18.906: *Algebraic Topology*

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The course will cover basic homological algebra, homotopy theory, and spectral sequences. Additional topics may include characteristic classes.

References

[1] A. Hatcher, *Algebraic topology*, Cambridge University Press.

[2] J. W. Milnor and J. D. Stasheff, *Characteristic classes*, Annals of Math. Studies, vol. 76, Princeton University Press.

Let k be a commutative (ground) ring, and let M, N, and P be k-modules. A map

$$M \times N \xrightarrow{f} P$$

is k-bilinear if $\forall\, m, m' \in M,\ n, n' \in N,\ a \in k$:

$$f(m+m', n) = f(m,n) + f(m', n)$$
$$f(a \cdot m, n) = a \cdot f(m,n) = f(m, a \cdot n)$$
$$f(m, n+n') = f(m,n) + f(m,n').$$

The tensor product, by definition, is a k-module $M \otimes_k N$ together with a k-bilin. map $M \times N \xrightarrow{\alpha} M \otimes_k N$ s.t. for all k-modules P and all k-bilinear maps $M \times N \xrightarrow{f} P$, there exists a unique k-lin. map $M \otimes_k N \xrightarrow{\tilde{f}} P$ s.t. $f = \tilde{f} \circ \alpha$, i.e.

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[dr, "\alpha"'] & P \\ & M \otimes_k N \arrow[u, dashed, "\exists!\, \tilde{f}"'] \end{tikzcd}$$

The tensor product, if it exists, is unique up to unique isomorphism. Indeed, suppose $M \times N \xrightarrow{\alpha} M \otimes_k N$ and $M \times N \xrightarrow{\alpha'} M \otimes'_k N$ both satisfy the definition. Then

$$\begin{tikzcd}
M \times N \arrow[rr, "\alpha"] \arrow[dd, "\alpha"] \arrow[dr, "\alpha'"] & & M \otimes_k N \\
& M \otimes'_k N \arrow[ur, dashed, "\exists!\, \tilde{\alpha}"'] & \\
M \otimes_k N \arrow[ur, dashed, "\exists!\, \tilde{\alpha}'"'] \arrow[uurr, bend right] & &
\end{tikzcd}$$

$\tilde{\alpha} \circ \tilde{\alpha}'$ and id both make the diagram commute: $\tilde{\alpha} \circ \tilde{\alpha}' = \mathrm{id}$

Similarly, $\tilde{\alpha}' \circ \tilde{\alpha} = \mathrm{id}$.

<u>Prop</u> The tensor product

$$M \times N \xrightarrow{\alpha} M \otimes_k N$$

exists.

<u>Pf</u> We define $M \otimes_k N$ to be the quotient of the free k-module generated by the set $M \times N$ by the sub-k-module gen. by

$$(m+m', n) - (m, n) - (m', n)$$
$$(m, n+n') - (m, n) - (m, n')$$
$$(a.m, n) - a.(m, n)$$
$$(m, a.n) - a.(m, n)$$

where $m, m' \in M$, $n, n' \in N$, $a \in k$. We write

$m \otimes n$ for the class of (m,n) in $M \otimes_k N$ and define $\alpha(m,n) = m \otimes n$. Then α is k-bilinear. Let P be a k-module. Any map of sets $M \times N \xrightarrow{f} P$ extends by linearity uniquely to a k-linear map

$$k\{M \times N\} \xrightarrow{f'} P$$

from the free k-module generated by the set $M \times N$. The map f' factors through the canonical projection

$$\begin{tikzcd} k\{M \times N\} \arrow[rr, "f'"] \arrow[dr] & & P \\ & M \otimes_k N \arrow[ur, dashed, "\tilde{f}"'] & \end{tikzcd}$$

if and only if f is k-bilinear. It it clear that $\tilde{f} \circ \alpha = f$ and that $\tilde{f}$ is unique with this property. //

The set $\mathrm{Hom}_k(M,N)$ is a k-module where

$$(f + f')(m) = f(m) + f'(m)$$

$$(a \cdot f)(m) = a \cdot f(m).$$

Here $a \cdot f$ is k-lin. since k is comm.

<u>Lemma</u> The following map is an iso.

$$\mathrm{Hom}_k(M \otimes_k N, P) \to \mathrm{Hom}_k(M, \mathrm{Hom}_k(N, P))$$

$$f \longmapsto f^{\#}$$

$$f^{\#}(m)(n) = f(m \otimes n). \qquad //$$

The <u>direct</u> <u>sum</u> of a family $\{M_\alpha\}_{\alpha \in \mathcal{A}}$ of k-modules is a k-module $\bigoplus_{\alpha \in \mathcal{A}} M_\alpha$ together with k-linear maps

$$M_\alpha \xrightarrow{i_\alpha} \bigoplus_{\alpha \in \mathcal{A}} M_\alpha \quad , \quad \alpha \in \mathcal{A},$$

s.t. for all k-modules P and all k-lin. maps $M_\alpha \xrightarrow{f_\alpha} P$, $\alpha \in \mathcal{A}$, there is a unique k-linear map

$$\bigoplus_{\alpha \in \mathcal{A}} M_\alpha \xrightarrow{f} P$$

s.t. $f_\alpha = f \circ i_\alpha$, for all $\alpha \in \mathcal{A}$, i.e.

$$\begin{tikzcd}
M_\alpha \arrow[rr, "f_\alpha"] \arrow[dr, "i_\alpha"'] & & P \\
& \bigoplus_{\alpha \in \mathcal{A}} M_\alpha \arrow[ur, dashed, "\exists!\, f"'] &
\end{tikzcd} \qquad \forall \alpha \in \mathcal{A}.$$

or equivalently, the map

$$\mathrm{Hom}_k\Big(\bigoplus_{\alpha\in\mathcal{U}} M_\alpha, P\Big) \xrightarrow{(\iota_\alpha^*)} \prod_{\alpha\in\mathcal{U}} \mathrm{Hom}_k(M_\alpha, P)$$

is an isomorphism.

Prop The canonical map

$$\bigoplus_{\alpha\in\mathcal{U}} (M_\alpha \otimes_k N) \longrightarrow \Big(\bigoplus_{\alpha\in\mathcal{U}} M_\alpha\Big) \otimes_k N$$

is an isomorphism

Pf For all k-modules P,

$$\mathrm{Hom}_k\Big(\bigoplus_{\alpha\in\mathcal{U}} (M_\alpha \otimes_k N), P\Big)$$

$$\xrightarrow{\sim} \prod_{\alpha\in\mathcal{U}} \mathrm{Hom}_k(M_\alpha \otimes_k N, P)$$

$$\xrightarrow{\sim} \prod_{\alpha\in\mathcal{U}} \mathrm{Hom}_k(M_\alpha, \mathrm{Hom}_k(N, P))$$

$$\xleftarrow{\sim} \mathrm{Hom}_k\Big(\bigoplus_{\alpha\in\mathcal{U}} M_\alpha, \mathrm{Hom}_k(N, P)\Big)$$

$$\xleftarrow{\sim} \mathrm{Hom}_k\Big(\Big(\bigoplus_{\alpha\in\mathcal{U}} M_\alpha\Big) \otimes N, P\Big).$$

Complete proof next time. //

Cor If M and N are free k-modules with bases $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ then $M\otimes_k N$ is a free k-module with basis $\{e_i\otimes f_j\}_{(i,j)\in I\times J}$.

pf Use prop. and $k\otimes_k k \xrightarrow[\sim]{\mu} k$. //

Prop If $M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is exact so are the induced sequences

$$M'\otimes_k N \xrightarrow{f\otimes N} M\otimes_k N \xrightarrow{g\otimes N} M''\otimes N \to 0$$

$$\mathrm{Hom}_k(M',N) \xleftarrow{f^*} \mathrm{Hom}_k(M,N) \xleftarrow{g^*} \mathrm{Hom}_k(M'',N) \leftarrow 0.$$

pf Since $(g\otimes N)\circ(f\otimes N) = (g\circ f)\otimes N = 0$, we have a map

$$C := \mathrm{coker}(f\otimes N) \xrightarrow{\bar{g}} M''\otimes N.$$

We must produce an inverse map. Given $m''\in M''$ we can find $m\in M$ s.t. $g(m'')=m$. We claim that the map

$$M''\otimes N \xrightarrow{h} C$$

$$m''\otimes n \longmapsto pr(m\otimes n)$$

is well-defined. We must show that the

assignment $(m'', n) \mapsto pr(m \otimes n)$ is well-defined and bilinear. If $m, \tilde{m} \in M$ and $g(m) = g(\tilde{m})$, then $m - \tilde{m} = f(m')$, for some $m' \in M'$. Hence,

$$m \otimes n - \tilde{m} \otimes n = f(m') \otimes n = (f \otimes N)(m' \otimes n),$$

so $pr(m \otimes n) = pr(\tilde{m} \otimes n)$. Bilinearity is now clear. The second statement of the prop. is left to the reader. //

<u>ex</u> The map induced by multiplication

$$\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \xrightarrow{\mu} \mathbb{Z}/(m,n)$$

is an isomorphism. Indeed:

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m \to 0 \quad \Rightarrow$$

$$\begin{tikzcd}
\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n \arrow[r, "m \otimes \mathrm{id}"] \arrow[d, "\mu", "\sim"'] & \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n \arrow[r] \arrow[d, "\mu", "\sim"'] & \mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \arrow[r] \arrow[d, "\mu"] & 0 \\
\mathbb{Z}/n \arrow[r, "m"] & \mathbb{Z}/n \arrow[r] & \mathbb{Z}/(m,n) \arrow[r] & 0
\end{tikzcd}$$

note: not injective unless $(m,n) = 1$.

Prop If F is a free k-module, then

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0 \quad \text{exact} \Rightarrow$$

$$0 \to M' \otimes_k F \xrightarrow{f \otimes F} M \otimes_k F \xrightarrow{g \otimes F} M'' \otimes_k F \to 0 \quad \text{exact.}$$

pf If $\{e_i\}_{i \in I}$ is a k-basis of F, then

$$\bigoplus_{i \in I} M \xleftarrow{\sim} \bigoplus_{i \in I} (M \otimes_k k)$$

$$\xrightarrow{\sim} M \otimes_k \Big(\bigoplus_{i \in I} k \Big) \xrightarrow{\sim} M \otimes_k F,$$

so

$$\begin{array}{ccccccccc} 0 \to & M' \otimes_k F & \to & M \otimes_k F & \to & M'' \otimes_k F & \to 0 \\ & \uparrow \wr & & \uparrow \wr & & \uparrow \wr & \\ 0 \to & \bigoplus_{i \in I} M' & \to & \bigoplus_{i \in I} M & \to & \bigoplus_{i \in I} M'' & \to 0 \end{array}$$

and the lower sequence is exact. //

Let k be a comm. ring, and let C_* and C'_* be chain cx. of k-mod. We define chain cx. $C_* \otimes_k C'_*$ and $\mathrm{Hom}_k(C_*, C'_*)$ by

$$(C_* \otimes_k C'_*)_n = \bigoplus_{s+t=n} C_s \otimes_k C'_t$$

$$d(x \otimes x') = dx \otimes x' + (-1)^{|x|} x \otimes dx'$$

$$\mathrm{Hom}_k(C_*, C'_*)_n = \prod_{t-s=n} \mathrm{Hom}_k(C_s, C'_t)$$

$$d(f(x)) = (df)(x) + (-1)^{|f|} f(dx).$$

We have canonical isomorphisms of chain complexes

$$\mathrm{Hom}_k(C_* \otimes_k C'_*, C''_*) \underset{\sim}{\xrightarrow{\phi}} \mathrm{Hom}_k(C_*, \mathrm{Hom}_k(C'_*, C''_*))$$

$$f \longmapsto f^\#$$

$$f^\#(x)(x') = f(x \otimes x')$$

$$C_* \otimes_k C'_* \underset{\sim}{\xrightarrow{\gamma}} C'_* \otimes_k C_*$$

$$x \otimes x' \longmapsto (-1)^{|x||x'|} x' \otimes x.$$

<u>Lemma</u> Let k be a <u>field</u>, let M be a k-vector space, and let C_* be a chain complex of k-vector spaces. Then the canonical maps

$$H_n(C_*) \otimes_k M \longrightarrow H_n(C_* \otimes_k M)$$

$$H_n(\mathrm{Hom}_k(C_*, M)) \longrightarrow \mathrm{Hom}_k(H_{-n}(C_*), M)$$

are isomorphisms.

<u>Pf</u> Let $Z_* \subset C_*$ (resp. $B_* \subset C_*$) be the cycles (resp. boundaries) considered as a complex with zero differential. Then we have a short-exact seq. of cx.

$$0 \to Z_* \longrightarrow C_* \xrightarrow{d} B_{*-1} \longrightarrow 0 .$$

The induced long-exact sequence on becomes

$$\cdots \longrightarrow B_n \xrightarrow{\partial = \mathrm{incl}} Z_n \xrightarrow{pr} H_n(C_*) \xrightarrow{0} B_{n-1} \xrightarrow{\partial = \mathrm{incl}} \cdots$$

Since k is a field, the induced sequences of complexes

$$0 \to Z_* \otimes_k M \longrightarrow C_* \otimes_k M \longrightarrow B_{*-1} \otimes_k M \to 0$$

$$0 \leftarrow \mathrm{Hom}_k(Z_*, M) \leftarrow \mathrm{Hom}_k(C_*, M) \leftarrow \mathrm{Hom}_k(B_{*-1}, M) \leftarrow 0$$

are again exact. The induced long-exact sequences on homology takes the form:

$$\begin{tikzcd}
\cdots \arrow[r, "0"] & B_n \otimes M \arrow[r, "\partial = \mathrm{ind} \otimes \mathrm{id}"] & Z_n \otimes M \arrow[r] \arrow[dr] & H_n(C_* \otimes M) \arrow[r, "0"] & B_{n-1} \otimes M \arrow[r] & \cdots \\
 & & & H_n(C_*) \otimes M \arrow[u, "\sim"'] \arrow[dr] & & \\
 & & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

injective since k a field

$$\begin{tikzcd}
\cdots & \mathrm{Hom}(B_{-n}, M) \arrow[l] & \mathrm{Hom}(Z_{-n}, M) \arrow[l, "0"'] & H_n(\mathrm{Hom}(C_*, M)) \arrow[l] \arrow[dl, "\sim"] & \cdots \arrow[l, "0"'] \\
 & & \mathrm{Hom}(H_{-n}(C_*), M) \arrow[u] \arrow[dl] & & \\
 & 0 & & 0 \arrow[ul] &
\end{tikzcd}$$

surjective since k a field

//

Write $H^n(-) := H_{-n}(-)$; then

$$H^n(\mathrm{Hom}_k(C_*, M)) \xrightarrow{\sim} \mathrm{Hom}(H_n(C_*), M).$$

Prop Let k be a field, and let C_* and C'_* be chain complexes. Then the canonical maps are isomorphisms:

$$H_*(C_*) \otimes_k H_*(C'_*) \xrightarrow{\sim} H_*(C_* \otimes_k C'_*)$$

$$H_*(\mathrm{Hom}_k(C_*, C'_*)) \xrightarrow{\sim} \mathrm{Hom}_k(H_*(C_*), H_*(C'_*)).$$

pf Since k is a field, the s.e.s.

$$0 \to Z'_* \to C'_* \to B'_{*-1} \to 0$$

induces s.e.s. of cx.

$$0 \to C_* \otimes Z'_* \to C_* \otimes C'_* \to C_* \otimes B'_{*-1} \to 0$$

$$0 \to \mathrm{Hom}(C_*, Z'_*) \to \mathrm{Hom}(C_*, C'_*) \to \mathrm{Hom}(C_*, B'_{*-1}) \to 0.$$

The induced long-exact sequences on homology take the form

$$\begin{tikzcd}
\cdots \arrow[r, "0"] & H_n(C_* \otimes B'_*) \arrow[r, "\partial"] & H_n(C_* \otimes Z'_*) \arrow[r] & H_n(C_* \otimes C'_*) \arrow[r, "0"] & \cdots \\
0 \arrow[r] & (H_*(C_*) \otimes B'_*)_n \arrow[u, "\sim"] \arrow[r] & (H_*(C_*) \otimes Z'_*)_n \arrow[u, "\sim"] \arrow[r] & (H_*(C_*) \otimes H_*(C'_*))_n \arrow[u, "\sim"] \arrow[r] & 0
\end{tikzcd}$$

injective since k is a field

//

Singular homology and cohomology. We define the <u>standard</u> n-<u>simplex</u> by

$$\Delta^n = \{(x_0, .., x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0 , \sum_{i=0}^{n} x_i = 1\}$$

The coface and codegeneracy operators are the maps

$$\Delta^{n-1} \xrightarrow{d^i} \Delta^n , \quad 0 \leq i \leq n ,$$

$$d^i(x_0, .., x_{n-1}) = (x_0, .., x_{i-1}, 0, x_i, .., x_{n-1}) ;$$

$$\Delta^{n+1} \xrightarrow{s^i} \Delta^n , \quad 0 \leq i \leq n ,$$

$$s^i(x_0, .., x_{n+1}) = (x_0, .., x_i + x_{i+1}, .., x_{n+1}) .$$

They satisfy the <u>cosimplicial</u> <u>relations</u>:

$$\begin{aligned}
d^j d^i &= d^i d^{j-1} , & i < j , \\
s^j s^i &= s^{j+1} s^i , & i \leq j , \\
s^j d^i &= d^i s^{j-1} , & i < j , \\
&= id , & i = j \text{ or } i = j+1 , \\
&= d^{i-1} s^j , & i > j+1 .
\end{aligned}$$

I.e. $\Delta^{\cdot}$ is a cosimplicial space.

If X is a space, we define the singular set (a simplicial set) by

$$\mathrm{Sin}_\cdot(X) := \mathrm{Map}(\Delta^\cdot, X),$$

i.e.

$$\begin{array}{ccc} \mathrm{Sin}_n(X) & = & \mathrm{Map}(\Delta^n, X) \\ d_i \big\downarrow \big\uparrow s_i & & d^{i*} \big\downarrow \big\uparrow s^{i*} \\ \mathrm{Sin}_{n-1}(X) & = & \mathrm{Map}(\Delta^{n-1}, X) \end{array}$$

The face and degeneracy operators satisfy the simplicial relations

$$\begin{aligned} d_i d_j &= d_{j-1} d_i, & i < j, \\ s_i s_j &= s_i s_{j+1}, & i \le j, \\ d_i s_j &= s_{j-1} d_i, & i < j, \\ &= \mathrm{id}, & i = j \text{ or } i = j+1, \\ &= s_j d_{i-1}, & i > j+1. \end{aligned}$$

The singular chain complex is

$$C_*(X; k) = k\{\mathrm{Sin}_\cdot(X)\}$$

$$d = \Sigma(-1)^i d_{i*}.$$

and the <u>singular</u> <u>cochain</u> <u>complex</u> is

$$C^*(X;k) = \mathrm{Hom}_k(C_*(X;k), k).$$

Singular homology and cohomology with coefficients in k is:

$$H_*(X;k) = H_*(C_*(X;k))$$

$$H^*(X;k) = H^*(C^*(X;k)).$$

Recall:

<u>Thm</u> A homotopy $H: X \times [0,1] \to Y$ from f to g induces a chain homotopy $s: C_*(X) \to C_{*+1}(Y)$ from f_* to g_*. //

<u>Cor</u> $f \simeq g: X \to Y \Rightarrow f_* = g_*: H_*(X) \to H_*(Y)$.

The complex $C_*(X;k)$ is <u>augmented</u> over k, i.e. there is a chain map

$$C_*(X;k) \xrightarrow{\varepsilon} k,$$

$$C_0(X;k) \xrightarrow{\varepsilon} k$$

$$\Sigma a_i a_i \longmapsto \Sigma a_i$$

Thm (Eilenberg - Zilber)

(i) There is a natural chain htpy. eq.

$$C_*(X\times Y;k) \xrightarrow{a} C_*(X;k)\otimes_k C_*(Y;k)$$

(resp. $C_*(X;k)\otimes_k C_*(Y;k) \xrightarrow{b} C_*(X\times Y;k)$)

which preserve the augmentation.

