to F(n) \to F(n+1) \otimes_{\mathcal{O}_S} E^\vee \to \cdots \to F(n+r) \otimes_{\mathcal{O}_S} \Lambda^r E^\vee \to 0$$

For $n \geq 0$, this is a seq. of $\mathcal{O}_X$-mod. acyclic for $R^q f_*$, $q > 0$. Using the sp. seq. of a double cx. applied to f_* of an inj. res. of this complex, we see that

$$0 \to f_*(F(n)) \to f_*(F(n+1) \underset{\mathcal{O}_S}{\otimes} E^\vee) \to \cdots \to f_*(F(n+r) \underset{\mathcal{O}_S}{\otimes} \Lambda^r E^\vee) \to 0$$

is exact. It follows, by downward ind. on n, that $f_*(F(n))$ is loc. free. //

Cor If F is a reg. loc. free $\mathcal{O}_X$-mod. then $T_n(F)$ is a loc. free $\mathcal{O}_S$-mod., $0 \leq n < r$. //

Get as before:

Thm Let S be a quasi-cpt. scheme, let E be a loc. free $\mathcal{O}_S$-mod. of rk r, and let $X = \mathrm{Proj}(SE) \xrightarrow{f} S$. Then

$$K_q(S)^r \xrightarrow{\sim} K_q(X)$$

$$(a_n)_{0 \leq n < r} \longmapsto \sum_{i=0}^{r-1} z^i \cdot f^*(a_i),$$

where $z \in K_0(X)$ is the class of the can. line bdl. $\mathcal{O}_X(-1)$.

pf Same as before; products ... //

4/11/02

$\mathcal{E}$ cat. w. sum ; $*$ initial ob.

$N_\bullet^\oplus \mathcal{E}$ simpl. cat. : let $\underline{n} = \{1, 2, \ldots, n\}$,

$$N_n^\oplus \mathcal{E} \subset \mathcal{E}^{\mathcal{P}(\underline{n})}$$

full subcat of functors $\mathcal{P}(\underline{n}) \xrightarrow{c} \mathcal{E}$ s.t.

(i) $c_\emptyset = *$

(ii) $c_S \vee c_T \xrightarrow{\cong} c_{S \amalg T}$

Simplicial structure:

$$[m] \xrightarrow{\theta} [n] \Rightarrow \mathcal{P}(\underline{m}) \xrightarrow{\theta_*} \mathcal{P}(\underline{n})$$

$$\theta_*(\{i\}) = \{ j \in \underline{n} \mid \theta(i-1) < j \leq \theta(i) \}$$

$$\theta_*(S \amalg T) = \theta_*(S) \amalg \theta_*(T).$$

ex an object of $N_2^\oplus \mathcal{E}$ is a sum-diagr:

$$\begin{tikzcd}
 & c_1 \arrow[r] \arrow[d] \arrow[dr] & c_{12} \arrow[d] & c_2 \arrow[l] \arrow[d] \arrow[dl] & \\
 & c_{13} \arrow[r] \arrow[dl, mapsto, "d_0"'] & c_{123} & c_{23} \arrow[l] \arrow[dr, mapsto, "d_2"] & \\
c_2 \to c_{23} \leftarrow c_3 & & c_3 \arrow[ul] \arrow[u] \arrow[ur] \arrow[d, mapsto, "d_1"] & & c_1 \to c_{12} \leftarrow c_2 \\
 & & c_{12} \to c_{123} \leftarrow c_3 & &
\end{tikzcd}$$

$w\mathcal{E} \subset \mathcal{E}$ cat. w. w.e. if $c \xrightarrow{\sim} c'$ and $d \xrightarrow{\sim} d'$ implies $c \vee c' \xrightarrow{\sim} d \vee d'$. Define

$$wN_n^\oplus \mathcal{E} = N_n^\oplus \mathcal{E} \cap (w\mathcal{E})^{\mathcal{P}(\underline{n})} .$$

Consider the "group-completion":

$$|N_\cdot wN_\cdot^\oplus \mathcal{E}| .$$

$\mathcal{E}$ cat. w. cof. and w.e.; $\mathcal{E}$ is a cat. w. sum and w.e. by neglect of structure. Map of simpl. cat.:

$$wN_\cdot^\oplus \mathcal{E} \longrightarrow w.S.\mathcal{E}$$

$$\begin{tikzcd}
\mathcal{P}(\underline{n}) \arrow[r, "c"] & \mathcal{E} \arrow[r, mapsto] & c_{\{1\}} \arrow[r, tail] & c_{\{1,2\}} \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] & c_{\{1,2,\dots,n\}} \arrow[d] \\
& & & c_{\{2\}} \arrow[r, tail] & \cdots \arrow[r, tail] & c_{\{2,\dots,n\}} \arrow[d] \\
& & & & & \vdots \arrow[d] \\
& & & & & c_{\{n\}}
\end{tikzcd}$$

The vertical maps on the right are:

$$\begin{tikzcd}
* \arrow[r, tail] \arrow[d] \arrow[rr, bend left] & c \arrow[r] \arrow[d] & * \arrow[d] \\
c' \arrow[r, tail] \arrow[rr, bend right, "\simeq"'] \arrow[ur, phantom, "\text{p.c.}"] & c \vee c' \arrow[r] & c''
\end{tikzcd} \implies \text{preferred retraction } c \vee c' \longrightarrow c'$$

Informally, the map is

$$\begin{array}{ccc} c_1, \ldots, c_n & & c_1 \rightarrowtail c_1 \vee c_2 \rightarrowtail \cdots \rightarrowtail c_1 \vee \cdots \vee c_n \\ + & \longmapsto & + \\ \text{choices of sums} & & \text{choices of sub-quot.} \end{array}$$

Thm If $\mathcal{E}$ is additive and if cofibrations split in $\mathcal{E}$, then

$$|N.iN.^{\oplus}\mathcal{E}| \xrightarrow{\sim} |N.iS.\mathcal{E}| .$$

Let $\mathcal{E}_X$ be the cat. of "cofibrant ob. under X".

ob $\mathcal{E}_X$: $X \rightarrowtail A$

mor $\mathcal{E}_X$:

$$\begin{tikzcd} & A \arrow[d] \\ X \arrow[ur, tail] \arrow[dr, tail] & \\ & A' \end{tikzcd}$$

Cat. w. sum : $(X \rightarrowtail A) \vee (X \rightarrowtail A')$

$$:= X \rightarrowtail A \cup_X A'$$

If $\mathcal{E}$ is additive :

$$A \oplus_X A \xrightarrow{\cong} A \oplus (A/X)$$

$$\begin{tikzcd}
X \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "p.o."] & A \arrow[d, "\sim"] \arrow[ddr, bend left, "i_1"] & \\
A \arrow[r, tail] \arrow[dd, "\Delta"'] & A \oplus_X A \arrow[dr, dashed] & \\
 & & A \oplus (A/X) \\
A \times A & A \oplus A \arrow[l, "\cong"'] \arrow[ur, "pr"] &
\end{tikzcd}$$

If $\mathcal{E}$ has a cylinder functor which satisfies ... one gets a chain of w.e.

$$\begin{tikzcd}
\Sigma A \vee_{\Sigma X} \Sigma A \arrow[r, phantom, "\sim"] & \Sigma A \vee (\Sigma A/\Sigma X) \\
\Sigma X \arrow[u, tail] & \Sigma X \arrow[u, tail]
\end{tikzcd}$$

In this case the theorem becomes :

$$\begin{tikzcd}
\operatorname{colim}_{\Sigma} |N.wN.^{\oplus}\mathcal{E}| \arrow[r, "\sim"] & \operatorname{colim}_{\Sigma} |N.wS.\mathcal{E}| \\
 & |N.wS.\mathcal{E}| \arrow[u, "\sim"']
\end{tikzcd}$$

$\mathcal{E} \xrightarrow{f} \mathcal{D}$ map of cat. w. sum and w-e.

$$\begin{tikzcd}
N_\cdot^\oplus(\mathcal{E} \xrightarrow{f} \mathcal{D}) \arrow[r] \arrow[d] \arrow[dr, phantom, "p.b."] & PN_\cdot^\oplus \mathcal{D} \arrow[d, "\varepsilon"] \\
N_\cdot^\oplus \mathcal{E} \arrow[r, "f"] & N_\cdot^\oplus \mathcal{D}
\end{tikzcd}$$

An object of $N_n^\oplus(\mathcal{E} \xrightarrow{f} \mathcal{D})$ is :

$$\mathcal{P}(\underline{n}) \xrightarrow{c} \mathcal{E} \quad + \quad \mathcal{P}(\underline{n+1}) \xrightarrow{d} \mathcal{D}$$

s.t.

$$(\mathcal{P}(\underline{n}) \xrightarrow{c} \mathcal{E} \xrightarrow{f} \mathcal{D}) =$$

$$(\mathcal{P}(\underline{n}) \longrightarrow \mathcal{P}(\underline{n+1}) \xrightarrow{d} \mathcal{D})$$

$$\underline{n} \xrightarrow{+1} \underline{n+1}$$

or, more informally,

$(c_1, -, c_n)$ + choices of sums

$(d_0, -, d_n)$ + choices of sums

s.t.

$$f(c_i) = d_i, \quad 1 \le i \le n,$$

compatibly w. choices of sums.

