18.904: Problem Set 2

Due: Thursday, March 20.

1. We fix a category $\mathcal{C}$ and consider objects and morphisms in this category. We say that a commutative (this means that $f \circ p = f' \circ p'$) diagram

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{f'} & Z \end{array}$$

is a *pull-back* diagram if for every commutative diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{f'} & Z, \end{array}$$

there exists a *unique* map $h\colon A \to X$ in $\mathcal{C}$ such that $g = f \circ h$ and $g' = f' \circ h$.

(i) Suppose that the following diagrams are both pull-back diagrams.

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{f'} & Z \end{array} \qquad\qquad \begin{array}{ccc} X' & \xrightarrow{q} & Y \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{f'} & Z. \end{array}$$

Show that the unique maps

$$h\colon X' \to X \qquad\qquad h'\colon X \to X'$$

with $p \circ h = q$ and $p' \circ h = q'$, and with $q \circ h' = p$ and $q' \circ h' = p'$ are each others inverses (and hence isomorphisms).

We say that the category $\mathcal{C}$ *has pull-backs* if every diagram of the form

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{f'} & Z \end{array}$$

can be completed to a pull-back diagram

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{f'} & Z. \end{array}$$

(ii) Show that each of the following three categories has pull-backs. The category of topological spaces and continuous maps. The category of topological spaces with a chosen base-point and base-point-preserving continuous maps. The category of groups and group homomorphisms.

(over)

(iii) Suppose that we are given a pull-back diagram of topological spaces with a chosen base-point and base-point-preserving continuous maps,

$$\begin{array}{ccc} (X, x_0) & \xrightarrow{p} & (Y, y_0) \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle f} \\ (Y', y_0') & \xrightarrow{f'} & (Z, z_0). \end{array}$$

Show that the diagram of fundamental groups and induced homomorphism,

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{p} & \pi_1(Y, y_0) \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle f} \\ \pi_1(Y', y_0') & \xrightarrow{f'} & \pi_1(Z, z_0), \end{array}$$

is a pull-back diagram of groups and group-homomorphisms.

(*Hint*: Use the explicit construction of the pull-back in the two cases from (ii).)

2. Please do the following exercises from pages 52-55 of the text: 4, 7, 14, and 19.