(ii) Any two natural chain maps

$$C_*(X\times Y;k) \longrightarrow C_*(X;k)\otimes_k C_*(Y;k)$$

(resp. $C_*(X;k)\otimes_k C_*(Y;k) \longrightarrow C_*(X\times Y;k)$).

which preserve the augmentation are naturally chain-homotopic. //

We now define the cross product

$$H^*(X;M)\otimes H^*(Y;N) \xrightarrow{\times} H^*(X\times Y;M\otimes N)$$

to be the composite map, where

$$C_*(X\times Y) \xrightarrow{a} C_*(X)\otimes C_*(Y)$$

is some chain homotopy equivalence:

$$
\begin{array}{c}
H^*(X; M) \otimes H^*(Y; N) \\
\parallel \\
H^*(\mathrm{Hom}(C_*(X), M)) \otimes H^*(\mathrm{Hom}(C_*(Y), N)) \\
\downarrow \mathrm{can} \\
H^*(\mathrm{Hom}(C_*(X); M) \otimes \mathrm{Hom}(C_*(Y), N)) \\
\downarrow \mathrm{can} = f_* \\
H^*(\mathrm{Hom}(C_*(X) \otimes C_*(Y), M \otimes N)) \\
\downarrow a^* \\
H^*(\mathrm{Hom}(C_*(X \times Y), M \otimes N)) \\
\parallel \\
H^*(X \times Y, M \otimes N)
\end{array}
$$

If $x \in H^m(X;M)$ and $y \in H^n(Y,N)$ are represented by the cocycles

$$C_m(X) \xrightarrow{\xi} M, \quad C_n(Y) \xrightarrow{\eta} N,$$

then $x \times y \in H^{m+n}(X \times Y; M \otimes N)$ is repr. by $(-1)^{mn}$ times the composite

$$C_{m+n}(X \times Y) \xrightarrow{a} (C_*(X) \otimes C_*(Y))_{m+n}$$

$$\xrightarrow{pr} C_m(X) \otimes C_n(Y) \xrightarrow{\xi \otimes \eta} M \otimes N.$$

The sign comes from the canonical map

$$\mathrm{Hom}(C_*, M) \otimes \mathrm{Hom}(C'_*, N) \xrightarrow{\varphi} \mathrm{Hom}(C_* \otimes C'_*, M \otimes N).$$

By definition, this is the adjoint of the composite $\varphi^\#$:

$$\mathrm{Hom}(C_*, M) \otimes \mathrm{Hom}(C'_*, N) \otimes C_* \otimes C'_*$$

$$\xrightarrow{id \otimes \gamma \otimes id} \mathrm{Hom}(C_*, M) \otimes C_* \otimes \mathrm{Hom}(C'_*, N) \otimes C'_*$$

$$\xrightarrow{ev \otimes ev} M \otimes N.$$

So if $\xi \in \mathrm{Hom}(C_m, M)$, $\eta \in \mathrm{Hom}(C'_n, N)$,

$x \in C_m$, and $y \in C_n'$, then

$$f^{\#}(\xi \otimes \eta \otimes x \otimes y)$$

$$:= (ev \otimes ev) \circ \gamma (\xi \otimes \eta \otimes x \otimes y)$$

$$= (-1)^{mn} (ev \otimes ev)(\xi \otimes x \otimes \eta \otimes y)$$

$$= (-1)^{mn} \xi(x) \otimes \eta(y)$$

so

$$f(\xi \otimes \eta)(x \otimes y) := f^{\#}(\xi \otimes \eta \otimes x \otimes y)$$

$$= (-1)^{mn} \xi(x) \otimes \eta(y).$$

/

We define the <u>cup-product</u>

$$H^*(X;k) \otimes H^*(X,k) \xrightarrow{\cup} H^*(X;k)$$

by

$$H^*(X;k) \otimes H^*(X;k) \xrightarrow{\times} H^*(X \times X, k \otimes k)$$

$$\xrightarrow{\Delta_X^*} H^*(X; k \otimes k) \underset{\sim}{\xrightarrow{\mu_*}} H^*(X; k).$$

Thm (Eilenberg - Zilber)

(i) There are natural chain htpy. eq.

$$C_*(X\times Y) \underset{b}{\overset{a}{\rightleftarrows}} C_*(X)\otimes C_*(Y),$$

which preserve augmentation.

(ii) Any two natural chain maps

$$C_*(X\times Y) \longrightarrow C_*(X)\otimes C_*(Y)$$

$$(\text{resp. } C_*(X)\otimes C_*(Y) \longrightarrow C_*(X\times Y)),$$

which preserve the augmentation, are naturally chain homotopic.

Pf We first construct the natural chain maps a and b in (i). We then prove (ii). This also shows that a and b are chain htpy eq.

We must produce natural transformations

$$C_n(X\times Y) \xrightarrow{a_n} (C_*(X)\otimes C_*(Y))_n$$

s.t. the following diagr. commute:

$$\begin{array}{ccc} C_0(X\times Y) & \xrightarrow{a_0} & C_0(X)\otimes C_0(Y) \\ \downarrow \varepsilon & & \downarrow \varepsilon\otimes\varepsilon \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}\otimes\mathbb{Z} \end{array}$$

$$1 \longmapsto 1\otimes 1$$

$$\begin{array}{ccc} C_n(X\times Y) & \xrightarrow{a_n} & (C_*(X)\otimes C_*(Y))_n \\ \downarrow d & & \downarrow d \\ C_{n-1}(X\times Y) & \xrightarrow{a_{n-1}} & (C_*(X)\otimes C_*(Y))_{n-1} \end{array}$$

We do this by induction on $n \geq 0$. To construct a_0, recall that

$$C_0(X\times Y) = \mathbb{Z}\{\mathrm{Map}(\Delta^0, X\times Y)\},$$

and let $(\Theta, \Psi) : \Delta^0 \to X\times Y$ be a gen. By naturality the following diagr. must commute:

$$\begin{array}{ccc} C_0(\Delta^0\times\Delta^0) & \xrightarrow{a_0} & C_0(\Delta^0)\otimes C_0(\Delta^0) \\ \downarrow (\Theta\times\Psi)_* & & \downarrow \Theta_*\otimes\Psi_* \\ C_0(X\times Y) & \xrightarrow{a_0} & C_0(X)\otimes C_0(Y). \end{array}$$

Let $\Delta : \Delta^\circ \to \Delta^\circ \times \Delta^\circ$ be the diagonal. Then

$$(\Theta, \Psi) = (\Theta \times \Psi)_* (\Delta)$$

so naturality implies

$$q(\Theta, \Psi) = q_0 \circ (\Theta \times \Psi)_* (\Delta)$$

$$= (\Theta_* \otimes \Psi_*) \circ q_0 (\Delta).$$

Hence, the natural transf. q_0 is completely determined by the element

$$q_0(\Delta) \in C_0(\Delta^\circ) \otimes C_0(\Delta^\circ).$$

In order that q preserve augmentation the following diagr. must commute

$$\begin{tikzcd}
C_0(\Delta^\circ \times \Delta^\circ) \arrow[r, "q_0"] \arrow[d, "\varepsilon", "\sim"'] & C_0(\Delta^\circ) \otimes C_0(\Delta^\circ) \arrow[d, "\varepsilon \otimes \varepsilon", "\sim"'] \\
\mathbb{Z} \arrow[r, "\sim"] & \mathbb{Z} \otimes \mathbb{Z}
\end{tikzcd}$$

This forces: $q_0(\Delta) = \mathrm{id} \otimes \mathrm{id}$.

Suppose now that q_i has been constructed for all $i < n$. We construct q_n. Again,

$$C_n(X \times Y) = \mathbb{Z}\{\mathrm{Map}(\Delta^n, X \times Y)\};$$

let $(\theta, \psi) : \Delta^n \to X \times Y$ be a generator. By naturality the diagr. must commute:

$$\begin{array}{ccc}
C_n(\Delta^n \times \Delta^n) & \xrightarrow{a_n} & (C_*(\Delta^n) \otimes C_*(\Delta^n))_n \\
\Big\downarrow (\theta \times \psi)_* & & \Big\downarrow \theta_* \otimes \psi_* \\
C_n(X \times Y) & \xrightarrow{a_n} & (C_*(X) \otimes C_*(Y))_n
\end{array}$$

so we must have

$$a_n(\theta, \psi) = a_n((\theta \times \psi)_*(\Delta))$$

$$= (\theta_* \otimes \psi_*)(a_n(\Delta)).$$

So a_n is completely determined by

$$a_n(\Delta) \in (C_*(\Delta^n) \otimes C_*(\Delta^n))_n .$$

In order that a be a chain map, the following diagram must commute

$$\begin{array}{ccc} C_n(\Delta^n \times \Delta^n) & \xrightarrow{a_n} & (C_*(\Delta^n) \otimes C_*(\Delta^n))_n \\ \downarrow d & & \downarrow d \\ C_{n-1}(\Delta^n \times \Delta^n) & \xrightarrow{a_{n-1}} & (C_*(\Delta^n) \otimes C_*(\Delta^n))_{n-1} \end{array}$$

so $a_n(\Delta)$ must satisfy the equation

$$(*) \qquad d(a_n(\Delta)) = a_{n-1}(d(\Delta)).$$

The right hand side is a cycle:

$$d\, a_{n-1}(d(\Delta)) = a_{n-2}\, dd(\Delta) = 0,$$

if $n > 1$, and is in the kernel of the augmentation, if $n = 1$. But

$$H_{n-1}(C_*(\Delta^n) \otimes C_*(\Delta^n))$$

$$\xleftarrow{\sim} (H_*(\Delta^n) \otimes H_*(\Delta^n))_{n-1}$$

$$\xrightarrow[\sim]{\varepsilon \otimes \varepsilon} \begin{cases} \mathbb{Z} \otimes \mathbb{Z}, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases}$$

Here $H_*(\Delta^n) \xrightarrow[\sim]{\varepsilon} \mathbb{Z}$, since $\Delta^n \simeq *$. So can solve $(*)$. Finally, the diagr.

$$\begin{array}{ccc} C_n(X\times Y) & \xrightarrow{a_n} & (C_*(X)\otimes C_*(Y))_n \\ \downarrow d & & \downarrow d \\ C_{n-1}(X\times Y) & \xrightarrow{a_{n-1}} & (C_*(X)\otimes C_*(Y))_{n-1} \end{array}$$

commutes, for all X, Y, because d is natural. The map is constructed analogously.

Suppose next that we are given two natural chain maps

$$C_*(X\times Y) \xrightarrow{a,\,a'} C_*(X)\otimes C_*(Y)$$

which preserve the augmentation. We already know, then, that $a_0 = a_0'$. Wish to produce natural maps

$$C_n(X\times Y) \xrightarrow{s_n} (C_*(X)\otimes C_*(Y))_{n+1}$$

s.t.

$$d s_n + s_{n-1} d = a_n - a_n'$$

Can take $s_0 = 0$. Suppose s_i has been constructed, $i<n$. By naturality it suffices to define s_n on the generator

$\Delta \in C_n(\Delta^n \times \Delta^n)$.

The elem. $s_n(\Delta) \in (C_*(\Delta^n) \otimes C_*(\Delta^n))_{n+1}$ must satisfy

$$(**) \quad d\, s_n(\Delta) = a_n(\Delta) - a_n'(\Delta) - s_{n-1}\, d(\Delta).$$

Since $H_n(C_*(\Delta^n) \otimes C_*(\Delta^n)) = 0$ it will suffice to know that the RHS is a cycle. Calc.:

$$d\,(a_n(\Delta) - a_n'(\Delta) - s_{n-1}\, d(\Delta))$$

$$= (a_{n-1} - a_{n-1}')(d(\Delta)) - d\, s_{n-1}\, d(\Delta)$$

$$= (d\, s_{n-1} + s_{n-2}\, d)\,(d(\Delta)) - d\, s_{n-1} d(\Delta)$$

$$= 0.$$

So can solve $(**)$. The map s_n will satisfy $d s_n + s_{n-1} d = a_n - a_n'$, for all X, Y, because d is natural.

Similarly, any two natural chain maps

$$C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$$

$$C_*(X \times Y) \to C_*(X \times Y)$$

$$C_*(X) \otimes C_*(Y) \to C_*(X) \otimes C_*(Y)$$

are naturally chain homotopic. In part., the maps $a \circ b$ and $id_{C_*(X) \otimes C_*(Y)}$ (resp. $b \circ a$ and $id_{C_*(X \times Y)}$) are naturally chain homotopic. So a and b are chain homotopy equivalences. //

The <u>cup product</u> is given by the composite in fig. 1.

<u>Prop</u> The cup product satisfies:

(i) $x \cup (y \cup z) = (x \cup y) \cup z$

(ii) $x \cup y = (-1)^{|x||y|} y \cup x$

(iii) $[\varepsilon] \cup x = x = x \cup [\varepsilon]$.

(In short: $H^*(X;k)$ is a graded commutative k-algebra with $1 = [\varepsilon]$.)

<u>Pf</u> Only prove (ii). The twist map

$$X \times Y \xrightarrow{tw} Y \times X$$
$$(x,y) \longmapsto (y,x)$$

induces a chain map

$$C_*(X \times Y) \xrightarrow{tw_*} C_*(Y \times X).$$

We also have the chain map

$$C_*(X) \otimes C_*(Y) \xrightarrow{\gamma} C_*(Y) \otimes C_*(X).$$

$$x \otimes y \longmapsto (-1)^{|x||y|} y \otimes x$$

Consider the diagr. of chain maps

$$\begin{tikzcd} C_*(X \times Y) \arrow[r, "a"] \arrow[d, "tw_*"] & C_*(X) \otimes C_*(Y) \arrow[d, "\gamma"] \\ C_*(Y \times X) \arrow[r, "a"] & C_*(Y) \otimes C_*(X) \end{tikzcd}.$$

It need not commute. But since all maps preserve augmentations, it commutes up to chain htpy, i.e., $a \circ tw_* \simeq \gamma \circ a$. Hence $tw^*(x \times y) = (-1)^{|x||y|} y \times x$. Take $X = Y$ and apply Δ^*. //

fig. 1

$$
\begin{array}{c}
H^*(X;k) \otimes H^*(X;k) \\
\| \\
H^*(\mathrm{Hom}(C_*(X),k)) \otimes H^*(\mathrm{Hom}(C_*(X),k)) \\
\big\downarrow \; \mathrm{can} \\
H^*(\mathrm{Hom}(C_*(X),k) \otimes \mathrm{Hom}(C_*(X),k)) \\
\big\downarrow \; \mathrm{can}_* \\
H^*(\mathrm{Hom}(C_*(X) \otimes C_*(X), k \otimes k)) \\
\big\downarrow \; \mathrm{Hom}(a,\mu) \\
H^*(\mathrm{Hom}(C_*(X \times X), k) \\
\big\downarrow \; \mathrm{Hom}(C_*(\Delta), id) \\
H^*(\mathrm{Hom}(C_*(X), k) \\
\| \\
H^*(X;k)
\end{array}
$$

18.906: Problem Set 1

Due: Thursday, February 20.

All chain complexes in problems 1.–3. are concentrated in non-negative degrees.

1. Let k be a commutative ring, and let C_* be a chain complex of free k-modules,

$$C_0 \xleftarrow{d} C_1 \xleftarrow{d} C_2 \leftarrow \cdots .$$

Show that there is a chain homotopy between 0 and id if and only if $H_*(C_*) = 0$.

2. If $f\colon X \to Y$ is a chain map, we define the *mapping cone* C_f to be the complex

$$(C_f)_n = Y_n \oplus X_{n-1}, \qquad d(y, x) = (dy - f(x), -dx),$$

and the suspension ΣX to be the cokernel of the inclusion $\iota\colon Y \to C_f$ of the first summand, that is,

$$(\Sigma X)_n = X_{n-1}, \qquad d_{\Sigma X}(x) = -d_X(x).$$

Then we have a short exact sequence of complexes

$$0 \to Y \xrightarrow{\iota} C_f \xrightarrow{\mathrm{pr}} \Sigma X \to 0,$$

where pr is the canonical projection. Show that the connecting homomorphism

$$H_{n-1}(X) = H_n(\Sigma X) \xrightarrow{\partial} H_{n-1}(Y)$$

is equal to $-H_{n-1}(f)$.

3. Prove the following result of J. H. C. Whitehead. Let $f\colon C_* \to C'_*$ be a chain map between chain complexes of of free k-modules. Then f is a chain homotopy equivalence if and only if $H_*(f)$ is an isomorphism. (*Hint*: use 1. and 2.)

4. We consider the following two functors from the category of pairs of spaces and pairs of continuous maps to the category of chain complexes and chain maps.

$$F(X, Y) = C_*(X \times Y),$$
$$G(X, Y) = C_*(X) \otimes C_*(Y).$$

We define a new chain complex denoted $\mathrm{Hom}(F, G)$ as follows. Let $\mathrm{Hom}(F, G)_n$ be the set of natural maps of graded abelian groups of degree n from F to G. That is, an element $f \in \mathrm{Hom}(F, G)_n$ is a sequence $f = (f_r)_{r\in\mathbb{Z}}$ of natural maps of abelian groups $f_r\colon F(X, Y)_r \to G(X, Y)_{r+n}$, but we do *not* assume that f is a chain map. The set $\mathrm{Hom}(F, G)_n$ is an abelian group with the sum is given by $(f + g)(\sigma) = f(\sigma) + g(\sigma)$. The differential

$$d\colon \mathrm{Hom}(F, G)_n \to \mathrm{Hom}(F, G)_{n-1}$$

is defined by the equation

$$d(f(\sigma)) = (df)(\sigma) + (-1)^{\deg(f)} f(d\sigma).$$

2/13/02

Let

$$S^{n-1} = \{ x \in \mathbb{R}^n \mid \|x\| = 1 \}$$

$$D^n = \{ x \in \mathbb{R}^n \mid \|x\| \leq 1 \}$$

with base point $s_0 = (1, 0, \text{—}, 0)$. We will need to choose a homeomorphism

$$D^n / S^{n-1} \xrightarrow[\approx]{p} S^n$$

and a comultiplication

$$S^n \xrightarrow{\psi} S^n \vee S^n .$$

If $X \supset A \ni x_0$, we define $\pi_n(X, A, x_0)$ to be the quotient of the set of maps

$$\begin{array}{ccc} D^n & \xrightarrow{f} & X \\ \cup & & \cup \\ S^{n-1} & \xrightarrow{g} & A \\ \cup & & \cup \\ s_0 & \longmapsto & x_0 \end{array}$$

modulo the homotopy relation. By a

homotopy we mean a map

$$\begin{array}{ccc} D^n \times [0,1] & \xrightarrow{F} & X \\ \uparrow & & \uparrow \\ S^{n-1} \times [0,1] & \xrightarrow{G} & A \\ \uparrow & & \uparrow \\ \{s_0\} \times [0,1] & \longrightarrow & \{x_0\} \end{array} .$$

We write $\pi_n(X, x_0) = \pi_n(X, \{x_0\}, x_0)$. The comultiplication γ makes $\pi_n(X, x_0)$ a group, if $n \geq 1$; it is abelian, if $n \geq 2$. /

We say that $A \subset X$ has the homotopy extension property (HEP) if given a map $X \xrightarrow{f} Y$ and a htpy: $A \times [0,1] \xrightarrow{G} Y$, there exists a htpy. $X \times [0,1] \xrightarrow{F} Y$ s.t. $F|_{X \times \{0\}} = f$ and $F|_{A \times [0,1]} = G$. Clearly:

<u>Lemma</u> $A \subset X$ has HEP if and only if $X \times \{0\} \cup A \times [0,1] \subset X \times [0,1]$ is a retract. //

<u>ex</u> $S^{n-1} \subset D^n$ has HEP; a retraction

of $D^n \times \{0\} \cup S^{n-1} \times [0,1] \subset D^n \times [0,1]$ is given as follows. Consider $D^n \times [0,1] \subset \mathbb{R}^{n+1}$, let $(x,s) \in D^n \times \{0\} \cup S^{n-1} \times [0,1]$, and let $t \in [0,1]$ be such that $t.(x,s) + (1-t).(0,2) \in D^n \times [0,1]$, i.e. $t \in [1/(2-s), 1]$. Then

$$r(t.(x,s) + (1-t).(0,2)) = (x,s).$$

//

Prop A CW-pair (X,A) has HEP.

pf We show the stronger statement that $X \times \{0\} \cup A \times [0,1]$ is a strong deformation retract of $X \times [0,1]$. Let

$$D^n \times [0,1] \xrightarrow{r} D^n \times \{0\} \cup S^{n-1} \times [0,1]$$

be the retraction from the example above and define

$$D^n \times [0,1] \times [0,1] \xrightarrow{F} D^n \times [0,1]$$

by $F = tr + (1-t)\,\mathrm{id}_{D^n \times [0,1]}$. Then F is a strong deformation retract of $D^n \times [0,1]$ on $D^n \times \{0\} \cup S^{n-1} \times [0,1]$. Using this simultaneously on the n-cells of (X,A) gives rise to a strong def. retraction

$$X^n \times [0,1] \times [0,1] \xrightarrow{F} X^n \times [0,1]$$

of $X^n \times [0,1]$ onto $X^n \times \{0\} \cup X^{n-1} \times [0,1]$. If we apply this homotopy in the interval $[2^{-(n+1)}, 2^{-n}]$, we get a strong def. retract.

$$X \times [0,1] \times [0,1] \xrightarrow{F} X \times [0,1]$$

of $X \times [0,1]$ onto $X \times \{0\} \cup A \times [0,1]$. //

Prop There is a natural exact sequence of pointed sets

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{p^*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \cdots$$

pf We prove exactness at $\pi_n(X, A, x_0)$. This means that $\partial \circ p^* = 0$ and that if $\partial(c) = 0$ then $c = p^* c'$. The map p^* takes the class of

$$\begin{array}{ccc} S^n & \xrightarrow{f} & X \\ \cup & & \cup \\ s_0 & \longmapsto & x_0 \end{array}$$

to the class of

$$\begin{array}{ccccc} D^n & \xrightarrow{p} & S^n & \xrightarrow{f} & X \\ \cup & & & & \cup \\ S^{n-1} & & \xrightarrow{x_0} & & A \\ \cup & & & & \cup \\ s_0 & & \longmapsto & & x_0 \end{array}.$$

The map ∂ takes the class of

$$\begin{array}{ccc} D^n & \xrightarrow{f} & X \\ \cup & & \cup \\ S^{n-1} & \xrightarrow{g} & A \\ \Psi & & \Psi \\ s_0 & \longmapsto & x_0 \end{array}$$

to the class of

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{g} & A \\ \Psi & & \Psi \\ s_0 & \longmapsto & x_0 \end{array}$$

It is clear that $\partial \circ p^* = 0$. Suppose that the class of g is trivial. This means that there exists a htpy.

$$\begin{array}{ccc} S^{n-1} \times [0,1] & \xrightarrow{G} & A \\ \cup & & \cup \\ \{s_0\} \times [0,1] & \longrightarrow & \{x_0\} \end{array}$$

s.t. $G|_{S^{n-1} \times \{0\}} = g$ and $G|_{S^{n-1} \times \{1\}} = x_0$.
Since $S^{n-1} \subset D^n$ has HEP we can find

$$D^n \times [0,1] \xrightarrow{F} X$$

s.t. $F|_{D^n \times \{0\}} = f$ and $F|_{S^{n-1} \times [0,1]} = G$.