Maps of simpl. cat. with sum and w.e.

$$\mathcal{D} \longrightarrow N_\cdot^\oplus(\mathcal{E} \xrightarrow{f} \mathcal{D}) \longrightarrow N_\cdot^\oplus \mathcal{E}$$

<u>Lemma</u> There is a fiber sequence

$$|N_\cdot i N_\cdot^\oplus \mathcal{D}| \to |N_\cdot i N_\cdot^\oplus N_\cdot^\oplus(\mathcal{E}\to\mathcal{D})| \to |N_\cdot i N_\cdot^\oplus N_\cdot^\oplus \mathcal{E}| .$$

pf It suffices to show that $\forall n \geq 0$,

$$|N_\cdot i N_\cdot^\oplus \mathcal{D}| \to |N_\cdot i N_n^\oplus N_\cdot^\oplus(\mathcal{E}\to\mathcal{D})| \to |N_\cdot i N_n^\oplus N_\cdot \mathcal{E}|$$

is a fiber sequence. Eq. of cat.

$$\begin{tikzcd}
\mathcal{D} \arrow[r] \arrow[d, equal] & N_n^\oplus(\mathcal{E}\to\mathcal{D}) \arrow[r] \arrow[d, "\sim"] & N_n^\oplus \mathcal{E} \arrow[d, "\sim"] \\
\mathcal{D} \arrow[r] & \mathcal{D}\times\mathcal{E}^n \arrow[r] & \mathcal{E}^n
\end{tikzcd}$$

so

$$|N_\cdot i \mathcal{D}| \to |N_\cdot i N_n^\oplus(\mathcal{E}\to\mathcal{D})| \to |N_\cdot i N_n^\oplus \mathcal{E}|$$

is eq. to a product-fibration; but the base is not connected. The extra $N_\cdot^\oplus$ makes the base connected. //

pf (of thm): in add. case this is easy:

$$\begin{tikzcd}
{|N_\cdot i N_\cdot^{\oplus} \mathcal{E}|} \arrow[r] \arrow[d, phantom, "\shortparallel"] & {|N_\cdot i S_\cdot \mathcal{E}|} \arrow[d, phantom, "\shortparallel"] \\
{|[n] \mapsto |N_\cdot i N_n^{\oplus} \mathcal{E}||} \arrow[r] & {|[n] \mapsto |N_\cdot^i S_n \mathcal{E}||}
\end{tikzcd}$$

and

$$\begin{tikzcd}
N_n^{\oplus} \mathcal{E} \arrow[rr] \arrow[dr, "\sim"] & & S_n \mathcal{E} \arrow[dl, "\sim"'] \\
& \mathcal{E}^n &
\end{tikzcd}$$

//

$\mathcal{E}$ cat. w. sum (and w.e.)

$$|N.w\mathcal{E}| = |N.wN_1^{\oplus}\mathcal{E}| \longrightarrow \Omega|N.wN_\cdot^{\oplus}\mathcal{E}| \, ;$$

on components:

$$\underset{\text{comm. monoid}}{([c], \nu)} \longrightarrow \underset{\text{Grothendieck grp.}}{K_0\big(([c], \nu)\big)} \, ;$$

on homology (if $w = i$):

$$\begin{tikzcd} \pi_0|N.i\mathcal{E}| \curvearrowright H_*(|N.i\mathcal{E}|) \arrow[r] \arrow[dr] & H_*(\Omega|N.iN_\cdot^{\oplus}\mathcal{E}|) \, ; \\ & (\pi_0|N.i\mathcal{E}|)^{-1} H_*(|N.i\mathcal{E}|) \arrow[u, "\sim"'] \end{tikzcd}$$

here

$$H_*(|N.i\mathcal{E}|) = \bigoplus_{[c]} H_*(B\mathrm{Aut}_{\mathcal{E}}(c)) .$$

<u>ex</u> $\mathcal{E} = \mathcal{P}(R)$; then

$$H_*(|N.i\mathcal{E}|) = \bigoplus_{[P]} H_*(B\mathrm{Aut}_R(P))$$

so

$$(\pi_0|N.i\mathcal{E}|)^{-1} H_*(|N.i\mathcal{E}|)$$

$$\xrightarrow{\sim} H_*(K_0(R) \times BGL(R)) .$$

4/18/02

$e_{ij}^{(n)}(\lambda)$ elem. matrix equal to I except for one off-diagonal entry λ in position (i,j).

$$A \rightsquigarrow A \cdot e_{ij}^{(n)}(\lambda)$$

j'th collumn $\longmapsto$ j'th collum $+$ (i'th collumn)$\cdot\lambda$

$E_n(R) \subset GL_n(R)$ subgroup gen. by elem. matrices

ex If $R = k$ is a field, we know from linear algebra:

$$\mathrm{diag}_n(k^*) \xrightarrow{\sim} GL_n(k)/E_n(k).$$

//

Note

$$e_{ij}^{(n)}(\lambda) \cdot e_{ij}^{(n)}(\mu) = e_{ij}(\lambda+\mu)$$

and

$$[e_{ij}^{(n)}(\lambda), e_{ke}^{(n)}(\mu)] = \begin{cases} 1 & , j \neq k, \ i \neq l \\ e_{i,e}^{(n)}(\lambda\mu) & , j = k, \ i \neq l \\ e_{k,j}^{(n)}(-\mu\lambda) & , j \neq k, \ i = l \end{cases}$$

Note: $[e_{i,j}^{(n)}(\lambda), e_{j,i}^{(n)}(\mu)] = ??$

Cor $E_n(R) = [E_n(R), E_n(R)]$, $n \geq 3$.

Lemma (Whitehead) For all $A \in GL_n(R)$,

$$\begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} \in E_{2n}(R)$$

pf This follows from

$$\begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ A^{-1} - I_n & I_n \end{bmatrix} \begin{bmatrix} I_n & I_n \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_n & 0 \\ A_n - I_n & I_n \end{bmatrix} \cdot \begin{bmatrix} I_n & -A^{-1} \\ 0 & I_n \end{bmatrix}$$

since for every $B \in GL_n(R)$,

$$\begin{bmatrix} I_n & B \\ 0 & I_n \end{bmatrix}, \begin{bmatrix} I_n & 0 \\ B & I_n \end{bmatrix} \in E_{2n}(R). \qquad //$$

Cor If $A, B \in GL_n(R)$, then

$$\begin{bmatrix} AB & 0 \\ 0 & I \end{bmatrix} \equiv \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \quad (\text{mod } E_{2n}(R)).$$

pf

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} AB & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} B^{-1} & 0 \\ 0 & B \end{bmatrix}. \qquad //$$

$$\begin{array}{ccc} GL(R) & = & \operatorname*{colim}_n GL_n(R) \\ \cup & & \cup \\ E(R) & = & \operatorname*{colim}_n E_n(R) \end{array}$$

$(=$ subgrp. gen. by $e_{i,j}(\lambda)$.)

<u>Cor</u> $E(R) = [E(R), E(R)] = [GL(R), GL(R)]$.

<u>Pf</u> Only right hand equality needs proof:

$$\begin{bmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} AB & 0 \\ 0 & (AB)^{-1} \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} B^{-1} & 0 \\ 0 & B \end{bmatrix}, \quad //$$

<u>Def</u> $K_1(R) = GL(R)^{ab} = GL(R)/E(R)$.

<u>ex</u> If $R = k$ is a field,

$$\begin{tikzcd} k^* = GL_1(k) \arrow[r, hook] \arrow[rr, bend right, "\sim"'] & GL(k) \arrow[r, two heads] & K_1(k) \end{tikzcd}$$

Indeed, we have already seen that every $A \in GL_n(k)$ is equiv. to a diag. matrix; and, modulo $E(k)$

$$\begin{bmatrix} a_1 & & \\ & a_2 & \\ & & \ddots \end{bmatrix} \equiv \begin{bmatrix} a_1 a_2 \cdots & & \\ & 1 & \\ & & \ddots \end{bmatrix}.$$

ex R commutative, then

$$\begin{tikzcd} R^* = GL_1(R) \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & K_1(R) \arrow[r, "\det"] & R^* \end{tikzcd} \implies$$

$$K_1(R) = R^* \times \underbrace{SL(R)/E(R)}_{SK_1(R)} .$$ //

—

Remember the easy commutator relations for elem. matrices:

$$x_{i,j}^{(n)}(\lambda) \cdot x_{i,j}^{(n)}(\mu) = x_{i,j}^{(n)}(\lambda+\mu)$$

$$[x_{i,j}^{(n)}(\lambda), x_{j,l}^{(n)}(\mu)] = x_{i,l}^{(n)}(\lambda\mu), \quad i \neq l$$

$$[x_{i,j}^{(n)}(\lambda), x_{k,l}^{(n)}(\mu)] = 1, \quad j \neq k, \ i \neq l ;$$

Let $St_n(R)$ be the group gen. by $x_{i,j}^{(n)}(\lambda)$, $1 \leq i \neq j \leq n$, $\lambda \in R$, subject to the above relations; $St_n(R)$ is called the Steinberg group.

Def (Milnor) $K_2(R) = \ker(St(R) \xrightarrow{\phi} E(R))$

$$x_{i,j}(\lambda) \longmapsto e_{i,j}(\lambda) .$$

Lemma $K_2(R)$ is the center of $St(R)$.

Pf First show $E(R)$ has trivial center, so the center of $St(R)$ contains $K_2(R)$. If (a_{ij}) commutes with $e^{(n)}_{k,l}(1)$ then $a_{kl} = 0$ and $a_{kk} = a_{ll}$. So

$$E_{n-1}(R) \subset E_n(R)$$

intersects the center trivially. Hence

$$E(R) = \underset{n}{\mathrm{colim}}\, E_n(R)$$

has trivial center.

Conversely, let $y \in K_2(R) \subset St(R)$. Must show that y lies in the center of $St(R)$, i.e. that y commutes with all $x_{ij}(\lambda)$; suppose

$$y \in St_{n-1}(R) \hookrightarrow St(R),$$

and let $P_n \subset St(R)$ be the subgrp. gen. by

$$x_{1,n}(\lambda), x_{2,n}(\lambda), \dots, x_{n-1,n}(\lambda);$$

Then P_n is commutative,

and every elem. of P_n can be written uniquely as a product

$$x_{1,n}(\lambda)\, x_{2,n}(\lambda) \text{ --- } x_{n-1,n}(\lambda),$$

so $P_n \overset{\phi}{\hookrightarrow} E(R)$.

$$x_{ij}(\lambda)\, P_n\, x_{ij}(-\lambda) \subset P_n$$

if $i, j < n$: check on gen. Since y is in $St_{n-1}(R)$,

$$y P_n y^{-1} \subset P_n.$$

But $\phi(y) = I$, so

$$\phi(ypy^{-1}) = \phi(y)\phi(p)\phi(y^{-1}) = \phi(p);$$

hence $ypy^{-1} = y$, since $P_n \overset{\phi}{\hookrightarrow} E(R)$.

We have shown that y commutes with $x_{i,n}(\lambda)$, $i < n$; similarly, one shows that y commutes with $x_{n,j}(\lambda)$, $j < n$. Hence y commutes with

$$[x_{i,n}(\lambda), x_{n,j}(1)] = x_{i,j}(\lambda)$$

for all $i \neq j$, $i, j < n$. But this is true for all n, so we are done.

