18.917: *Topics in algebraic topology*

The course will give an introduction to higher algebraic K-theory following [**1**] and [**2**]. To give an idea of what this is about, suppose first that k is a field. The determinant $\det\colon GL_n(k) \to k^*$ factors through the group homomorphisms

$$GL_n(k) \hookrightarrow GL(k) \twoheadrightarrow GL(k)^{\mathrm{ab}},$$

and the induced map $GL(k)^{\mathrm{ab}} \xrightarrow{\sim} k^*$, it turns out, is an isomorphism. If k is any ring, we still have the group homomorphisms above, and therefore, we can define the "determinant" of $M \in GL_n(k)$ to be the image $[M] \in GL(k)^{\mathrm{ab}}$. The abelian group $K_1(k) = GL(k)^{\mathrm{ab}}$ is the first algebraic K-group or the Whitehead group.

There is no algebraic definition definition of the higher K-groups. Instead, let $BGL(k)$ be the classifying space with fundamental group $GL(k)$ and with all higher homotopy groups trivial. There is a space $K(k)$ which can be thought of as the "abelianization" of $BGL(k)$. The fundamental group of $K(k)$ is the group $K_1(k) = GL(k)^{\mathrm{ab}}$, but the construction also introduces higher homotopy groups. These, by definition, are the higher K-groups, $K_q(k) = \pi_q K(k)$.

Clearly, the groups $K_q(k)$ depend only on the homotopy type of $K(k)$, and it is useful to consider many different homeomorphism types within this homotopy type. One example of a result we will prove is the so-called localization sequence: Let V be discrete valuation ring with field of fractions K and residue field k. Then there is a long-exact sequence of K-groups

$$\cdots \to K_q(k) \to K_q(V) \to K_q(K) \xrightarrow{\partial} K_{q-1}(k) \to \dots,$$

which for $q = 1$ is isomorphic to the sequence

$$0 \to V^* \to K^* \xrightarrow{v} \mathbb{Z} \to 0.$$

References

[1] D. Quillen, *Higher algebraic K-theory I*, Algebraic K-theory I: Higher K-theories (Battelle Memorial Inst., Seattle, Washington, 1972), Lecture Notes in Mathematics, vol. 341, Springer-Verlag, 1973.

[2] F. Waldhausen, *Algebraic K-theory of spaces*, Algebraic and geometric topology, Lecture Notes in Mathematics, vol. 1126, Springer-Verlag, 1985, pp. 318–419.