Hence, F is a homotopy from (f, g) to a map of the form

$$\begin{array}{ccc}
D^n & \xrightarrow{f'} & X \\
\uparrow & & \cup \\
S^{n-1} & \xrightarrow{x_0} & A \\
\psi & & \psi \\
s_0 & \longmapsto & x_0
\end{array}$$

This shows that the class of (f, g) is in the image of p^*. The other two cases are similar. //

Def A map $X \xrightarrow{f} Y$ is a <u>weak</u> <u>equivalence</u> if for all $n \geq 0$ and all $x_0 \in X$, the induced map

$$\pi_n(X, x_0) \xrightarrow{f_*} \pi_n(Y, f(x_0))$$

is an isomorphism.

<u>Theorem</u> (Whitehead) Let $X \xrightarrow{f} Y$ be a weak equivalence and suppose that X and Y are CW-complexes. Then f is a homotopy equivalence.

<u>Lemma</u> Let (X, A) be a CW-pair and let (Y, B) be any pair with $B \neq \emptyset$. Suppose for all $n \geq 0$ s.t. (X, A) has cells of dim. n that $\pi_n(Y, B, y_0) = 0$, for all $y_0 \in B$. Then every map

$$(X, A) \xrightarrow{f} (Y, B)$$

is homotopic rel A to a map $X \to B$.

Pf Suppose inductively that a homotopy rel A has been constructed from f to a map $(X, X^{n-1}) \xrightarrow{f'} (Y, B)$. Consider

$$\begin{array}{ccccc}
\coprod_\alpha S^{n-1} & \xrightarrow{\varphi} & X^{n-1} & \xrightarrow{f'} & B \\
\downarrow & & \downarrow & & \downarrow \\
\coprod_\alpha D^n & \xrightarrow{\varphi} & X^n & \xrightarrow{f'} & Y
\end{array}$$

Since $\pi_n(Y, B, y_0) = 0$, the map $f'\varphi$ is homotopic rel X^{n-1} to a map $X^n \xrightarrow{f''} B$. And since $X^n \subset X$ has HEP we can extend this homotopy to a homotopy rel X^{n-1} of $(X, X^{n-1}) \xrightarrow{f'} (Y, B)$ to a map

$(X, X^n) \xrightarrow{f''} (Y, B)$. We apply this htpy on $[2^{-(n+1)}, 2^{-n}]$. Then in all we get a homotopy rel A of $(X, A) \xrightarrow{f} (Y, B)$ to a map $X \to B$ as desired. //

Cor Let (X, A) be a CW-pair and suppose that the inclusion $A \hookrightarrow X$ is a weak equivalence. Then A is a strong deformation retract of X.

Pf Since $A \xrightarrow{\sim} X$, $\pi_n(X, A, x_0) = 0$, for all $n \geq 0$ and all $x_0 \in A$. (For $n = 0$ this means that (X, A) is connected.) Hence, the lemma applied to

$$(X, A) \xrightarrow{id} (X, A)$$

gives a htpy. rel. A of id_X to a map $X \to A$. //

In particular, the theorem holds if $X \xrightarrow{f} Y$ is the inclusion of a sub-CW-complex. To bring ourselves in this situation, we consider the (unreduced) mapping cylinder

$$M_f = Y \underset{\substack{f \nwarrow \quad \nearrow i \\ X}}{\cup} X \times [0,1]$$

Clearly, Y is a strong deformation retract of M_f.

<u>Lemma</u> Let $X \xrightarrow{f} Y$ be a w.e. and suppose that X and Y are CW-cx. Then X is a strong deformation retract of M_f.

<u>Pf</u> We do not assume that the f is cellular. Therefore, there is no obvious CW-structure on M_f. However, the pair $(M_f, Y \amalg X)$ <u>is</u> a CW-pair, so $Y \amalg X \subset M_f$ has HEP. This will do.

We first apply the lemma to

$$(Y \amalg X, X) \hookrightarrow (M_f, X)$$

It gives a homotopy rel X to a map $Y \amalg X \to X$, and since $Y \amalg X \subset M_f$ has HEP, we can extend this map to a homotopy rel. X of id_{M_f} to a map $M_f \xrightarrow{g} M_f$ with $g(Y) \subset X$. We then apply the lemma to the composite

$$(Y \amalg X \times [0,1],\ Y \amalg X \times \{0,1\})$$

$$\xrightarrow{\text{collaps}} (M_f,\ Y \amalg X) \xrightarrow{g} (M_f,\ X).$$

We get a htpy. rel. $Y \amalg X \times \{0,1\}$ to a map $Y \amalg X \times [0,1] \to X$. Since this map agrees with the original map on $Y \amalg X \times \{0,1\}$, it factors to give a map $M_f \to X$. Putting the two homotopies together, we get a homotopy rel. X of id_{M_f} to a map $M_f \to X$. This is the statement of the lemma. //

<u>pf of thm.</u> Indeed, we have

$$\begin{tikzcd}
X \arrow[r, hook, "i", "\simeq"'] \arrow[rr, bend right, "f"'] & M_f \arrow[r, "r", "\simeq"'] & Y
\end{tikzcd}$$

(lemma)

so f is a htpy. equivalence. //

A relative CW-cx. is the following data: a sequence of spaces

$$A = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X = \bigcup_{n \geq 0} X^n$$

and for all $n \geq 0$, a push-out diagram

$$\begin{tikzcd} \coprod_\alpha S^{n-1} \arrow[r, "\varphi"] \arrow[d, hook] \arrow[dr, phantom, "p.o."] & X^{n-1} \arrow[d, hook] \\ \coprod_\alpha D^n \arrow[r, "\varphi"] & X^n \end{tikzcd}$$

We give X the topology where $U \subset X$ is open if and only if $U \cap X^n \subset X^n$. Call $X^n \subset X$ the n-skeleton and say that X^n is obtained from X^{n-1} by adjoining n-cells. Call $e^n_\alpha := \varphi_\alpha(D^n_\alpha)$ an n-cell of X. Note that φ_α induces a homeomorphism $D^n_\alpha \setminus S^{n-1}_\alpha \longrightarrow e^n_\alpha \setminus \partial e^n_\alpha$.

<u>Lemma</u> Let (X, A) be a CW-pair. Then $A \times [0,1] \cup X \times \{0\}$ is a strong deformation retract of $X \times [0,1]$. //

A pair (Y, B) is n-connected, $n \geq 0$, if $\pi_0(B) \xrightarrow{\approx} \pi_0(Y)$ and $\pi_i(Y, B, y_0) = 0$, for all $1 \leq i \leq n$ and all $y_0 \in B$.

<u>Lemma</u> Let (X, A) be a relative CW-cx. of dim. $\leq n$ and let (Y, B) be an n-connected pair. Then every map

$$(X, A) \xrightarrow{f} (Y, B)$$

is homotopic rel. A to a map $X \to B$. //

<u>Cor</u> Let (X, A) be a rel. CW-cx. and let (Y, B) be a n-connected, for all $n \geq 0$. Then every map

$$(X, A) \xrightarrow{f} (Y, B)$$

is homotopic rel. A to a map $X \to B$. //

The following result will use the fund. fact that for all $n \geq 1$ and all $0 \leq k < n$,

$$\pi_k(S^n, s_0) = 0,$$

or equivalently,

$$\pi_k(D^n, S^{n-1}, s_0) = 0.$$

We shall prove this later.

<u>Lemma</u> If X is obtained from A by adjoining n-cells, for some $n \geq 1$, then (X, A) is $(n-1)$-connected.

pf Let $0 \leq k < n$ and let f be a map.

$$\begin{array}{ccc} D^k & \xrightarrow{f} & X \\ \cup & & \cup \\ S^{k-1} & \longrightarrow & A . \end{array}$$

We show that f is homotopic rel A to a map $D^k \to A$. Since $f(D^k)$ is cpt., we have

$$f(D^k) \subset A \cup e_1 \cup \cdots \cup e_m =: X'$$

Pick $x_i \in e_i \setminus \partial e_i$, $1 \leq i \leq m$. Then the open sets

$$U = A \cup (e_1 \setminus \{x_1\}) \cup \cdots \cup (e_m \setminus \{x_m\})$$

$$V_i = e_i \setminus \partial e_i \quad , 1 \leq i \leq m,$$

cover X'. Hence the sets $f^{-1}(U)$ and $f^{-1}(V_i)$, $1 \leq i \leq m$, form an open cover of D^k. One can find a simplicial triangulation $D^k = |K|$ s.t. for all simplices $\sigma \in K$ either $f(|\sigma|) \subset U$ or $f(|\sigma|) \subset e_i \setminus \partial e_i$, some $1 \le i \le m$. Define

$$B := |\sigma : f(|\sigma|) \subset U|$$

$$Y_i := |\sigma : f(|\sigma|) \subset V_i|$$

$$\partial Y_i := Y_i \cap B.$$

Then

$$S^{k-1} \subset B$$

$$D^k = B \cup Y_1 \cup \dots \cup Y_m$$

$$(Y_i \setminus \partial Y_i) \cap (Y_j \setminus \partial Y_j) = \emptyset \quad \text{if } i \neq j$$

$(Y_i, \partial Y_i)$ rel. CW-cx. with cells of dim. $\le k$.

Consider

$$(Y_i, \partial Y_i) \xrightarrow{f} (e_i \setminus \partial e_i,\ e_i \setminus (\partial e_i \cup \{x_i\}))$$

rel. CW-cx. dim $\le k < n$ | n-connected

$\uparrow$ assume

By previous lemma, $f|_{Y_i}$ is homotopic rel ∂Y_i to a map $f': Y_i \to U$. Since $(Y_i \setminus \partial Y_i) \cap (Y_j \setminus \partial Y_j) = \emptyset$ if $i \neq j$, $f: D^k \to X'$ is homotopic rel B to a map $f': D^k \to U$. But A is a strong deformation retract of U, so f' is homotopic rel. A to a map $f'': D^k \to A$.

Return to assumption. We have iso.

$$\pi_i(e_i \setminus \partial e_i, e_i \setminus (\partial e_i \cup \{x_i\}), \varphi(s_0))$$

$$\underset{\approx}{\xleftarrow{\varphi_*}} \pi_i(D^n \setminus S^{n-1}, D^n \setminus (S^{n-1} \cup \{0\}), s_0)$$

$$\underset{\approx}{\xrightarrow{\text{incl}_*}} \pi_i(D^n, D^n \setminus \{0\}, s_0)$$

$$\underset{\approx}{\xleftarrow{\text{incl}_*}} \pi_i(D^n, S^{n-1}, s_0)$$

We use (without proof) that this is zero for $i < n$. //

Cor Let (X, A) be a rel. CW-cx., and let $n \geq 0$. Then (X, X^n) is n-connected.

pf Let $(D^k, S^{k-1}) \xrightarrow{f} (X, X^n)$, $k \leq n$. Since D^k is cpt, $f(D^k) \subset X^m$, some $m \geq n$. By easy induction using the previous lemma, we see that f is homotopic rel S^{k-1} to a map $D^k \to X^n$. //

A map between rel. CW-cx.

$$(X, A) \xrightarrow{f} (X', A')$$

is <u>cellular</u> if for all $n \geq 0$, $f(X^n) \subset X'^n$.

<u>Theorem</u> (Cellular approximation) Every map between relative CW-complexes

$$(X, A) \xrightarrow{f} (X', A')$$

is homotopic rel A to a cellular map.

Pf Suppose, inductively, that $f(X^k) \subset X'^k$, for all $k < n$. Since (X^n, X^{n-1}) has dim. n and since (X', X'^n) is n-conn., we can find a homotopy rel X^{n-1} of $f|_{X^n}$ to a map $f: X^n \to X'^n$. And since $X^n \subset X$ has HEP we can extend this to a homotopy rel. X^{n-1} of f to a map f' with $f'(X^n) \subset X'^n$. We combine the homotopies as usual. //

<u>Addendum</u> If two cellular maps

$$(X, A) \xrightarrow{f, f'} (X', A')$$

are homotopic rel A, then there exists a cellular homotopy rel A between them. //

18.906: Problem Set 2

Due: Thursday, February 27.

1. Let X and Y be objects in a category $\mathcal{C}$, and let $*$ and $*'$ be two composition laws on the set of morphisms $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. Assume that $*$ and $*'$ are mutually distributive and have a common two-sided identity element. Show that $*$ and $*'$ are equal, and that each is commutative and associative.

2. Let (X, e) be an H-space with multiplication $\mu\colon X \times X \to X$ (see Hatcher p. 281). Show that for all $n \geq 1$, the group structure on $\pi_n(X, e)$ defined by

$$(f * g)(x) = \mu(f(x), g(x))$$

is equal to the usual group structure. Show further that $\pi_1(X, e)$ is an abelian group.

3. Hatcher, chap. 4, §1, exercise 11.

4. Hatcher, chap. 4, §1, exercise 23.

A <u>simplicial complex</u> is a set V of <u>vertices</u> and a set $K \subset \mathcal{P}(V)$ of <u>simplices</u> s.t.

(i) $\forall \sigma \in K$: $\sigma \neq \emptyset$ and finite.

(ii) $\forall v \in V$: $\{v\} \in K$.

(iii) $\forall \emptyset \neq \sigma' \subset \sigma \in K$: $\sigma' \in K$.

We call $\sigma' \subset \sigma$ a <u>face</u> of σ. We say that $\sigma = \{v_0, v_1, \ldots, v_q\} \in K$ is a <u>q-simplex</u>. A <u>simplicial map</u>

$$K \xrightarrow{\varphi} K'$$

is a map $V \xrightarrow{\varphi} V'$ s.t. for all $\sigma \in K$, $\varphi(\sigma) \in K$.

<u>ex</u> Let $V = \{e_0, e_1, \ldots, e_n\}$. The standard n-simplex and its boundary are:

$$\Delta[n] = \{\sigma \in \mathcal{P}(V) \mid \sigma \neq \emptyset\}$$

$$\partial\Delta[n] = \{\sigma \in \mathcal{P}(V) \mid \emptyset \neq \sigma \neq V\}.$$

The inclusion $\partial\Delta[n] \hookrightarrow \Delta[n]$ is a simplicial map.

We define the <u>space</u> $|K|$ of a simplicial complex K as follows. The set $|K|$ consists of the maps

$$V \xrightarrow{\alpha} [0,1]$$

s.t.

(i) $\{v \in V \mid \alpha(v) \neq 0\} \in K$

(ii) $\sum_{v \in V} \alpha(v) = 1.$

We call $\alpha(v)$ the barycentric coordinate of α w.r.t. v. For $\sigma \in K$, the <u>closed</u> <u>simplex</u> $|\sigma| \subset |K|$ is:

$$|\sigma| = \{\alpha \in |K| \mid \alpha(v) \neq 0 \Rightarrow v \in \sigma\}.$$

Clearly,

$$|K| = \bigcup_{\sigma \in K} |\sigma|.$$

Let $\mathbb{R}\{V\}$ be the real vectorspace with basis V. We give $\mathbb{R}\{V\}$ the usual metric topology. There is an injective map

$$|K| \longrightarrow \mathbb{R}\{V\} .$$

$$\alpha \longmapsto \sum_{v \in V} \alpha(v) \cdot v$$

We give $|\sigma| \subset |K|$ the subspace topology from $\mathbb{R}\{V\}$, and give $|K|$ the topology of the union. Then $|K|$ can be given the structure of a CW-cx. with one n-cell for each n-simplex $\sigma \in K$. A <u>triangulation</u> of a space X is a simplicial complex K and a homeo.

$$|K| \xrightarrow[\approx]{f} X .$$

A <u>polyhedron</u> is a space X that has a triangulation.

<u>Note</u> If K is finite then $|K|$ is cpt. by Heine-Borel. The converse is also true. Indeed, choosing an interior point of $|\sigma|$, for all $\sigma \in K$, we get a discrete subset of $|K|$ that is bijective to K. And a cpt. discrete space is finite.

The <u>barycentric</u> <u>subdivision</u> of K is the simplicial complex whose set of vertices is the set K and whose set of simpl.

$$sd(K) \subset \mathcal{P}(K)$$

consists of the subsets of the form

$$\{\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{q-1}, \sigma_q\},$$

where $\sigma_{i-1} \subset \sigma_i$, $1 \leq i \leq q$. For example:

$\Delta[2]$ $\qquad\qquad\qquad$ $sd(\Delta[2])$

If $\sigma = \{v_0, v_1, \ldots, v_q\}$, then the <u>barycenter</u> of $|\sigma|$ is the point with barycentric coordinates $\alpha(v_i) = 1/(q+1)$, $0 \leq i \leq q$. We define the (non-simplicial) map

$$|sd(K)| \xrightarrow{D} |K|$$

to be the unique affine map that

the vertex σ of $sd(K)$ to the bary-center of $|\sigma| \subset |K|$.