So :

$$1 \to K_2(R) \to St(R) \to E(R) \to 1$$

is a *central* extension (in particular $K_2(R)$ is abelian). In fact, this is the *universal* central ext. of $E(R)$. A group G has a universal central ext. if and only if G is perfect. It is then given as follows: pick a presentation

$$1 \to R \to F \to G \to 1 ;$$

then the universal central ext. is

$$\begin{tikzcd}
1 \arrow[r] & (R\cap[F,F])/[R,F] \arrow[r] \arrow[d, equal] & {[F,F]}/[R,F] \arrow[r] & G \arrow[r] & 1 . \\
 & H_2(G,\mathbb{Z}) & & &
\end{tikzcd}$$

prove this:

$$1 \to R \to F \to G \to 1 \overset{H.-S.}{\Longrightarrow}$$

$$E^2_{s,t} = H_s(G, H_t(R,\mathbb{Z})) \Rightarrow H_{s+t}(F,\mathbb{Z})$$

$$\begin{array}{lll} H_0(G, H_1(R, \mathbb{Z})) & * & \\ H_0(G, H_1(R, \mathbb{Z})) & * & \\ \mathbb{Z} & 0 & H_2(G, \mathbb{Z}) \end{array}$$

since free

$$\begin{tikzcd}
H_2(F, \mathbb{Z}) \arrow[r] \arrow[d, phantom, "\sim \|"] & H_2(G, \mathbb{Z}) \arrow[r, "d^2"] \arrow[d, "\sim", "d^2"'] & H_1(R, \mathbb{Z}) \arrow[r] \arrow[d, phantom, "\|"] & H_1(F, \mathbb{Z}) \arrow[r] \arrow[d, phantom, "\|"] & 0 \\
0 \arrow[r] & (R \cap [F,F]) / [R,F] \arrow[r] & R / [R,F] \arrow[r] & F / [F,F] &
\end{tikzcd}$$

4/23/02

k field, A semi-simple k-algebra. Then there are division algebras $D_1, \ldots, D_k$ and integers $r_1, \ldots, r_k$ s.t.

$$A \cong \prod_{i=1}^{k} M_{r_i}(D_i).$$

$$Z(A) \cong \prod_{i=1}^{k} Z(D_i)$$

So A is simple iff $Z(A)$ is a field. If A is a central simple k-alg. ($Z(A) = k$), then

$$\dim_k A = n^2,$$

some $n \geq 1$.

ex C_n cyclic group of order n; then

$$\mathbb{Q}[C_n] \xrightarrow{\sim} \prod_{d \mid n} \mathbb{Q}(\xi_d)$$

$$\mathbb{R}[C_n] \xrightarrow{\sim} \begin{cases} \mathbb{R} \oplus \mathbb{C}(\xi_n) \oplus \cdots \oplus \mathbb{C}(\xi_n^m), & n = 2m+1 \\ \mathbb{R} \oplus \mathbb{C}(\xi_n) \oplus \cdots \oplus \mathbb{C}(\xi_n^m) \oplus \mathbb{R}_-, & n = 2m+2 \end{cases}$$

$$\mathbb{C}[C_n] \xrightarrow{\sim} \prod_{i=0}^{n-1} \mathbb{C}(\xi_n^i).$$

A central simple F-algebra; $Z(A) = F$.
Pick E/F fin. ext. s.t.

$$E \underset{F}{\otimes} A \underset{\sim}{\xrightarrow{\varphi}} M_n(E),$$

Any two such iso. differ by an inner automorphism of $M_n(E)$ by Skolem - Noether; hence

$$\text{Nrd.}(a) := \det(\varphi(1 \otimes a)) \in E^*$$

is well-defined. If E/F is Galois, then $\text{Gal}(E/F)$ fixes $\text{Nrd}(a) \in E^*$, hence $\text{Nrd}(a) \in F^*$; it is is independent of the choice of E/F. So there is a well-defined homomorphism

$$\text{nrd.}: A^* \longrightarrow F^*.$$

Let $n = \dim_F (A)^{1/2}$. Then

(i) $\text{nrd}(a)^n = \det_F (A \xrightarrow{a \cdot} A)$.

(ii) $\text{nrd}(u) = u^n$, if $u \in F^*$

(iii) $\text{nrd} = \det : A^* \to F^*$, if

$$A \cong M_n(F).$$

A central simple F-alg. $\Rightarrow$

$M_n(A)$ central simple F-alg.

$$\begin{tikzcd} A^* \arrow[r, "nrd"] \arrow[d, hook] & F^* \arrow[d, equal] \\ M_n(A)^* \arrow[r, "nrd"] & F^* \end{tikzcd} \rightsquigarrow$$

$$nrd : GL(A) \longrightarrow F^* \rightsquigarrow$$

$$K_1(A) \xrightarrow{Nrd} F^* = Z(A)^*$$

If A is semi-simple,

$A = A_1 \times \cdots \times A_k$, A_i simple,

let

$$\begin{tikzcd} K_1(A) \arrow[r, "Nrd"] \arrow[d, "\sim"] & Z(A)^* \arrow[d, "\sim"] \\ \prod_{i=1}^{k} K_1(A_i) \arrow[r, "\prod Nrd"] & \prod_{i=1}^{k} Z(A_i)^* \end{tikzcd}$$

Let A be a simple $\mathbb{Q}$-algebra, let $F = Z(A)$, and let $R \subset F$ be the integral closure of $\mathbb{Z}$ in F. Let $\Lambda \subset A$ be a $\mathbb{Z}$-order in A. Then

$$\begin{tikzcd}
K_1(A) \arrow[r, "\mathrm{Nrd}"] & F^* \\
K_1(\Lambda) \arrow[u] \arrow[r, dashed] & R^* \arrow[u, hook]
\end{tikzcd}$$

Indeed, if $u \in GL_k(\Lambda)$, then

$$\mathrm{nrd}(u)^n = \det_F\left(A^k \xrightarrow{u\cdot} A^k\right)$$

$$= \det_\Lambda\left(\Lambda^k \xrightarrow{u\cdot} \Lambda^k\right) \in R^*,$$

and hence $\mathrm{nrd}(u) \in R^*$, $n = \dim_F(A)^{1/2}$.

<u>Thm</u> Let A be a semi-simple $\mathbb{Q}$-algebra, and let $R \subset Z(A)$ be the product of the rings of integers in the component fields of $Z(A)$. Then for any $\mathbb{Z}$-order $\Lambda \subset A$, the reduced norm

$$K_1(\Lambda) \xrightarrow{\mathrm{Nrd}} R^*$$

has finite kernel and cokernel. //

Cor let G be a finite group, and let r (resp. q) be the number of simple $\mathbb{R}[G]$-modules (resp. $\mathbb{Q}[G]$)-modules. Then

$$\mathrm{rk}_{\mathbb{Z}} K_1(\mathbb{Z}[G]) = r - q. \qquad //$$

ex $G = C_n$; then

$$\left.\begin{array}{l} r = \left[\frac{n+2}{2}\right] \\ \\ q = n - \varphi(n) \end{array}\right\} \Rightarrow$$

$$\mathrm{rk}_{\mathbb{Z}} K_1(\mathbb{Z}[C_n]) = \left[\frac{n+2}{2}\right] + \varphi(n) - n. \qquad //$$

4/25/02

Thm Let (X, x_0) be a pointed connected space, and let $P \subset \pi_1(X, x_0)$ be a perfect normal subgroup. Then there exists a map

$$(X, x_0) \xrightarrow{f} (X^+, x_0^+)$$

s.t.

(i) $\pi_1(X, x_0)/P \underset{\sim}{\xrightarrow{f_*}} \pi_1(X^+, x_0^+)$;

(ii) for every local coeff. system L on X^+,

$$H_*(X, f^*L) \underset{\sim}{\xrightarrow{f_*}} H_*(X^+, L) ;$$

(iii) every map $(X, x_0) \xrightarrow{g} (Y, y_0)$ with $g_*(P) = 0$ factors uniquely, up to homotopy,

$$\begin{tikzcd}
(X, x_0) \arrow[rr, "g"] \arrow[dr, "f"] & & (Y, y_0) \\
& (X^+, x_0^+) \arrow[ur, dashed] &
\end{tikzcd}$$

.

pf First kill P: choose $(S^1, s_0) \xrightarrow{\gamma_\alpha} (X, x_0)$, $\alpha \in \mathcal{A}$, s.t. $e_\alpha = [\gamma_\alpha]$, $\alpha \in \mathcal{A}$, generate P as a normal subgroup. Define

$$\begin{tikzcd}
\bigvee_{\alpha \in \mathcal{A}} S^1 \arrow[r, "\vee \gamma_\alpha"] \arrow[d] \arrow[dr, phantom, "\text{p.o.}"] & X \arrow[d, "q"] \\
\bigvee_{\alpha \in \mathcal{A}} D^2 \arrow[r] & X_1
\end{tikzcd}$$

.

Then, assuming that $x_0 \in X$ is "good",

$$\pi_1(X,x_0)/P \xrightarrow[\sim]{g} \pi_1(X_1,x_0).$$

Consider the covering spaces

$$\begin{tikzcd}
X' \arrow[r, "\tilde{g}"] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & \tilde{X}_1 \arrow[d, "\pi"] & \text{univ. cov.} \\
X \arrow[r, "g"] & X_1 .
\end{tikzcd}$$

Then $(\tilde{X}_1, X')$ is a rel. $\pi_1(X_1)$-CW-cx.:

$$\begin{tikzcd}
\bigvee_{\alpha\in\mathcal{B}} S^1 \wedge \pi_1(X_1)_+ \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.o.}"] & X' \arrow[d, "\tilde{g}"] \\
\bigvee_{\alpha\in\mathcal{B}} D^2 \wedge \pi_1(X_1)_+ \arrow[r] \arrow[d] & \tilde{X}_1 \arrow[d] ; \\
\bigvee_{\alpha\in\mathcal{B}} S^2 \wedge \pi_1(X_1)_+ \arrow[r, "\cong"] & \tilde{X}_1 / X'
\end{tikzcd}$$

so

$$\bigoplus_{\alpha\in\mathcal{B}} \mathbb{Z}[\pi_1(X_1)] \xrightarrow{\sim} H_2(\tilde{X}_1, X'; \mathbb{Z}).$$

$$\begin{tikzcd}
 & & & & P \arrow[d, equal] & 0 \arrow[d, equal] & \\
\cdots \arrow[r] & \pi_2(X') \arrow[r] \arrow[d] & \pi_2(\tilde{X}_1) \arrow[r] \arrow[d, "\sim"] & \pi_2(\tilde{X}_1, X') \arrow[r] \arrow[d] & \pi_1(X') \arrow[r] \arrow[d] & \pi_1(\tilde{X}_1) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_2(X') \arrow[r] & H_2(\tilde{X}_1) \arrow[r, "j"] & H_2(\tilde{X}_1, X') \arrow[r] & H_1(X') \arrow[r] \arrow[d, equal] & H_1(\tilde{X}_1) \arrow[r] & \cdots \\
 & & & & P^{ab} \arrow[d, equal] \arrow[r, phantom, "- \text{ perfect}"] & & \\
 & & & & 0 & &
\end{tikzcd}$$

Choose $(S^2, s_0) \xrightarrow{\tilde{\rho}_\alpha} (\tilde{X}_1, \tilde{x})$ s.t. the images of $[\rho_\alpha]$ form a basis of $H_2(\tilde{X}_1, X'; \mathbb{Z})$ as a $\mathbb{Z}[\pi_1(X_1)]$ - module; define

$$\begin{tikzcd}
\bigvee_{\alpha \in \mathcal{A}} S^2 \arrow[r, "\vee \rho_\alpha"] \arrow[d] \arrow[dr, phantom, "\text{p.o.}"] & X_1 \arrow[d, "h"] \\
\bigvee_{\alpha \in \mathcal{A}} D^3 \arrow[r] & X^+
\end{tikzcd}$$

$\rho_\alpha = \pi \circ \tilde{\rho}_\alpha$

then $\pi_1(X_1, x_0) \xrightarrow{\sim} \pi_1(X^+, x_0)$. Consider

$$\begin{tikzcd}
X' \arrow[r, "\tilde{g}"] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & \tilde{X}_1 \arrow[r, "\tilde{h}"] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & \tilde{X}^+ \arrow[d] \\
X \arrow[r, "g"] & X_1 \arrow[r, "h"] & X^+
\end{tikzcd}$$

and let L be a local coeff. sys. on X^+, i.e. a $\pi_1(X^+)$ - module. Then

$$\cdots \to H_n(X;L) \xrightarrow{i_*} H_n(X^+;L) \to H_n(X^+,X;L) \xrightarrow{\partial} H_{n-1}(X,L) \to$$

and

$$H_*(X^+,X;L) \cong H_*(C_*(\tilde{X}^+,X') \otimes_{\pi_1(X^+)} L).$$

The rel. cellular cx. is conc. in deg. 2 and 3 :

$$\begin{tikzcd}
0 \arrow[r] & C_3(\tilde{X}^+,X') \arrow[r, "\partial"] \arrow[d, equal] & C_2(\tilde{X}^+,X') \arrow[r] \arrow[d, equal] & 0 \\
 & H_3(\tilde{X}^+,\tilde{X}_1) \arrow[r, "\partial"] & H_2(\tilde{X}_1,X') &
\end{tikzcd}$$

$$\begin{tikzcd}
\pi_3(\tilde{X}^+,\tilde{X}_1) \arrow[r, "\partial"] \arrow[d] & \pi_2(\tilde{X}_1) \arrow[r] \arrow[d, "\sim"] & \pi_2(\tilde{X}_1,X') \arrow[d] \\
H_3(\tilde{X}^+,\tilde{X}_1) \arrow[r, "\partial"] & H_2(\tilde{X}_1) \arrow[r, "j"] & H_2(\tilde{X}_1,X') \\
\bigoplus_{\alpha\in\mathcal{A}} \mathbb{Z}[\pi_1(X_1)] \arrow[u, "\sim"] & & \bigoplus_{\alpha\in\mathcal{A}} \mathbb{Z}[\pi_1(X_1)] \arrow[u, "\sim"]
\end{tikzcd}$$

Recall $[\tilde{\rho}_\alpha] \in \pi_2(\tilde{X}_1)$, $S^2 \xrightarrow{\tilde{\rho}_\alpha} \tilde{X}_1$, whose images in $H_2(\tilde{X}_1, X')$ form a basis; by the def. of $\tilde{X}^+$ these extend to

$$(D^3,S^2) \xrightarrow{\bar{\rho}_\alpha} (\tilde{X}_+,\tilde{X}_1).$$

The classes of these maps live in the

upper left hand term; their images in the lower left hand term form a $\mathbb{Z}[\pi_1(X_1)]$-basis; and they map to $[\tilde{\rho}_\alpha]$ in $\pi_2(\tilde{X}_1)$. It follows that

$$C_*(\tilde{X}^+, X')$$

is acyclic. The proof of (iii) is by obstruction theory. //

Kill $E(R) \subset GL(R)$ to get:

$$BGL(R) \longrightarrow BGL(R)^+;$$

let $F(R)$ be the htpy fiber.

Prop (i) $H_q(F(R), \mathbb{Z}) = 0, \quad q > 0.$

(ii) $\pi_1 F(R) = St(R)$.

(iii) $\pi_1 F(R)$ acts trivially on $\pi_q F(R)$, $q \geq 2$.

pf By the construction of $BGL(R)^+$:

$$\begin{tikzcd}
F(R)' \arrow[r] \arrow[d, "\sim"] & BGL(R)' \arrow[r, "\tilde{f}"] \arrow[d] \arrow[rd, phantom, "p.b."] & \widetilde{BGL(R)^+} \arrow[d] \\
F(R) \arrow[r] & BGL(R) \arrow[r, "f"] & BGL(R)^+
\end{tikzcd}$$

and $\tilde{f}$ induces iso. on homology with local coeff. Hence, the edge-homomorphism of the Serre sp. seq.

$$E^2_{s,t} = H_s(\overline{BGL(R)^+}, H_t(F(R), \mathbb{Z}))$$

$$\Rightarrow H_{s+t}(BGL(R)', \mathbb{Z})$$

is an iso. It follows that $E^2_{s,t} = 0$, if $t > 0$. Hence

$$E^2_{0,t} = H_0(\overline{BGL(R)^+}, H_t(F(R), \mathbb{Z}))$$

$$= H_t(F(R), \mathbb{Z}) = 0.$$

On htpy groups,

$$1 \to \pi_2 BGL(R)^+ \to \pi_1 F(R) \to GL(R) \to K_1(R) \to 1$$

and

$$\pi_{q+1} BGL(R)^+ \xrightarrow{\sim} \pi_q F(R), \quad q \geq 2.$$

It follows that $\pi_1 F(R)$ acts trivially on $\pi_q F(R)$, $q \geq 2$, and that

$$\begin{tikzcd} 1 \arrow[r] & \pi_2 BGL(R)^+ \arrow[r] & \pi_1 F(R) \arrow[r] \arrow[d, phantom, "\parallel"] & E(R) \arrow[r] & 1 \\ & & G & & \end{tikzcd}$$

is a central extension. Consider universal covering

$$\widetilde{F(R)} \to F(R) \to BG$$

and Serre sp. seq.

$$E^2_{s,t} = H_s(G, H_t(\widetilde{F(R)}, \mathbb{Z}))$$

$$\Rightarrow H_{s+t}(F(R), \mathbb{Z}) = 0, \text{ if } s+t \geq 0.$$

Since $H_1(\widetilde{F(R)}, \mathbb{Z}) = 0$, it follows that $E^2_{0,1} = 0$, and hence, $E^2_{1,0} = E^2_{2,0} = 0$, or

$$\begin{array}{ccccc} H_1(G, \mathbb{Z}) & = & H_2(G, \mathbb{Z}) & = & 0 \\ \Updownarrow & & \Updownarrow & & \\ G \text{ perfect} & & \text{every central ext.} & & \\ & & \text{of } G \text{ is split} & & \end{array}$$

It follows that

$$1 \to \pi_2 BGL(R)^+ \to \pi_1 F(R) \to E(R) \to 1$$

is the universal central ext. //

<u>Cor</u> $\pi_2 BGL(R)^+ = K_2(R)$. //

4/30/02

Monoidal cat.:

$$X\otimes(Y\otimes Z)\xrightarrow{\alpha_{X,Y,Z}}(X\otimes Y)\otimes Z$$

$$\mathbb{1}\otimes X\xrightarrow{\lambda_X}X\xleftarrow{\rho_X}X\otimes\mathbb{1}$$

nat'l transf. s.t.

$$\begin{tikzcd}
& (X\otimes Y)\otimes(Z\otimes W) \arrow[dr, "\alpha"] & \\
X\otimes(Y\otimes(Z\otimes W)) \arrow[ur, "\alpha"] \arrow[d, "1\otimes\alpha"'] & & ((X\otimes Y)\otimes Z)\otimes W \\
X\otimes((Y\otimes Z)\otimes W) \arrow[rr, "\alpha"] & & (X\otimes(Y\otimes Z))\otimes W \arrow[u, "\alpha\otimes 1"']
\end{tikzcd}$$

$$\begin{tikzcd}
X\otimes(\mathbb{1}\otimes Y) \arrow[rr, "\alpha"] & & (X\otimes\mathbb{1})\otimes Y \\
& X\otimes Y \arrow[ul, "X\otimes\lambda_Y"] \arrow[ur, "\rho_X\otimes Y"'] &
\end{tikzcd}$$

commute.

Symmetric monoidal $\mathcal{C}$ in add.

$$X \otimes Y \xrightarrow{\gamma_{X,Y}} Y \otimes X$$

s.t.

$$\gamma_{X,Y} \circ \gamma_{Y,X} = \mathrm{id}$$

$$\lambda_Y \circ \gamma_{\mathbb{1},Y} = \rho_Y$$

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "\alpha"] \arrow[d, "X \otimes \gamma"] & (X \otimes Y) \otimes Z \arrow[r, "\gamma"] & Z \otimes (X \otimes Y) \arrow[d, "\alpha"] \\
X \otimes (Z \otimes Y) \arrow[r, "\alpha"] & (X \otimes Z) \otimes Y \arrow[r, "\gamma \otimes Y"] & (Z \otimes X) \otimes Y
\end{tikzcd}$$

=

monoid :

commutative monoid :

Index cat. Σ :

$$\mathrm{ob}\ \Sigma = \mathbb{N}_0$$

$$\mathrm{Hom}_{\Sigma}(m,n) = \begin{cases} \Sigma_m & , \text{if } m=n \\ \emptyset & , \text{else} . \end{cases}$$

$\mathcal{S}_* =$ pointed spaces ;

$\mathcal{S}_*^{\Sigma} =$ functor cat.