Prop D is a homeomorphism.

pf The proof is by induction $\dim(K) = n$, the case $n=0$ being trivial. So assume the statement for $n-1$. If $\sigma \in K$, we let $\bar{\sigma} \subset K$ and $\partial\sigma \subset K$ denote simpl. cx. that consist of all faces and all proper faces of σ, respectively. Then $|\bar{\sigma}| = |\sigma| \subset |K|$. It suffices to show that for all $\sigma \in K$ of dim. n,

$$|sd\,\bar{\sigma}| \xrightarrow[\approx]{D} |\bar{\sigma}|.$$

Inductively,

$$|sd\,\partial\sigma| \xrightarrow[\approx]{D} |\partial\sigma|$$

We need to discuss cones. If L is a simpl. cx. with vertices V, then the simpl. cone is the simplicial cx. with vertices $V \amalg \{\infty\}$ and simpl.

$$C(L) = \left\{ \sigma \amalg \sigma' \;\middle|\; \begin{array}{l} \sigma \in K \cup \{\emptyset\},\ \sigma' \subset \{\infty\}, \\ \sigma \amalg \sigma' \neq \emptyset \end{array} \right\}$$

The (unreduced) cone of a space X is the space

$$C(X) = \frac{X \times [0,1]}{X \times \{1\}} .$$

There is a canonical natural homeo.

$$C(|L|) \xrightarrow[\approx]{\psi} |C(L)|$$

In the case at hand, we have a simplicial iso. $C(sd(\partial\sigma)) \xrightarrow{\cong} sd(\bar{\sigma})$ and a commutative diagr.

$$\begin{tikzcd}
C(|sd\,\partial\sigma|) \arrow[r, "C(D)", "\approx"'] \arrow[d, "\psi", "\approx"'] & C(|\partial\sigma|) \arrow[dd, "\approx"] \\
|C(sd\,\partial\sigma)| \arrow[d, "\approx"] & \\
|sd\,\bar{\sigma}| \arrow[r, "D"] & |\bar{\sigma}|
\end{tikzcd}$$

This proves the induction step. The case of an infinite dimensional simpl. cx. follows since $|K|$ has the topology of the union. //

Suppose K is finite s.t. $|K|$ has the metric topology from $\mathbb{R}\{V\}$. Define

$$\text{mesh}(K) = \max\{\text{diam}(|\sigma|) \mid \sigma \in K\}.$$

Lemma Let K be a finite simpl. cx. of dim. m. Then

$$\text{mesh}(\text{sd}\, K) \leq \frac{m}{m+1}\, \text{mesh}(K).$$

pf It suffices to show that for all $\sigma \in K$ all $\sigma' \in \text{sd}\, \bar{\sigma}$,

$$\text{diam}(|\sigma'|) \leq \frac{q}{q+1}\, \text{diam}(|\sigma|),$$

where $q = \dim \sigma$. This is easy. //

Thm Let $\mathcal{U}$ be an open covering of a compact polyhedron X. Then X has a triangulation $|K| \xrightarrow{\cong} X$ s.t. for all $\sigma \in K$, $|\sigma| \subset U$, for some $U \in \mathcal{U}$.

pf Let $|K'| \xrightarrow{f} X$ be a triangulation of X; then K' is finite. Let $\varepsilon > 0$ be a Lebesgue number of the covering

$$f^{-1}\mathcal{U} = \{f^{-1}(U) \mid U \in \mathcal{U}\}$$

of $|K'|$ w.r.t. to the metric from

$\mathbb{R}\{V\}$. This means that if $A \subset |K'|$ and $\operatorname{diam}(A) < \varepsilon$, then $A \subset f^{-1}(U)$, some $U \in \mathcal{U}$. Let $m = \dim K'$ and choose $N \geq 0$ s.t.

$$\left(\frac{m}{m+1}\right)^N \cdot \operatorname{mesh}(K) < \varepsilon .$$

Then $K = \operatorname{sd}^N K'$ and

$$|K| \xrightarrow[\approx]{D^N} |K'| \xrightarrow[\approx]{f} X$$

is the desired triangulation. //

For $\sigma \in K$, the <u>open</u> <u>simplex</u> $\langle\sigma\rangle \subset |K|$ is

$$\langle\sigma\rangle = \{\alpha \in |K| \mid \alpha(v) \neq 0 \iff v \in \sigma\}$$

$$= |\sigma| \setminus |\partial\sigma| ,$$

and for $v \in V$, the <u>star</u> <u>of</u> v is

$$\operatorname{st}(v) = \{\alpha \in |K| \mid \alpha(v) \neq 0\}$$

$$= \bigcup_{\sigma \ni v} \langle\sigma\rangle .$$

Since $\alpha \mapsto \alpha(v)$ is cts, $\operatorname{st}(v) \subset |K|$ is an open subset. Hence, $\{\operatorname{st}(v)\}_{v \in V}$ is an open covering of K.

<u>Addendum</u> Let $\mathcal{U}$ be an open covering of a compact polyhedron X. Then X has a triangulation $|K| \xrightarrow{\cong} X$ s.t. $\{st(v)\}_{v\in V}$ is a refinement of $\mathcal{U}$. //

<u>Def</u> Let $|K| \xrightarrow{f} |K'|$ be a cts. map. A simpl. map $K \xrightarrow{\varphi} K'$ is a <u>simplicial approximation</u> of f if

$$f(\alpha) \in |\sigma'| \implies |\varphi|(\alpha) \in |\sigma'| .$$

<u>Lemma</u> Let $|K| \xrightarrow{f} |K'|$ be a cts. map which on some subcomplex $L \subset K$ is induced by a simpl. map, and let $K \xrightarrow{\varphi} K'$ be a simpl. approx. of f. Then

(i) $f|_{|L|} = |\varphi||_{|L|}$.

(ii) $f \simeq |\varphi|$ rel. $|L|$.

<u>pf</u> Since φ is a simpl. approx. of f,

$|\varphi|(v) = f(v)$, $\forall v \in V$. Since f is simplicial on $|L|$, (i) follows. To prove (ii), note that the htpy

$$|K| \times [0,1] \longrightarrow |K'|$$

$$(\alpha, t) \longmapsto t|\varphi|(\alpha) + (1-t) f(\alpha)$$

is well-defined since $f(\alpha) \in |\sigma'|$ implies $|\varphi|(\alpha) \in |\sigma|$. //

<u>Theorem</u> Let K be a finite simpl. cx., and let $|K| \xrightarrow{f} |K'|$ be a cts. map. Then there exists $n \geq 0$ s.t. the map

$$|sd^n K| \xrightarrow[\approx]{D^n} |K| \xrightarrow{f} |K'|$$

has a simpl. approximation.

pf (sketch) A map $|K| \xrightarrow{f} |K'|$ has a simplicial approx. if (and only if) for all $v \in V$, there exists $v' \in V'$ s.t.

$$f(st(v)) \subset st(v').$$

If $v \in V$, choose $v' = \varphi(v) \in V'$ with this property. Then $V \xrightarrow{\varphi} V'$ is a simpl. app. Can achieve this by subdivision. //

Cor $\pi_k(S^n, s_0) = 0$, $k < n$.

pf We show that every cts. map

$$(|\partial\Delta[k+1]|, v_0) \xrightarrow{f} (|\partial\Delta[n+1]|, v_0)$$

is homotopic rel. v_0 to the constant map. Pick $N \geq 0$ s.t. the composite

$$(|sd^N \partial\Delta[k+1]|, v_0)$$

$$\xrightarrow[\simeq]{D^N} (|\partial\Delta[k+1]|, v_0)$$

$$\xrightarrow{f} (|\partial\Delta[n+1]|, v_0)$$

has a simpl. approx. φ. Then $fD^N \simeq |\varphi|$ rel. v_0. But the image of $|\varphi|$ is contained in $sk_k|\partial\Delta[n+1]|$. Hence, if $\sigma \in \partial\Delta[n+1]$ is an m-simplex, $m > k$, then $\langle\sigma\rangle \cap \mathrm{im}(|\varphi|) = \emptyset$. Pick $\alpha \in \langle\sigma\rangle$. Since $|\partial\Delta[n+1]| \setminus \{\alpha\} \simeq v_0$, the claim follows. //

The (unreduced) cone of X is

$$CX = \frac{X \times [0,1]}{X \times \{1\}}.$$

and the (unreduced) <u>mapping</u> <u>cone</u> of a map $X \xrightarrow{f} Y$ is

$$C_f = Y \underset{f \nwarrow \; X \; \nearrow i_0}{\cup} CX.$$

<u>Lemma</u> If $A \overset{i}{\hookrightarrow} X$ has HEP then the canonical map

$$X \underset{A}{\cup} CA \longrightarrow X/A$$

is a homotopy equivalence.

<u>pf</u> We produce the homotopy inverse. It is easy to see that $CA \hookrightarrow X \underset{A}{\cup} CA$ has HEP. Let $CA \times [0,1] \xrightarrow{h} CA$ be a null-htpy that contracts CA to the cone-point.

$$\begin{tikzcd}
(X \underset{A}{\cup} CA) \times \{0\} \underset{CA \times \{0\}}{\cup} CA \times [0,1] \arrow[r, "\mathrm{id} \cup h"] \arrow[d, hook] & X \underset{A}{\cup} CA \\
(X \underset{A}{\cup} CA) \times [0,1] \arrow[ur, dashed, "\exists \bar{h}"'] &
\end{tikzcd}$$

The composite

$$X \xrightarrow{i} X\times[0,1] \hookrightarrow (X \cup_A CA)\times[0,1] \xrightarrow{\bar{h}} X \cup_A CA$$

maps $A \subset X$ to the cone-point, so get an induced map

$$X/A \longrightarrow X \cup_A CA \ .$$

This is a htpy inverse of the can. map. //

Let $1 \in C_f$ be the cone-point, and let

$$\tilde{H}_*(C_f) := H_*(C_f, \{1\})$$

be the <u>reduced</u> homology.

<u>Prop</u> There is a natural exact seq.

$$\cdots \to H_n(X) \xrightarrow{f_*} H_n(Y) \to \tilde{H}_n(C_f) \xrightarrow{\partial} H_{n-1}(X) \to \cdots$$

<u>pf</u> Apply Mayer-Victoris to

$$\begin{array}{ccc} X & \longrightarrow & M_f \\ \big\downarrow & & \big\downarrow \\ CX & \longrightarrow & C_f \end{array} .$$

It follows that

$$\begin{tikzcd} \cdots \arrow[r] & H_n(X) \arrow[r] \arrow[d] & H_n(M_f) \arrow[r] \arrow[d] & H_n(M_f, X) \arrow[r] \arrow[d, "\sim"] & \cdots \\ \cdots \arrow[r] & H_n(CX) \arrow[r] & H_n(C_f) \arrow[r] & H_n(C_f, CX) \arrow[r] & \cdots \end{tikzcd}$$

Since $CX \simeq *$, we further have

$$\begin{tikzcd} \cdots \arrow[r] & H_n(CX) \arrow[r] & H_n(C_f) \arrow[r] & H_n(C_f, CX) \arrow[r] & \cdots \\ \cdots \arrow[r] & H_n(\{1\}) \arrow[r] \arrow[u, "\sim"'] & H_n(C_f) \arrow[r] \arrow[u, equal] & H_n(C_f, \{1\}) \arrow[r] \arrow[u, "\sim"'] & \cdots \end{tikzcd}$$

Finally, Y is a strong def. retract of M_f. //

<u>Cor</u> If $A \overset{i}{\hookrightarrow} X$ has HEP then there is a natural exact sequence

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \to \tilde{H}_n(X/A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

//

A map $E \xrightarrow{p} B$ has the *homotopy lifting property* (HLP) w.r.t. to (X, A) if

$$\begin{tikzcd} X \times \{0\} \cup A \times [0,1] \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r] \arrow[ur, dashed, "\exists"] & B \end{tikzcd}.$$

It has the *left lifting property* w.r.t. to (X, A) if

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ X \arrow[r] \arrow[ur, dashed, "\exists"] & B \end{tikzcd}.$$

Lemma A map $E \xrightarrow{p} B$ has HLP w.r.t. all relative CW-pairs (X, A) if and only if it has LLP w.r.t. to all rel. CW-pairs (X, A) s.t. $A \xrightarrow{\sim} X$.

Pf If (X, A) is a rel. CW-cx. then so is $(X \times [0,1], X \times \{0\} \cup A \times [0,1])$ and

$X \times \{0\} \cup A \times [0,1]$ is a strong def. retract of $X \times [0,1]$. So "if" is clear. Conversely, if (X,A) is a rel. CW-cx. and if $A \overset{\sim}{\hookrightarrow} X$ is a w-e. then A is a strong def. retract of X; let

$$X \times [0,1] \xrightarrow{h} X$$

be a deformation retraction of X onto A. Consider

$$\begin{tikzcd}
A \arrow[r, hook, "i_1"] \arrow[d, hook, "\sim"'] & X \times \{1\} \cup A \times [0,1] \arrow[r, "h"] \arrow[d, hook] & A \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\
X \arrow[r, "i_0"] & X \times [0,1] \arrow[r, "h"] \arrow[urr, dashed, "\exists F"] & X \arrow[r] & B
\end{tikzcd}$$

The lift exist by HLP for (X,A). //

<u>Lemma</u> A map $E \xrightarrow{p} B$ has HLP w.r.t. all rel. CW-cx. (X,A) if and only if it has HLP w.r.t. (D^n, S^{n-1}), for all $n \geq 0$.

Pf exercise. //

<u>Lemma</u> A map $E \xrightarrow{p} B$ has HLP w.r.t. (D^n, S^{n-1}) if and only if it has HLP w.r.t. $D^n = (D^n, \emptyset)$.

<u>pf</u> The pairs $(D^n \times [0,1], D^n \times \{0\} \cup S^{n-1} \times [0,1])$ and $(D^n \times [0,1], D^n \times \{0\})$ are homeomorphic. //

A map $E \xrightarrow{p} B$ which satisfies one of the equivalent properties of the three lemmas above is called a <u>Serre fibration</u>.

<u>Prop</u> Let $E \xrightarrow{p} B$ be a Serre fibration, let $b_0 \in B$, and let $x_0 \in F = p^{-1}(\{b_0\})$. Then for all $n \geq 1$,

$$\pi_n(E, F, x_0) \underset{\sim}{\xrightarrow{p_*}} \pi_n(B, b_0).$$

<u>pf</u> To prove surjectivity, we use that p has LLP w.r.t. $(D^n, \{s_0\})$:

$$\begin{tikzcd} \{s_0\} \arrow[rr, "x_0"] \arrow[d, hook, "\sim"] & & E \arrow[d, "p"] \\ D^n \arrow[r, "pr"] \arrow[urr, dashed, "\tilde{f}"] & S^n \arrow[r, "f"] & B \end{tikzcd}$$

Since $f \circ pr(S^{n-1}) = b_0$, the lift $\tilde{f}$ takes S^{n-1} to $F = p^{-1}(\{b_0\})$. So

$$\begin{tikzcd}
D^n \arrow[r, "\tilde{f}"] & E \\
S^{n-1} \arrow[u, hook] \arrow[r, "\tilde{f}"] & F \arrow[u, hook] \\
s_0 \arrow[u, phantom, "\in" sloped] \arrow[r, mapsto] & x_0 \arrow[u, phantom, "\in" sloped]
\end{tikzcd}$$

To prove injectivity, let such a map be given and suppose that f,

$$\begin{tikzcd}
D^n \arrow[r, "\tilde{f}"] \arrow[d, "pr"] & E \arrow[d, "p"] \\
S^n \arrow[r, "f"] & B
\end{tikzcd}$$

is homotopic rel. $\{s_0\}$ to the const. map; let $h : S^n \times [0,1] \to B$ be the htp.

$$\begin{tikzcd}
D^n \times \{0\} \cup \{s_0\} \times [0,1] \arrow[rr, "\tilde{f} \cup x_0"] \arrow[d, "\simeq"] & & E \arrow[d, "p"] \\
D^n \times [0,1] \arrow[r] \arrow[urr, dashed, "\exists \tilde{h}"] & S^n \times [0,1] \arrow[r, "h"] & B
\end{tikzcd}$$

The lift $\tilde{h}$ is a homotopy

$$\begin{array}{ccc} D^n \times [0,1] & \xrightarrow{\tilde{h}} & E \\ \cup & & \cup \\ S^{n-1} \times [0,1] & \xrightarrow{\tilde{h}} & F \\ \cup & & \cup \\ \{s_0\} \times [0,1] & \xrightarrow{x_0} & \{x_0\} \end{array}$$

of $\tilde{f}$ to a map $D^n \to F \hookrightarrow E$. //

Cor There is a natural exact seq.

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots$$

A map $E \xrightarrow{p} B$ is a Hurewicz fibration if it has HLP w.r.t. all spaces X. We shall now show that every map can be replaced "up to homotopy" by a Hurewicz fibration. The construction is "dual" to the mapping cylinder. We write Y^X for the set $\mathrm{Map}(X, Y)$ of all cts. maps from X to Y equipped with the compact-open topology. Then

$$\mathrm{Map}(X \times Y, Z) \longrightarrow \mathrm{Map}(X, Z^{Y})$$

$$f \longmapsto f^{\#}$$

$$f^{\#}(x)(y) = f(x,y)$$

is a bijection, provided that Y is locally compact.

Let $X \xrightarrow{f} Y$ be a map and recall

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, hook, "i_0"'] \arrow[dr, phantom, "\text{p.o.}"] & Y \arrow[d, hook] \arrow[ddr, bend left, "\mathrm{id}"] & \\
X \times [0,1] \arrow[r] \arrow[drr, bend right, "f \circ pr"'] & M_f \arrow[dr, dashed, "\exists !"] & \\
& & Y
\end{tikzcd} \quad \Longrightarrow$$

$$\begin{tikzcd}
X \arrow[r, hook, "i", "\mathrm{HEP}"'] \arrow[dr, "f"'] & M_f \arrow[d, "\sim"] & Y \arrow[l, hook', "\sim"'] \arrow[dl, equal] \\
& Y &
\end{tikzcd}$$

mapping cylinder

$$
\begin{tikzcd}
X \arrow[r, hook, "i"] \arrow[d] \arrow[dr, phantom, "\text{p.o.}"] & M_f \arrow[d] & Y \arrow[l, hook', "\sim"'] \arrow[dl] \\
\text{pt.} \arrow[r] & C_f &
\end{tikzcd}
$$

mapping cone

$$\cdots \to H_n(X) \xrightarrow{f_*} H_n(Y) \to \tilde{H}_n(C_f) \to H_{n-1}(X) \to \cdots$$

Dually,

$$
\begin{tikzcd}
X & Y \arrow[l, "f"'] & \\
X^{[0,1]} \arrow[u, "ev_0"] \arrow[ur, phantom, "\text{p.b.}"] & E_f \arrow[l] \arrow[u] & \\
& & Y \arrow[ul, dashed] \arrow[uul, bend right, "id"'] \arrow[ull, bend left, "c\circ f"]
\end{tikzcd}
\qquad \Rightarrow
$$

$$
\begin{tikzcd}
X & E_f \arrow[l, "ev_1"', "HLP"] \arrow[r, "\sim"] & Y \\
& Y \arrow[ul, "f"] \arrow[u, "\sim"'] \arrow[ur, equal] &
\end{tikzcd}
$$

?

$$\begin{tikzcd} X & E_f \arrow[l, "ev_1"'] \arrow[r, "\sim"] & Y \\ pt. \arrow[u, "x_0"] & F_f \arrow[l] \arrow[u] \arrow[ur] & \end{tikzcd}$$

homotopy fiber at $x_0 \in X$.

$$\cdots \to \pi_n(F_f, z_0) \to \pi_n(Y, y_0) \to \pi_n(X, x_0) \to \pi_{n-1}(F_f, z_0) \to \cdots$$

$z_0 = (y_0, \text{const. path at } x_0 = f(y_0))$.

18.906: Problem Set 3

Due: Thursday, March 6.

1. Let K and K' be simplicial complexes with vertex sets V and V', respectively, and let $f\colon |K| \to |K'|$ be a continuous map. Show that f has a simplicial approximation if and only if for all $v \in V$, there exists $v' \in V'$ such that

$$f(\mathrm{st}(v)) \subset \mathrm{st}(v').$$

(*Hint*: For all $v \in V$, choose $v' = \varphi(v) \in V$ such that $f(\mathrm{st}(v)) \subset \mathrm{st}(v')$. Show that the map $\varphi\colon V \to V'$ is a simplicial map and a simplicial approximation to f.)

2. Let $f\colon Y \to X$ be a map and consider the following pull-back diagram.

$$\begin{tikzcd} X & Y \arrow[l, "f"'] \\ X^{[0,1]} \arrow[u, "\mathrm{ev}_0"] & E_f . \arrow[l, "q"] \arrow[u, "p_0"'] \end{tikzcd}$$

The identity map on Y and the map $Y \to X^{[0,1]}$ given as the composite of $f\colon Y \to X$ and the map $c\colon X \to X^{[0,1]}$, which takes x to the constant path at x, give rise to a map $i\colon Y \to E_f$. We also let $p_1\colon E_f \to X$ be the composite of $q\colon E_f \to X^{[0,1]}$ and $\mathrm{ev}_1\colon X^{[0,1]} \to X$. We then have the following commutative diagram.

$$\begin{tikzcd} X & E_f \arrow[l, "p_1"'] \arrow[r, "p_0"] & Y \\ & Y \arrow[ul, "f"] \arrow[u, "i"'] \arrow[ur, equal] & \end{tikzcd}$$

(i) Show that p_1 is a Hurewicz fibration.

(ii) Show further that the following composite is homotopic to the identify.

$$E_f \xrightarrow{p_0} Y \xrightarrow{i} E_f.$$

We define the *homotopy fiber* of f at $x_0 \in X$ by the following pull-back diagram.

$$\begin{tikzcd} X & E_f \arrow[l, "p_1"'] \\ * \arrow[u, "x_0"] & F_f . \arrow[l] \arrow[u] \end{tikzcd}$$

Let $y_0 \in f^{-1}(x_0)$, and let $z_0 \in F_f$ be the point such that $p_0(z_0) = y_0$ and such that $q(z_0)$ is the constant path at x_0.