Define

$$\mathcal{S}_*^{\Sigma} \times \mathcal{S}_*^{\Sigma} \xrightarrow{\otimes} \mathcal{S}_*^{\Sigma}$$

$$(X \otimes Y)_n = \bigvee_{k+l=n} (\Sigma_n)_+ \wedge_{\Sigma_k \times \Sigma_l} (X_k \wedge Y_l) .$$

This is a symmetric monoidal product with twist isomorphism

$$X \otimes Y \xrightarrow{\gamma_{X,Y}} Y \otimes X$$

given by

$$(\Sigma_n)_+ \wedge_{\Sigma_k \times \Sigma_l} (X_k \wedge Y_l) \longrightarrow (\Sigma_n)_+ \wedge_{\Sigma_l \times \Sigma_k} (Y_l \wedge X_k)$$

$$(\alpha, x, y) \longmapsto (\alpha \circ \sigma_{k,l}, y, x)$$

with $\sigma_{k,\ell}(i) = i+k \pmod{n}$. The unit for the tensor product is:

$$(S^0, *, *, *, \dots)$$

Let $\mathbb{S} \in \mathcal{S}_*^{\Sigma}$ be the seq.

$$(S^0, S^1, S^2, S^3, \dots)$$

where

$$S^n = \underbrace{S^1 \wedge \dots \wedge S^1}_{n}$$

Then $\mathbb{S}$ is a commutative monoid. A symmetric spectrum is a (left) $\mathbb{S}$-module.

Spell out: for all $m, n \geq 0$:

X_n Σ_n-space +

$S^m \wedge X_n \xrightarrow{\lambda_{m,n}} X_{m+n}$ $\Sigma_m \times \Sigma_n$-equiv.

s.t. $\lambda_{0,n} = \lambda_{X_n}$ and

$$\begin{tikzcd}
S^k \wedge S^m \wedge X_n \arrow[r, "S^k \wedge \lambda_{m,n}"] \arrow[d, "can"'] \arrow[dr, phantom, "\circlearrowleft"] & S^k \wedge X_{m+n} \arrow[d, "\lambda_{k,m+n}"] \\
S^{k+m} \wedge X_n \arrow[r, "\lambda_{k+m,n}"] & X_{k+m+n}
\end{tikzcd}$$

.

Define right $\mathbb{S}$-action by:

$$X \otimes \mathbb{S} \xrightarrow{\gamma} \mathbb{S} \otimes X \xrightarrow{\alpha} X \ ;$$

since $\mathbb{S}$ is commutative this makes X a right $\mathbb{S}$-module, and in fact, an $\mathbb{S}$-$\mathbb{S}$-bimodule. Define

$$\begin{tikzcd} X \otimes \mathbb{S} \otimes Y \arrow[r, shift left, "m \otimes Y"] \arrow[r, shift right, "X \otimes m"'] & X \otimes Y \arrow[r] & X \wedge Y \ ; \end{tikzcd}$$

coequalizer

then $(\wedge, \mathbb{S})$ is a symmetric monoidal structure on the category of symmetric spectra.

Symmetric ring spectrum

= monoid in the symmetric monoidal cat. of symm. spectra.

Spell out: for all $m, n \geq 0$:

E_n (left) Σ_n-space +

$S^n \xrightarrow{\eta_n} E_n$ Σ_n-equiv. +

$E_m \wedge E_n \xrightarrow{\mu_{m,n}} E_{m+n}$ $\Sigma_m \times \Sigma_n$-equiv.

s.t.

(i)

$$\begin{tikzcd}
S^m \wedge S^n \arrow[r, "\eta_m \wedge \eta_n"] \arrow[d, "can"] & E_m \wedge E_n \arrow[d, "\mu_{m,n}"] \\
S^{m+n} \arrow[r, "\eta_{m+n}"] & E_{m+n}
\end{tikzcd}$$

(ii)

$$\begin{tikzcd}
S^m \wedge E_n \arrow[r, "\eta_m \wedge E_n"] \arrow[d, "tw"] & E_m \wedge E_n \arrow[r, "\mu_{m,n}"] & E_{m+n} \arrow[d, "\sigma_{m,n}"] \\
E_n \wedge S^m \arrow[r, "E_n \wedge \eta_m"] & E_n \wedge E_m \arrow[r, "\mu_{n,m}"] & E_{n,m}
\end{tikzcd}$$

(iii)

$$\begin{tikzcd}
E_k \wedge E_m \wedge E_n \arrow[r, "\mu_{k,m} \wedge E_n"] \arrow[d, "E_k \wedge \mu_{m,n}"] & E_{k+m} \wedge E_n \arrow[d] \\
E_k \wedge E_{m+n} \arrow[r, "\mu_{k,m+n}"] & E_{k+m+n}
\end{tikzcd}$$

(iv) $\mu_{0,n} \circ (\eta_0 \wedge E_n) = \text{can.} = \mu_{n,0} \circ (E_n \wedge \eta_0)$.

commutative ⅅ

(v)
$$\begin{tikzcd} E_m \wedge E_n \arrow[r, "\mu_{m,n}"] \arrow[d, "tw"] & E_{m+n} \arrow[d, "\sigma_{m,n}"] \\ E_n \wedge E_m \arrow[r, "\mu_{n,m}"] & E_{n+m} \end{tikzcd}$$
.

5/9/02

$$\begin{tikzcd}
X_F \arrow[r, hook] \arrow[d] & X \arrow[d, "f\text{-type}"] & X_E \arrow[l, hook'] \arrow[d] \\
\operatorname{Spec} F \arrow[r, hook] & \operatorname{Spec} \mathcal{O} & \operatorname{Spec} E \arrow[l, hook']
\end{tikzcd}$$

$$\begin{tikzcd}
 & & 0 \arrow[d, phantom, "\parallel"] \arrow[r, phantom, "\text{if } F = \bar{F}"] & {} \\
K_i'(X_E, \mathbb{Z}/n) \otimes \mathcal{O}^* \arrow[rr, "\partial \otimes \mathrm{red}"] \arrow[d] & \circlearrowleft & K_{i-1}'(X_F, \mathbb{Z}/n) \otimes F^* \arrow[d, "\mu"] \\
K_i'(X_E, \mathbb{Z}/n) \otimes E^* \arrow[r, "\mu"] \arrow[d, "1 \otimes v_E"] & K_{i+1}'(X_E, \mathbb{Z}/n) \arrow[r, "\partial"] & K_i'(X_F, \mathbb{Z}/n) \\
K_i'(X_E, \mathbb{Z}/n) \otimes \mathbb{Z} \arrow[u, bend left] \arrow[urr, dashed, "s"'] & &
\end{tikzcd}$$

specialization

$$K_i'(X_E, \mathbb{Z}/n) \xrightarrow{s} K_i'(X_F, \mathbb{Z}/n).$$

<u>Lemma</u> The composite is zero:

$$K_{i-1}'(X, \mathbb{Z}/n) \otimes E^* \xrightarrow{\mu} K_i'(X_E, \mathbb{Z}/n) \xrightarrow{s} K_i'(X_F, \mathbb{Z}/n).$$

~~pf~~ let $t \in K_{i-1}'(X, \mathbb{Z}/n)$ and $a \in E^*$; let

π be a unif. By def.,

$$s(t.\ell(a)) = \partial(t.\ell(a).\ell(\pi))$$

$$= \pm\ \bar{t}\ .\ \partial(\ell(a).\ell(\pi))$$

$$\in K'_{i-1}(X_F, \mathbb{Z}/n) \cdot K_1(F) = c. \qquad //$$

Let $\mathcal{O}'$ be a finite normal $\mathcal{O}$-algebra. Then $\mathcal{O}'$ is semi-local and the local rings $\mathcal{O}'_j$, $j=1,\dots,k$, are DVRs. Let F'_j be the residue field of $\mathcal{O}'_j$, and let e_j be the ramification index of $\mathcal{O}'_j / \mathcal{O}$. The diagr. comm.

$$\begin{tikzcd}
K'_{i+1}(X_{E'}, \mathbb{Z}/n) \arrow[r, "N_{E'/E}"] \arrow[d, "(\partial_j)"] & K'_{i+1}(X_E, \mathbb{Z}/n) \arrow[d, "\partial"] \\
\bigoplus_{j=1}^{k} K'_i(X_{F'_j}, \mathbb{Z}/n) \arrow[r, "\Sigma N_{F'_j/F}"] & K'_i(X_F, \mathbb{Z}/n)
\end{tikzcd}$$

Suppose $F = \bar{F}$; then each $F'_j = F$, and we have specializations

$$K'_i(X_{E'}, \mathbb{Z}/n) \xrightarrow{s_j} K'_i(X_F, \mathbb{Z}/n).$$

<u>Lemma</u> $s N_{E'/E} = \sum_{j=1}^{k} e_j \cdot s_j$

Pf Let $\pi \in \mathcal{O}$ be a uniformizer s.t. $e_j = v_{\mathcal{O}'_j}(\pi)$. If $t \in K'_i(X_{E'}, \mathbb{Z}/n)$, then

$$s(N_{E'/E}(t)) := \partial(N_{E'/E}(t) . l(\pi))$$

$$= \partial N_{E'/E}(t . l(\pi))$$

$$(\text{diagr.}) \quad = \sum_{j=1}^{k} N_{F'_j/F}(\partial(t . l(\pi)))$$

$$= \sum_{j=1}^{k} \partial(t . l(\pi_j)^{e_j} . l(u_j)) \qquad u_j \in \mathcal{O}_j^*$$

$$= \sum_{j=1}^{k} e_j \, \partial(t . l(\pi_j)) . \cancel{\overline{l(u_j)}}^{1}$$

$$= \sum_{j=1}^{k} e_j \cdot s_j(t)$$

//

$$\begin{tikzcd} E = F(C) & C \arrow[d, "f"] \\ E_0 = F(\mathbb{P}^1_F) \arrow[u] & \mathbb{P}^1_F \end{tikzcd}$$

$$s_0(N_{E/E_0}(t)) = \sum_{c \in f^{-1}(0)} e_c \cdot s_c(t)$$

$$s_\infty(N_{E/E_0}(t)) = \sum_{c \in f^{-1}(\infty)} e_c \cdot s_c(t)$$

$$q(t \otimes (f)) = \sum_{c \in f^{-1}(0)} e_c \cdot s_c(t) - \sum_{c \in f^{-1}(\infty)} e_c s_c(t)$$

$$= s_0(N(t)) - s_\infty(N(t)) \underset{\substack{| \\ \text{calc. for } \mathbb{P}^1_F}}{=} 0$$

$$\begin{tikzcd}
K_i'(X_E, \mathbb{Z}/n) \otimes H^0(C, \mathcal{M}^*) \arrow[d, "1 \otimes (-)"] \arrow[dr, bend left, "0"] & \\
K_i'(X_E, \mathbb{Z}/n) \otimes \mathrm{Div}^0(C) \arrow[r, "\varphi"] \arrow[d] & K_i'(X, \mathbb{Z}/n) \\
K_i'(X_E, \mathbb{Z}/n) \otimes \mathrm{Pic}^0(C) \arrow[ur, dashed] \arrow[d, equal, "\text{— since } \mathrm{Pic}^0(C) \text{ is divisible.}"] & \\
0 &
\end{tikzcd}$$

In particular

$$t \otimes (c_0 - c_1) \longmapsto s_0(t) - s_1(t)$$

is zero. This is the rigidity thm.

Both of s_0 and s_1 vanish on the latter summands by the first lemma. And both are ± 1 on the first summand.

In general, we can assume that C is complete. Consider

$$K_i'(X_E, \mathbb{Z}/n) \otimes \mathrm{Div}^0(C) \xrightarrow{q} K_i'(X, \mathbb{Z}/n)$$

$$t \otimes \sum n_i c_i' \longmapsto \sum n_i' s_{c_i'}(t).$$

We wish to show that this is the zero map.

5/14/02

A pair (R, I) of a ring R and an ideal I is <u>henselian</u> if :

$$\begin{tikzcd} R/I & B \arrow[l] \arrow[dl, dashed, "\exists"'] \\ R \arrow[u, "\pi"] \arrow[r, equal] & R \arrow[u, "\text{étale}"'] \end{tikzcd}$$

Equivalently, (R, I) is henselian if for every $f \in R[T]$ and every root $\bar{\alpha} \in R/I$ of $\bar{f} \in R/_I[T]$ s.t. $\bar{f}'(\bar{\alpha})$ is a unit, there exists a root $\alpha \in R$ of f s.t. $\alpha + I = \bar{\alpha}$.

<u>ex</u> Let V be a DVR with maximal ideal m and suppose V is m-adically complete :

$$V \xrightarrow{\sim} \lim_{n} V/m^n .$$

Then (V, m) is henselian. //

<u>Thm</u> (Gabber - Suslin) If (R, I) is henselian and if m is invertible in R/I, then

$$K_*(R, \mathbb{Z}/m) \xrightarrow{\sim} K_*(R/I, \mathbb{Z}/m) .$$

//

ex Let $(\mathcal{O}, m, k)$ be a henselian local ring; let $n \neq \text{char } k$. Then

$$K_*(\mathcal{O}, \mathbb{Z}/n) \xrightarrow{\sim} K_*(k, \mathbb{Z}/n).$$ //

Lemma Let A be a uniquely m-divisible abelian group, and let P be a trivial A-module. Suppose that for all $x \in P$, there exists $N \geq 0$ s.t. $m^N . x = 0$. Then

$$H_i(A, P) = 0, \quad i > 0.$$

pf Since $H_i(-,-)$ commutes with filtered colimits, we can assume that A (resp. P) is a f.g. $\mathbb{Z}[\frac{1}{m}]$-module (resp. $\mathbb{Z}$-mod.). Then we can find $N \geq 0$ s.t. $m^N P = 0$. Replacing m by m^N, we can assume $mP = 0$. Hence, $mH_i(A,P) = 0$. Suppose A is finite of order n. Then $(n,m) = 1$. Moreover, $nH_i(A,P) = 0$, if $i > 0$:

$$\begin{tikzcd} H_i(A,P) \arrow[r, "trf"] \arrow[rr, bend right, "\cdot n"'] & \overset{= 0}{H_i(\{1\},P)} \arrow[r] & H_i(A,P) \end{tikzcd}$$

Since $(n,m)=1$, $H_i(A,P)=0$, $i>0$. It remains to consider $A=\mathbb{Z}[\frac{1}{m}]$. Since

$$\mathbb{Z}[\tfrac{1}{m}] = \mathrm{colim}\,(\mathbb{Z}\xrightarrow{m}\mathbb{Z}\xrightarrow{m}\mathbb{Z}\xrightarrow{m}\cdots)$$

and since $H_i(\mathbb{Z},P)=0$, $i\geq 2$, and

$$H_1(A,P)\xleftarrow[\text{since } P \text{ trivial}]{\sim} H_1(A,\mathbb{Z})\otimes P \xleftarrow[\text{since } A \text{ abelian}]{\sim} A\otimes P = 0.$$

//

Cor Let $A\xrightarrow{\phi}B$ be a map of abelian groups and suppose $\ker\phi$ and $\mathrm{coker}\,\phi$ are uniquely m-divisible. Then for P as above,

$$H_i(A,P)\xrightarrow[\sim]{\phi_*}H_i(B,P)$$

is an iso, for all $i\geq 0$.

Pf Suppose ϕ is injective with cokernel C, and consider the Hochschild-Serre:

$$E^2_{s,t}=H_s(C,H_t(A,P))\Rightarrow H_{s+t}(B,P).$$

By the lemma, $E^2_{s,t}=0$, if $s>0$. So the edge homomorphism is an iso.:

$$H_t(A,P)\xrightarrow{\sim}H_t(B,P),\quad t\geq 0.$$

Same arg. for ϕ epi. //

Prop Let $I \subset R$ be a two-sided ideal, and let m be an integer invertible in R/I. Then the action of $GL(R)$ by conjugation on $H_*(GL(R,I), \mathbb{Z}/m)$ is trivial.

Pf Recall that

$$1 \to GL(R,I) \underset{\text{def}}{\longrightarrow} GL(R) \to GL(R/I) \to 1.$$

Write $\Gamma = GL(R,I)$ and $\Gamma_n = GL_n(R,I)$, and let

$$\Gamma_n' = \left[\begin{array}{ccc|c} & & & I \\ & \Gamma_n & & \vdots \\ & & & I \\ 0 & - & 0 & 1 \end{array}\right]$$

$$\Gamma_n'' = \left[\begin{array}{ccc|c} & & & R \\ & \Gamma_n & & \vdots \\ & & & R \\ 0 & - & 0 & 1 \end{array}\right]$$

s.t. we have extensions

$$\begin{tikzcd}
1 \arrow[r] & I^n \arrow[r] \arrow[d] & \Gamma_n' \arrow[r] \arrow[d] & \Gamma_n \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & R^n \arrow[r] & \Gamma_n'' \arrow[r] & \Gamma_n \arrow[r] & 1
\end{tikzcd}$$

We get a map of sp. seq.

$$\begin{tikzcd}
{}'E^2_{s,t} = H_s(\Gamma_n, H_t(I^n, \mathbb{Z}/m)) \arrow[r, Rightarrow] \arrow[d, "\sim"] & H_{s+t}(\Gamma_n', \mathbb{Z}/m) \arrow[d, "\sim"] \\
{}''E^2_{s,t} = H_s(\Gamma_n, H_t(R^n, \mathbb{Z}/m)) \arrow[r, Rightarrow] & H_{s+t}(\Gamma_n'', \mathbb{Z}/m)
\end{tikzcd}$$

and the map of E^2-terms is an iso. by the cor. since R/I is m-divisible. Hence

$$H_*(\Gamma_n', \mathbb{Z}/m) \xrightarrow{\sim} H_*(\Gamma_n'', \mathbb{Z}/m).$$

The elem. matrices $e_{i,n+1}(r)$, $r \in R$, $1 \le i \le n$, are contained in Γ_n'', so act trivially on the right. But then they also act trivially on the left. Since

$$\Gamma_n \hookrightarrow \Gamma_n' \hookrightarrow \Gamma_{n+1} ,$$

it follows that $e_{i,n+1}(r)$ acts trivially on the image of $H_*(\Gamma_n, \mathbb{Z}/m)$ in $H_*(\Gamma_{n+1}, \mathbb{Z}/m)$. Similar for $e_{n+1,i}(r)$. Since $e_{i,n+1}(r)$ and $e_{n+1,i}(r)$ generate $E_{n+1}(R)$, this group acts trivially on the image of $H_*(\Gamma_n, \mathbb{Z}/m)$ in $H_*(\Gamma, \mathbb{Z}/m)$. But

$$H_*(\Gamma, \mathbb{Z}/m) \xleftarrow{\sim} \operatorname{colim}_n H_*(\Gamma_n, \mathbb{Z}/m),$$

so $E(R)$ acts trivially on $H_*(\Gamma, \mathbb{Z}/m)$. Finally, if $\alpha \in GL_n(R)$, then both

$$\begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \in GL_{2n}(R) \quad \text{and}$$

$$\begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} \in E_{2n}(R)$$

act trivially on the image of $H_*(\Gamma_n, \mathbb{Z}/m)$ in $H_*(\Gamma_{2n}, \mathbb{Z}/m)$. Hence, so does

$$\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} \qquad //$$

Lemma Let X be a conn. H-space, let $X' \xrightarrow{p} X$ be a conn. covering space w. grp. G, and suppose that G is uniquely m-divisible. Then

$$\pi_*(X', \mathbb{Z}/m) \xrightarrow{\sim} \pi_*(X, \mathbb{Z}/m)$$

$$H_*(X', \mathbb{Z}/m) \xrightarrow{\sim} H_*(X, \mathbb{Z}/m).$$

pf The fiber sequence

$$X' \longrightarrow X \longrightarrow BG$$

gives rise to

$$E^2_{s,t} = H_s(G, H_t(X', \mathbb{Z}/m)) \Rightarrow H_t(X, \mathbb{Z}/m).$$

Since G acts trivially on $H_*(X', \mathbb{Z}/m)$, the lemma shows $E^2_{s,t} = 0$, if $s > 0$. Finally,