(iii) Conclude that there is a long-exact sequence of pointed sets

$$\dots \to \pi_n(F_f, z_0) \xrightarrow{p_{0*}} \pi_n(Y, y_0) \xrightarrow{f_*} \pi_n(X, x_0) \xrightarrow{\partial} \pi_{n-1}(F_f, z_0) \to \dots$$

$(E^r_{*,*}, d^r)$ differential bi-graded mod.

$$E^{r+1}_{*,*} = H(E^r_{*,*}, d^r)$$

$$E^r_{s,t} \xrightarrow{d^r} E^r_{s-r,t+r-1}.$$

—

$$\cdots \subset Fil_{s-1} C_* \subset Fil_s C_* \subset Fil_{s+1} C_* \subset \cdots$$

$$\cdots \subset Fil_{s-1} H_*(C_*) \subset Fil_s H_*(C_*) \subset Fil_{s+1} H_*(C_*) \subset \cdots$$ induced filtr.

$$Fil_s H_*(C_*) := \operatorname{im}(H_*(Fil_s C_*) \to H_*(C_*)).$$

—

$$E^1_{s,t} = H_{s+t}(gr_s C_*)$$

$$E^\infty_{s,t} = gr_s H_{s+t}(C_*)$$

<u>Theorem</u> A filtered chain cx. determines a spectral sequence with

$$E^1_{s,t} = H_{s+t}(gr_s C_*).$$

If the filtr. is bounded, then

$$E^\infty_{s,t} \approx gr_s H_{s+t}(C_*).$$

pf We let

$$Z^r_{s,t} = \{x \in Fil_s C_{s+t} \mid dx \in Fil_{s-r} C_{s+t-1}\}, \quad r \geq 0,$$

$$B^r_{s,t} = \{dx \in Fil_s C_{s+t} \mid x \in Z^{r-1}_{s+r-1,t-r+2}\}, \quad r \geq 1,$$

and define

$$E^r_{s,t} = Z^r_{s,t} \big/ (B^r_{s,t} + Z^{r-1}_{s-1,t+1})$$

$$\xrightarrow{\sim} (Z^r_{s,t} + Fil_{s-1} C_{s+t}) \big/ (B^r_{s,t} + Fil_{s-1} C_{s+t}).$$

Prove . Recall Noether iso:

$$A/A \cap B \xrightarrow{\sim} A+B/B$$

$$
\begin{tikzcd}
0 & 0 \\
Z^r_{s,t}/(B^r_{s,t}+Z^{r-1}_{s-1,t+1}) \arrow[u] \arrow[r] & (Z^r_{s,t}+\mathrm{Fil}_{s-1}C_{s+t})/(B^r_{s,t}+\mathrm{Fil}_{s-1}C_{s+t}) \arrow[u] \\
Z^r_{s,t}/Z^{r-1}_{s-1,t+1} \arrow[u] \arrow[r, "\sim"] & (Z^r_{s,t}+\mathrm{Fil}_{s-1}C_{s+t})/\mathrm{Fil}_{s-1}C_{s+t} \arrow[u] \\
(B^r_{s,t}+Z^{r-1}_{s-1,t+1})/Z^{r-1}_{s-1,t+1} \arrow[u] \arrow[r] & (B^r_{s,t}+\mathrm{Fil}_{s-1}C_{s+t})/\mathrm{Fil}_{s-1}C_{s+t} \arrow[u] \\
0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}
$$

$$
\begin{tikzcd}
(B^r_{s,t}+Z^{r-1}_{s-1,t+1})/Z^{r-1}_{s-1,t+1} \arrow[r] & (B^r_{s,t}+\mathrm{Fil}_{s-1}C_{s+t})/\mathrm{Fil}_{s-1}C_{s+t} \\
B^r_{s,t}/B^r_{s,t}\cap Z^{r-1}_{s-1,t+1} \arrow[u, "\sim"'] & B^r_{s,t}/B^r_{s,t}\cap \mathrm{Fil}_{s-1}C_{s+t} \arrow[u, "\sim"']
\end{tikzcd}
$$

$$B^r_{s,t}/B^r_{s,t}\cap Z^{r-1}_{s-1,t+1} = B^r_{s,t}/B^r_{s,t}\cap \mathrm{Fil}_{s-1}C_{s+t},$$

$$\left.\begin{array}{lll} x \in Z^r_{s,t} & \Rightarrow & dx \in Z^r_{s-r,\,t+r-1} \\ x \in B^r_{s,t} & \Rightarrow & dx = 0 \\ x \in Z^{r-1}_{s-1,\,t+1} & \Rightarrow & dx \in B^r_{s-r,\,t+r-1} \end{array}\right\} \Rightarrow$$

The ~~differential~~ d induces a diff.

$$E^r_{s,t} \xrightarrow{d^r} E^r_{s-r,\,t+r-1} .$$

<u>Lemma</u> There is a canonical iso.

$$H(E^r_{s+r,\,t-r+1} \xrightarrow{d^r} E^r_{s,t} \xrightarrow{d^r} E^r_{s-r,\,t+r-1})$$

$$\xrightarrow{\sim} E^{r+1}_{s,t} .$$

pf Let $x \in Z^r_{s,t}$ repr. $[x] \in E^r_{s,t}$, and suppose $d^r[x] = 0$. Then

$$dx \in B^r_{s-r,\,t+r-1} + Z^{r-1}_{s-r-1,\,t+r}$$

say $dx = y + z$. Now $y = dx'$, $x' \in Z^{r-1}_{s-1,\,t+1}$, and since $[x] = [x - x']$, we can assume $y = 0$. But then

$$\left.\begin{array}{l} x \in \mathrm{Fil}_s C_{s+t} \\ dx \in Z^{r-1}_{s-r-1,\,t+r} \subset \mathrm{Fil}_{s-(r+1)} C_{s+t-1} \end{array}\right\} \Rightarrow x \in Z^{r+1}_{s,t}$$

so we get a map

$$\begin{array}{ccc} \ker\left(E^r_{s,t} \xrightarrow{d^r} E^r_{s-r,\,t+r-1}\right) & \longrightarrow & E^{r+1}_{s,t} \\ \cup & & \cup \\ [x] & \longmapsto & [x-x'] \end{array}$$

It is clear that this map is surj. Moreover, if $z \in Z^r_{s+r,\,t-r+1}$ repr. $[z] \in E^r_{s+r,\,t-r+1}$, then $dz \in B^{r+1}_{s,t}$, so we have induced surj.

$$H\left(E^r_{s+r,\,t-r+1} \xrightarrow{d^r} E^r_{s,t} \xrightarrow{d^r} E^r_{s-r,\,t+r-1}\right)$$

$$\longrightarrow\!\!\!\!\rightarrow \quad E^r_{s,t}$$

Suppose $x \in Z^{r+1}_{s,t} \subset Z^r_{s,t}$ repr. zero in $E^{r+1}_{s,t}$. Then

$$x \in B^{r+1}_{s,t} + Z^r_{s-1,\,t+1},$$

say $x = y + z$. Since $y \in B^{r+1}_{s,t}$, $y = da$, $a \in Z^r_{s+r,\,t-r+1}$, and hence y also repr. zero in the homology group. Finally,

$z \in Z^r_{s-1,t+1} \subset \mathrm{Fil}_{s-1} C_{s+t}$ repr. zero in

$$E^r_{s,t} \xrightarrow{\sim} (Z^r_{s,t} + \mathrm{Fil}_{s-1} C_{s+t}) \Big/ (B^r_{s,t} + \mathrm{Fil}_{s-1} C_{s+t}).$$

This concludes the pf. of the lemma. /

The filtr. is <u>bounded</u> if for all n, there exists $n_0, n_1 \in \mathbb{Z}$ s.t.

$$\mathrm{Fil}_s C_n = 0, \quad s < n_0,$$

$$\mathrm{Fil}_s C_n = C_n, \quad s \geq n_1.$$

Often, $n_0 = 0$ and $n_1 = n$.

<u>Lemma</u> If the filtr. is bounded then

$$E^r_{s,t} \approx gr^s H_{s+t}(C_*)$$

for $r >> 0$.

pf If $r >> 0$, then

$$Z^r_{s,t} = Z_{s+t} \cap \mathrm{Fil}_s C_{s+t},$$

$$B^r_{s,t} = B_{s+t} \cap \mathrm{Fil}_s C_{s+t},$$

so

$$E^r_{s,t} = \frac{Z_{s+t} \cap \mathrm{Fil}_s C_{s+t}}{B_{s+t} \cap \mathrm{Fil}_s C_{s+t} + Z_{s+t} \cap \mathrm{Fil}_{s-1} C_{s+t}}$$

Also,

$$\mathrm{Fil}_s H_{s+t}(C_*) = \frac{Z_{s+t} \cap \mathrm{Fil}_s C_{s+t}}{B_{s+t} \cap \mathrm{Fil}_s C_{s+t}}$$

so

$$gr_s H_{s+t}(C_*) = \frac{Z_{s+t} \cap \mathrm{Fil}_s C_{s+t}}{B_{s+t} \cap \mathrm{Fil}_s C_{s+t} + Z_{s+t} \cap \mathrm{Fil}_{s-1} C_{s+t}}. \quad //$$

We write the theorem as

$$E^1_{s,t} = H_{s+t}(gr_s C_*)$$

$$\Longrightarrow H_{s+t}(C_*)$$

and say that the spectral seq. <u>converges</u> to $H_*(C_*)$.

ex X CW-cx, $C_* = C_*(X)$ singular chain cx. on X. Define

$$\mathrm{Fil}_s C_* = C_*(X^s)$$

$$\mathrm{gr}_s C_* =: C_*(X^s, X^{s-1}).$$

Then

$$E^1_{s,t} := H_{s+t}(\mathrm{gr}_s C_*)$$

$$=: H_{s+t}(X^s, X^{s-1})$$

$$=: \begin{cases} C^{cw}_s(X) & , t=0, \\ 0 & , t \neq 0. \end{cases}$$

Moreover,

$$\begin{array}{ccc} E^1_{s-1,0} & \xleftarrow{d^1} & E^1_{s,t} \\ \| & & \| \\ C^{cw}_{s-1}(X) & \xleftarrow{d^{cw}} & C^{cw}_s(X) \end{array}$$

so

$$E^2_{s,t} = \begin{cases} H^{cw}_s(X) & , t=0, \\ 0 & , t \neq 0. \end{cases}$$

For degree reasons, $E^r \approx E^2$, $r \geq 2$.

$$\begin{array}{ccccccccc}
0 \subset \mathrm{Fil}_0 H_s(X) & \subset & \mathrm{Fil}_1 H_s(X) & \subset \cdots \subset & \mathrm{Fil}_s H_s(X) & \subset & \mathrm{Fil}_{s+1} H_s(X) & \subset \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
gr_0 H_s(X) & & gr_1 H_s(X) & & gr_s H_s(X) & & gr_{s+1} H_s(X) & \\
\wr\wr & & \wr\wr & & \wr\wr & & \wr\wr & \\
E^\infty_{0,s} & & E^\infty_{1,s-1} & & E^\infty_{s,0} & & E^\infty_{-1,s+1} & \\
\| & & \| & & \wr\wr & & \| & \\
0 & & 0 & & H_s^{CW}(X) & & 0 &
\end{array}$$

so conclude:

$$H_s(X) \xleftarrow{\sim} \mathrm{Fil}_s H_s(X) \xrightarrow{\sim} gr_s H_s(X) \approx H_s^{CW}(X).$$

A <u>bi-complex</u> $(C_{*,*}, d', d'')$ is a bi-graded module $C_{*,*}$ with maps

$$C_{p,q} \xrightarrow{d'} C_{p-1,q}$$

$$C_{p,q} \xrightarrow{d''} C_{p,q-1}$$

s.t.

$$d'd' = d''d'' = d'd'' + d''d' = 0.$$

The associated <u>total complex</u> is the chain cx.

$$\mathrm{Tot}(C)_n = \bigoplus_{s+t=n} C_{s,t}$$

$$d = d' + d''.$$

$$\begin{tikzcd}
& \arrow[d] & \arrow[d] & \arrow[d] & \\
\cdots & C_{s-1,t+1} \arrow[l] \arrow[d, "d''"] & C_{s,t+1} \arrow[l, "d'"'] \arrow[d, "d''"] & C_{s+1,t+1} \arrow[l, "d'"'] \arrow[d, "d''"] & \cdots \arrow[l] \\
\cdots & C_{s-1,t} \arrow[l] \arrow[d, "d''"] & C_{s,t} \arrow[l, "d'"'] \arrow[d, "d''"] & C_{s+1,t} \arrow[l, "d'"'] \arrow[d, "d''"] & \cdots \arrow[l] \\
& C_{s-1,t-1} & C_{s,t-1} \arrow[l, "d'"'] & C_{s+1,t-1} \arrow[l, "d'"'] &
\end{tikzcd}$$

There are two canonical filtrations of the total cx.:

$$\mathrm{Fil}'_s \, \mathrm{Tot}(C)_n = \bigoplus_{i \leq s} C_{i,n-i}$$

$$\mathrm{Fil}''_s \, \mathrm{Tot}(C)_n = \bigoplus_{i \leq s} C_{n-i,i}$$

and canonical isomorphisms of cx.

$$\mathrm{gr}'_s \, \mathrm{Tot}(C)_* \xrightarrow{\sim} C_{s,*-s} ,$$

$$\mathrm{gr}''_s \, \mathrm{Tot}(C)_* \xrightarrow{\sim} C_{*-s,s} .$$

Hence, the corresponding spectral seq. have

$$E'^1_{s,t} = H_t(C_{s,*} ; d'')$$

$$E''^1_{s,t} = H_t(C_{*,s} ; d')$$

and both converge to $H_*(\mathrm{Tot}(C_*))$. The d^1-differential is equal to the map induced from d' and d'', resp.

Hence,

$$E'^2_{s,t} = H_s(H_t(C_{*,*}; d''); d')$$

$$E''^2_{s,t} = H_s(H_t(C_{*,*}; d'); d'').$$

Defined projective modules, (projective) resolutions; showed that every module has a proj. res. and that any two such are chain-homotopy equivalent by a unique chain-homotopy class of chain maps. Defined

$$\mathrm{Tor}^A_*(M,N) = H_*(P_* \otimes_A N)$$

and showed that

$$H_*(P_* \otimes_A N) \xleftarrow{\sim} H_*(P_* \otimes_A Q_*) \xrightarrow{\sim} H_*(M \otimes_A Q_*),$$

for proj. res. $P_* \xrightarrow{\sim} M$ and $Q_* \xrightarrow{\sim} N$.

ex Let A be an abelian grp., and

$$P_* \xrightarrow[-]{\varepsilon} A$$

a projective resolution. Let $C_*(X) = C_*(X, \mathbb{Z})$ be the singular cx. of a space X, and recall that

$$C_*(X, A) := C_*(X) \otimes_{\mathbb{Z}} A .$$

We consider $C_*(X) \otimes_{\mathbb{Z}} P_*$ as a bi-cx.:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
C_0 \otimes P_2 \arrow[d, "1\otimes d"] & C_1 \otimes P_2 \arrow[l, "d\otimes 1"'] \arrow[d, "-1\otimes d"] & C_2 \otimes P_2 \arrow[l, "d\otimes 1"'] \arrow[d, "1\otimes d"] & \cdots \arrow[l] \\
C_0 \otimes P_1 \arrow[d, "1\otimes d"] & C_1 \otimes P_1 \arrow[l, "d\otimes 1"'] \arrow[d, "-1\otimes d"] & C_2 \otimes P_1 \arrow[l, "d\otimes 1"'] \arrow[d] & \cdots \arrow[l] \\
C_0 \otimes P_0 & C_1 \otimes P_0 \arrow[l, "d\otimes 1"'] & C_2 \otimes P_0 \arrow[l, "d\otimes 1"'] & \cdots \arrow[l]
\end{tikzcd}$$

This gives two spectral sequences:

$${}'E^1_{s,t} = H_t(C_s(X) \otimes P_*, 1 \otimes d)$$

$$\Rightarrow H_{s+t}(\mathrm{Tot}(C_*(X) \otimes P_*))$$

and

$$''E^1_{s,t} = H_t(C_*(X) \otimes P_s, d \otimes 1)$$

$$\Rightarrow H_{s+t}(\mathrm{Tot}(C_*(X) \otimes P_*)).$$

By def$^{\underline{n}}$,

$$'E^1_{s,t} = \mathrm{Tor}^{\mathbb{Z}}_t(C_s(X), A)$$

$$= \begin{cases} C_s(X, A) & , t = 0, \\ 0 & , t \neq 0, \end{cases} \Rightarrow$$

$$'E^2_{s,t} = \begin{cases} H_s(X, A) & , t = 0, \\ 0 & , t \neq 0. \end{cases}$$

For degree-reasons, $'E^r = {}'E^2$, $r \geq 2$, so we get

$$H_s \mathrm{Tot}(C_*(X) \otimes P_*) \approx H_s(X, A).$$

$$''E^1_{*,t} = H_t(X) \otimes P_* \Longrightarrow$$

$$''E^2_{s,t} = \mathrm{Tor}^{\mathbb{Z}}_s(H_t(X), A).$$

Since a subgrp. of a free ab. grp. is free, every ab. grp. has a free res. of length 1. So the sp. seq. $''E^2$ looks:

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
H_2(X)\otimes A & \mathrm{Tor}^{\mathbb{Z}}_1(H_2(X),A) & 0 & \cdots \\
H_1(X)\otimes A & \mathrm{Tor}^{\mathbb{Z}}_1(H_1(X),A) & 0 & \cdots \\
H_0(X)\otimes A & \mathrm{Tor}^{\mathbb{Z}}_1(H_0(X),A) & 0 & \cdots
\end{array}$$

For degree-reasons $''E^r = ''E^2$, $r \geq 2$.

$$\begin{array}{ccccccc}
0 \subset \mathrm{Fil}_0 H_s(X,A) & \subset & \mathrm{Fil}_1 H_s(X,A) & \subset & \mathrm{Fil}_2 H_s(X,A) & \subset & \cdots \\
\downarrow\wr & & \downarrow & & \downarrow & & \\
\mathrm{gr}_0 H_s(X,A) & & \mathrm{gr}_1 H_s(X,A) & & \mathrm{gr}_2 H_s(X,A) & & \\
\wr\wr & & \wr\wr & & \parallel & & \\
H_s(X)\otimes A & & \mathrm{Tor}^{\mathbb{Z}}_1(H_{s-1}(X),A) & & 0 & &
\end{array}$$

$$\Rightarrow \quad 0 \to H_s(X)\otimes A \to H_s(X,A) \to \mathrm{Tor}^{\mathbb{Z}}_1(H_{s-1}(X),A) \to 0.$$

18.906: Problem Set 4

Due: Thursday, March 13.

1. Let (X, A) be a pair of locally compact spaces, and let Y be any space. Suppose that (X, A) has the homotopy extension property with respect to all spaces. Show that the map induced by the inclusion

$$Y^X \xrightarrow{i^*} Y^A$$

has the homotopy lifting property with respect to all spaces, *i.e.* that i^* is a Hurewicz fibration.

2. Let $f: C \to D$ and $g: D \to C$ be functors. We say that (f, g) is an adjoint pair (or that f is left adjoint to g or that g is right adjoint to f) if there exists a natural (with respect to maps in both variables) bijection

$$\varphi: \mathrm{Hom}_D(f(c), d) \xrightarrow{\sim} \mathrm{Hom}_C(c, g(d)).$$

The natural transformation $\eta: c \to g(f(c))$ given by $\eta = \eta_c = \varphi(\mathrm{id}_{f(c)})$ is called the *unit* of the adjunction. The *counit* is the natural transformation $\epsilon: f(g(d)) \to d$ given by $\epsilon = \epsilon_d = \varphi^{-1}(\mathrm{id}_{g(d)})$.

$H_*(X) \otimes A$

(i) Let $a: f(c) \to d$ and $b: c \to g(d)$ be morphisms in D and C, respectively. Show that the morphisms $\varphi(a): c \to g(d)$ and $\varphi^{-1}(b): f(c) \to d$ are given by the following composites, respectively.

$$c \xrightarrow{\eta_c} g(f(c)) \xrightarrow{g(a)} g(d),$$
$$f(c) \xrightarrow{f(b)} f(g(d)) \xrightarrow{\epsilon_d} d.$$

(ii) Consider the following diagram of categories and functors.

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[d, shift left, "h"] & B \arrow[l, shift left, "f'"] \arrow[d, shift left, "g"] \\
C \arrow[r, shift left, "k"] \arrow[u, shift left, "h'"] & D. \arrow[l, shift left, "k'"] \arrow[u, shift left, "g'"]
\end{tikzcd}$$

Suppose that f (resp. g, resp. h, resp. k) is left adjoint to f' (resp. g', resp. h', resp. k'). Show that $gf = kh$ if and only if $f'g' = h'k'$.

3. Let $f: A \to B$ be a ring homomorphism, let M be a (right) A-module, and let N be a (left) B-module. The ring homomorphism f allows us to view N as a (left) A-module. Show that there is a spectral sequence

$$E^2_{s,t} = \mathrm{Tor}^B_s(\mathrm{Tor}^A_t(M, B), N) \Rightarrow \mathrm{Tor}^A_{s+t}(M, N).$$

(*Hint*: Choose a resolution $P_* \to M$ by projective (right) A-modules and a resolution $Q_* \to N$ by projective (left) B-modules. Then consider the two spectral sequences associated with the bi-complex $P_* \otimes_A Q_* \approx (P_* \otimes_A B) \otimes_B Q_*$.)

Let $E \xrightarrow{f} B$ be a Serre fibration, and let Δ^p be the standard p-simplex, $p \geq 0$. Then the induced map of mapping spaces

$$E^{\Delta^p} \xrightarrow{f_*} B^{\Delta^p}$$

is again a Serre fibration. We denote by F_g the fiber over $g \in B^{\Delta^p}$. Let

$$F_g \xrightarrow{i_0^*} F_{g \circ i_0}$$

be the map induced by the inclusion of the 0'th vertex $\Delta^0 \xhookrightarrow{i_0} \Delta^p$.

<u>Lemma</u> The map i_0^* induces a chain homotopy equivalence

$$C_*(F_g) \underset{\sim}{\xrightarrow{i_0^*}} C_*(F_{g \circ i_0}).$$

<u>pf</u> We first construct a chain map

$$C_*(F_g) \xleftarrow{\varphi} C_*(F_{g \circ i_0})$$

such that $i_0^* \circ \varphi = \mathrm{id}$. Suppose φ has been defined in degrees $< q$. We can identify $C_q(F_g)$ with the free k-module generated by the set of commutative diagrams of the form

$$\begin{tikzcd} \Delta^p \times \Delta^q \arrow[r] \arrow[d, "pr_1"] & E \arrow[d, "f"] \\ \Delta^p \arrow[r, "g"] & B \end{tikzcd},$$

and similarly for $C_q(F_{g\circ i_0})$. To define φ on the generator

$$\begin{tikzcd} \Delta^0 \times \Delta^q \arrow[r, "u"] \arrow[d, "pr_1"] & E \arrow[d, "f"] \\ \Delta^0 \arrow[r, "g\circ i_0"] & B \end{tikzcd}$$

is the same as finding a lifting

$$\begin{tikzcd} \Delta^0 \times \Delta^q \cup \Delta^p \cup \partial\Delta^q \arrow[rr, "u \cup \varphi(\partial u)"] \arrow[d, "\sim"] & & E \arrow[d, "f"] \\ \Delta^p \times \Delta^q \arrow[r, "pr_1"] \arrow[rru, dashed, "\exists\, \varphi(u)"] & \Delta^p \arrow[r, "g"] & B \end{tikzcd}.$$

This exists since f is a Serre fibration. We must still show that $\varphi \circ i_0^* \simeq \mathrm{Id}$. To this end, we construct a chain map h which makes the following diagram commute.

$$\begin{tikzcd}
 & & C_*(F_g) \\
C_*(F_g) \arrow[urr, "id"] \arrow[rr, "h"] \arrow[drr, "\varphi \circ i_0^*"'] & & C_*(F_g^{[0,1]}) \arrow[u, "ev_{0*}"'] \arrow[d, "ev_{1*}"] \\
 & & C_*(F_g)
\end{tikzcd}$$

Suppose again that h has been constr. in degrees $< q$. To define h on

$$\begin{tikzcd}
\Delta^p \times \Delta^q \arrow[r, "u"] \arrow[d, "pr_1"] & E \arrow[d, "f"] \\
\Delta^p \arrow[r, "g"] & B
\end{tikzcd}$$

the amounts to solving the lifting problem

$$\begin{tikzcd}
\Delta^p \times \Delta^q \times \{0,1\} \cup \Delta^p \times \partial\Delta^q \times [0,1] \cup \Delta^0 \times \Delta^q \times [0,1] \arrow[r] \arrow[d, hook, "\sim"] & E \arrow[d, "f"] \\
\Delta^p \times \Delta^q \times [0,1] \arrow[r, "g \circ pr_1"] \arrow[ur, dashed, "h(u)^\#"] & B
\end{tikzcd}$$

We can do this since f is a Serre fibration.

Let $q : \Delta^1 \to [0,1]$ be a cts. map with $q(e_0) = 0$ and $q(e_1) = 1$. Then the desired chain homotopy is given by

$$C_*(F_g) \longrightarrow C_*(F_g) \otimes C_*([0,1])$$

$$\sigma \longmapsto \sigma \otimes q$$

composed with

$$C_*(F_g) \otimes C_*([0,1]) \xrightarrow{\text{hol}} C_*(F_g^{[0,1]}) \otimes C_*([0,1])$$

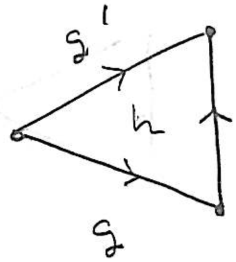
$$\xrightarrow[\sim]{b} C_*(F_g^{[0,1]} \times [0,1]) \xrightarrow{ev_*} C_*(F_g),$$

where b is the chain homotopy eq. from the Eilenberg-Zilber theorem. (We only use here that b is a chain map.) //

A path $\Delta^1 \xrightarrow{g} B$ induces isomorphisms

$$H_*(F_{g\circ i_0}) \xrightarrow[\sim]{i_0^*} H_*(F_g) \xleftarrow[\sim]{i_1^*} H_*(F_{g\circ i_1}),$$

and the composite isomorphism only depends on the homotopy class of the path:

$g \circ i_0 = g' \circ i_0$  $g \circ i_1 = g' \circ i_1$

Hence, if B is simply connected, then for all $b, b' \in B$, $H_*(F_b)$ and $H_*(F_{b'})$ are canonically isomorphic.

Theorem (Serre) Let $E \xrightarrow{f} B$ be a Serre fibration with B simply connected. Then there is a spectral sequence

$$E^2_{p,q} = H_p(B, H_q(F)) \Rightarrow H_{p+q}(E),$$

where $H_*(F)$ is the homology of the fiber.

pf We follow A. Dress and consider the two spectral sequences associated with the following bi-complex $C_{*,*}(f)$. We let $C_{p,q}(f)$ be the free k-module gen. by all commutative diagrams of the form

$$\begin{tikzcd} \Delta^p \times \Delta^q \arrow[r] \arrow[d, "pr_1"] & E \arrow[d, "f"] \\ \Delta^q \arrow[r] & B \end{tikzcd}.$$

The two differentials

$$C_{p-1,q}(f) \xleftarrow{d'} C_{p,q}(f) \xrightarrow{d''} C_{p,q-1}(f)$$

are given by

$$d' = \sum_{i=0}^{p} (-1)^p d_i' ,$$

$$d'' = \sum_{i=0}^{q} (-1)^{p+i} d_i'' ,$$

where d_i' and d_i'' are the maps induced from the face maps

$$\Delta^{m-1} \xrightarrow{d^i} \Delta^m , \quad 0 \le i \le m.$$

We first evaluate

$$E''^2_{q,p} = H_q (H_p (C_{*,*}(f) ; d') ; d'').$$

Lemma The projection $\Delta^q \xrightarrow{pr} \Delta^0$ induces a chain homotopy equivalence

$$C_{*,q}(f) \underset{\sim}{\xrightarrow{pr^*}} C_{*,0}(f) \approx C_*(E).$$

pf Note that we have a can- iso.

$$C_{*,q}(f) \approx C_*(E^{\Delta^q} \times_{B^{\Delta^q}} B)$$

where

$$\begin{tikzcd} E^{\Delta^q} \times_{B^{\Delta^q}} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & E^{\Delta^q} \arrow[d, "f_*"] \\ B \arrow[r, "pr^*"] & B^{\Delta^q} \end{tikzcd}$$

.

The maps $\Delta^q \xrightarrow{pr} \Delta^0$ and $\Delta^0 \xrightarrow{i_0} \Delta^q$ induce

$$C_{*,q}(f) \approx C_*(E^{\Delta^q} \times_{B^{\Delta^q}} B) \underset{i_0^*}{\overset{pr^*}{\rightleftarrows}} C_*(E),$$

and $i_0^* \circ pr^* = (pr \circ i_0)^* = id$. We show that $pr^* \circ i_0^* \simeq id$. Pick a homotopy

$$\begin{tikzcd}
\Delta^q \arrow[d, hook, "i_0"] \arrow[dr, bend left, "id"] & \\
\Delta^q \times [0,1] \arrow[r, "h"] & \Delta^q \\
\Delta^q \arrow[u, hook, "i_1"'] \arrow[ur, bend right, "i_0 \circ pr"'] &
\end{tikzcd} \Rightarrow$$

$$\begin{tikzcd}
& E^{\Delta^q} \times_{B^{\Delta^q}} B \\
E^{\Delta^q} \times_{B^{\Delta^q}} B \arrow[ur, bend left, "id"] \arrow[r, "h^*"] \arrow[dr, bend right, "pr^* \circ i_0^*"'] & E^{\Delta^q \times [0,1]} \times_{B^{\Delta^q \times [0,1]}} B \arrow[u, "i_0^*"'] \arrow[d, "i_1^*"] \\
& E^{\Delta^q} \times_{B^{\Delta^q}} B
\end{tikzcd} \Rightarrow$$

$$C_*(E^{\Delta^q} \underset{B^{\Delta^q}}{\times} B) \otimes C_*([0,1]) \xrightarrow[\sim]{b} C_*(E^{\Delta^q} \underset{B^{\Delta^q}}{\times} B \times [0,1])$$

$$\xrightarrow{h^*} C_*(E^{\Delta^q \times [0,1]} \underset{B^{\Delta^q \times [0,1]}}{\times} B \times [0,1])$$

$$\xrightarrow{\omega_*} C_*(E^{\Delta^q} \underset{B^{\Delta^q}}{\times} B)$$

The required chain-homotopy from id to $pr^* \circ i_0^*$ is obtained by precomposing by

$$C_*(E^{\Delta^q} \underset{B^{\Delta^q}}{\times} B) \longrightarrow C_*(E^{\Delta^q} \underset{B^{\Delta^q}}{\times} B) \otimes C_*([0,1]).$$

$$\sigma \longmapsto \sigma \otimes 4$$ //

It follows that

$$\begin{array}{ccc} E''^{1}_{q,p} = H_p(C_{*,q}(f); d'') & \xleftarrow{pr^*} & H_p(E) \\ \Big\downarrow d' & & \Big\downarrow \sum_{i=0}^{q} (-1)^{p+i} \\ E''^{1}_{q-1,p} = H_p(C_{*,q-1}(f); d'') & \xleftarrow{pr^*} & H_p(E) \end{array}$$

and hence

$$E''^2_{p,q} = H_q(H_p(C_{*,*}(f); d''); d')$$

$$\approx \begin{cases} H_p(E) & , q = 0 , \\ 0 & , q > 0 . \end{cases}$$

So the (two) spectral sequences converge to $H_*(E)$ as stated.

We next consider

$$E'^2_{p,q} = H_p(H_q(C_{*,*}(f); d'); d'') .$$

Note that we have a canonical iso.

$$C_{p,*}(f) \approx \bigoplus_{\Delta^p \xrightarrow{g} B} C_*(F_g) ,$$

where F_g is the fiber of $E^{\Delta^p} \xrightarrow{f_*} B^{\Delta^p}$ over g. It follows that

$$H_q(C_{p,*}(f)) \approx \bigoplus_{\Delta^p \xrightarrow{g} B} H_q(F_g)$$

$$\xrightarrow{\sim} \bigoplus_{\Delta^p \xrightarrow{g} B} H_q(F_{g \circ i_0}) .$$

The map

$$H_q(C_{p,*}(f);d'') \xrightarrow{d_i} H_q(C_{p-1,*}(f);d'')$$

takes the summand $\Delta^p \xrightarrow{g} B$ to the summand

$$\Delta^{p-1} \xhookrightarrow{d^i} \Delta^p \xrightarrow{g} B$$

by a map

$$H_q(F_{g\circ i_0}) \xrightarrow{\alpha_i} H_q(F_{g\circ d^i\circ i_0})$$

For $1 \leq i \leq p$, this map is the identity, since

$$\begin{tikzcd} \Delta^{p-1} \arrow[rr, hook, "d^i"] & & \Delta^p \\ & \Delta^0 \arrow[ul, "i_0"] \arrow[ur, "i_0"'] & \end{tikzcd}$$

But for $i=0$, we instead have

$$\begin{tikzcd} \Delta^{p-1} \arrow[rr, hook, "d^i"] & & \Delta^p \\ & \Delta^0 \arrow[ul, "i_0"] \arrow[ur, "i_1"'] & \end{tikzcd},$$

so α_0 is the isomorphism corresponding to the path from the 0'th to the 1'st vertex of $\Delta^p \xrightarrow{g} B$. However, if B is simply-connected, any two paths between two points of B give rise to the same iso. of the homology of the corresponding fibers. So if we pick a point $b \in B$ and identify $F_{g \circ i}$ with $F = F_b$, for all $\Delta^p \xrightarrow{g} B$, then we get an isomorphism

$$E'^2_{p,q} = H_p(H_q(C_{*,*}(f); d''); d')$$

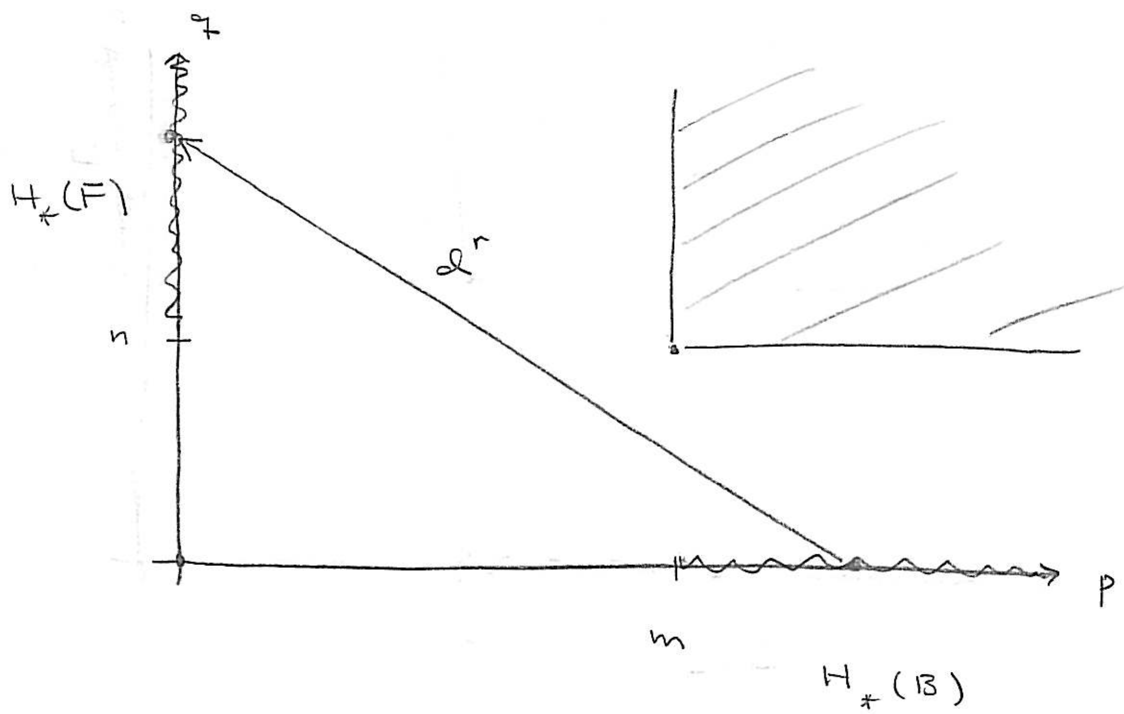
$$\approx H_p(B, H_q(F)). \qquad //$$

3/13/03

Let $E \xrightarrow{f} B$ be a Serre fibration with B simply-connected, and let $F = F_b$ be the fiber over $b \in B$. Then we have the Serre spectral sequence

$$E^2_{p,q} = H_p(B, H_q(F)) \Rightarrow H_{p+q}(E).$$

Suppose that $\tilde{H}_p(B) = 0$, for $p < m$, and $\tilde{H}_q(F) = 0$, $q < n$. Then the E^2-term is located in the following way.



It follows that, as long as differentials do not hit the are $p \geq m$ and $q \geq n$, the homology groups of F, E, and B form an exact sequence. The first

Thm The following sequence is exact, for $p < m+n$.

$$H_p(F) \xrightarrow{i_*} H_p(E) \xrightarrow{f_*} H_p(B) \xrightarrow{d^p} H_{p-1}(F) \xrightarrow{i_*} H_{p-1}(E).$$

pf The first differential that involve the groups $E^r_{p,q}$, $p \geq m$ and $q \geq n$, is:

$$E^m_{m,n} \xrightarrow{d^m} E^m_{m,m+n-1} \approx H_{m+n-1}(F).$$

The statement is now clear, except for the claim that $H_p(F) \to H_p(E)$ is induced by the inclusion $F \hookrightarrow E$. Recall the bi-complex $C_{*,*}(f)$ that gives rise to the spectral sequence. We identify the homology of the total complex with the homology of E by means of the qis.

$$C_*(E) \underset{\sim}{\xrightarrow{\alpha}} \mathrm{Tot}(C_{*,*}(f))$$

$$\Delta^p \to E \quad \longmapsto \quad \begin{array}{ccc} \Delta^p \times \Delta^0 & \xrightarrow{u} & E \\ \downarrow pr_1 & & \downarrow f \\ \Delta^p & \xrightarrow{f \circ u} & B \end{array}$$

The composite

$$H_p(\mathrm{Tot}(C_{*,*}(f))) \twoheadrightarrow E^\infty_{p,0} \hookrightarrow E^2_{p,0} \approx H_p(B)$$

is induced by the chain map

$$\mathrm{Tot}\,(C_{*,*}(f)) \xrightarrow{\beta} C_*(B)$$

$$\begin{array}{ccc} \Delta^p \times \Delta^q & \xrightarrow{u} & E \\ \big\downarrow pr_1 & & \big\downarrow f \\ \Delta^p & \xrightarrow{s} & B \end{array} \quad \longmapsto \quad \begin{cases} \Delta^p \xrightarrow{s} B & , q = 0, \\ 0 & , q > 0. \end{cases}$$

Hence, the map $H_p(E) \to H_p(B)$ of the seq. of the statement is induced from f as stated. Similarly, the composite

$$H_q(F_b) \approx E^2_{0,q} \twoheadrightarrow E^\infty_{0,q} \hookrightarrow H_q(\mathrm{Tot}\,(C_{*,*}(f)))$$

is induced by the chain map

$$C_*(F_b) \xrightarrow{\gamma} \mathrm{Tot}\,(C_{*,*}(f))$$

$$\Delta^q \xrightarrow{u} F_b \quad \longmapsto \quad \begin{array}{ccc} \Delta^0 \times \Delta^q & \xrightarrow{i \circ u} & E \\ \big\downarrow pr_1 & & \big\downarrow f \\ \Delta^0 & \xrightarrow{b} & B \end{array},$$

However, γ does not land in the image of α, so it is not clear that $H_p(F) \to H_p(E)$ is induced by i.