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & 0 \arrow[d] & & \\
\cdots \arrow[r] & \pi_2 X' \arrow[r, "m"] \arrow[d, "\sim"] & \pi_2 X' \arrow[r] \arrow[d, "\sim"] & \pi_2(X', \mathbb{Z}/m) \arrow[r] \arrow[d] & \pi_1 X' \arrow[r, "m"] \arrow[d] & \pi_1 X' \arrow[r] \arrow[d] & \pi_1(X', \mathbb{Z}/m) \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & \pi_2 X \arrow[r, "\sim"] & \pi_2 X \arrow[r] & \pi_2(X, \mathbb{Z}/m) \arrow[r] & \pi_1 X \arrow[r, "m"] \arrow[d] & \pi_1 X \arrow[r] \arrow[d] & \pi_1(X, \mathbb{Z}/m) \arrow[r] & 0 \\
 & & & & G \arrow[r, "m", "\sim"'] \arrow[d] & G \arrow[d] & & \\
 & & & & 0 & 0 & &
\end{tikzcd}$$

shows $\pi_*(X', \mathbb{Z}/m) \xrightarrow{\sim} \pi_*(X, \mathbb{Z}/m)$. //

Prop Let $X \xrightarrow{f} Y$ be a cts. map and suppose both X and Y are conn. H-spaces. Let m be s.t. both the kernel and coker. of $\pi_1(f)$ are uniquely m-divisible. Then the following are equivalent:

(a) $\pi_*(f, \mathbb{Z}/m)$ is an iso.

(b) $H_*(f, \mathbb{Z}/m)$ is an iso.

pf Let $G = \operatorname{im}(\pi_1(f)) \subset \pi_1 Y$, and let $Y' \xrightarrow{p} Y$ be the corresponding covering. By lemma, $\pi_*(p, \mathbb{Z}/m)$ and $H_*(p, \mathbb{Z}/m)$ are iso. Choosing a base-pt. in Y',

$$\begin{tikzcd}
 & Y' \arrow[d, "p"] \\
X \arrow[ur, "\exists"] \arrow[r, "f"] & Y
\end{tikzcd},$$

so we can assume $\pi_1(f)$ is surjective.

Consider the covering spaces

$$\begin{tikzcd}
X'' \arrow[r, "H"] \arrow[d, "\pi_1(X)"'] \arrow[rr, bend left, "f''"] & X' \arrow[r, "f'"] \arrow[d, "G"'] \arrow[dr, phantom, "p.b."] & Y'' \arrow[d, "G = \pi_1(Y)"] \\
X \arrow[r, equal] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

$H = \ker(\pi_1(f))$ uniquely m-divisible, so by lemma:

$(a)'' \Longleftrightarrow (a)'$ and $(b)'' \Longleftrightarrow (b)'$.

And since X'' and Y'' are 1-conn.,

$(a)'' \Longleftrightarrow (b)''$.

So enough to show

(a) ⟺ (a)' and (b) ⟺ (b)'.

Prove ⤴ : we have $\pi_i(f', \mathbb{Z}/m)$ iso, if $i \geq 3$, so consider $i = 1, 2$.

$$\begin{tikzcd}
& & & H \arrow[r, "m", "\sim"'] \arrow[d, equal] & H \arrow[d, equal] \arrow[r, phantom, "\Rightarrow"] & 0 \arrow[d, equal] \\
\pi_2 X' \arrow[r, "m"] \arrow[d] & \pi_2 X' \arrow[r, two heads] \arrow[d] & \pi_2(X', \mathbb{Z}/m) \arrow[r] \arrow[d] & \pi_1 X' \arrow[r, "m"] \arrow[d] & \pi_1 X' \arrow[r, two heads] \arrow[d] & \pi_1(X', \mathbb{Z}/m) \arrow[d, "\sim"] \arrow[d, bend left=60, Rightarrow] \\
\pi_2 Y'' \arrow[r, "m"] & \pi_2 Y'' \arrow[r, two heads] & \pi_2(Y'', \mathbb{Z}/m) \arrow[r] & \pi_1 Y'' \arrow[r] \arrow[d, equal] & \pi_1 Y'' \arrow[r, two heads] \arrow[d, equal] & \pi_1(Y'', \mathbb{Z}/m) \\
& & & 0 & 0 &
\end{tikzcd}$$

$$\begin{tikzcd}
\pi_2(X)/m \arrow[d, "f_*"] & \pi_2(X')/m \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "f'_*"] & \pi_2(X', \mathbb{Z}/m) \arrow[d, "f'_*"] \\
\pi_2(Y)/m & \pi_2(Y'')/m \arrow[l, "\sim"'] \arrow[r, "\sim"] & \pi_2(Y'', \mathbb{Z}/m)
\end{tikzcd}$$

Consider

$$\begin{tikzcd}
1 \arrow[r] & \pi_2(X)/m \arrow[r] \arrow[d] & \pi_2(X, \mathbb{Z}/m) \arrow[r] \arrow[d] & {}_m\pi_1(X) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi_2(Y)/m \arrow[r] & \pi_2(Y, \mathbb{Z}/m) \arrow[r] & {}_m\pi_1(Y) \arrow[r] & 1
\end{tikzcd}$$

Since X and Y are H-spaces, the two rows are exact seq. of groups. Hence the natural action of $\pi_2(Y)$ (resp. $\pi_2(X)$) on $\pi_2(Y, \mathbb{Z}/m)$ (resp. $\pi_2(X, \mathbb{Z}/m)$) facters through $\pi_2(Y)/m$ (resp. $\pi_2(X)/m$), and the induced action on the fibers of the right hand horizontal map is free and transitive. Since ${}_m\pi_1(X') = 0$, we get (a) $\Leftrightarrow$ (a)$'$.

Prove (b) $\Leftrightarrow$ (b)$'$: replacing f by the mapping cylinder, we can assume that (Y, X) (resp. (Y'', X')) is a relative CW-cx. We wish to show:

$$H_*(Y, X; \mathbb{Z}/m) = 0 \iff H_*(Y'', X'; \mathbb{Z}/m) = 0.$$

There is a rel. Serre sp. seq.

$$E^2_{s,t} = H_s(G, H_t(Y'', X'; \mathbb{Z}/m)) \Rightarrow H_{s+t}(Y, X; \mathbb{Z}/m),$$

so $(\Leftarrow)$ is clear. To prove $(\Rightarrow)$, let $j \geq 0$ be minimal with $H_j(Y'', X'; \mathbb{Z}/m) = 0$. Then $H_0(G, H_j(Y'', X'; \mathbb{Z}/m)) = 0$. Consider

$$H_j(Y'', \mathbb{Z}/m) \to H_j(Y'', X'; \mathbb{Z}/m) \to H_{j-1}(X', \mathbb{Z}/m).$$

Since X and Y are H-spaces, the G-action on the extreme groups is trivial. It follows that $H_j(Y'', X'; \mathbb{Z}/m) = 0$. //

Cor Let (R, I) be a henselian pair, and let m be an integer invertible in R/I. Then the following are equivalent:

(a) $K_*(R, \mathbb{Z}/m) \xrightarrow{\sim} K_*(R/I, \mathbb{Z}/m)$

(b) $H_*(GL(R), \mathbb{Z}/m) \xrightarrow{\sim} H_*(GL(R/I), \mathbb{Z}/m)$

(c) $H_*(GL(R, I), \mathbb{Z}/m) = 0$.

pf The map $K_0(R) \to K_0(R/I)$ is 1-1, since $I \subset \operatorname{rad}(R)$ and onto since (R, I) is henselian. Since $I \subset \operatorname{rad}(R)$,

$$K_1(R) \longrightarrow K_1(R/I)$$

is surjective with kernel $1+I$. But this is a uniquely m-divisible group, so applying the proposition to

$$BGL(R)^+ \longrightarrow BGL(R/I)^+$$

shows (a) $\Leftrightarrow$ (b). Finally (b) $\Leftrightarrow$ (c) follows from a sp. seq. arg. since $GL(R)$ acts trivially on $H_*(GL(R,I), \mathbb{Z}/m)$. //

5/16/02

let $k = \mathbb{R}, \mathbb{C}$; cts. map

$$\begin{tikzcd} BGL(k) \arrow[rr] \arrow[dr] & & BGL(k)^{top} \\ & BGL(k)^{+} \arrow[ur, dashed] & \end{tikzcd}$$

Thm (Suslin) The canonical map

$$BGL(k)^{+} \longrightarrow BGL(k)^{top}$$

induces iso. of $\pi_*(-, \mathbb{Z}/m)$ and $H_*(-, \mathbb{Z}/m)$. ,

Setup: let G be a Lie group with fin. many conn. comp.; fix left inv. Riem. metric; let $G_\varepsilon \subset G$ be the ε-ball around $e \in G$; let

$$N_\cdot G_\varepsilon \subset N_\cdot G$$

be the sub-simplicial set of the nerve of G given by

$$(g_1, \dots, g_t) \in N_t G_\varepsilon \underset{def}{\iff}$$

$$G_\varepsilon \cap g_1 G_\varepsilon \cap \dots \cap g_1 g_2 \cdots g_t G_\varepsilon \neq \emptyset .$$

note $\underbrace{G_\varepsilon \times \dots \times G_\varepsilon}_{t} \subset N_t G_\varepsilon \subset \underbrace{G_{\varepsilon'} \times \dots \times G_{\varepsilon'}}_{t}$

Lemma If ε is small enough, then

$$BG_\varepsilon \longrightarrow BG \longrightarrow BG^{top}$$

is a fibration up to htpy.

note (1) this means that BG_ε is (non-can.) equivalent to the htpy fiber of the map on the right.