This would be clear, if instead of γ, we had the chain map

$$C_*(F_b) \xrightarrow{\gamma'} \mathrm{Tot}(C_{*,*}(f))$$

$$\left(\Delta^p \xrightarrow{u} F_b\right) \longmapsto \begin{array}{ccc} \Delta^p \times \Delta^0 & \xrightarrow{iou} & E \\ \downarrow pr_1 & & \downarrow f \\ \Delta^p & \xrightarrow{b} & B \end{array} .$$

So we need to show that γ and γ' are chain-homotopic. The maps γ and γ' factor through the chain map

$$\mathrm{Tot}(C_{*,*}(f')) \longrightarrow \mathrm{Tot}(C_{*,*}(f))$$

induced from

$$\begin{array}{ccc} F_b & \overset{i}{\hookrightarrow} & E \\ \downarrow f' & & \downarrow f \\ * & \xrightarrow{b} & B \end{array}$$

So we can replace f by f'. In that case γ and γ' are both qis. We define a <u>natural</u> chain homotopy

$$C_*(F_b) \xrightarrow{s} \mathrm{Tot}(C_{*,*}(F_b))_{*+1}$$

Suppose s_i has been constructed, $i<p$. (We take $s_{-1} = 0$.) Let $\Delta^p \xrightarrow{u} F_b$ be a generator of $C_p(F_b)$. By naturality, the following diagram must commute

$$\begin{array}{ccc}
C_p(F_b) & \xrightarrow{s_p} & \mathrm{Tot}(C_{*,*}(F_b))_{p+1} \\
\uparrow u_* & & \uparrow u_* \\
C_p(\Delta^p) & \xrightarrow{s_p} & \mathrm{Tot}(C_{*,*}(\Delta^p))_{p+1}
\end{array}$$

so it suffices to define s_p on the identity map of Δ^p. We wish to find $s_p(id) \in \mathrm{Tot}(C_{*,*}(\Delta^p))_{p+1}$ s.t.

$$(*) \qquad d s_p(id) = \gamma(id) - \gamma'(id) - s_{p-1} d(id).$$

Since $H_{p+1}(\mathrm{Tot}(C_{*,*}(\Delta^p))) = 0$, it suffices to show that RHS of $(*)$ is a cycle. This is easy:

$$d\gamma - d\gamma' - ds_{p-1}d = (\gamma - \gamma')d - ds_{p-1}d$$
$$= ds_{p-1}d - dds_{p-2} - ds_{p-1}d = 0. \qquad //$$

Let $[S^n] \in H_n(S^n)$ be the fundamental class corresponding to a fixed orientation of S^n. (We have to fix this.) Then the Hurewicz homomorphism is given by

$$\pi_n(X, x_0) \xrightarrow{h} H_n(X, \mathbb{Z})$$

$$[S^n \xrightarrow{f} X] \longmapsto f_*([S^n]).$$

<u>Addendum</u> The following diagram commutes (possibly up to a sign) for $p < m+n$.

$$\begin{tikzcd}
H_p(F) \arrow[r, "i_*"] & H_p(E) \arrow[r, "f_*"] & H_p(B) \arrow[r, "d^p"] & H_{p-1}(F) \\
\pi_p(F) \arrow[r, "i_*"] \arrow[u, "h"] & \pi_p(E) \arrow[r, "f_*"] \arrow[u, "h"] & \pi_p(B) \arrow[r, "\partial"] \arrow[u, "h"] \arrow[ur, phantom, "(\pm 1)"] & \pi_{p-1}(F) \arrow[u, "h"]
\end{tikzcd}$$

Pf The left and middle sq. commutes since H_p is a functor. Let

$$(S^p, s_0) \xrightarrow{g} (B, b)$$

repr. $[g] \in \pi_p(B, b)$. Then $\partial[g] \in \pi_{p-1}(F_b, e)$ is the class of $w|_{S^{p-1}}$, where

$$\begin{tikzcd}
\{s_0\} \arrow[rr, "e"] \arrow[d, "\sim"] & & E \arrow[d, "f"] \\
D^p \arrow[r, two heads] \arrow[urr, dashed, "\exists w"] & S^p \arrow[r, "g"] & B
\end{tikzcd}$$

.

We identify $(D^p, S^{p-1}) \approx (\Delta^p, \partial\Delta^p)$. Then the class of the following element of $C_p(B)$ repr. $\pm h([g])$.

$$\tilde{g} : \Delta^p \twoheadrightarrow \Delta^p/\partial\Delta^p \longrightarrow B .$$

The following diagram is an element of $C_{p,0}(f)$ whose image under β is the element $\tilde{g} \in C_p(B)$ above.

$$\begin{array}{ccc} \Delta^p \times \Delta^0 & \xrightarrow{u} & E \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle f} \\ \Delta^p & \xrightarrow{g} & B \end{array}$$

The d'-boundary $\xi' \in C_{p-1,0}(f)$ is the alternating sum of the elements given by the following diagrams, $0 \leq i \leq p$.

$$\begin{array}{ccc} \Delta^{p-1} \times \Delta^0 & \xrightarrow{u \circ d^i} & E \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle f} \\ \Delta^{p-1} & \xrightarrow{b} & B \end{array} .$$

To calculate $d^p[\tilde{g}]$ means the following. The element $\xi' \in C_{p-1,0}(f)$ is homologous in $\mathrm{Tot}(C_{*,*}(\mathcal{E}))$ to $\xi \in C_{0,p-1}(\mathcal{E})$ and then

$$d^p[\tilde{g}] = [\xi].$$

On the other hand, $h(\partial[g])$ is repr. by

$$v = \sum_{i=0}^{p} (-1)^i v_i$$

$$\begin{tikzcd} \Delta^{p-1} \arrow[r, "v_i"] \arrow[rr, bend right, "u \circ d^i"'] & F_b \arrow[r, hook, "i"] & E \end{tikzcd}$$

and, clearly, $\gamma'(v) = \xi'$. Since

$$\gamma, \gamma' : C_*(F_b) \to \mathrm{Tot}(C_{*,*}(\mathcal{E}))$$

are chain-homotopic, it follows that ξ' is homologous to

$$\xi = \gamma(v) \in C_{0,p-1}(\mathcal{E}).$$

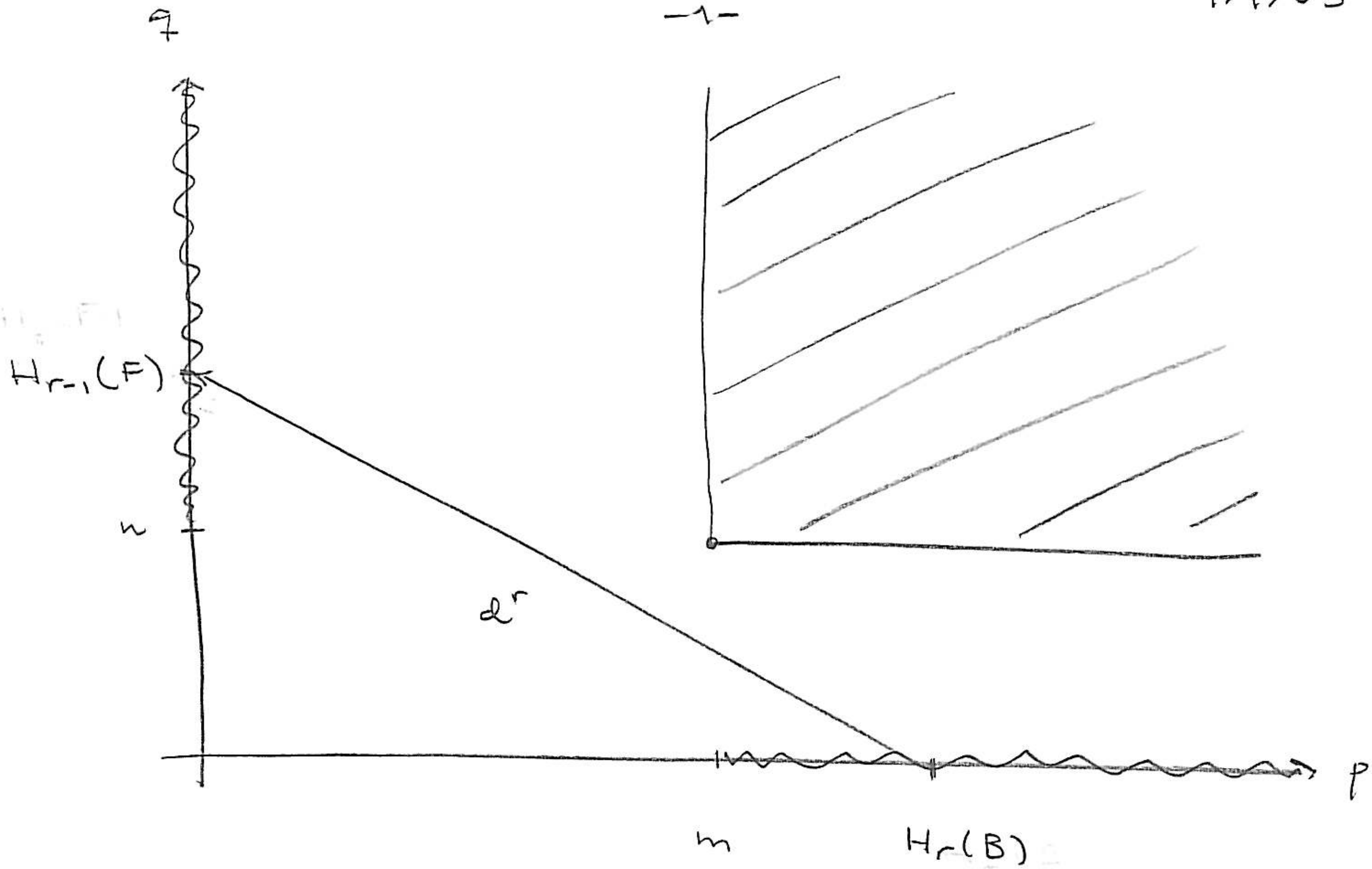
We are done. //

Prop If X is connected, then

$$\pi_1(X, x_0)^{ab} \xrightarrow{\sim} H_1(X, \mathbb{Z}).$$

pf Reading assignment. //

4/1/03



$$
\begin{array}{ccccccccccc}
\cdots \to H_r(F) & \xrightarrow{i_*} & H_r(E) & \xrightarrow{f_*} & H_r(B) & \xrightarrow{d^r} & H_{r-1}(F) & \xrightarrow{f_*} & H_{r-1}(E) & \xrightarrow{i_*} & \cdots \\
\uparrow h & & \uparrow h & & \uparrow h & & \uparrow h & & \uparrow h & & \\
\cdots \to \pi_r(F) & \xrightarrow{i_*} & \pi_r(E) & \xrightarrow{f_*} & \pi_r(B) & \longrightarrow & \pi_{r-1}(F) & \xrightarrow{f_*} & \pi_{r-1}(E) & \xrightarrow{i_*} & \cdots
\end{array}
$$

Top row exact from $H_{m+n-1}(F)$. Apply to path-space fibration

$$\Omega(X, x_0) \longrightarrow P(X, x_0) \xrightarrow{ev_1} X$$

$$\shortparallel$$

$$\{ [0,1] \xrightarrow{\omega} X \mid \omega(0) = x_0 \}$$

$$\cong$$

$$*$$

Prop If X is (path-) connected, then

$$\pi_1(X, x_0)^{ab} \xrightarrow[\sim]{h} H_1(X, \mathbb{Z}).$$ //

Thm Let X be an $(n-1)$-connected space, $n \geq 2$. Then for all $1 \leq i \leq n$,

$$\pi_i(X, x_0) \xrightarrow[\sim]{h} H_i(X, \mathbb{Z}).$$

pf The statement for $i=1$ follows from the prop. Proof by induction on $n \geq 2$. The homotopy l.e.s.

$$\cdots \to \underset{\substack{\| \\ 0}}{\pi_i(PX)} \to \pi_i(X) \xrightarrow{\partial} \pi_{i-1}(\Omega X) \to \underset{\substack{\| \\ 0}}{\pi_{i-1}(PX)} \to \cdots$$

shows that for all $i \geq 1$,

$$\pi_i(X) \xrightarrow[\sim]{\partial} \pi_{i-1}(\Omega X).$$

So if X is $(n-1)$-conn., ΩX is $(n-2)$-conn. If $n=2$, we have ΩX conn. so by the proposition:

$$\pi_1(\Omega X) = \pi_1(\Omega X)^{ab} \xrightarrow{\sim} H_1(\Omega X).$$

The thm. for $n = 2$ follows:

$$\begin{tikzcd} H_2(X) \arrow[r, "d^2", "\sim"'] & H_1(\Omega X) \\ \pi_2(X) \arrow[u, "h"] \arrow[r, "\partial", "\sim"'] & \pi_1(\Omega X) \arrow[u, "\sim", "h"'] \end{tikzcd} .$$

Suppose X is $(n-1)$-conn., $n \geq 2$, s.t. ΩX is $(n-2)$-conn. Inductively,

$$\pi_i(\Omega X) \xrightarrow[\sim]{h} H_i(\Omega X),$$

for $1 \leq i \leq n-1$. In particular, $H_i(\Omega X) = 0$, for $1 \leq i < n-1$. Also, X is $(n-1)$-conn., hence $(n-2)$-conn., so by induction

$$0 = \pi_i(X) \xrightarrow[\sim]{h} H_i(X),$$

for $1 \leq i \leq n-1$. Induction step:

$$\begin{tikzcd} H_i(X) \arrow[r, "d^i", "\sim \atop 1 \leq i < 2n-1"'] & H_{i-1}(\Omega X) \\ \pi_i(X) \arrow[u] \arrow[r, "\partial", "\sim \atop 1 \leq i"'] & \pi_{i-1}(\Omega X) \arrow[u, "h", "\sim \quad 1 \leq i-1 \leq n-1"'] \end{tikzcd} .$$

//

Cor Suppose that X is simply-conn. and that $H_i(X) = 0$, for $1 \le i < n$. Then

$$\pi_i(X) \xrightarrow[\sim]{h} H_i(X),$$

for $1 \le i \le n$.

pf By induction; $n = 2$ o.k. Suppose that $H_i(X) = 0$, for $1 \le i < n$. By induction, $\pi_i(X) = 0$, for $1 \le i < n$. By thm.

$$\pi_i(X) \xrightarrow[\sim]{h} H_i(X),$$

for $1 \le i \le n$. //

ex S^n is $(n-1)$-connected and

$$\pi_n(S^n) \xrightarrow[\sim]{\deg} \mathbb{Z}.$$ //

the Hurewicz thm., $H_i(F)=0$, $1 \le i < n$, and since $H_i(Y)=0$, $1 \le i < 2$, the following seq. is exact.

$$
\begin{aligned}
& H_{n+1}(X) \xrightarrow{f_*} H_{n+1}(Y) \xrightarrow{\partial^{n+1}} H_n(F) \\
\to\ & H_n(X) \xrightarrow{f_*} H_n(Y) \xrightarrow{\partial^n} \underset{\|}{H_{n-1}(F)} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \\
\to\ & H_{n-1}(X) \xrightarrow{f_*} H_{n-1}(Y) \xrightarrow{\partial^{n-1}} \underset{\|}{H_{n-2}(F)} \to \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \\
& \qquad\qquad \vdots
\end{aligned}
$$

So (ii) follows.

(ii) $\Rightarrow$ (i) Argue by induction on $n \geq 1$; $n=1$ is O.K. by assumption. Assume O.K. for $n-1$. By induction, $\pi_i(F)=0$, for $1 \le i < n-1$, and by Hurewicz, it follows that $H_i(F)=0$, $1 \le i < n-1$. Also $\pi_i(Y)=0$, $1 \le i < 2$, so $H_i(Y)=0$, $1 \le i < 2$. Hence the homology seq. above is exact. We conclude from (ii) that $H_i(F)=0$, $1 \le i \le n-1$, and by Hurewicz that $\pi_i(F)=0$, $1 \le i \le n-1$. So (i) follows. //

Thm (Whitehead) Let $X \xrightarrow{f} Y$ be a map between simply-conn. spaces. Then the following are equivalent:

(i) The induced map

$$\pi_i(X) \xrightarrow{f_*} \pi_i(Y)$$

is iso., for $i < n$, and epi., for $i \leq n$.

(ii) The induced map

$$H_i(X, \mathbb{Z}) \xrightarrow{f_*} H_i(Y, \mathbb{Z})$$

is iso., for $i < n$, and epi., for $i \leq n$.

pf We can assume that f is a fibration with fiber F (which is conn. since X and Y are simply-conn.)

(i) $\Rightarrow$ (ii) The htpy. seq.

$$\cdots \to \pi_i(F) \to \pi_i(X) \xrightarrow{f_*} \pi_i(Y) \xrightarrow{\partial} \pi_{i-1}(F) \to \cdots$$

shows that F is $(n-1)$-conn. By

18.906: Problem Set 5

Due: Thursday, April 10.

Let $f\colon E \to B$ be a Serre fibration with B simply-connected. This problem set concerns the cohomological Serre spectral sequence. It takes the form

$$E_2^{s,t} = H^s(B, H^t(F)) \Rightarrow H^{s+t}(E),$$

where $F = F_b = f^{-1}(b)$ is the fiber over $b \in B$. (We recall that the cohomology of the fibers F_b are canonically isomorphic.) The upper indexing indicates that the spectral sequence is *cohomology type*, so the differentials are maps

$$d_r\colon E_r^{s,t} \to E_r^{s+r,t-r+1}.$$

This spectral sequence is *multiplicative*. Hence, (E_r, d_r) is an anti-commutative differential-bigraded-ring, which means that there is a product

$$E_r^{s,t} \otimes E_r^{s',t'} \xrightarrow{\cup} E_r^{s+s',t+t'}$$

which satisfies

$$\omega \cup \omega' = (-1)^{\deg \omega \cdot \deg \omega'} \omega' \cup \omega,$$
$$d_r(\omega \cup \omega') = d_r(\omega) \cup \omega' + (-1)^{\deg \omega} \omega \cup d_r(\omega'),$$

where deg is the total degree. In addition, the isomorphism $E_{r+1} \approx H^*(E_r, d_r)$ is multiplicative.

There is an isomorphism of (anti-commutative) bigraded rings

$$E_2^{*,*} = H^*(B, H^*(F))$$

with the product on the right given by the cup-product. Finally, the spectral sequence converges to $H^*(E)$ as an algebra. This meaning of this is as follows. Firstly, there is, associated with the spectral sequence, a natural descending filtration by graded ideals

$$H^*(E) = \mathrm{Fil}^0 H^*(E) \supset \mathrm{Fil}^1 H^*(E) \supset \mathrm{Fil}^2 H^*(E) \supset \dots$$

and the cup-product restricts to maps

$$\mathrm{Fil}^s H^*(E) \otimes \mathrm{Fil}^{s'} H^*(E) \xrightarrow{\cup} \mathrm{Fil}^{s+s'} H^*(E).$$

It follows that the cup-product on $H^*(E)$ induces a product

$$\mathrm{gr}^s H^i(E) \otimes \mathrm{gr}^{s'} H^{i'}(E) \xrightarrow{\cup} \mathrm{gr}^{s+s'} H^{i+i'}(E)$$

which makes $\mathrm{gr}^* H^*(E)$ an anti-commutative bi-graded ring. Secondly, there is a natural isomorphism of anti-commutative bigraded rings

$$E_\infty^{*,*} \approx \mathrm{gr}^* H^*(E).$$

---

1. Consider the cohomological Serre spectral sequence for the path-space fibration associates with the n-sphere with $n \geq 2$,

$$\Omega(S^n) \to P(S^n) \to S^n. \qquad \text{(over)}$$

$$H_*(X,\mathbb{Z}) \xrightarrow{\text{with}} \pi_*(X).$$

Suppose X is connected. Define

$$X_0 = X$$
$$T_1 = \tilde{X}_0$$
$$X_1 = \Omega(T_1)$$
$$T_2 = \tilde{X}_1$$
$$X_2 = \Omega(T_2)$$
$$\vdots$$

Then X_n is connected, $n \geq 0$, and

$$\pi_{n+i}(X) \approx \pi_i(X_n) \quad , \; i \geq 1.$$

In particular, for $n \geq 1$,

$$\pi_{n+1}(X) \approx \pi_1(X_n) \approx H_1(X_n, \mathbb{Z}).$$

X_n H-space, $n \geq 1$.