(2) the statement is true as soon as G_ε is geodesically convex. //

Prop let $n, t \geq 1$; if ε is small enough,

$$H_t(BGL_n(k)_\varepsilon, \mathbb{Z}/m) \xrightarrow{\iota_*} H_t(BGL(k), \mathbb{Z}/m)$$

is zero.

pf Consider the map on cellular chains:

$$C_*(GL_n(k)_\varepsilon, \mathbb{Z}/m) \xrightarrow{\iota_*} C_*(GL(k), \mathbb{Z}/m).$$

Given $N \geq 1$, we will construct maps

$$C_t(GL_n(k)_\varepsilon, \mathbb{Z}/m) \xrightarrow{s_t} C_{t+1}(GL(k), \mathbb{Z}/m)$$

s.t. $ds_t + s_{t-1}d = \iota_t$, $1 \leq t \leq N$. This will prove the proposition.

Let

$$X^{cont}_{n,t} = \underbrace{GL_n(k)^{top} \times \cdots \times GL_n(k)^{top}}_{t}$$

$$= N_t(GL_n(k)^{top})$$

$\mathcal{O}^{cont}_{n,t}$ = ring of germs of cts. fct.

$$X^{cont}_{n,t} \longrightarrow k^{top}$$

defined in a nbhd of $(e, -, e)$.

Note:

$GL_r(\mathcal{O}^{cont}_{n,t})$ = germs of cts. fcts

$$X^{cont}_{n,t} \longrightarrow GL_r(k)^{top}$$

def. around $(e, -, e)$.

Hence a chain

$$\tilde{c} \in C_q(GL_r(\mathcal{O}^{cont}_{n,t}), \mathbb{Z}/m)$$

gives a (cts.) map

$$\underbrace{GL_n(k)_\varepsilon \times \cdots \times GL_n(k)_\varepsilon}_{t} \longrightarrow C_q(GL_r(k), \mathbb{Z}/m),$$

for some $\varepsilon > 0$, and hence a map of abelian groups

$$C_t(GL_n(k)_\varepsilon, \mathbb{Z}/m) \xrightarrow{c} C_q(GL_r(k), \mathbb{Z}/m) ,$$

for some (possibly smaller) $\varepsilon > 0$, So to construct the maps s_t, we may instead construct chains

$$\tilde{s}_t \in C_{t+1}(GL(\mathcal{O}^{cont}_{n,t}), \mathbb{Z}/m).$$

Translate requirement $ds_t + s_{t-1}d = \iota_t$:

$X^{cont}_{n,\bullet} = N_\bullet(GL_n(k)^{top})$ simpl. space $\rightsquigarrow$

$C_q(GL(\mathcal{O}^{cont}_{n,\bullet}), \mathbb{Z}/m)$ cosimplicial ab. grp. $\rightsquigarrow$

— " — cochain complex.

Let $\delta = \Sigma(-1)^i d^i$ be the coboundary operator. Then

$$ds_t + s_{t-1}d = \iota_t \iff d\tilde{s}_t + \delta\tilde{s}_{t-1} = \tilde{\iota}_t$$

$$d\iota_t = \iota_{t-1}d \iff d\tilde{\iota}_t = \delta\tilde{\iota}_{t-1}.$$

Set $\tilde{s}_0 = 0$ and assume, inductively, that $\tilde{s}_i$, $0 \le i < t$, have been constructed.

Calculate:

$$d(\tilde{\iota}_t - \delta \tilde{s}_{t-1}) = \delta \tilde{\iota}_{t-1} - d\delta \tilde{s}_{t-1}$$

$$(\text{ind.}) = \delta d \tilde{s}_{t-1} + \delta\delta \tilde{s}_{t-2} - d\delta \tilde{s}_{t-1} = 0,$$

so $\tilde{\iota}_t - \delta \tilde{s}_{t-1} \in C_t(GL(\mathcal{O}^{cont}_{n,t}), \mathbb{Z}/m)$ is a cycle. We have

$$1 \to GL(\mathcal{O}^{cont}_{n,t}, m^{cont}_{n,t}) \to GL(\mathcal{O}^{cont}_{n,t}) \to GL(k) \to 1,$$

and can assume w.l.o.g.

$$\tilde{\iota}_t - \delta \tilde{s}_{t-1} \in C_t(GL(\mathcal{O}^{cont}_{n,t}, m^{cont}_{n,t}), \mathbb{Z}/m).$$

Indeed, this is true for $\tilde{\iota}_t$ and $\tilde{s}_0 = 0$, and will be true inductively for $\tilde{s}_{t-1}$.

<u>Claim</u> $H_*(GL(\mathcal{O}^{cont}_{n,t}, m^{cont}_{n,t}), \mathbb{Z}/m) = 0, \; * > 0.$

This shows that $\tilde{\iota}_t - \delta \tilde{s}_{t-1}$ is a bd.; pick $\tilde{s}_t \in C_{t+1}(GL(\mathcal{O}^{cont}_{n,t}, m^{cont}_{n,t}), \mathbb{Z}/m)$ s.t.

$$d\tilde{s}_t = \tilde{\iota}_t - \delta \tilde{s}_{t-1}.$$

It remains to prove the claim. From what we did last time, it suffices to show:

Lemma The pair $(\mathcal{O}^{cont}_{n,t}, m^{cont}_{n,t})$ is henselian.

pf Let

$$p(T) = g_m T^m + \cdots + g_1 T + g_0,$$

where

$$GL_n(k)^{top}_{\varepsilon} \times \cdots \times GL_n(k)^{top}_{\varepsilon} \xrightarrow{g_i} k^{top}$$

some $\varepsilon > 0$, and suppose

$$\bar{p}(T) = \bar{g}_m T^m + \cdots + \bar{g}_0 \in k[T],$$

$\bar{g}_i = g_i(e, \dots, e)$, has a simple root $\bar{\alpha} \in k$. We must find $\alpha \in \mathcal{O}^{cont}_{n,t}$ s.t. $p(\alpha) = 0$ and $\alpha(e, \dots, e) = \bar{\alpha}$.
Cts. map:

$$U = GL_n(k)_{\varepsilon} \times \cdots \times GL(k)_{\varepsilon} \xrightarrow{\gamma} k^{m+1}$$

$$y \longmapsto (g_0(y), \dots, g_m(y)).$$

There exists $\delta, \gamma > 0$ s.t. all

$$\bar{q}(T) \in \bar{p}(T) + B_\delta(0)[T]$$

has a unique root in $\bar{\alpha} + B_\gamma(0)$. Let

$$V = q(e, \dots, e) + B_\delta(0)^{m+1} \xrightarrow{\rho} k$$

be the map which takes $(a_0, \dots, a_m)$ to the unique root $\bar{\beta} \in \bar{\alpha} + B_\gamma(0)$ of $a_m T^m + \dots + a_1 T + a_0$. Then

$$\alpha : \bar{q}^{-1}(V) \xrightarrow{q} V \xrightarrow{\rho} k$$

is the desired root. //

Same is true for SL:

$$\begin{tikzcd}
H_t(BSL_n(k)_\varepsilon, \mathbb{Z}/m) \arrow[r, "i_* = 0"] \arrow[d] & H_t(BSL(k), \mathbb{Z}/m) \arrow[d, shift right, "\int"'] \\
H_t(BGL_n(k)_\varepsilon, \mathbb{Z}/m) \arrow[r, "i_* = 0"] \arrow[ur, phantom, "\Uparrow"] & H_*(BGL(k), \mathbb{Z}/m) \arrow[u, shift right]
\end{tikzcd}$$

Cor For small ε and $0 < t \leq \frac{n-1}{2}$,

$$H_i(BSL_n(k)_\varepsilon, \mathbb{Z}/m) = 0.$$

pf This uses two facts:

(1) (Milnor) $H_*(BSL_n(k), \mathbb{Z}/m) \twoheadrightarrow H_*(BSL_n(k)^{top}, \mathbb{Z}/m)$.

(2) (Suslin, Charney) If $0 \leq i \leq \frac{n-1}{2}$,

$$H_i(BSL_n(k), \mathbb{Z}/m) \xrightarrow{\sim} H_i(BSL(k), \mathbb{Z}/m).$$

Let i_0 be minimal w. $H_{i_0}(BSL_n(k)_\varepsilon, \mathbb{Z}/m) \neq 0$.

The Serre sp. seq.

$$E^2_{s,t} = H_s(BSL_n(k)^{top}, H_t(BSL_n(k)_\varepsilon, \mathbb{Z}/m))$$

$$\Rightarrow H_{s+t}(BSL_n(k), \mathbb{Z}/m)$$

gives

$$\begin{tikzcd}
H_{i_0+1}(BSL_n(k), \mathbb{Z}/m) \arrow[r] & H_{i_0+1}(BSL_n(k)^{top}, \mathbb{Z}/m) \arrow[dl, "d^{i_0+1}"] & \\
H_{i_0}(BSL_n(k)_\varepsilon, \mathbb{Z}/m) \arrow[r] & H_{i_0}(BSL_n(k), \mathbb{Z}/m) \arrow[r] & \cdots
\end{tikzcd}$$

so by (1), $\nearrow$ is 1-1. Hence, the prop. and (2) shows $i_0 > \frac{n-1}{2}$. //

Cor The can. maps

$$BSL(k)^+ \longrightarrow BSL(k)^{top}$$

$$(\text{resp. } BGL(k)^+ \longrightarrow BGL(k)^{top})$$

induces an iso. on $\pi_*(-,\mathbb{Z}/m)$ and $H_*(-,\mathbb{Z}/m)$.

pf We have $H_t(BSL(k)_\varepsilon, \mathbb{Z}/m) = 0$, for $t>0$, so by Serre sp. seq.

$$BSL(k)^+ \longrightarrow BSL(k)^{top}$$

induces iso on $H_*(-,\mathbb{Z}/m)$. Since both spaces are simply-conn., same is true on $\pi_*(-,\mathbb{Z}/m)$. Finally,

$$\begin{tikzcd} BGL(k)^+ \arrow[d] & Bk^* \times BSL(k)^+ \arrow[l, "\sim"'] \arrow[d] \\ BGL(k)^{top} & Bk^{*\,top} \times BSL(k)^{top} \arrow[l, "\sim"'] \end{tikzcd}$$

. //

Thm $K_*(\mathbb{C},\mathbb{Z}/m) \xleftarrow{\sim} S_{\mathbb{Z}/m}\{\underset{\substack{|\\2}}{b}\}$.