Use

$$E^2_{s,t} = H_s(T_n, H_t(X_n)) \Rightarrow H_{s+t}(P(T_n)),$$

$$E^2_{s,t} = H_s(\pi_1(X_n), H_t(T_{n+1})) \Rightarrow H_{s+t}(X_n).$$

Conclude that

$$H_*(\mathrm{Tot}(P_* \otimes_{k[G]} C_*(\tilde{X})) \approx H_*(X).$$

The other sp. seq. has

$$\begin{aligned} E'^1_{s,t} &= H_t(P_s \otimes_{k[G]} C_*(\tilde{X})) \\ &\xleftarrow{\sim} P_s \otimes_{k[G]} H_t(C_*(\tilde{X})) \\ &= P_s \otimes_{k[G]} H_t(\tilde{X}) \end{aligned}$$

so

$$\begin{aligned} E'^2_{s,t} &= H_s(H_t(P_* \otimes_{k[G]} C_*(\tilde{X}), d''), d') \\ &\approx H_s(P_* \otimes_{k[G]} H_t(\tilde{X})) \\ &=: \mathrm{Tor}_s^{k[G]}(k, H_t(\tilde{X})) \end{aligned}$$

If G is a group and M a $k[G]$-mod., one defines the group homology of G with coefficients in M by

$$H_i(G, M) := \mathrm{Tor}_i^{k[G]}(k, M).$$

ex Let G be the cyclic group of order r and let $t \in G$ be a generator. A res. of $\mathbb{Z}$ by free $\mathbb{Z}[G]$-modules is:

$$0 \leftarrow \mathbb{Z} \xleftarrow{\varepsilon} \mathbb{Z}[G] \xleftarrow{t-1} \mathbb{Z}[G] \xleftarrow{N} \mathbb{Z}[G] \xleftarrow{t-1} \cdots$$

where

$$N = 1+t+t^2+\cdots+t^{r-1} \in \mathbb{Z}[G]$$

is the norm element. Hence $H_*(G,M)$ is the homology of the cx.

$$\begin{array}{ccccccc}
\mathbb{Z}[G]\otimes_{\mathbb{Z}[G]} M & \xleftarrow{t-1} & \mathbb{Z}[G]\otimes_{\mathbb{Z}[G]} M & \xleftarrow{N} & \mathbb{Z}[G]\otimes_{\mathbb{Z}[G]} M & \xleftarrow{t-1} & \cdots \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\
M & \xleftarrow{t-1} & M & \xleftarrow{N} & M & \xleftarrow{t-1} & \cdots
\end{array}$$

//

These two examples and the Künneth formula shows:

Lemma Let A be a f-g. abelian group, and let M be a $\mathbb{Z}[A]$-module that as an abelian group is f-g. Then for all $i \geq 0$, $H_i(A,M)$ is a f-g. abelian group. //

The second spectral seq. is that of a covering space. This actually is a Serre sp. seq., but we give independent construction.

Suppose that the group G acts properly discontinuously on the space $\tilde{X}$ s.t.

$$\tilde{X} \xrightarrow{f} X = \tilde{X}/G$$

is a covering projection. The unique lifting property shows that

$$\mathrm{Sin}_n(\tilde{X}) = \mathrm{Map}(\Delta^n, \tilde{X})$$

is a <u>free</u> G-set. Therefore

$$C_n(\tilde{X}, k) = k\{\mathrm{Sin}_n(\tilde{X})\}$$

is a free $k[G]$-module, and hence the map f induces an iso.

$$k \otimes_{k[G]} C_*(\tilde{X}, k) \xrightarrow{\sim} C_*(X, k).$$

Let $P_* \underset{\sim}{\xrightarrow{\varepsilon}} k$ be a res. of k by projective $k[G]$-modules and consider the two sp. seq. associated with the bi-complex

$$P_* \otimes_{k[G]} C_*(\tilde{X}, k).$$

Since $C_s(\tilde{X}, k)$ is a free $k[G]$-mod.,

$$E''^1_{s,t} := H_t(P_* \otimes_{k[G]} C_s(\tilde{X}), d')$$

$$\xleftarrow{\sim} H_t(P_*) \otimes_{k[G]} C_s(\tilde{X})$$

$$\xrightarrow[\sim]{\varepsilon} \begin{cases} k \otimes_{k[G]} C_s(\tilde{X}) & , t=0, \\ 0 & , t>0, \end{cases}$$

so

$$E''^2_{s,t} := H_s(H_t(P_* \otimes_{k[G]} C_*(\tilde{X}), d'), d'')$$

$$\approx \begin{cases} H_s(k \otimes_{k[G]} C_*(\tilde{X})) & , t=0, \\ 0 & , t>0 \end{cases}$$

$$\xrightarrow[\sim]{f_*} \begin{cases} H_s(X) & , t=0, \\ 0 & , t>0. \end{cases}$$

So we have a spectral sequence

$$E^2_{s,t} = H_s(G, H_t(\tilde{X})) \Rightarrow H_{s+t}(X).$$

$\underline{\text{ex}}$ Let G be an infinite cyclic group, and let $t \in G$ be a generator. Then

$$\mathbb{Z}[G] = \mathbb{Z}[t^{\pm 1}].$$

A res of $\mathbb{Z}$ by free $\mathbb{Z}[G]$-mod. is:

$$0 \leftarrow \mathbb{Z} \xleftarrow{\varepsilon} \mathbb{Z}[t^{\pm 1}] \xleftarrow{t-1} \mathbb{Z}[t^{\pm 1}] \leftarrow 0$$

so $H_*(G, M)$ is the homology of

$$\begin{array}{ccc} \mathbb{Z}[t^{\pm 1}] \otimes_{\mathbb{Z}[t^{\pm 1}]} M & \xleftarrow{(t-1)\otimes M} & \mathbb{Z}[t^{\pm 1}] \otimes_{\mathbb{Z}[t^{\pm 1}]} M \\ \downarrow \wr & & \downarrow \wr \\ M & \xleftarrow{t-1} & M \end{array}$$

i.e.

$$H_i(G, M) = \begin{cases} M_G & , i = 0, \\ M^G & , i = 1, \\ 0 & , i \geq 2. \end{cases}$$

//

<u>Thm</u> Let X be a simply-connected space and suppose that $H_i(X, \mathbb{Z})$ is f.g., $i \geq 0$. Then $\pi_i(X)$ is f.g., $i \geq 0$.

<u>pf</u> Recall the spaces

$$X_0 = X$$

$$T_1 = \tilde{X}_0 = X \qquad \text{since } \pi_1(X) = 0$$

$$X_1 = \Omega(T_1)$$

$$T_2 = \tilde{X}_1$$

$$X_2 = \Omega(T_2)$$

$$\vdots$$

Also recall that

$$\pi_n(X) \approx \pi_1(X_{n-1}) \approx H_1(X_{n-1}, \mathbb{Z}).$$

We show by induction on $n \geq 1$, that $H_i(X_n, \mathbb{Z})$ and $H_i(T_n, \mathbb{Z})$ are f.g., for all $i \geq 0$. The case $n=1$ follows from our assumptions. So suppose $n-1$ O.K.

To show that $H_i(T_n, \mathbb{Z})$ is f.g., $i \geq 0$, we consider the sp. seq. associated with the universal covering

$$T_n = \tilde{X}_{n-1} \longrightarrow X_{n-1}$$

with group $\pi_1(X_{n-1}) = \pi_n(X_n)$,

$$E^2_{s,t} = H_s(\pi_n(X), H_t(T_n))$$

$$\Rightarrow H_{s+t}(X_{n-1}).$$

One can show that if Y is an H-space, then $\pi_1(Y)$ acts trivially on $H_*(\tilde{Y}, \mathbb{Z})$. Since $X_{n-1} = \Omega(T_{n-1})$ is an H-space, $n \geq 2$, it follows that $\pi_1(X_{n-1}) = \pi_n(X)$ acts trivially on $H_*(T_n, \mathbb{Z})$. Hence,

$$E^2_{0,t} = H_t(T_n, \mathbb{Z}).$$

Suppose that for some $i \geq 2$, $H_i(T_n, \mathbb{Z})$ was not f.g.; let i be smallest with this property. Inductively,

$$\pi_n(X) = H_1(X_{n-1}, \mathbb{Z})$$

is a f.g. ab. group, so for $0 \leq t < i$

$$E^2_{s,t} = H_s(\pi_n(X), H_t(T_n, \mathbb{Z}))$$

is a f-g. abelian group. It follows that $E^r_{0,i}$ is not f-g., for all $r \geq 2$. But $E^\infty_{0,i}$ is a subgroup of $H_i(X_{n-1}, \mathbb{Z})$, which, by induction, is f-g. $\lightning$. So $H_i(T_n, \mathbb{Z})$ is f-g., for all $i \geq 0$.

To show that $H_i(X_n, \mathbb{Z})$ is f-g. for all $i \geq 0$, we consider

$$E^2_{s,t} = H_s(T_n, H_t(X_n, \mathbb{Z}))$$

$$\Rightarrow H_{s+t}(P(T_n), \mathbb{Z}) = \begin{cases} \mathbb{Z} & , s=t=0 \\ 0 & , \text{else}. \end{cases}$$

The same arg. as before works. //

In order to apply the comparison theorem to the cohomological Serre sp. seq. we need:

Prop There is a natural exact sequence

$$0 \to H^p(X,\mathbb{Z})\otimes A \to H^p(X,A) \to \mathrm{Tor}_1^{\mathbb{Z}}(H^{p+1}(X,\mathbb{Z}),A) \to 0$$

provided that either A or $H_p(X,\mathbb{Z})$, $p \geq 0$, is finitely generated.

Pf Let $C = C_* = C_*(X,\mathbb{Z})$. It suffices to show that the canonical map

$$\mathrm{Hom}(C,\mathbb{Z})\otimes A \xrightarrow{\lambda} \mathrm{Hom}(C,A)$$

is a quasi-isomorphism. Indeed, the cohomology groups of the LHS sit in a s.e.s. as in the statement. (Pick a proj. res. $P \xrightarrow{\sim} A$ and consider the two sp. seq. assoc. with

$$\mathrm{Hom}(C,\mathbb{Z}) \otimes P \quad)$$

Suppose first that A is f-g. We show that λ is an isomorphism. This is clearly true if $A = \mathbb{Z}$, and more generally, if A is f-g. free Since A

is f.g. we can find

$$0 \to P_1 \to P_0 \to A \to 0$$

with P_0 and P_1 f.g. free. We have

$$\begin{tikzcd}
 & 0 \arrow[d] \\
\mathrm{Hom}(C_i, \mathbb{Z}) \otimes P_1 \arrow[r, "\lambda", "\sim"'] \arrow[d] & \mathrm{Hom}(C_i, P_1) \arrow[d] \\
\mathrm{Hom}(C_i, \mathbb{Z}) \otimes P_0 \arrow[r, "\lambda", "\sim"'] \arrow[d] & \mathrm{Hom}(C_i, P_0) \arrow[d] \\
\mathrm{Hom}(C_i, \mathbb{Z}) \otimes A \arrow[r, "\lambda"] \arrow[d] & \mathrm{Hom}(C_i, A) \arrow[d] \\
0 & 0
\end{tikzcd}$$

— since C_i is free

so O.K. in this case.

If C was a cx. of f.g. free ab. groups, the map λ would clearly also be an iso. Hence, the case $H_p(X, \mathbb{Z}) = H_p(C)$ f.g., $p \geq 0$, follows from the lemma below.

//

<u>Lemma</u> Let C be a chain cx. of free ab. grps and suppose that $H_i(C)$ is a f.g. ab. grp., for all i. Then there exists a quasi-isomorphism

$$C' \xrightarrow{\sim} C$$

from a cx. C' of f.g. free ab. grps.

pf We pick for all i, a f.g. (free) subgrp. $F_i \subset Z_i$ s.t. the composite

$$F_i \hookrightarrow Z_i \twoheadrightarrow H_i$$

is surjective. Let F_i' be the kernel, i.e.

$$0 \to F_i' \to F_i \to H_i \to 0 .$$

We view F_i' and F_i, $i \in \mathbb{Z}$, as chain cx. F' and F with zero differential and define C' to be the mapping cone of the inclusion $F' \hookrightarrow F$. Then C' is a cx. of f.g. free ab. grps., we have a canonical chain map

$$C' \longrightarrow C ,$$

and this map is a quasi-iso. 4

Here is an example of how the comparison theorem works.

Prop Let X be a simply-connected space with $H_i(X, \mathbb{Z})$ f.g., $i \geq 0$. Suppose that $H^*(X, \mathbb{Z}) = S_{\mathbb{Z}}\{x\}$, $|x| = n$. Then

$$H^*(\Omega X, \mathbb{Z}) = \Lambda_{\mathbb{Z}}\{y\}, \quad |y| = n-1.$$

Pf Let $(E_r, d_r) \Rightarrow \mathbb{Z}$ be the coh. Serre spectral sequence assoc. with

$$\Omega X \longrightarrow PX \longrightarrow X.$$

We define an abstract spectral seq.

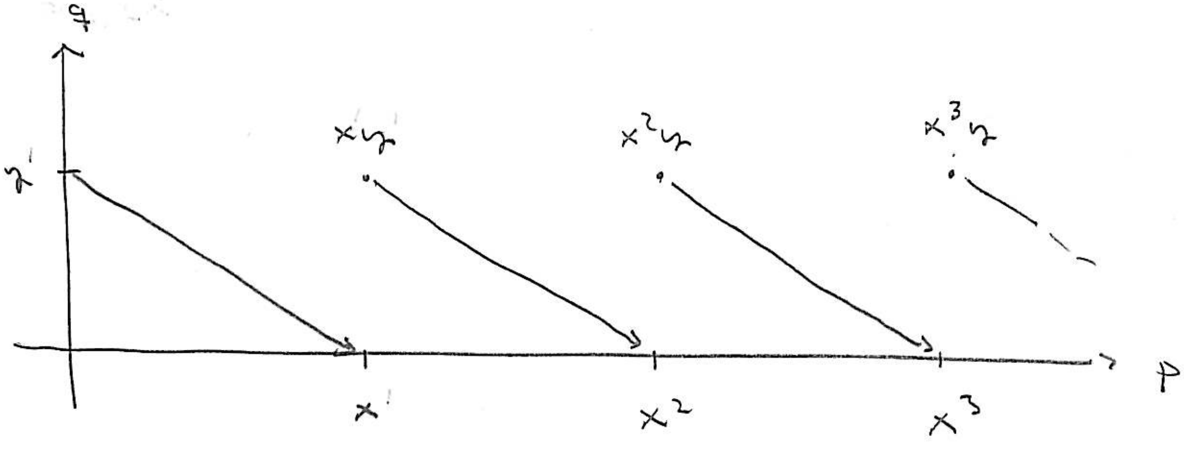
$$(E'_r, d'_r) \Rightarrow \mathbb{Z}$$

as follows:

$$E'_r = \begin{cases} S_{\mathbb{Z}}\{x\} \otimes \Lambda_{\mathbb{Z}}\{y\}, & 2 \leq r \leq n, \\ \mathbb{Z}, & n < r \leq \infty, \end{cases}$$

with $\deg x = (n, 0)$ and $\deg(y) = (0, n-1)$.

The diff. d'_r is zero, unless $r = n$, and d'_n is a derivation with $d'_n(y) = x$:



In the Serre sp. seq., we must have $E_r^{0,q} = 0$, $0 < q < n-1$, and

$$E_n^{0,n-1} \xrightarrow[\sim]{d_n} E_n^{n,0} = \mathbb{Z} \cdot x .$$

Hence, there is a unique class $y \in E_n^{0,n-1} \approx E_2^{0,n-1}$ s.t. $d_n y = x$. And since the Serre sp. seq. is multiplicative, there is a unique map of sp. seq.

$$\begin{tikzcd}
(E'_r, d'_r) \arrow[r, Rightarrow] \arrow[d, "f"] & \mathbb{Z} \arrow[d, equal] \\
(E_r, d_r) \arrow[r, Rightarrow] & \mathbb{Z}
\end{tikzcd}$$

that takes x to x and y to y. Since both $E_2'^{p,0} \xrightarrow{f} E_2^{p,0}$, $p \geq 0$, and $\mathbb{Z} \xrightarrow{f} \mathbb{Z}$ are isomorphisms, the comparison thm. shows that also $E_2'^{0,q} \xrightarrow{f} E_2^{0,q}$, $q \geq 0$, is an iso., i.e.

$$\Lambda_{\mathbb{Z}}\{y\} \xrightarrow{\sim} H^*(\Omega X, \mathbb{Z}). \qquad //$$

Rem The same proof shows that

$$H^*(X, k) = S_k\{x\} \implies$$

$$H^*(\Omega X, k) = \Lambda_k\{y\},$$

for any comm. ring k. //

Prop Let X be a simply-connected space and suppose $H^*(X,k) = \Lambda_k\{x\}$, $|x| = n$. Then

$$H^*(\Omega X, k) = \Gamma_k\{y\}, \quad |y| = n-1. \qquad //$$

Here $\Gamma_k\{y\}$ is the divided power algebra which as a k-module is free on generators $y^{[r]}$, $r \geq 0$, with mult.

$$y^{[r]} \cdot y^{[s]} = \binom{r+s}{r} y^{[r+s]}.$$

If $k = \mathbb{Q}$ (or a $\mathbb{Q}$-alg.), then

$$S_k\{y\} \xrightarrow{\sim} \Gamma_k\{y\}$$

$$y^r \longmapsto r!\, y^{[r]}$$

Using the two prop., we conclude:

$$H^*(S^n, \mathbb{Q}) = \Lambda_{\mathbb{Q}}\{x_n\} \Rightarrow$$

$$H^*(\Omega S^n, \mathbb{Q}) = S_{\mathbb{Q}}\{x_{n-1}\} \Rightarrow$$

$$H^*(\Omega^2 S^n, \mathbb{Q}) = \Lambda_{\mathbb{Q}}\{x_{n-2}\}$$

i.e. for $n \geq 3$,

$$H^*(\Omega^2 S^n, \mathbb{Q}) \approx H^*(S^{n-2}, \mathbb{Q}).$$

Let $n \geq 3$ be odd. Then, inductively,

$$H^*(\Omega^{n-1} S^n, \mathbb{Q}) \approx H^*(S^1, \mathbb{Q}) = \Lambda_{\mathbb{Q}}\{x_1\}.$$

Since, for every space X,

$$H^i(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{Hom}(H_i(X, \mathbb{Q}), \mathbb{Q}),$$

we conclude

$$H_i(\Omega^{n-1} S^n, \mathbb{Q}) \approx \begin{cases} \mathbb{Q} & , i = 0 \text{ or } i = 1, \\ 0 & , i > 1. \end{cases}$$

We consider the universal cover

$$T = \widetilde{\Omega^{n-1} S^n}$$

and the spectral seq.

$$E^2_{s,t} = H_s(\pi_1(\Omega^{n-1} S^n), H_t(T, \mathbb{Q}))$$

$$\Rightarrow H_{s+t}(\Omega^{n-1} S^n, \mathbb{Q}).$$

The group

$$\pi_1(\Omega^{n-1} S^n) \approx \pi_n(S^n) \approx \mathbb{Z}$$

acts trivially on $H_*(T, \mathbb{Q})$ (since $\Omega^{n-1} S^n$ is an H-space), so

$$E^2_{s,t} \approx \begin{cases} H_t(T, \mathbb{Q}), & s = 0 \text{ or } s = 1, \\ 0, & s \geq 2. \end{cases}$$

Hence, we can conclude that

$$H_i(T, \mathbb{Q}) = 0, \quad i > 0.$$

The (proof of the) Hurewicz thm shows

$$\pi_i(T) \otimes \mathbb{Q} = 0, \quad i > 0.$$

But for all $i \geq 2$,

$$\pi_i(T) \approx \pi_i(\Omega^{n-1} S^n) \approx \pi_{i+n-1}(S^n),$$

so $\pi_i(S^n) \otimes \mathbb{Q} = 0$, $i > n$. But $\pi_i(S^n)$ is a f.g. ab. group, so we conclude:

<u>Thm</u> (Serre) If n is odd, and if $m > n$, then $\pi_m(S^n)$ is a finite group. //

$$(E^r, d^r) \Rightarrow A$$

assume :

- $E^r_{p,q} = 0$ if $p<0$ or $q<0$ or both.

- there is an exact sequence

$$0 \to E^2_{p,0} \otimes E^2_{0,q} \to E^2_{p,q} \to \mathrm{Tor}_1^{\mathbb{Z}}(E^2_{p-1,0}, E^2_{0,q}) \to 0$$

=

$$\begin{array}{ccc} (E^r, d^r) & \Rightarrow & A \\ \downarrow f & & \downarrow f \\ (E'^r, d'^r) & \Rightarrow & A' \end{array}$$

Clearly :

$$\left.\begin{array}{ll} E^2_{p,0} \xrightarrow[\sim]{f} E'^2_{p,0} & , p \geq 0 \\ E^2_{0,q} \xrightarrow[\sim]{f} E'^2_{0,q} & , q \geq 0 \end{array}\right\} \Rightarrow$$

$$E^2_{p,q} \xrightarrow[\sim]{f} E'^2_{p,q} \quad , p,q \geq 0 \quad \Rightarrow$$

$$E^r_{p,q} \xrightarrow[\sim]{f} E'^r_{p,q} \quad p,q \geq 0, \quad 2 \leq r \leq \infty \Rightarrow$$

$$A_n \xrightarrow[\sim]{f} A'_n \quad n \geq 0.$$

Thm Suppose that two of following three maps are isomorphisms.

$$E^2_{p,0} \longrightarrow E'^2_{p,0} \quad p \geq 0$$

$$E^2_{0,q} \longrightarrow E'^2_{0,q} \quad q \geq 0$$

$$A_n \longrightarrow A'_n \quad n \geq 0.$$

Then so is the third.